SPECTRAL APPROXIMATION TO A TRANSMISSION EIGENVALUE PROBLEM AND ITS APPLICATIONS TO AN INVERSE PROBLEM

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Abstract. We first develop an efficient spectral-Galerkin method and an rigorous error analysis for the generalized eigenvalue problems associated to a transmission eigenvalue problem. Then, we present an iterative scheme, based on computation of the first transmission eigenvalue, to estimate the index of refraction of an inhomogeneous medium. We present ample numerical results to demonstrate the effectiveness and accuracy of our approach.

1. Introduction

We consider in this paper the interior transmission eigenvalue problem for the scattering of acoustic waves by a bounded inhomogeneous medium $D \subset \mathbb{R}^d$ ($d = 2, 3$): Find $k \in \mathbb{C},\ w, v \in L^2(D),\ w - v \in H_0^2(D)$ such that

$$\Delta w + k^2 n(x) w = 0, \qquad in\ D, \tag{1.1}$$

$$\Delta v + k^2 v = 0, \qquad in\ D, \tag{1.2}$$

$$w - v = 0, \qquad on\ \partial D, \tag{1.3}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \qquad on\ \partial D, \tag{1.4}$$

where ν is the unit outward normal to the boundary ∂D, and the index of refraction $n(x)$ is positive. In the above, k is called a transmission eigenvalue if, for such k, there exists a nontrivial solution (w, v) to (1.1)-(1.4). The above interior transmission problem arises in inverse scattering theory for inhomogeneous media, and the associated transmission eigenvalue problem plays an important role in inverse scattering theory [12, 11]. In particular, the transmission eigenvalues associated to

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the interior transmission problem can be used to estimate the material properties of the scattering object [8, 4, 5, 6, 11].

Mathematical properties of the transmission eigenvalue problem has been well studied in [26, 25, 10, 21, 9]. There exist also a few numerical investigations on how to approximate the transmission eigenvalues. In [13, 14, 20], finite element methods, including the Argyris element, continuous and mixed finite elements, were applied to solve the transmission eigenvalue problem. In [32], the author reformulated the coupled second-order problem (1.1)-(1.4) to a fourth-order problem, and developed a finite-element approximation to the fourth-order problem. On the other hand, we developed in [1] an efficient spectral-element method to compute the transmission eigenvalues from (1.1)-(1.4) for problems with a radially stratified media; and in [2], we combined the algorithms developed in [1] and [6] to construct a new algorithm for estimating the index of refraction. However, the algorithms developed in [1] and [2] are somewhat restricted to problems with radially stratified media, and due to the non-standard variational forms, we were only able to derive error estimates for the eigenvalue approximation in terms of errors for the eigenfunction approximation, whose estimates are still elusive. The main goal of this paper is three-fold: (i) developing an efficient spectral-Galerkin method to compute the eigenvalues of the generalized eigenvalue problem; (ii) deriving a rigorous error estimate using the spectral theory of completely continuous operators, for the eigenvalues and eigenfunctions of the generalized eigenvalue problem; and (ii) combining the algorithm for transmission eigenvalue problem and an iterative scheme to estimate the index of refraction of an inhomogeneous medium.

To simplify the presentation, we shall restrict ourselves to the case where D is a rectangular domain so an efficient spectral-Galerkin method can be used. However, the general framework we used here is, in principal, applicable to general domains with a conforming finite-element method for fourth-order problems [24].

We now briefly describe the contents in the remainder of the paper. In §2, we present the generalized eigenvalue problem, and its spectral-Galerkin approximation, associated to the transmission eigenvalue problem, and derive error estimates using the spectral theory of completely continuous operators. In §3, we describe an efficient implementation of the Spectral-Galerkin approximation. We present in §4 an iterative scheme to estimate the index of refraction. We then present in §5 several numerical results to demonstrate the accuracy and efficiency of our algorithms, and conclude with a summary.

2. Approximation of Transmission Eigenvalues

We first formulate (1.1)-(1.4) as a fourth-order eigenvalue problem, and then introduce a generalized eigenvalue problem, in the weak form and operator form, which will be used to determine the transmission eigenvalues. Then, we present a spectral-Galerkin method of the generalized eigenvalue problem and derive its error estimates.

2.1. A generalized eigenvalue problem associated to the transmission eigenvalue problem.

We first rewrite (1.1)-(1.4) as an equivalent fourth order eigenvalue problem. Define

$$H_0^2(D) = \{u \in H^2(D) : u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D\}.$$

Let $u = w - v \in H_0^2(D)$. Subtracting (1.2) from (1.1), we obtain

$$(\Delta + k^2)u = -k^2(n(x) - 1)w.$$

Dividing $n(x) - 1$ and applying $(\Delta + k^2 n(x))$ to both sides of the above equation, we obtain

$$(\Delta + k^2 n(x))\frac{1}{n(x) - 1}(\Delta + k^2)u = 0.$$

Then the weak formulation for the transmission eigenvalue problems can be stated as follows: find $(k^2 \neq 0, u) \in \mathbb{C} \times H_0^2(D)$ such that

$$\int_D \frac{1}{n(x) - 1}(\Delta u + k^2 u)(\Delta \bar{v} + k^2 n(x)\bar{v})dx = 0, \text{ for all } v \in H_0^2(D). \tag{2.1}$$

We now introduce an associated generalized eigenvalue problems (see [10, 32, 9]) for more details). Let us denote $\tau = k^2$ and $(\cdot, \cdot)_D$ be the $L^2(D)$ inner product. We define

$$A_\tau(u, v) := (\frac{1}{n(x) - 1}(\Delta u + \tau u), (\Delta v + \tau v))_D + \tau^2(u, v)_D,$$

for $n(x) > 1$, and

$$\begin{aligned}\tilde{A}_\tau(u, v) &:= (\frac{1}{1 - n(x)}(\Delta u + \tau n(x)u), (\Delta v + \tau n(x)v))_D + \tau^2(n(x)u, v)_D \\ &= (\frac{n(x)}{1 - n(x)}(\Delta u + \tau u), (\Delta v + \tau v))_D + (\Delta u, \Delta v)_D,\end{aligned}$$

for $n(x) < 1$. We also define

$$B(u, v) := (\nabla u, \nabla v)_D.$$

Then (2.1) can be written as either

$$A_\tau(u, v) - \tau B(u, v) = 0 \text{ for all } v \in H_0^2(D), \tag{2.2}$$

or

$$\tilde{A}_\tau(u,v) - \tau B(u,v) = 0 \text{ for all } v \in H_0^2(D). \tag{2.3}$$

The associated generalized eigenvalue problem is (cf. [9]): Find $\lambda(\tau) \in \mathbb{C}$ and $u \in H_0^2(D)$ such that

$$A_\tau(u,v) - \lambda(\tau)B(u,v) = 0 \text{ for all } v \in H_0^2(D), \tag{2.4}$$

or

$$\tilde{A}_\tau(u,v) - \lambda(\tau)B(u,v) = 0 \text{ for all } v \in H_0^2(D), \tag{2.5}$$

where $\lambda(\tau)$ is a continuous function of τ. From (2.2)-(2.3), we know that a transmission eigenvalue is the root of

$$f(\tau) := \lambda(\tau) - \tau. \tag{2.6}$$

Since we will use the real transmission eigenvalues to estimate the index of refraction, we will only consider the generalized eigenvalue problems in real Hilbert space.

Lemma 2.1. *Let $n(x) \in L^\infty(D)$ satisfying*

$$1 + \alpha \le n_* \le n(x) \le n^* < \infty, \tag{2.7}$$

or

$$0 < n_* \le n(x) \le n^* < 1 - \beta, \tag{2.8}$$

for some $\alpha > 0$ and $\beta > 0$ positive constants. Then A_τ or $\tilde{A}_\tau$ is a continuous and coercive sesquilinear form on $H_0^2(D) \times H_0^2(D)$, i.e.,

$$|A_\tau(u,v)| \text{ or } |\tilde{A}_\tau(u,v)| \le M\|u\|_{H^2(D)}\|v\|_{H^2(D)}, \tag{2.9}$$

$$A_\tau(u,u) \text{ or } \tilde{A}_\tau(u,u) \ge C_\tau\|u\|^2_{H^2(D)}, \tag{2.10}$$

where $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$, $M = \frac{1}{\alpha}(1+\tau)^2 + \tau^2$, C_τ is a constant depending on τ.

Proof. The proof of (2.10) is the same as in Section 3 of [10] (see also Lemma 2.1 in [32]), so we shall only sketch the proof for (2.9) in the case of $1 + \alpha \le n_* \le n(x) \le$

$n^* < \infty$. Indeed, we have

$$
\begin{aligned}
|A_\tau(u,v)| &= |(\frac{1}{n(x)-1}(\Delta u + \tau u), (\Delta v + \tau v))_D + \tau^2 (u,v)_D| \\
&= |(\frac{1}{n(x)-1}\Delta u, \Delta v)_D + \tau(\frac{1}{n(x)-1}\Delta u, v)_D \\
&+ \tau(\frac{1}{n(x)-1}u, \Delta v)_D + \tau^2(\frac{1}{n(x)-1}u, v)_D + \tau^2(u,v)_D| \\
&\le \|\frac{1}{n(x)-1}\Delta u\|_{L^2(D)}\|\Delta v\|_{L^2(D)} + \tau\|\frac{1}{n(x)-1}\Delta u\|_{L^2(D)}\|v\|_{L^2(D)} \\
&+ \tau\|\frac{1}{n(x)-1}u\|_{L^2(D)}\|\Delta v\|_{L^2(D)} + \tau^2\|\frac{1}{n(x)-1}u\|_{L^2(D)}\|v\|_{L^2(D)} + \\
&\tau^2\|u\|_{L^2(D)}\|v\|_{L^2(D)} \le \frac{1}{\alpha}\|\Delta u\|_{L^2(D)}\|\Delta v\|_{L^2(D)} + \frac{\tau}{\alpha}\|\Delta u\|_{L^2(D)}\|v\|_{L^2(D)} \\
&+ \frac{\tau}{\alpha}\| u\|_{L^2(D)}\|\Delta v\|_{L^2(D)} + \tau^2(\frac{1}{\alpha}+1)\|u\|_{L^2(D)}\|v\|_{L^2(D)} \\
&\le M\|u\|_{H^2(D)}\|v\|_{H^2(D)}.
\end{aligned}
$$

□

Thanks to the Poincaré inequality, it is obvious that $B(u,v)$ is a continuous and coercive bilinear form on $H_0^1(D) \times H_0^1(D)$, i.e.,

$$B(u,u) \ge M_1\|u\|^2_{H^1(D)}, \text{ for all } u \in H_0^1(D), \tag{2.11}$$

$$|B(u,v)| \le \|u\|_{H^1(D)}\|v\|_{H^1(D)}, \text{ for all } u,v \in H_0^1(D), \tag{2.12}$$

where M_1 is a constant.

Now we derive the operator formulation of generalized eigenvalue problem. For the sake of brevity, we only take (2.4) and (2.17) into account. For (2.5) and (2.18), we can treat similarly.

According to Lax-Milgram theorem, we can define the following bounded linear operators T:

$$A_\tau(Tu,v) = B(u,v), \text{ for all } u \in H_0^1(D), v \in H_0^2(D). \tag{2.13}$$

Lemma 2.2. *$T : H_0^1(D) \to H_0^1(D)$ and $T : H_0^2(D) \to H_0^2(D)$ are self-adjoint compact operators.*

Proof. By taking $v = Tu$ in (2.13), we can obtain

$$A_\tau(Tu,Tu) = B(u,Tu).$$

From Lemma 2.1 and (2.11)-(2.12), we can derive

$$\begin{aligned} C_\tau \|Tu\|^2_{H^2(D)} &\le A_\tau(Tu,Tu) = B(u,Tu) \\ &\le \|u\|_{H^1(D)}\|Tu\|_{H^1(D)} \\ &\le \|u\|_{H^1(D)}\|Tu\|_{H^2(D)}. \end{aligned}$$

Then we have

$$\|Tu\|_{H^2(D)} \le \frac{1}{C_\tau}\|u\|_{H^1(D)}. \tag{2.14}$$

Assuming E is a bounded set in $H^1_0(D)$, From (2.14) we know that TE is the bounded set in $H^2_0(D)$. Since $H^2_0(D)$ is compactly embedded in $H^1_0(D)$, then TE is a sequentially compact set in $H^1_0(D)$. So $T : H^1_0(D) \to H^1_0(D)$ is a compact operator.

On the other hand, assuming E is a bounded set in $H^2_0(D)$, then E is a sequentially compact set in $H^1_0(D)$. From (2.14) we know that TE is a compact set in $H^2_0(D)$. So $T : H^2_0(D) \to H^2_0(D)$ is a compact operator.

From Lemma 2.1 and (2.11)-(2.12), we know that $A_\tau(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are inner products in $H^2_0(D)$ and $H^1_0(D)$, respectively. From the symmetry of $A_\tau(\cdot,\cdot)$ and $B(\cdot,\cdot)$ and (2.13) we can derive

$$A_\tau(Tu,v) = B(u,v) = B(v,u) = A_\tau(Tv,u) = A_\tau(u,Tv), \text{ for all } u,v \in H^2_0(D),$$
$$B(Tu,v) = B(v,Tu) = A_\tau(Tv,Tu) = A_\tau(Tu,Tv) = B(u,Tv), \text{ for all } u,v \in H^1_0(D).$$

Hence, $T : H^2_0(D) \to H^2_0(D)$ and $T : H^1_0(D) \to H^1_0(D)$ are self-adjoint operators with inner products $A_\tau(\cdot,\cdot)$ and $B(\cdot,\cdot)$, respectively. □

From (2.13), we know that an equivalent operator formulation of (2.4) is:

$$Tu = \lambda(\tau)^{-1}u.$$

Let $(\lambda(\tau), u)$ be an eigenpair of (2.4), then $A_\tau(u,u) = \lambda(\tau)B(u,u)$. We then derive from Lemma 2.1 and (2.11)-(2.12) that

$$A_\tau(u,u) \ge C_\tau\|u\|^2_{H^2(D)} \ge C_\tau\|u\|^2_{H^1(D)} \ge C_\tau B(u,u). \tag{2.15}$$

Therefore,

$$\lambda(\tau) = \frac{A_\tau(u,u)}{B(u,u)} \ge C_\tau > 0. \tag{2.16}$$

Therefore, from the spectral theory of completely continuous operator we know that all eigenvalue of T are real and have finite algebraic multiplicity. We arrange the eigenvalues of T by increasing order:

$$0 < \lambda_1(\tau) \le \lambda_2(\tau) \le \lambda_3(\tau) \le \cdots \nearrow +\infty.$$

The eigenfunctions corresponding to two arbitrary different eigenvalues of T must be orthogonal. And there must exist a standard orthogonal basis with respect to $\|\cdot\|_{A_\tau} = \sqrt{A_\tau(u,u)}$ in eigenspace corresponding to the same eigenvalue. Hence, we can construct a complete orthonormal system of $H_0^2(D)$ by using the eigenfunctions of T corresponding to $\{\lambda_j(\tau)\}$.

2.2. **Spectral-Galerkin approximation and error estimates.** From (2.6) we know that the accuracy of transmission eigenvalue k depends on the accuracy of generalized eigenvalue $\lambda(\tau)$. Therefore, the effectiveness of the above method rests on having an efficient and robust algorithm for computing the generalized eigenvalue problem (2.4) or (2.5). For simplicity, we consider $D = (-1,1)^d$ $(d = 2,3,4,\cdots)$ and present below a Legendre-Galerkin approximation.

Let us denote

$$P_N =\{L_0(x), L_1(x), \cdots, L_N(x)\},\ S_N = P_N \cap H_0^2(I),\ X_N = S_N^d,$$

where $I = (-1,1)$ and $L_n(x)$ is the Legendre polynomial of degree n. Then the Legendre-Galerkin approximation of (2.4) is: Find $\lambda_N(\tau) \in \mathbb{C}, u_N \in X_N$ with $\|u_N\|_{L^2(D)} = 1$ such that

$$A_\tau(u_N, v_N) - \lambda_N(\tau)B(u_N, v_N) = 0 \text{ for all } v_N \in X_N. \tag{2.17}$$

The Legendre-Galerkin approximation of (2.5) is: Find $\lambda_N(\tau) \in \mathbb{C}, u_N \in X_N$ with $\|u_N\|_{L^2(D)} = 1$ such that

$$\tilde{A}_\tau(u_N, v_N) - \lambda_N(\tau)B(u_N, v_N) = 0 \text{ for all } v_N \in X_N. \tag{2.18}$$

Similarly as in the space continuous case, we can define the following bounded linear operators T_N:

$$A_\tau(T_N u, v) = B(u,v), \text{ for all } u \in H_0^1(D), v \in X_N. \tag{2.19}$$

It's obvious that $T_N : H_0^1(D) \to X_N$ and $T_N : H_0^2(D) \to X_N$ are all finite rank operators. From (2.19) we know that the equivalent operator formulation of (2.17) is :

$$T_N u_N = \lambda_N(\tau)^{-1} u_N.$$

Define the projection operator $\Pi_N^{2,0} : H_0^2(D) \to X_N$ satisfying

$$A_\tau(u - \Pi_N^{2,0} u, v) = 0, \text{ for all } u \in H_0^2(D), v \in X_N. \tag{2.20}$$

Lemma 2.3. *Let T and T_N be linear bounded operator defined by* (2.13) *and* (2.19), *respectively. Then the following equality holds:*

$$T_N = \Pi_N^{2,0} T.$$

Proof. For any $u \in H_0^1(D)$, $v \in X_N$, we have

$$A_\tau(\Pi_N^{2,0}Tu - T_N u, v) = A_\tau(\Pi_N^{2,0}Tu - Tu, v) + A_\tau(Tu - T_N u, v) = 0. \tag{2.21}$$

Taking $v = \Pi_N^{2,0}Tu - T_N u$, we obtain

$$A_\tau(\Pi_N^{2,0}Tu - T_N u, \Pi_N^{2,0}Tu - T_N u) = 0.$$

We find from (2.10) that

$$T_N = \Pi_N^{2,0}T.$$

□

It is clear that

$$T_N|_{X_N} : X_N \to X_N$$

is a self-adjoint finite rank operator with respect to the inner product $A_\tau(\cdot,\cdot)$, and the eigenvalues of (2.17) can be arranged as

$$0 < \lambda_{1N}(\tau) \le \lambda_{2N}(\tau) \le \lambda_{3N}(\tau) \le \cdots \le \lambda_{KN}(\tau) \ (K = \dim(X_N)).$$

Lemma 2.4. *Let* $(\lambda(\tau), u)$ *and* $(\lambda_N(\tau), u_N)$ *be the* k*-th eigenpair of* (2.4) *and* (2.17), *respectively. Then,*

$$\lambda_N(\tau) - \lambda(\tau) = \frac{\|u_N - u\|_{A_\tau}^2}{\|\nabla u_N\|_{L^2(D)}^2} - \lambda(\tau)\frac{\|\nabla(u_N - u)\|_{L^2(D)}^2}{\|\nabla u_N\|_{L^2(D)}^2}. \tag{2.22}$$

Proof. We derive from (2.4) that

$$\begin{aligned}
&A_\tau(u_N - u, u_N - u) - \lambda(\tau)B(u_N - u, u_N - u) = A_\tau(u_N, u_N)\\
&- 2A_\tau(u_N, u) + A_\tau(u, u) - \lambda(\tau)B(u_N, u_N) + 2\lambda(\tau)B(u_N, u)\\
&- \lambda(\tau)B(u, u) = A_\tau(u_N, u_N) - 2\lambda(\tau)B(u_N, u) + \lambda(\tau)B(u, u)\\
&- \lambda(\tau)B(u_N, u_N) + 2\lambda(\tau)B(u_N, u) - \lambda(\tau)B(u, u)\\
&= A_\tau(u_N, u_N) - \lambda(\tau)B(u_N, u_N)
\end{aligned}$$

Dividing $B(u_N, u_N)$ and applying (2.17) to both sides of the above equation, we obtain

$$\lambda_N(\tau) - \lambda(\tau) = \frac{\|u_N - u\|_{A_\tau}^2}{\|\nabla u_N\|_{L^2(D)}^2} - \lambda(\tau)\frac{\|\nabla(u_N - u)\|_{L^2(D)}^2}{\|\nabla u_N\|_{L^2(D)}^2}.$$

□

It is clear that we have

$$\eta_N(\tau) = \sup_{u \in H_0^2(D),\, \|u\|_{A_\tau}=1} \inf_{v \in X_N} \|Tu - v\|_{A_\tau} \to 0 \ (N \to \infty). \tag{2.23}$$

Theorem 2.1. *There holds*

$$\lim_{N\to\infty} \|T - T_N\|_{A_\tau} = 0. \tag{2.24}$$

Proof. By the definition of operator norm we have

$$\begin{aligned}\|T - T_N\|_{A_\tau} &= \sup_{u\in H_0^2(D),\, \|u\|_{A_\tau}=1} \|(T - T_N)u\|_{A_\tau} \\ &= \sup_{u\in H_0^2(D),\, \|u\|_{A_\tau}=1} \|Tu - \Pi_N^{2,0} Tu\|_{A_\tau} \\ &= \sup_{u\in H_0^2(D),\, \|u\|_{A_\tau}=1} \inf_{v\in X_N} \|Tu - v\|_{A_\tau} = \eta_N(\tau).\end{aligned}$$

Then the desired result follows from (2.23). □

Let $M(\lambda(\tau))$ denote the eigenfunctions space of (2.4) corresponding to the eigenvalue $\lambda(\tau)$.

Theorem 2.2. *Let* $(\lambda(\tau), u)$ *and* $(\lambda_N(\tau), u_N)$ *be the k-th eigenpair of* (2.4) *and* (2.17)*, respectively. Then there holds*

$$\|u - u_N\|_{A_\tau} \le \sup_{u\in M(\lambda(\tau)),\, \|u\|_{A_\tau}=1} \frac{C}{\lambda(\tau)} \|u - \Pi_N^{2,0} u\|_{A_\tau}, \tag{2.25}$$

$$\lambda_N(\tau) - \lambda(\tau) \le \sup_{u\in M(\lambda(\tau)),\, \|u\|_{A_\tau}=1} \frac{C}{\lambda(\tau)^2} \frac{\|u - \Pi_N^{2,0} u\|_{A_\tau}^2}{\|\nabla u_N\|_{L^2(D)}^2}, \tag{2.26}$$

where C is a constant independent of N, and it is different in different place.

Proof. From Theorem 2.1 we know that $\|T - T_N\|_{A_\tau} \to 0$ $(N \to \infty)$. According to Theorem 7.4 in [23], we have

$$\|u - u_N\|_{A_\tau} \le C\|(T - T_N)|_{M(\lambda(\tau))}\|_{A_\tau}. \tag{2.27}$$

Therefore, for any $u \in M(\lambda(\tau))$, $\|u\|_{A_\tau} = 1$, we deduce that

$$\|(T - T_N)u\|_{A_\tau} = \|Tu - \Pi_N^{2,0} Tu\|_{A_\tau} = \frac{1}{\lambda(\tau)} \|u - \Pi_N^{2,0} u\|_{A_\tau},$$

$$\|(T - T_N)|_{M(\lambda(\tau))}\|_{A_\tau} = \sup_{u\in M(\lambda(\tau)),\, \|u\|_{A_\tau}=1} \|(T - T_N)u\|_{A_\tau},$$

combining the above two relations with (2.27), we derive (2.25). By Lemma 2.4, we obtain

$$\lambda_N(\tau) - \lambda(\tau) \le \frac{\|u_N - u\|_{A_\tau}^2}{\|\nabla u_N\|_{L^2(D)}^2}. \tag{2.28}$$

We can then derive (2.26) from the above and (2.25). □

It remains to estimate $\|u_N - u\|_{A_\tau}$. To this end, we need to use the approximation results for generalized Jacobi polynomials with negative integers (cf. [17]). Let $J_n^{\alpha,\beta}(x)$ be the Jacobi polynomials which are orthogonal with respect to the Jacobi weight function $\omega^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$ over $I := (-1,1)$, namely,

$$\int_{-1}^{1} J_n^{\alpha,\beta}(x) J_m^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)dx = \gamma_n^{\alpha,\beta}\delta_{mn}, \tag{2.29}$$

where $\gamma_n^{\alpha,\beta} = \|J_n^{\alpha,\beta}\|^2_{\omega^{\alpha,\beta}}$.

The classical Jacobi polynomials are only defined for $\alpha,\beta > -1$. In [17], the range of Jacobi polynomials is extended to α or β being negative integers as follows:

$$J_n^{k,l}(x) = \begin{cases} (1-x)^{-k}(1+x)^{-l} J_{n-n_0}^{-k,-l}(x), & \text{if } k,l \le -1, \\ (1-x)^{-k} J_{n-n_0}^{-k,l}(x), & \text{if } k \le -1, l > -1, \\ (1+x)^{-l} J_{n-n_0}^{k,-l}(x), & \text{if } k > -1, l \le -1, \end{cases}$$

where $n \ge n_0$ with $n_0 := -(k+l), -k, -l$ for the above three cases, respectively. We note in particular that with this extension, (2.29) is still valid for α or β being negative integers.

We now define the d-dimensional tensorial generalized Jacobi polynomials and Jacobi weight functions as

$$\mathbf{J}_{\mathbf{n}}^{-\mathbf{2},-\mathbf{2}}(\mathbf{x}) = \prod_{j=1}^{d} \hat{J}_{n_j}^{-2,-2}(x_j),\ \omega^{-\mathbf{2},-\mathbf{2}}(\mathbf{x}) = \prod_{j=1}^{d} \omega^{-2,-2}(x_j),$$

where $\hat{J}_n^{-2,-2}$ are normalized generalized Jacobi polynomials, i.e., $\|\hat{J}_n^{-2,-2}\|_{L^2_{\omega^{-2,-2}}} = 1, \mathbf{n} = (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d$. Then the d-dimensional tensorial generalized Jacobi polynomials $\mathbf{J}_{\mathbf{n}}^{-\mathbf{2},-\mathbf{2}}(\mathbf{x})$ form a complete orthogonal system in $L^2_{\omega^{-\mathbf{2},-\mathbf{2}}}(I^d)$. Hence, we may define the d-dimensional polynomial space of degree N as

$$Q_N^{-\mathbf{2},-\mathbf{2}} := \text{span}\{\mathbf{J}_{\mathbf{n}}^{-\mathbf{2},-\mathbf{2}}(\mathbf{x}) : |\mathbf{n}|_\infty \le \mathbf{N} \ \text{ with } \ |\mathbf{n}|_\infty = \max_{\mathbf{1}\le \mathbf{j} \le \mathbf{d}} \mathbf{n_j}\}.$$

We now define the orthogonal projection: $\Pi_N^{-\mathbf{2},-\mathbf{2}} : L^2_{\omega^{-\mathbf{2},-\mathbf{2}}}(I^d) \to Q_N^{-\mathbf{2},-\mathbf{2}}$ by

$$\int_{I^d} (\Pi_N^{-\mathbf{2},-\mathbf{2}} u - u) v_N \omega^{-\mathbf{2},-\mathbf{2}} d\mathbf{x} = 0,\ \forall v_N \in Q_N^{-\mathbf{2},-\mathbf{2}},$$

and we define the d-dimensional non-uniformly Jacobi-weighted Sobolev space:

$$B^m_{-\mathbf{2},-\mathbf{2}}(I^d) := \{u : \partial_{\mathbf{x}}^{\mathbf{k}} \in L^2_{\omega^{\mathbf{k}-\mathbf{2},\mathbf{k}-\mathbf{2}}}(I^d), 0 \le |\mathbf{k}|_1 \le m\},$$

equipped with the norm and semi-norm

$$\|u\|_{B^m_{-\mathbf{2},-\mathbf{2}}(I^d)} := \Big(\sum_{0\le|\mathbf{k}|_1\le m} \|\partial^{\mathbf{k}}_{\mathbf{x}} u\|^2_{L^2_{\omega^{\mathbf{k}-\mathbf{2},\mathbf{k}-\mathbf{2}}}(I^d)}\Big)^{\frac{1}{2}},$$

$$|u|_{B^m_{-\mathbf{2},-\mathbf{2}}(I^d)} := \Big(\sum_{j=1}^{d} \|\partial^m_{x_j} u\|^2_{L^2_{\omega^{m-2,m-2}}(I^d)}\Big)^{\frac{1}{2}},$$

where

$$\mathbf{k} = (k_1, k_2, \cdots, k_d) \in \mathbb{N}^d,\ |\mathbf{k}|_1 = \sum_{j=1}^{d} k_j,\ \partial^{\mathbf{k}}_{\mathbf{x}} u = \partial^{k_1}_{x_1} \cdots \partial^{k_d}_{x_d} u.$$

From Theorem 8.1 and Remark 8.14 in [30] we have the following result:

Lemma 2.5. *For any $u \in B^m_{-\mathbf{2},-\mathbf{2}}(I^d)$, we have*

$$|\Pi^{-\mathbf{2},-\mathbf{2}}_N u - u|_{B^2_{-\mathbf{2},-\mathbf{2}}(I^d)} \le CN^{2-m}|u|_{B^m_{-\mathbf{2},-\mathbf{2}}(I^d)}.$$

We are now in position to prove the following results:

Theorem 2.3. *Let $(\lambda(\tau), u)$ and $(\lambda_N(\tau), u_N)$ be the kth eigenpair of (2.4) and (2.17), respectively. for any $u \in B^m_{-\mathbf{2},-\mathbf{2}}(I^d)$, we have*

$$\|u - u_N\|_{A_\tau} \le \sup_{u\in M(\lambda(\tau)),\|u\|_{A_\tau}=1} \frac{C}{\lambda(\tau)} N^{2-m}|u|_{B^m_{-\mathbf{2},-\mathbf{2}}(I^d)},$$

$$\lambda_N(\tau) - \lambda(\tau) \le \sup_{u\in M(\lambda(\tau)),\|u\|_{A_\tau}=1} \frac{C}{\lambda(\tau)^2\|\nabla u_N\|^2_{L^2(I^d)}} N^{2(2-m)}|u|^2_{B^m_{-\mathbf{2},-\mathbf{2}}(I^d)}.$$

Proof. From (2.9) and (2.20) we can derive

$$\begin{aligned}
\|u - \Pi^{2,0}_N u\|^2_{A_\tau} &= A_\tau(u - \Pi^{2,0}_N u, u - \Pi^{2,0}_N u)\\
&= \inf_{\phi_N\in X_N} A_\tau(u - \phi_N, u - \phi_N)\\
&\le M \inf_{\phi_N\in X_N} \|u - \phi_N\|^2_{H^2(I^d)}.
\end{aligned}$$

By using the Poincaré inequality we have that

$$\begin{aligned}
\|u - \Pi^{2,0}_N u\|^2_{A_\tau} &\le M \inf_{\phi_N\in X_N} \|u - \phi_N\|^2_{H^2(I^d)}\\
&\le CM \inf_{\phi_N\in X_N} |u - \phi_N|^2_{H^2(I^d)}\\
&\le CM|u - \Pi^{-\mathbf{2},-\mathbf{2}}_N u|^2_{H^2(I^d)}\\
&\le CM|\Pi^{-\mathbf{2},-\mathbf{2}}_N u - u|^2_{B^2_{-\mathbf{2},-\mathbf{2}}(I^d)}
\end{aligned}$$

From Lemma 2.5 we can obtain

$$\begin{aligned}
\|u - \Pi^{2,0}_N u\|^2_{A_\tau} &\le CM|\Pi^{-\mathbf{2},-\mathbf{2}}_N u - u|^2_{B^2_{-\mathbf{2},-\mathbf{2}}(I^d)}\\
&\le CN^{2(2-m)}|u|^2_{B^m_{-\mathbf{2},-\mathbf{2}}(I^d)}.
\end{aligned}$$

From the above and Theorem 2.2, we derive the desired results. □

3. Efficient implementation of the Legendre-Galerkin approximation

We describe in this section how to solve the problems (2.17) efficiently. Firstly, we start by constructing a set of basis functions for X_N.
Let $\phi_k(x) := d_k(L_k(x) - \frac{2(2k+5)}{2k+7}L_{k+2}(x) + \frac{2k+3}{2k+7}L_{k+4}), k = 0, 1, \cdots, N-4$, where $d_k = \frac{1}{\sqrt{2(2k+3)^2(2k+5)}}$. We recall that $\{\phi_k\}_{k=0}^{N-4}$ form a basis for S_N (see, e.g., [27]). In fact it is easy to see that $\phi_k(x)$ is proportional to $J_{k+4}^{-2,-2}(x)$. Let $A = (a_{kj})$ with $a_{kj}(\phi_j'', \phi_k'')$, $B = (b_{kj})$ with $b_{kj} = (\phi_j, \phi_k)$, and $C = (c_{kj})$ with $c_{kj} = (\phi_j', \phi_k')$, then it is easy to see that A is a diagonal matrix, B and C sparse matrices whose non-zeroes entries can be computed explicitly using the properties of Legendre polynomials.

Next we will consider the matrix form for the discrete scheme (2.17).

• Case $d = 2$: In this case, $X_N = \text{span}\{\phi_i(x)\phi_j(y) : i, j = 0, 1, \cdots, N-4\}$. Hence, we shall look for

$$u_N = \sum_{i,j=0}^{N-4} u_{ij}\phi_i(x)\phi_j(y). \tag{3.1}$$

Let us denote

$$U = \begin{pmatrix} u_{00} & u_{01} & \cdots & u_{0,N-4} \\ u_{10} & u_{11} & \cdots & u_{1,N-4} \\ \vdots & \vdots & \cdots & \vdots \\ u_{N-4,0} & u_{N-4,1} & \cdots & u_{N-4,N-4} \end{pmatrix}.$$

We use $\bar{u}$ to denote the vector formed by the columns of U. Now, plugging the expressions of (3.1) in (2.17), and taking v_N through all the basis functions in X_N, we can reduce the Legendre-Galerkin approximation to the system (2.17) in two dimension case to:

$$A_\tau \bar{u} = \lambda_N(\tau) B_\tau \bar{u}. \tag{3.2}$$

For constant n, we can deduce the matrix form based on the tensor-product for A_τ, B_τ , i.e.,

$$\begin{aligned} A_\tau =& \frac{1}{n-1}(B \otimes A + A \otimes B + 2C \otimes C) - \frac{2\tau}{n-1}(B \otimes C \\ &+ C \otimes B) + \frac{n}{n-1}\tau^2 B \otimes B, \\ B_\tau =& B \otimes C + C \otimes B, \\ A =& \{a_{ij}\}_{i,j=0}^{N-4}, \ B = \{b_{ij}\}_{i,j=0}^{N-4}, \ C = \{c_{ij}\}_{i,j=0}^{N-4}, \end{aligned}$$

and $\otimes$ is the tensor product operator.

• Case $d = 3$: Here, $X_N = \text{span}\{\phi_i(x)\phi_j(y)\phi_k(z), i, j, k = 0, 1, \cdots, N-4\}$. Hence, we shall look for

$$u_N = \sum_{i,j,k=0}^{N-4} u_{ijk}\phi_i(x)\phi_j(y)\phi_k(z). \tag{3.3}$$

Let us denote

$$U^k = \begin{pmatrix} u_{00}^k & u_{01}^k & \cdots & u_{0,N-4}^k \\ u_{10}^k & u_{11}^k & \cdots & u_{1,N-4}^k \\ \vdots & \vdots & \cdots & \vdots \\ u_{N-4,0}^k & u_{N-4,1}^k & \cdots & u_{N-4,N-4}^k \end{pmatrix}.$$

We use $\overline{U^k}$ to denote the vector formed by the columns of U^k. Let $U = (\overline{U^0}, \overline{U^1}, \ldots, \overline{U^{N-4}})$, and let $\bar{u}$ to denote the vector formed by the columns of U. Now, plugging the expressions of (3.3) in (2.17), and taking v_N through all the basis functions in X_N, we can obtain once again a matrix system of the form (3.2). For constant n, we can deduce the matrix form based on the tensor-product for A_τ, B_τ, i.e.,

$$\begin{aligned} A_\tau = & \frac{1}{n-1}(B \otimes B \otimes A + B \otimes A \otimes B + A \otimes B \otimes B + 2B \otimes C \otimes C \\ & + 2C \otimes B \otimes C + 2C \otimes C \otimes B) - \frac{2\tau}{n-1}(B \otimes B \otimes C + B \otimes C \otimes B \\ & + C \otimes B \otimes B) + \frac{n}{n-1}\tau^2 B \otimes B \otimes B; \\ B_\tau = & (B \otimes B \otimes C + B \otimes C \otimes B + C \otimes B \otimes B). \end{aligned}$$

If n is constant, A_τ and B_τ are all sparse and can be evaluated exactly using the properties of Legendre polynomials (cf. [27, 30] for more details). For general media n, the matrix A_τ and B_τ are usually full, and it is expensive to form them explicitly. However, their product with vectors can be efficiently computed. So when combined with a suitable matrix-free preconditioned iterative method, one can still solve (3.2) efficiently.

4. Estimation of the index of refraction

In this section, we shall present an algorithm, based on the work in [6] and [31], to estimate the index of refraction $n(x)$ using the first transmission eigenvalue.

Theorem 4.1. *Let $k_1(D, n)$ be the first transmission eigenvalue for* (2.1)*, and let α and β be positive constants. Denote by $k_1(D, \underline{n})$ and $k_1(D, \overline{n})$ the first transmission eigenvalue for* (2.1) *for $n(x) \equiv \underline{n}$ and $n(x) \equiv \overline{n}$ respectively. Then, we have the* *following results:*

(*i*) *if* $\overline{n} \geq n(x) \geq \underline{n} \geq \alpha > 1$, *then* $0 < k_1(D, \overline{n}) \leq k_1(D, n) \leq k_1(D, \underline{n})$;

(*ii*) *if* $0 < \underline{n} \leq n(x) \leq \overline{n} \leq 1 - \beta$, *then* $0 < k_1(D, \underline{n}) \leq k_1(D, n) \leq k_1(D, \overline{n})$.

The above results can be proved by using an argument similar to the proof of Theorem 3.3 in [6], namely, by replacing $\nabla \times \nabla \times u - \tau u$ by $\Delta u + \tau u$, and $\|\nabla \times u\|$ by $\|\nabla u\|$. The detail of the proof is left to the interested reader.

We recall that $k_1(D, n)$ can be estimated from far field data (cf., for instance, [31]). We also recall that a lower bound for $\sup_D n(x)$ can be estimated using the Faber-Krahn type inequality by ([11]):

$$\sup_D n(x) > \frac{\lambda_0(D)}{k_1^2(D, n)}, \tag{4.1}$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue.

We shall look for a constant n_0 minimizing the difference between $k_1(D, n_0)$ and $k_1(D, n)$. when $n > 1$ (the case of $0 < n < 1$ can be treated the same way), Theorem 4.1 shows that the transmission eigenvalues for n being a constant are monotonically decreasing with respect to n. Since $k_1(D, n)$ is a continuous function of n, we can estimate n_0 using the following algorithm, which is a slight modification of the algorithm presented in [31], such that the computed lowest transmission eigenvalue $k_1(D, n_0)$ coincides with the value $k_1(D, n)$ obtained from the far field data.

AlgorithmN $n_0 = algorithmN(k_1(D, n), tol)$
(i) Estimate an interval a and b using (4.1) such that $k_1(D, n)$ lies between $k_1(D, a)$ and $k_1(D, b)$
(ii) compute $k_1(D, a)$ and $k_1(D, b)$ by using the algorithm presented in the last section

```
while abs(a − b) > tol
        c = (a + b)/2 and compute k1(D, c)
    if k1(D, c) > k1(D, n) then
        a = c
    else
        b = c
    end if
end
n0 = c.
```

5. Numerical Results and summary

We present below some numerical results using the algorithms developed above.

5.1. **Generalized eigenvalue problem.** We consider first the approximate generalized eigenvalue problem (2.17) whose matrix form is (3.2).

Since one is mostly interested in a few smallest eigenvalues, it is most efficient to solve (3.2) using shifted inverse power method (cf., for instance, [16]) which requires solving, repeatedly for different righthand side $\bar{f}$,

$$A_\tau \bar{u} - \lambda_{\tau,a} B_\tau \bar{u} = \bar{f}, \tag{5.1}$$

where $\lambda_{\tau,a}$ is some approximate value for the eigenvalue λ_τ .

For problems with constant media n, the linear system (5.1) can be efficiently solved by using a capacitance matrix approach as in [27]. For problems with general media $n(x)$, we can use a preconditioned iterative method with a suitable constant coefficient problem as preconditioner.

Example 1. The generalized eigenvalue problem (2.17)

We choose $n(x) = 4+e^{(x_1+x_2)}$, $\tau = 8$, and $D = (-1/2, 1/2)\times(-1/2, 1/2)$, and use the system matrix with $n(x) = 4$ as the preconditioner. The first four eigenvalues computed with $tol = 10^{-6}$ are listed in Table 5.1.

N	number of iterations	1st	2nd	3rd	4th
N=10	21	11.59353	18.67219	19.69486	27.22014
N=15	21	11.59352	18.67198	19.69478	27.21861
N=20	21	11.59352	18.67198	19.69478	27.21861

TABLE 5.1. The first four eigenvalues for $n(x) = 4 + e^{(x_1+x_2)}$ and $\tau = 8$.

From Table 5.1, we observe that the eigenvalues achieve at least seven-digit accuracy with $N \le 15$.

5.2. **Transmission eigenvalue problems.** With an efficient algorithm for computing $\lambda_N(\tau)$, we can then compute the root of $f_N(\tau) := \lambda_N(\tau) - \tau$ by using a standard bisection method or secant method (cf. [32]).

Example 2. Transmission eigenvalues with constant $n(x)$

We first consider the case when the index of refraction is constant. Here we choose $n(x) = 16$ and D is a unit square given by $(-1/2, 1/2) \times (-1/2, 1/2)$. The first four transmission eigenvalues computed with $tol = 10^{-10}$ are listed in Table 5.2.

From Table 5.2, we observe that the approximate eigenvalues achieve about ten-digit accuracy with $N \le 25$. As a comparison, we list in Table 5.3 the results obtained by three different finite element methods in [13].

Next we consider again $n(x) = 16$ but with a three-dimensional unit cube $D = (-1/2, 1/2)^3$. When the iterative tolerance $tol \le 10^{-10}$, The first five eigenvalues

N	1st	$2nd$	3rd	4th
N=15	1.879591178	2.444236101	2.444236101	2.866439117
N=20	1.879591174	2.444236100	2.444236100	2.866439111
N=25	1.879591173	2.444236099	2.444236099	2.866439110
N=30	1.879591173	2.444236099	2.444236099	2.866439110

TABLE 5.2. The first four transmission eigenvalues for $n = 16$.

	1st	$2nd$	3rd	4th
Argyris method ($M = 2684$)	1.8651	2.4255	2.4271	2.8178
Continuous method ($M = 330$)	1.9094	2.5032	2.5032	2.9679
Mixed method ($M = 513$)	1.8954	2.4644	2.4658	2.8918

TABLE 5.3. The first four transmission eigenvalues for $n = 16$ computed by three finite element methods with M being the total number of unknowns.

computed with $tol = 10^{-10}$ are listed in Table 5.4. From Table 5.4, we observe that the approximate eigenvalues achieve about eight-digit accuracy with $N \leq 15$. However, we are not aware of other 3-D numerical results in the literature.

N	1st	$2nd$	3rd	4th	5th
N=10	2.067228335	2.584868678	2.584868678	2.584868678	2.987064163
N=15	2.067227678	2.584856763	2.584856763	2.584856763	2.987043164
N=20	2.067227672	2.584856755	2.584856755	2.584856755	2.987043138

TABLE 5.4. The first five transmission eigenvalues for $n = 16$ in the unit cube.

Example 3. Transmission eigenvalues with general $n(x)$

We choose $n(x) = 8 + x_1 - x_2$ and D to be the unit square.

The first four transmission eigenvalues computed with $tol = 10^{-10}$ are listed in Table 5.5.

N	1st	$2nd$	3rd	4th
N=15	2.820406823	3.535465075	3.536705877	4.121732935
N=20	2.820406804	3.535465073	3.536705874	4.121732916
N=25	2.820406802	3.535465071	3.536705873	4.121732915
N=30	2.820406802	3.535465071	3.536705872	4.121732914

TABLE 5.5. The first four transmission eigenvalues for $n(x) = 8 + x_1 - x_2$.

From Table 5.5, we observe that the approximate eigenvalues achieve about nine-digit accuracy with $N \leq 25$. As a comparison, we list in Table 5.6 the results obtained by a mixed finite element method in [20].

Domain	Index of refraction n	1st	2nd	3rd	4th
Unit square	$8 + x_1 - x_2$	2.8373	3.5632	3.5642	4.1582

TABLE 5.6. The first four transmission eigenvalues for $n(x) = 8 + x_1 - x_2$ computed in [20].

5.3. **Estimation of index of refraction.** Let $k_{1,D,n}$ be the first transmission eigenvalue measured from far field data, we aim to find a constant n_0, using the algorithm given in Section 4, that gives the same first transmission eigenvalue, i.e.,

$$k_{1,D,n_0} = k_{1,D,n},$$

and k_{1,D,n_0} is the first transmission eigenvalue for $n = n_0$.

We take $D = (-1/2, 1/2)^2$, so we have $\lambda_0(D) = 2\pi^2$. We consider two cases studied in [31]. In the first case, the unknown index of refraction is $n(x) = 16$. Given $k_{1,D,n} = 1.76$ (cf. [31]), we obtain from (4.1) that a lower bound for $\sup_D n(x)$ is given by 6.37. By using the algorithm given in Section 4, we can get $n_0 = 18.06189631$.

When $n(x)$ is not a constant, we can still seek a constant n_0 which approximates $n(x)$ using the algorithm in Section 4. Consider the case where the unknown index of refraction is $n(x) = 8 + x - y$. Given $k_{1,D,n} = 2.90$ (cf. [31]), we obtain from (4.1) that a lower bound for $\sup_D n(x)$ is given by 2.35. Then, by using the algorithm given in Section 4, we can get $n_0 = 7.684616595$.

These results are essentially in agreement with those presented in [31]. Note that the estimation of index of refraction depends essentially on the accuracy of the estimated $k_{1,D,n}$. With more accurate estimate on $k_{1,D,n}$, we will be able to get more accurate estimation of index of refraction.

5.4. **Summary.** We considered in this paper approximation of the transmission eigenvalue problems, and its application to an inverse problem of determining the index of refraction from far field data.

We presented first an efficient spectral-Galerkin method for computing the generalized eigenvalue problems associated to the transmission eigenvalue problem. By using the spectral theory of completely continuous operator, we derived rigorous error estimates for the approximate eigenvalues and eigenfunctions. This method enables us to compute the first few transmission eigenvalues efficiently with high accuracy. We then presented an algorithm which use the first transmission eigenvalue to estimate the index of refraction.

While we have restricted our attention in this paper to rectangular domains, the approach presented in this paper can be extended to more general domains by using a spectral-element method.

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