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Error estimate of Gauge–Uzawa methods for incompressible flows with variable density ☆

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ABSTRACT

In this paper, we construct a positivity-preserving Gauge–Uzawa method for the semi-discrete-in-time scheme of incompressible viscous flows with variable density, and establish its stability and error estimates. We also construct and implement a fully discrete scheme with finite elements in space and derive its positivity-preserving and stability result.

1. Introduction

This paper deals with the error estimates for a numerical approximation of the incompressible viscous flows with variable density, which are governed by the time-dependent Navier–Stokes equations:

$$\rho_t + u\cdot\nabla\rho = 0, \quad \text{in } \ \Omega\times(0,T], \tag{1.1}$$

$$\rho(u_t + u\cdot\nabla u) + \nabla p - \mu\Delta u = f, \quad \text{in } \ \Omega\times(0,T], \tag{1.2}$$

$$\nabla\cdot u = 0, \quad \text{in } \ \Omega\times(0,T], \tag{1.3}$$

supplemented by the following initial and boundary conditions for u and ρ:

$$\rho(x,0) = \rho_0(x), \qquad \rho(x,t)|_{\Gamma^-} = a(x,t), \tag{1.4}$$

$$u(x,0) = u_0(x), \qquad u(x,t)|_{\Gamma} = b(x,t), \tag{1.5}$$

and pressure mean-value $\int_\Omega p = 0$ where $\Gamma = \partial\Omega$ and Γ^- is the inflow boundary defined by $\Gamma^- = \{x\in\Gamma : \ u\cdot\vec{n} < 0\}$. In the above, the primitive variables are the (vector) velocity u, the (scalar) pressure p and the (scalar) density ρ, μ is the

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dynamic viscosity, Ω is an open bounded polyhedral domain in R^d, with $d = 2$ or 3. As in [1], we shall assume that the boundary is impermeable, i.e. $\Gamma^- = \phi$ and $b(x, t) = 0$.

It is challenging to construct stable and efficient numerical schemes for the system (1.1)–(1.3), since in addition to all the difficulties with the incompressible flows with constant density, it involves a transport equation for the density ρ which enforces that the mass density remains unchanged during the fluid motion [2] and preserves the positivity of the density.

It is now well established [3] that the difficulties associated with the incompressibility can be effectively handled by using a suitable projection type scheme, originally proposed by Chorin [4] and Temam [5], and very popular in the computational fluid dynamics community. Over the years, a large amount of efforts have been devoted to develop more accurate and efficient projection type schemes, we refer to [3] for comprehensive and up-to-date review on this subject. This approach has also been used in [6], among others, for incompressible flows with variable density. However, for the more accurate version which is based on the incremental projection scheme (i.e., the pressure-correction scheme) presented in [6], two projection steps (i.e., two pressure-Poisson solvers) are needed to preserve the stability of the scheme. Since the pressure-Poisson solver consumes a significant part of the total computational effort, this approach could increase the total computational cost significantly as opposed to the schemes with only one projection step.

On the other hand, the Gauge–Uzawa method[GUM] has been constructed in [7] to solve Navier–Stokes equations with variable density, through a Boussinesq approximation in [8] and directly in [9,10]. In [9] two Gauge–Uzawa schemes for incompressible flows with variable density is presented. It is shown in [9] that the GUM has many advantages over the original Gauge method and the pressure-correction method. More precisely, these two schemes only involve one projection step and have been proved unconditionally stable.

While the error analysis for the incompressible flows with constant density has been well studied (cf., for instance, [3] and the references therein), the case with variable densities is much more difficult and very few results are available. In [1] the error estimates for the momentum equation (1.2) are obtained under the assumption that the numerical density is obtained from another approach and remains to be bounded. On the other hand, no error estimates are given for the schemes presented in [9] and no fully discrete scheme was constructed for the convective Gauge–Uzawa scheme in [9].

The aim of this paper is to provide an error analysis for the Gauge–Uzawa schemes introduced in [9] without making bounded assumption on the numerical density. Based on the stability results for the two Gauge–Uzawa schemes for incompressible flows with variable density [9], we prove the density approximations for the semi-discrete schemes is L^∞ bounded and positivity-preserving, then we derive the corresponding error estimates for each variable (see Theorems 1 and 2). As far as we know, this is the first error estimates for the system (1.1)–(1.3). We also construct a new finite element algorithm for the convective Gauge–Uzawa scheme in [9] and prove that it is positivity-preserving and unconditionally stable (see Theorem 3).

The paper is organized as follows. In Section 2, we recall the convective GUM and the stability results in [9], then we prove that the numerical density of the schemes is L^∞ bounded and positivity-preserving. We will establish a first error estimate for the velocity in Section 3, followed by an improved error estimate for the velocity in Section 4 which allows us to derive an error estimate for the pressure in Section 5. In Section 6, we construct and analyze a new fully discrete algorithm for the convective Gauge–Uzawa scheme in [9]. Finally, some numerical experiments will be presented in Section 7.

We now describe some notations to be used hereafter. Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$). We denote by $H^s(\Omega)$ and $H^s_0(\Omega)$ the usual Sobolev spaces. We set $\mathbf{L}^2(\Omega) := (L^2(\Omega))^d$ and $\mathbf{H}^s(\Omega) := (H^s(\Omega))^d$, and denote by $L^2_0(\Omega)$ the subspace of $L^2(\Omega)$ of functions with vanishing mean-value. We use $\|\cdot\|_s$ to denote the norm in $H^s(\Omega)$ and $(\cdot, \cdot)$ to denote the L^2 inner product. For any sequence $\{v^n\}$, we denote $D_\tau v^n := \frac{v^n - v^{n-1}}{\tau}$.

2. The convective Gauge–Uzawa scheme and its properties

We first recall the convective GUM presented in [9].

Set $\rho^0 = \rho_0, u^0 = u_0, s^0 = 0$. Assuming ρ^n, u^n, s^n are known, we determine ρ^{n+1}, u^{n+1}, s^{n+1} as follows:

Step 1. Find ρ^{n+1} as the solution of

$$D_\tau \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} = 0. \tag{2.1}$$

Step 2. Find $\hat{u}^{n+1}$ as the solution of

$$\rho^n \frac{\hat{u}^{n+1} - u^n}{\tau} + \rho^{n+1}(u^n \cdot \nabla)\hat{u}^{n+1} + \mu \nabla s^n - \mu \Delta \hat{u}^{n+1} = f^{n+1}, \tag{2.2}$$

$$\hat{u}^{n+1}|_\Gamma = 0. \tag{2.3}$$

Step 3. Find ϕ^{n+1} as the solution of

$$-\nabla \cdot (\frac{1}{\rho^{n+1}} \nabla \phi^{n+1}) = \nabla \cdot \hat{u}^{n+1}, \tag{2.4}$$

$$\partial_n \phi^{n+1}|_\Gamma = 0. \tag{2.5}$$

Step 4. Update u^{n+1} and s^{n+1} by

$$u^{n+1} = \hat{u}^{n+1} + \frac{1}{\rho^{n+1}} \nabla \phi^{n+1}, \tag{2.6}$$

$$s^{n+1} = s^n - \nabla \cdot \hat{u}^{n+1}. \tag{2.7}$$

where

$$p^{n+1} = \mu s^{n+1} - \frac{1}{\tau} \phi^{n+1}. \tag{2.8}$$

For the sake of simplicity, we shall consider only the homogeneous Dirichlet boundary condition for the velocity, i.e. $u|_\Gamma = 0$. It is shown in [9, Theorem 3.1] that the following results hold:

Lemma 1. *The convective GUM (2.1)–(2.7) is unconditionally stable in the sense that, for all $\tau > 0$ and $1 \le N \le T/\tau$, the following a priori bounds hold:*

$$\|\rho^N\|^2 + \sum_{k=0}^{N-1} \|\rho^{k+1} - \rho^k\|^2 = \|\rho^0\|^2, \tag{2.9}$$

and

$$\begin{aligned} &\|\sigma^N \hat{u}^N\|^2 + \mu\tau \|s^N\|^2 + \sum_{k=0}^{N-1} \left(\|\sigma^k (\hat{u}^{k+1} - u^k)\|^2 + \|\frac{1}{\sigma^k} \nabla \phi^k\|^2 + \frac{\mu}{2} \tau \|\nabla \hat{u}^{k+1}\|^2 \right) \\ &\quad \le \|\sigma^0 \hat{u}^0\|^2 + C\mu\tau \sum_{k=0}^{N-1} \|f^{k+1}\|_{-1}^2, \end{aligned} \tag{2.10}$$

where $\sigma = \sqrt{\rho}$.

However, for the above results to make sense, it implicitly assumes that $\rho^n > 0$ for all n. Below, we shall first show that this is indeed true.

Lemma 2. *Assume that there exists two constants $c, C > 0$ such that $c \le \rho^0(x) \le C, \ \forall x \in \bar{\Omega}$. Then, the numerical density ρ^n determined from (2.1) satisfies:*

$$c \le \rho^n(x) \le C, \quad \forall x \in \bar{\Omega}, \ \forall n. \tag{2.11}$$

Proof. We prove the boundedness of ρ^n by mathematical induction. When $n = 0$, $\rho^0 = \rho_0$ satisfies (2.11). In the following we shall prove that if (2.11) holds for $0 \le n \le m$, then it also holds for $n = m+1$. On one hand, assume that ρ^{m+1} achieves the maximum at $x^* \in \Omega$, then it follows that $\nabla \rho^{m+1}(x^*) = 0$, which from (2.1) implies that

$$\rho^{m+1}(x^*) = \rho^m(x^*) \le C. \tag{2.12}$$

Similarly it holds that if ρ^{m+1} achieves the minimum at $x^{**} \in \Omega$, then

$$\rho^{m+1}(x^{**}) \ge c. \tag{2.13}$$

On the other hand, if ρ^{m+1} achieves the maximum at $x^* \in \Gamma$, it follows that the tangential component of $\nabla \rho^{m+1}(x^*)$ and the normal component of $u^m(x^*)$ both vanish by using (2.3), (2.5) and (2.6). Since we can divide both $\nabla \rho^{m+1}(x^*)$ and $u^m(x^*)$ into their tangential and normal components, by the computation of the inner product we further have $\left(u^m \cdot \nabla \rho^{m+1}\right)(x^*) = 0$. By (2.1) again, it implies (2.12) is still true. We then obtain (2.13) is also true if ρ^{m+1} achieves the minimum at $x^{**} \in \Gamma$. Consequently, this completes the mathematical induction and we have the desired result (2.11) for any n. □

An immediate consequence of (2.10) and (2.11) is:

$$\|u^N\| \le C \quad \forall n. \tag{2.14}$$

3. A first error estimate

Our purpose in this section is to show that u^n and $\hat{u}^n$ are both order 1/2 approximations to $u(t_n)$ in $\mathbf{L}^2(\Omega)$, which is the same as the constant density case [7,11].

We first recall some inequalities that will be used in the sequel:

$$((u \cdot \nabla)v, w) \le C \begin{cases} \|u\|_1 \|v\|_1 \|w\|_1, \\ \|u\|_2 \|\nabla v\| \|w\|, \\ \|u\| \|v\|_2 \|\nabla w\|, \\ \|\nabla u\| \|v\|_2 \|w\|, \\ \|u\|_2 \|v\| \|\nabla w\|. \end{cases} \tag{3.1}$$

$$\|\operatorname{div} v\| \le \|\nabla v\| \quad \forall v \in H^1(\Omega) \text{ and } v\cdot n|_\Gamma = 0. \tag{3.2}$$

$$(u\cdot\nabla v, v) = 0 \quad \forall u \in H := \{u \in L^2(\Omega)^d : \nabla\cdot u = 0, u\cdot n|_\Gamma = 0\} \text{ and } v \in H^1(\Omega)^d. \tag{3.3}$$

Let us denote

$$\hat{e}_u^n = u(t_n) - \hat{u}^n, \qquad e_u^n = u(t_n) - u^n, \qquad e_\rho^n = \rho(t_n) - \rho^n, \qquad e_p^n = p(t_n) - p^n. \tag{3.4}$$

Before showing the error estimates, we list the following useful formulae which can be proved directly using (2.3), (2.4) and (2.6):

$$\hat{e}_u^n = 0, \quad \text{on } \partial\Omega, \tag{3.5}$$

$$\operatorname{div} e_u^n = 0, \qquad \hat{e}_u^{n+1} = e_u^{n+1} + \frac{1}{\rho^{n+1}}\nabla\phi^{n+1}, \quad \text{in } \Omega, \tag{3.6}$$

$$(e_u^n, \nabla\phi^n) = 0, \qquad (\rho^{n+1}\hat{e}_u^m, e_u^n) = (\rho^{n+1}e_u^m, e_u^n), \quad \text{in } \Omega. \tag{3.7}$$

Furthermore, we also have

$$\|e_u^{n+1}\|_1^2 \le C\left(\|\hat{e}_u^{n+1}\|_1^2 + \|\frac{1}{\rho^{n+1}}\nabla\phi^{n+1}\|_1^2\right), \tag{3.8}$$

$$\begin{aligned}\|\sigma^n e_u^n\|^2 &= (\rho^n e_u^n, e_u^n) = (\rho^n\hat{e}_u^n - \nabla\phi^n, e_u^n) = (\rho^n\hat{e}_u^n, \hat{e}_u^n - \frac{1}{\rho^n}\nabla\phi^n)\\ &= \|\sigma^n\hat{e}_u^n\|^2 - (e_u^n + \frac{1}{\rho^n}\nabla\phi^n, \nabla\phi^n) = \|\sigma^n\hat{e}_u^n\|^2 - \|\frac{1}{\sigma^n}\nabla\phi^n\|^2.\end{aligned} \tag{3.9}$$

We first show that the semi-discrete solution u^{n+1} converges to $u(t_{n+1})$ with order 1/2. More precisely, we have the following lemma.

Lemma 3. *Assuming the exact solutions of the problem (1.1)–(1.3) have the following regularity:*

$$\rho \in L^\infty(W^{1,\infty}) \cap W^{2,\infty}(L^2), \quad u \in L^\infty(\mathbf{H}^2) \cap W^{1,\infty}(\mathbf{L}^2) \cap W^{2,\infty}(\mathbf{H}^{-1}), \quad p \in L^\infty(H^1), \tag{3.10}$$

we have

$$\begin{aligned}&\|e_\rho^N\|^2 + \|\hat{e}_u^N\|^2 + \|e_u^N\|^2 + \mu\tau\|s^N\|^2 + \tau\mu\sum_{n=0}^{N-1}\left(\|\nabla e_u^{n+1}\|^2 + \|\nabla\hat{e}_u^{n+1}\|^2\right)\\ &+\sum_{n=0}^{N-1}\left(\|e_\rho^{n+1} - e_\rho^n\|^2 + \|e_u^{n+1} - e_u^n\|^2 + \|\frac{1}{\sigma^{n+1}}\nabla\phi^{n+1}\|^2\right) \le C\tau, \quad \forall 1 \le N \le T/\tau.\end{aligned} \tag{3.11}$$

Proof. Subtracting (2.1)–(2.2) from (1.1)–(1.2) respectively, it is easy to see that e_ρ^n, e_u^n satisfies

$$D_\tau e_\rho^{n+1} = -u^n\cdot\nabla e_\rho^{n+1} - \nabla\rho(t_{n+1})\cdot e_u^n + R_\rho^{n+1}, \tag{3.12}$$

and

$$\begin{aligned}\rho^n\left(\frac{\hat{e}_u^{n+1} - e_u^n}{\tau}\right) - \mu\Delta\hat{e}_u^{n+1} = &-u(t_{n+1})\cdot\nabla u(t_{n+1})e_\rho^{n+1} - \rho^{n+1}e_u^n\cdot\nabla\hat{u}^{n+1} - \rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1}\\ &+\mu\nabla s^n - \nabla p(t_{n+1}) - \rho^{n+1}\big(u(t_{n+1}) - u(t_n)\big)\cdot\nabla\hat{u}^{n+1} + R_u^{n+1}\end{aligned} \tag{3.13}$$

where R_ρ^n and R_u^n are truncation errors which satisfy

$$\|R_\rho^{n+1}\| = \| - \nabla\rho(t_{n+1})\cdot\big(u(t_{n+1}) - u(t_n)\big) + D_\tau\rho(t_{n+1}) - \rho_t(t_{n+1})\| \le C\tau, \tag{3.14}$$

and

$$\begin{aligned}\|R_u^{n+1}\|_{-1} = &\|\rho^n D_\tau u(t_{n+1}) - \rho(t_{n+1})u_t(t_{n+1})\|_{-1} = \| -(\rho(t_{n+1}) - \rho(t_n))u_t(t_{n+1})\\ &+\rho^n(D_\tau u(t_{n+1}) - u_t(t_{n+1})) - e_\rho^n u_t(t_{n+1})\,\|_{-1} \le C(\tau + \|e_\rho^n\|).\end{aligned} \tag{3.15}$$

We now take inner product of (3.12) with $2\tau e_\rho^{n+1}$ and use the identity

$$(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2. \tag{3.16}$$

Then from (3.3) we have

$$\begin{aligned} &\|e_\rho^{n+1}\|^2 - \|e_\rho^n\|^2 + \|e_\rho^{n+1} - e_\rho^n\|^2 \\ =& -2\tau(u^n \cdot \nabla e_\rho^{n+1}, e_\rho^{n+1}) - 2\tau(\nabla\rho(t_{n+1}) \cdot e_u^n, e_\rho^{n+1}) + 2\tau(R_\rho^{n+1}, e_\rho^{n+1}) \\ =& -2\tau(\nabla\rho(t_{n+1}) \cdot e_u^n, e_\rho^{n+1}) + 2\tau(R_\rho^{n+1}, e_\rho^{n+1}). \end{aligned} \tag{3.17}$$

Taking inner product of (3.13) with $2\tau\hat{e}_u^{n+1}$, then by (3.9) we derive

$$\begin{aligned} &\|\sigma^n\hat{e}_u^{n+1}\|^2 - \|\sigma^n e_u^n\|^2 + \|\sigma^n(\hat{e}_u^{n+1} - e_u^n)\|^2 + 2\tau\mu\|\nabla\hat{e}_u^{n+1}\|^2 \\ =& -2\tau\left(u(t_{n+1}) \cdot \nabla u(t_{n+1})e_\rho^{n+1}, \hat{e}_u^{n+1}\right) - 2\tau(\rho^{n+1}e_u^n \cdot \nabla\hat{u}^{n+1}, \hat{e}_u^{n+1}) \\ & - 2\tau(\rho^{n+1}u(t_{n+1}) \cdot \nabla\hat{e}_u^{n+1}, \hat{e}_u^{n+1}) + 2\mu\tau(\nabla s^n, \hat{e}_u^{n+1}) - 2\tau(\nabla p(t_{n+1}), \hat{e}_u^{n+1}) \\ & - 2\tau\left(\rho^{n+1}\left(u(t_{n+1}) - u(t_n)\right) \cdot \nabla\hat{u}^{n+1}, \hat{e}_u^{n+1}\right) + 2\tau\left(R_u^{n+1}, \hat{e}_u^{n+1}\right). \end{aligned} \tag{3.18}$$

We take the inner product of (2.1) with a scalar function $\tau\hat{e}_u^{n+1} \cdot \hat{e}_u^{n+1}$ to get

$$(\rho^{n+1} - \rho^n, \hat{e}_u^{n+1} \cdot \hat{e}_u^{n+1}) = -\tau(\nabla \cdot (\rho^{n+1}u^n), \hat{e}_u^{n+1} \cdot \hat{e}_u^{n+1}), \tag{3.19}$$

which can be rewritten as

$$\|\sigma^{n+1}\hat{e}_u^{n+1}\|^2 - \|\sigma^n\hat{e}_u^{n+1}\|^2 - 2\tau(\rho^{n+1}u^n \cdot \nabla\hat{e}_u^{n+1}, \hat{e}_u^{n+1}) = 0. \tag{3.20}$$

We combine (3.17), (3.18) with (3.20) to obtain

$$\begin{aligned} &\|e_\rho^{n+1}\|^2 - \|e_\rho^n\|^2 + \|e_\rho^{n+1} - e_\rho^n\|^2 + \|\sigma^{n+1}\hat{e}_u^{n+1}\|^2 \\ &\quad -\|\sigma^n e_u^n\|^2 + \|\sigma^n(\hat{e}_u^{n+1} - e_u^n)\|^2 + 2\tau\mu\|\nabla\hat{e}_u^{n+1}\|^2 = \sum_{i=1}^{5} A_i, \end{aligned} \tag{3.21}$$

with

$$\begin{aligned} A_1 &= -2\tau(\nabla\rho(t_{n+1}) \cdot e_u^n, e_\rho^{n+1}) - 2\tau\left(u(t_{n+1}) \cdot \nabla u(t_{n+1})e_\rho^{n+1}, \hat{e}_u^{n+1}\right), \\ A_2 &= -2\tau(\nabla p(t_{n+1}), \hat{e}_u^{n+1}), \\ A_3 &= 2\tau(R_\rho^{n+1}, e_\rho^{n+1}) + 2\tau\left(R_u^{n+1}, \hat{e}_u^{n+1}\right), \\ A_4 &= 2\mu\tau(\nabla s^n, \hat{e}_u^{n+1}), \\ A_5 &= -2\tau(\rho^{n+1}(u(t_{n+1}) - u(t_n)) \cdot \nabla u(t_{n+1}), \hat{e}_u^{n+1}) - 2\tau(\rho^{n+1}e_u^n \cdot \nabla u(t_{n+1}), \hat{e}_u^{n+1}). \end{aligned}$$

We now analyze each term on the right hand side of (3.21) as follows. First, we have

$$A_1 \le C\tau(\|e_u^n\|^2 + \|e_\rho^{n+1}\|^2) + \frac{1}{4}\mu\tau\|\nabla\hat{e}_u^{n+1}\|^2.$$

From (3.6), we derive

$$A_2 \le C\tau^2 + \varepsilon\|\frac{1}{\rho^{n+1}}\nabla\phi^{n+1}\|^2.$$

By (3.14) and (3.15),

$$A_3 \le C\tau^3 + \frac{1}{4}\mu\tau\|\nabla\hat{e}_u^{n+1}\|^2 + C\tau(\|e_\rho^n\|^2 + \|e_\rho^{n+1}\|^2).$$

Making use of (2.7), (3.2) and (3.4),

$$\begin{aligned} A_4 &= 2\mu\tau(\nabla s^n, \hat{e}_u^{n+1}) = -2\mu\tau(s^n, \nabla \cdot (u(t_{n+1}) - \hat{u}^{n+1})) = 2\mu\tau(s^n, \nabla \cdot \hat{u}^{n+1}) \\ &= 2\mu\tau(s^n, s^n - s^{n+1}) = \mu\tau\left(\|s^n\|^2 - \|s^{n+1}\|^2 + \|s^n - s^{n+1}\|^2\right) \\ &\le \mu\tau(\|s^n\|^2 - \|s^{n+1}\|^2) + \mu\tau\|\nabla\hat{e}_u^{n+1}\|^2. \end{aligned}$$

As a consequence of (2.11) and (3.1),

$$A_5 \le C\tau^3 + C\tau\|e_u^n\|^2 + \frac{1}{4}\mu\tau\|\nabla\hat{e}_u^{n+1}\|^2.$$

Inserting the above estimates for each term A_i, $1 \le i \le 5$ into (3.21) and in view of (2.11) and (3.9), by choosing sufficiently small ε we have

$$\begin{aligned} &\|e_\rho^{n+1}\|^2 - \|e_\rho^n\|^2 + \|e_\rho^{n+1} - e_\rho^n\|^2 + \|\sigma^{n+1} e_u^{n+1}\|^2 - \|\sigma^n e_u^n\|^2 + \|\sigma^n (e_u^{n+1} - e_u^n)\|^2 \\ &\quad + \|\frac{1}{\sigma^{n+1}} \nabla \phi^{n+1}\|^2 + \mu\tau(\|s^{n+1}\|^2 - \|s^n\|^2) + \frac{1}{4}\tau\mu\|\nabla \hat{e}_u^{n+1}\|^2 \\ &\quad \le C\tau(\|e_u^n\|^2 + \|e_\rho^{n+1}\|^2 + \|e_\rho^n\|^2 + \|e_u^{n+1}\|^2) + C\tau^2. \end{aligned} \tag{3.22}$$

Summing (3.22) over n from 0 to $N-1$ gives

$$\begin{aligned} &\|e_\rho^N\|^2 + c\|e_u^N\|^2 + \mu\tau\|s^N\|^2 + \sum_{n=0}^{N-1}\big(\|e_\rho^{n+1} - e_\rho^n\|^2 + \|e_u^{n+1} - e_u^n\|^2 \\ &+ \|\frac{1}{\sigma^{n+1}}\nabla\phi^{n+1}\|^2 + \frac{1}{2}\tau\mu\|\nabla\hat{e}_u^{n+1}\|^2\big) \\ \le\ & \|e_\rho^0\|^2 + C\|e_u^0\|^2 + \mu\tau\|s^0\|^2 + C\tau\sum_{n=0}^{N}(\|e_u^n\|^2 + \|e_\rho^n\|^2) + C\tau, \end{aligned} \tag{3.23}$$

where we have used (2.11). If we choose the initial data as the time discrete approximation at the initial time, i.e. $e_\rho^0 = e_u^0 = 0$, then by applying the discrete Gronwall lemma to (3.23), we have

$$\begin{aligned} &\|e_\rho^N\|^2 + \|e_u^N\|^2 + \mu\tau\|s^N\|^2 \\ &\quad + \sum_{n=0}^{N-1}\big(\|e_\rho^{n+1} - e_\rho^n\|^2 + \|e_u^{n+1} - e_u^n\|^2 + \|\frac{1}{\sigma^{n+1}}\nabla\phi^{n+1}\|^2 + \frac{1}{2}\tau\mu\|\nabla\hat{e}_u^{n+1}\|^2\big) \le C\tau, \end{aligned}$$

for all $0 \le N \le T/\tau$. Finally, thanks to (3.8) and (3.9), we also have

$$\|\hat{e}_u^N\|^2 + \tau\sum_{n=0}^{N-1}\mu\|\nabla e_u^{n+1}\|^2 \le C\tau, \quad \forall 1 \le N \le T/\tau, \tag{3.24}$$

which completes the proof. □

4. Optimal error estimate for velocity in $L^2(\mathbf{L}^2)$

We first introduce the following subspace of $H^1(\Omega)$:

$$V(\Omega) := \{z \in H_0^1(\Omega) : \operatorname{div} z = 0\},$$

and denote by $V(\Omega)^*$ the dual space of $V(\Omega)$ with $\|\cdot\|_*$ to be the corresponding norm. We consider the stationary Stokes equations which will be used in a duality argument:

$$\begin{aligned} &-\Delta v + \nabla q = g, \quad \text{in } \Omega, \\ &\operatorname{div} v = 0, \quad \text{in } \Omega, \\ &v = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

In the following we assume that the unique solution $(v, q) \in H_0^1(\Omega) \times L_0^2(\Omega)$ of the problem (4.1) satisfies

$$\|v\|_2 + \|q\|_1 \le C\|g\|. \tag{4.2}$$

We notice that this assumption is valid provided $\partial\Omega$ is of class C^2, or if Ω is a convex polygonal domain [7]. Under this assumption, it immediately follows that

$$\|g\|_* \le \sup_{w\in V}\frac{(g, w)}{\|w\|_1} = \sup_{w\in V}\frac{(-\Delta v, w)}{\|w\|_1} \le \|\nabla v\|. \tag{4.3}$$

Theorem 1. *Under the same assumptions as Lemma 3, then there exists a constant $C > 0$ such that for all $1 \le N \le T/\tau$*

$$\|e_u^N\|_*^2 + \|e_\rho^N\|^2 + \sum_{n=0}^{N-1}\left\{\|e_u^{n+1} - e_u^n\|_*^2 + \|e_\rho^{n+1} - e_\rho^n\|^2 + \tau\mu\|e_u^{n+1}\|^2 + \tau\mu\|\hat{e}_u^{n+1}\|^2\right\} \le C\tau^2. \tag{4.4}$$

Proof. For all $1 \le n \le N$, let (v^n, q^n) be solutions of the system (4.1) with $g = \rho^n e_u^n$, then from (4.2) and (2.11) it follows that

$$\|v^n\|_2 + \|q^n\|_1 \le C\|\rho^n e_u^n\| \le C\|e_u^n\|. \tag{4.5}$$

Now taking $0 \le n \le N-1$, since v^{n+1} is divergence free, then we have from (3.6)

$$\begin{aligned}(\rho^n(\hat{e}_u^{n+1} - e_u^n), v^{n+1}) &= -((\rho^{n+1} - \rho^n)\hat{e}_u^{n+1}, v^{n+1}) + (\rho^{n+1}e_u^{n+1} - \rho^n e_u^n, v^{n+1}) \\ &= -((\rho^{n+1} - \rho^n)\hat{e}_u^{n+1}, v^{n+1}) + (\nabla(v^{n+1} - v^n), \nabla v^{n+1}),\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}-\mu(\Delta\hat{e}_u^{n+1}, v^{n+1}) &= -\mu(\hat{e}_u^{n+1}, \Delta v^{n+1}) = -\mu(\hat{e}_u^{n+1}, -\rho^{n+1}e_u^{n+1} + \nabla q^{n+1}) \\ &= \mu(\|\sigma^{n+1}e_u^{n+1}\|^2 + (e_u^{n+1}, \nabla\phi^{n+1}) - (e_u^{n+1}, \nabla q^{n+1}) - (\frac{1}{\rho^{n+1}}\nabla\phi^{n+1}, \nabla q^{n+1})) \\ &= \mu(\|\sigma^{n+1}e_u^{n+1}\|^2 - (\frac{1}{\rho^{n+1}}\nabla\phi^{n+1}, \nabla q^{n+1})).\end{aligned} \tag{4.7}$$

Taking the inner product of (3.13) with $2\tau v^{n+1}$ and using (4.6)–(4.7), we combine the resultant equality with (3.17) to get

$$\begin{aligned}&\|\nabla v^{n+1}\|^2 - \|\nabla v^n\|^2 + \|\nabla(v^{n+1} - v^n)\|^2 + 2\mu\tau\|\sigma^{n+1}e_u^{n+1}\|^2 + \|e_\rho^{n+1}\|^2 - \|e_\rho^n\|^2 + \|e_\rho^{n+1} - e_\rho^n\|^2 \\ &= 2\mu\tau(\frac{1}{\rho^{n+1}}\nabla\phi^{n+1}, \nabla q^{n+1}) + \left(2((\rho^{n+1} - \rho^n)\hat{e}_u^{n+1}, v^{n+1}) - 2\tau\left(\rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}\right)\right) \\ &\quad - 2\tau\left(u(t_{n+1})\cdot\nabla u(t_{n+1})e_\rho^{n+1}, v^{n+1}\right) - 2\tau\left(\rho^{n+1}e_u^n\cdot\nabla\hat{u}^{n+1}, v^{n+1}\right) \\ &\quad -2\tau\left(\rho^{n+1}\big(u(t_{n+1}) - u(t_n)\big)\cdot\nabla\hat{u}^{n+1}, v^{n+1}\right) + 2\tau(R_u^{n+1}, v^{n+1}) \\ &\quad - 2\tau(\nabla\rho(t_{n+1})\cdot e_u^n, e_\rho^{n+1}) + 2\tau(R_\rho^{n+1}, e_\rho^{n+1}) = \sum_{i=1}^{8} B_i.\end{aligned} \tag{4.8}$$

We now estimate B_1 to B_8 separately. First, from (2.11) and (4.5) we obtain

$$B_1 \le C\mu\tau\|\nabla\phi^{n+1}\|^2 + \frac{\mu\tau}{6}\|\sigma^{n+1}e_u^{n+1}\|^2.$$

In view of (2.1) and $\operatorname{div} u^n = \operatorname{div} u(t^n) = 0$, then by integration by parts we arrive at

$$\begin{aligned}B_2 &= -2\tau\left((u^n\cdot\nabla\rho^{n+1})\hat{e}_u^{n+1}, v^{n+1}\right) - 2\tau(\rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}) \\ &= -2\tau(\nabla\cdot(\rho^{n+1}u^n)\hat{e}_u^{n+1}, v^{n+1}) - 2\tau(\rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}) \\ &= 2\tau(\nabla\cdot(\rho^{n+1}e_u^n)\hat{e}_u^{n+1}, v^{n+1}) - 2\tau(\nabla\cdot(\rho^{n+1}u(t_n))\hat{e}_u^{n+1}, v^{n+1}) \\ &\quad - 2\tau(\rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}) \\ &= 2\tau(\nabla\cdot(\rho^{n+1}e_u^n)\hat{e}_u^{n+1}, v^{n+1}) - 2\tau\left(\rho^{n+1}(u(t_{n+1}) - u(t_n))\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}\right) \\ &\quad + 2\tau(\rho^{n+1}u(t_n)\cdot\nabla v^{n+1}, \hat{e}_u^{n+1}) \\ &= -2\tau((\rho^{n+1}e_u^n)\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}) - 2\tau((\rho^{n+1}e_u^n)\cdot\nabla v^{n+1}, \hat{e}_u^{n+1}) \\ &\quad - 2\tau\left(\rho^{n+1}(u(t_{n+1}) - u(t_n))\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}\right) + 2\tau(\rho^{n+1}u(t_n)\cdot\nabla v^{n+1}, \hat{e}_u^{n+1}) \\ &\le C\tau(\|e_u^n\| + \tau)\|\nabla\hat{e}_u^{n+1}\|\|v^{n+1}\|_2 + \frac{\mu\tau}{12}\|\sigma^{n+1}e_u^{n+1}\|^2 + C\tau\|\frac{1}{\sigma^{n+1}}\nabla\phi^{n+1}\|^2 + C\tau\|\nabla v^{n+1}\|^2 \\ &\le C\tau^2\|\nabla\hat{e}_u^{n+1}\|^2 + \frac{\mu\tau}{6}\|\sigma^{n+1}e_u^{n+1}\|^2 + C\tau\|\frac{1}{\sigma^{n+1}}\nabla\phi^{n+1}\|^2 + C\tau\|\nabla v^{n+1}\|^2,\end{aligned}$$

where we have already used (2.11), (2.14), (3.1), (3.9), (3.11) and (4.5).

By (3.1) again,

$$B_3 \le C\tau\|e_\rho^{n+1}\|^2 + C\tau\|\nabla v^{n+1}\|^2.$$

We further split B_4 by

$$\begin{aligned}B_4 &= 2\tau\left(\rho^{n+1}e_u^n\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}\right) + 2\tau\left(\rho^{n+1}(e_u^{n+1} - e_u^n)\cdot\nabla u(t_{n+1}), v^{n+1}\right) \\ &\quad - 2\tau\left(\rho^{n+1}e_u^{n+1}\cdot\nabla u(t_{n+1}), v^{n+1}\right)\end{aligned}$$

For the first term of B_4, by (2.11) and (3.1) we have

$$\tau(\rho^{n+1}e_u^n\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}) \le C\tau\|e_u^n\|^2\|\nabla\hat{e}_u^{n+1}\|^2 + \epsilon\tau\|v^{n+1}\|_2^2. \tag{4.9}$$

From (4.5), there exists a constant C_1 such that

$$\|v^{n+1}\|_2^2 \le C_1\|\sigma^{n+1}e_u^{n+1}\|^2.$$

Then as a consequence of (3.11), it follows that $\|e_u^n\|^2 \le C\tau$. Substituting the above two estimates into (4.9) and choosing sufficiently small ϵ, we have $\epsilon C_1\tau \le \frac{\mu\tau}{12}$ and thus

$$\tau(\rho^{n+1}e_u^n\cdot\nabla\hat{e}_u^{n+1}, v^{n+1}) \le C\tau^2\|\nabla\hat{e}_u^{n+1}\|^2 + \frac{\mu\tau}{12}\|\sigma^{n+1}e_u^{n+1}\|^2.$$

We can obtain similar bounds for the rest terms in B_4. Hence, we obtain

$$B_4 \le C\tau^2\|\nabla\hat{e}_u^{n+1}\|^2 + \frac{\mu\tau}{6}\|\sigma^{n+1}e_u^{n+1}\|^2 + \tau\|e_u^{n+1}-e_u^n\|^2 + C\tau\|\nabla v^{n+1}\|^2.$$

Thanks to (3.11) again,

$$\|\nabla\hat{u}^{n+1}\| \le \|\nabla u(t^{n+1})\| + \|\nabla\hat{e}_u^{n+1}\| \le C, \tag{4.10}$$

which together with (2.11), (4.5) and (3.1) implies

$$B_5 \le C\tau^2\int_{t^n}^{t^{n+1}}\|u_t\|^2dt + \frac{\mu\tau}{6}\|\sigma^{n+1}e_u^{n+1}\|^2.$$

As a consequence of (3.15) and (3.11),

$$B_6 \le C\tau^3 + C\tau\|e_\rho^n\|^2 + C\tau\|\nabla v^{n+1}\|^2.$$

In light of (3.1),

$$\begin{aligned} B_7 &= 2\tau(\nabla\rho(t_{n+1})\cdot(e_u^{n+1}-e_u^n), e_\rho^{n+1}) - 2\tau(\nabla\rho(t_{n+1})\cdot e_u^{n+1}, e_\rho^{n+1}) \\ &\le \tau\|e_u^{n+1}-e_u^n\|^2 + \frac{\mu\tau}{6}\|\sigma^{n+1}e_u^{n+1}\|^2 + C\tau\|e_\rho^{n+1}\|^2. \end{aligned}$$

For the last term, (3.14) produces

$$B_8 \le C\tau^3 + C\tau\|e_\rho^{n+1}\|^2.$$

Inserting the above estimates for $B_1 - B_8$ into (4.8) and summing over n from zero to $N-1$, we derive by discrete Gronwall lemma and (3.11) that

$$\|\nabla v^N\|^2 + \|e_\rho^N\|^2 + \sum_{n=0}^{N-1}\|\nabla(v^{n+1}-v^n)\|^2 + \mu\tau\|\sigma^{n+1}e_u^{n+1}\|^2 + \|e_\rho^{n+1}-e_\rho^n\|^2 \le C\tau^2.$$

Hence we arrive at (4.4) upon invoking (4.3). Finally we can use the above and (3.11), (3.9) to derive the desired bound for $\hat{e}_u^n$. The proof is thus complete. □

5. Error estimate for the pressure

In this section, we shall derive an error estimate for the pressure, which is based on the following error estimate for the difference of velocity.

5.1. Error estimate for the difference of velocity

From now on, for any function sequence v^n we denote the difference by $\delta v^n = v^n - v^{n-1}$.

Lemma 4. *Under the same assumptions as Lemma 3, there exists a constant $C > 0$ such that for all $1 \le N \le T/\tau$, we have*

$$\|\delta(\rho^N e_u^N)\|_*^2 + \sum_{n=0}^{N-1}\left\{\|\delta(\rho^{n+1}e_u^{n+1}) - \delta(\rho^n e_u^n)\|_*^2 + \tau\mu\|\delta e_u^{n+1}\|^2 + \tau\mu\|\delta\hat{e}_u^{n+1}\|^2\right\} \le C\tau^2. \tag{5.1}$$

Proof. Let (v^n, q^n) be solutions of the system (4.1) with $g = \rho^n e_u^n$. From (4.5) and (2.11), it immediately follows that

$$\|\delta v^n\|_2 + \|\delta q^n\|_1 \le C\|\delta(\rho^n e_u^n)\| \le C(\|\delta e_u^n\| + \|e_u^{n-1}\|). \tag{5.2}$$

Since by (3.6),

$$\delta(\rho^{n+1}\hat{e}_u^{n+1}) = \delta(\rho^{n+1}e_u^{n+1}) + \nabla(\delta\phi^{n+1}),$$

it holds that

$$
\begin{aligned}
&(\rho^n\left(\hat{e}_u^{n+1}-e_u^n\right),\delta v^{n+1})-(\rho^{n-1}\left(\hat{e}_u^n-e_u^{n-1}\right),\delta v^{n+1})=-((\rho^{n+1}-\rho^n)\hat{e}_u^{n+1},\delta v^{n+1})\\
&\quad+((\rho^n-\rho^{n-1})\hat{e}_u^n,\delta v^{n+1})+(\delta(\rho^{n+1}e_u^{n+1})-\delta(\rho^n e_u^n),\delta v^{n+1})=-((\rho^{n+1}-\rho^n)\hat{e}_u^{n+1},\delta v^{n+1})\\
&\quad+((\rho^n-\rho^{n-1})\hat{e}_u^n,\delta v^{n+1})+(\nabla(\delta v^{n+1}-\delta v^n),\nabla(\delta v^{n+1})),
\end{aligned}
$$

and

$$
\begin{aligned}
(\Delta\delta\hat{e}_u^{n+1},\delta v^{n+1}) &= (\delta\hat{e}_u^{n+1},\Delta(\delta v^{n+1}))=(\delta\hat{e}_u^{n+1},-\delta(\rho^{n+1}e_u^{n+1})+\nabla\delta q^{n+1})\\
&= -\|\sigma^{n+1}\delta e_u^{n+1}\|^2-((\rho^{n+1}-\rho^n)e_u^n,\delta e_u^{n+1})-(\frac{\rho^{n+1}-\rho^n}{\rho^{n+1}}e_u^n,\nabla\phi^{n+1})+(\frac{\rho^{n+1}}{\rho^n}e_u^{n+1},\nabla\phi^n)\\
&\quad+(\frac{1}{\rho^{n+1}}\nabla\phi^{n+1},\nabla\delta q^{n+1})-(\frac{1}{\rho^n}\nabla\phi^n,\nabla\delta q^{n+1}).
\end{aligned}
$$

Subtracting two consecutive expressions in (3.13) yields

$$
\begin{aligned}
&\rho^n\left(\frac{\hat{e}_u^{n+1}-e_u^n}{\tau}\right)-\rho^{n-1}\left(\frac{\hat{e}_u^n-e_u^{n-1}}{\tau}\right)-\mu\Delta\delta\hat{e}_u^{n+1}=-u(t_{n+1})\cdot\nabla u(t_{n+1})e_\rho^{n+1}-\rho^{n+1}e_u^n\cdot\nabla\hat{u}^{n+1}\\
&-\rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1}+u(t_n)\cdot\nabla u(t_n)e_\rho^n+\rho^n e_u^{n-1}\cdot\nabla\hat{u}^n+\rho^n u(t_n)\cdot\nabla\hat{e}_u^n+\mu\nabla\delta s^n-\nabla\delta p(t_{n+1})\\
&-\rho^{n+1}\big(u(t_{n+1})-u(t_n)\big)\cdot\nabla\hat{u}^{n+1}+\rho^n\big(u(t_n)-u(t_{n-1})\big)\cdot\nabla\hat{u}^n+R_u^{n+1}-R_u^n.
\end{aligned}
\tag{5.3}
$$

Taking inner product of (5.3) with $2\tau\delta v^{n+1}$, we arrive at

$$
\begin{aligned}
&\|\nabla\delta v^{n+1}\|^2-\|\nabla\delta v^n\|^2+\|\nabla\delta v^{n+1}-\nabla\delta v^n\|^2+2\mu\tau\|\sigma^{n+1}\delta e_u^{n+1}\|^2\\
=\;&2((\rho^{n+1}-\rho^n)\hat{e}_u^{n+1},\delta v^{n+1})-2((\rho^n-\rho^{n-1})\hat{e}_u^n,\delta v^{n+1})-2\mu\tau((\rho^{n+1}-\rho^n)e_u^n,\delta e_u^{n+1})\\
&-2\mu\tau(\frac{\rho^{n+1}-\rho^n}{\rho^{n+1}}e_u^n,\nabla\phi^{n+1})+2\mu\tau(\frac{\rho^{n+1}}{\rho^n}e_u^{n+1},\nabla\phi^n)+2\mu\tau(\frac{1}{\rho^{n+1}}\nabla\phi^{n+1},\nabla\delta q^{n+1})\\
&-2\mu\tau(\frac{1}{\rho^n}\nabla\phi^n,\nabla\delta q^{n+1})-2\tau(u(t_{n+1})\cdot\nabla u(t_{n+1})e_\rho^{n+1},\delta v^{n+1})\\
&-2\tau(\rho^{n+1}e_u^n\cdot\nabla\hat{u}^{n+1},\delta v^{n+1})-2\tau(\rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1},\delta v^{n+1})\\
&+2\tau(u(t_n)\cdot\nabla u(t_n)e_\rho^n,\delta v^{n+1})+2\tau(\rho^n e_u^{n-1}\cdot\nabla\hat{u}^n,\delta v^{n+1})\\
&+2\tau(\rho^n u(t_n)\cdot\nabla\hat{e}_u^n,\delta v^{n+1})+2\tau(\mu\nabla\delta s^n-\nabla\delta p(t_{n+1}),\delta v^{n+1})\\
&-2\tau(\rho^{n+1}\big(u(t_{n+1})-u(t_n)\big)\cdot\nabla\hat{u}^{n+1},\delta v^{n+1})+2\tau(\rho^n\big(u(t_n)-u(t_{n-1})\big)\cdot\nabla\hat{u}^n,\delta v^{n+1})\\
&+2\tau(R_u^{n+1}-R_u^n,\delta v^{n+1})=\sum_{i=1}^{17}M_i.
\end{aligned}
$$

Like (3.19), it easily holds that

$$
\begin{aligned}
M_1+M_{10} &= -2\tau(\nabla\cdot(\rho^{n+1}u^n)\cdot\hat{e}_u^{n+1},\delta v^{n+1})-2\tau(\rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1},\delta v^{n+1})\\
&= 2\tau(\nabla\cdot(\rho^{n+1}e_u^n),\hat{e}_u^{n+1}\cdot\delta v^{n+1})-2\tau(\rho^{n+1}(u(t_{n+1})-u(t_n))\cdot\nabla\hat{e}_u^{n+1},\delta v^{n+1})+2\tau(\rho^{n+1}u(t_n)\cdot\nabla\delta v^{n+1},\hat{e}_u^{n+1})\\
&\le C\tau(\tau+\|e_u^n\|)\|\nabla\hat{e}_u^{n+1}\|\|\delta v^{n+1}\|_2+C\tau\|\hat{e}_u^{n+1}\|^2+C\tau\|\nabla\delta v^{n+1}\|^2\\
&\le C\tau^2\|\nabla\hat{e}_u^{n+1}\|^2+C\tau(\|\hat{e}_u^{n+1}\|^2+\|e_u^n\|^2+\|\nabla\delta v^{n+1}\|^2)+\frac{\mu\tau}{6}\|\sigma^{n+1}\delta e_u^{n+1}\|^2
\end{aligned}
$$

where we have already used (3.1), (2.11), (2.14), (3.11) and (5.2). Similarly, we can bound

$$
\begin{aligned}
&M_2\le C\tau^2\|\nabla\hat{e}_u^n\|^2+C\tau(\|\hat{e}_u^n\|^2+\|e_u^n\|^2+\|\nabla\delta v^{n+1}\|^2)+\frac{\mu\tau}{6}\|\sigma^{n+1}\delta e_u^{n+1}\|^2,\\
&M_3,M_9\le C\tau\|e_u^n\|^2+\frac{\mu\tau}{6}\|\sigma^{n+1}\delta e_u^{n+1}\|^2,\quad M_4\le C\tau\|e_u^n\|^2+C\tau\|\nabla\phi^{n+1}\|^2,\\
&M_5\le C\tau\|e_u^{n+1}\|^2+C\tau\|\nabla\phi^n\|^2,\quad M_6\le C\tau(\|\nabla\phi^{n+1}\|^2+\|e_u^n\|^2)+\frac{\mu\tau}{6}\|\sigma^{n+1}\delta e_u^{n+1}\|^2,\\
&M_7\le C\tau(\|\nabla\phi^n\|^2+\|e_u^n\|^2)+\frac{\mu\tau}{6}\|\sigma^{n+1}\delta e_u^{n+1}\|^2,\quad M_8\le C\tau(\|e_\rho^{n+1}\|^2+\|\nabla\delta v^{n+1}\|^2),\\
&M_{11}\le C\tau(\|e_\rho^n\|^2+\|\nabla\delta v^{n+1}\|^2),\quad M_{12}\le C\tau(\|e_u^{n-1}\|^2+\|e_u^n\|^2)+\frac{\mu\tau}{6}\|\sigma^{n+1}\delta e_u^{n+1}\|^2,\\
&M_{13}\le C\tau(\|\hat{e}_u^n\|^2+\|\nabla\delta v^{n+1}\|^2),\quad M_{15},M_{16}\le C\tau^3+C\tau\|\nabla\delta v^{n+1}\|^2,\\
&M_{17}\le C\tau(\tau^2+\|e_\rho^n\|^2+\|e_\rho^{n-1}\|^2)+C\tau\|\nabla\delta v^{n+1}\|^2,
\end{aligned}
$$

where we also used (4.10). Finally the only remaining term M_{14} vanishes since δv^{n+1} is divergence free. Combining all the error estimates for M_i, $1 \le i \le 17$, we get

$$\begin{aligned} &\|\nabla\delta v^{n+1}\|^2 - \|\nabla\delta v^n\|^2 + \|\nabla\delta v^{n+1} - \nabla\delta v^n\|^2 + 2\mu\tau\|\sigma^{n+1}\delta e_u^{n+1}\|^2 \\ \le\ & C\tau^3 + C\tau^2(\|\nabla\hat{e}_u^n\|^2 + \|\nabla\hat{e}_u^{n+1}\|^2) + C\tau(\|\hat{e}_u^n\|^2 + \|\hat{e}_u^{n+1}\|^2 + \|e_u^{n-1}\|^2 + \|e_u^n\|^2 + \|e_u^{n+1}\|^2 \\ &+ \|\nabla\phi^{n+1}\|^2 + \|\nabla\phi^n\|^2 + \|e_\rho^{n-1}\|^2 + \|e_\rho^n\|^2 + \|e_\rho^{n+1}\|^2 + \|\nabla\delta v^{n+1}\|^2) + \mu\tau\|\sigma^{n+1}\delta e_u^{n+1}\|^2. \end{aligned}$$

Then summing over n from zero to $N-1$, thanks to (3.11), (3.24) and (4.4), we derive from the discrete Gronwall lemma that

$$\|\nabla\delta v^N\|^2 + \sum_{i=0}^{N-1}\|\nabla\delta v^{n+1} - \nabla\delta v^n\|^2 + \mu\tau\|\sigma^{n+1}\delta e_u^{n+1}\|^2 \le C\tau^2.$$

We derive from (3.6) that

$$\delta\hat{e}_u^{n+1} = \delta e_u^{n+1} + \frac{1}{\rho^{n+1}}\nabla\phi^{n+1} - \frac{1}{\rho^n}\nabla\phi^n.$$

We derive from the above and (3.11) that

$$\tau\sum_{n=0}^{N-1}\|\delta\hat{e}_u^{n+1}\|^2 \le C\tau\sum_{n=0}^{N-1}\left(\|\delta e_u^{n+1}\|^2 + \|\frac{1}{\rho^{n+1}}\nabla\phi^{n+1}\|^2 + \|\frac{1}{\rho^n}\nabla\phi^n\|^2\right) \le C\tau^2.$$

Therefore we arrive at (5.1) thanks to (4.3). □

5.2. Error estimate for the pressure

We now estimate the pressure error in $L^2(0,T;L^2)$. This hinges on the error estimate for the time difference of velocity in Lemma 4.

Theorem 2. *Under the same assumptions as Lemma 3, there exists a constant $C > 0$ such that*

$$\tau\sum_{n=0}^{N-1}\|p(t_n) - p^n\|^2_{L^2/R} \le C\tau, \quad \forall 1 \le N \le T/\tau. \tag{5.4}$$

Proof. In view of (3.6), we replace e_u^n in (3.13) by $\hat{e}_u^n - \frac{1}{\rho^n}\nabla\phi^n$ to get

$$\begin{aligned} \nabla\left(p(t_n) - p^n\right) =\ & -\nabla\left(p(t_{n+1}) - p(t_n)\right) - \rho^n\left(\frac{\hat{e}_u^{n+1} - \hat{e}_u^n}{\tau}\right) + \mu\Delta\hat{e}_u^{n+1} - u(t_{n+1})\cdot\nabla u(t_{n+1})e_\rho^{n+1} \\ & - \rho^{n+1}e_u^n\cdot\nabla\hat{u}^{n+1} - \rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1} - \rho^{n+1}\big(u(t_{n+1}) - u(t_n)\big)\cdot\nabla\hat{u}^{n+1} + R_u^{n+1}, \end{aligned} \tag{5.5}$$

where we have used the definition (2.8), i.e. $p^n = \mu s^n - \frac{\phi^n}{\tau}$.

Since for all $v \in V$ we have from (3.1)

$$\begin{aligned} &(u(t_{n+1})\cdot\nabla u(t_{n+1})e_\rho^{n+1}, v) \le C\|e_\rho^{n+1}\|\|v\|_1, \\ &(\rho^{n+1}e_u^n\cdot\nabla\hat{u}^{n+1}, v) \le \|\nabla e_u^n\|\|v\|_1, \\ &(\rho^{n+1}u(t_{n+1})\cdot\nabla\hat{e}_u^{n+1}, v) \le C\|\nabla\hat{e}_u^{n+1}\|\ \|\ \|v\|_1, \\ &(\rho^{n+1}\big(u(t_{n+1}) - u(t_n)\big)\cdot\nabla\hat{u}^{n+1}, v) \le C\tau\|v\|_1, \\ &(\nabla\left(p(t_{n+1}) - p(t_n)\right), v) \le C\tau\|v\|_1. \end{aligned}$$

We have also, for all $v \in V$,

$$(-\rho^n\left(\frac{\hat{e}_u^{n+1} - \hat{e}_u^n}{\tau}\right) + \mu\Delta\hat{e}_u^{n+1} + R_u^{n+1}, v) \le (\frac{1}{\tau}\|\hat{e}_u^{n+1} - \hat{e}_u^n\|_{-1} + \|R_u^{n+1}\|_{-1} + \mu\|\nabla\hat{e}_u^{n+1}\|)\|v\|_1.$$

From (4.4) and (3.15) we derive

$$\|p(t_n) - p^n\|_{L^2/R} \le \sup_{v\in H_0^1(\Omega)}\frac{(p(t_n) - p^n, \operatorname{div} v)}{\|\nabla v\|} \le (\frac{1}{\tau}\|\hat{e}_u^{n+1} - \hat{e}_u^n\|_{-1} + \|\nabla\hat{e}_u^{n+1}\| + \|\nabla e_u^n\|) + C\tau.$$

What remains now is to square, multiply by τ, and sum over n from 0 to $N-1$. Recalling (3.11), (3.24) and (5.1), assertion (5.4) follows immediately. This concludes the proof. □

6. A finite element discretization and its stability

The scheme (2.1)–(2.8) in its semi-discrete form was introduced in [9]. However, it was mentioned in **Remark 3.2** of [9] that "How to design a suitable space discretization and prove its stability is a more complicate issue". We shall construct below a finite element method for the scheme (2.1)–(2.8).

Let $\mathcal{T}_h = \{K\}$ be a shape regular quasi-uniform partition of Ω with mesh size h. We define

$$V_h^k = \{v \in C(\Omega) : v|_K \in \mathcal{P}^k,\ k \geq 1,\ \forall K \in \mathcal{T}_h\},$$

where $\mathcal{P}^k$ denotes the space consisting polynomials with order less or equal to k. Moreover, we denote the Raviart–Thomas element (or, Nédélec face element in 3D) and discontinuous element of degree k by RT^k and DG^k respectively, which satisfies the discrete inf–sup condition for the mixed formulation of Laplacian [12–16]. We denote $\mathbf{V}_h^k := (V_h^k)^d$, $\mathring{\mathbf{V}}_h^k = \mathbf{V}_h^k \cap \mathbf{H}_0^1$, and $\mathring{RT}^k := RT^k \cap H_0(\text{div})$ where $H_0(\text{div}) = \{v \in H(\text{div}) :\ v \cdot \vec{n} = 0 \text{ on } \partial\Omega\}$. Our finite element scheme is described as follows:

FEM for convective GUM. Let ρ_{0h} and u_{0h} be a suitable approximation of ρ_0 and u_0 respectively. Take $\rho_h^0 = \rho_{0h}$, $u_h^0 = u_{0h}$ and $s_h^0 = 0$; repeat for $0 \leq n \leq N-1$ $(1 \leq N \leq T/\tau)$:

Step 1. Find $\rho_h^{n+1} \in V_h^{2k}$ as the solution of

$$(D_\tau \rho_h^{n+1} + u_h^n \cdot \nabla \rho_h^{n+1}, \psi) = 0, \quad \forall \psi \in V_h^{2k}. \tag{6.1}$$

Step 2. Find $\hat{u}_h^{n+1} \in \mathring{\mathbf{V}}_h^k$ as the solution of

$$(\rho_h^n \frac{\hat{u}_h^{n+1} - u_h^n}{\tau}, w) + (\rho_h^{n+1}(u_h^n \cdot \nabla)\hat{u}_h^{n+1}, w) - (\mu s_h^n, \nabla \cdot w) + (\mu \nabla \hat{u}_h^{n+1}, \nabla w) = (f^{n+1}, w). \quad \forall w \in \mathring{\mathbf{V}}_h^k. \tag{6.2}$$

Step 3. Find $\omega_h^{n+1} \in \mathring{RT}^k$, $\phi_h^{n+1} \in DG^k$ as the solution of

$$(\rho_h^{n+1}\omega_h^{n+1}, \chi) - (\phi_h^{n+1}, \nabla \cdot \chi) = 0, \quad \forall \chi \in \mathring{RT}^k, \tag{6.3}$$

$$(\nabla \cdot \omega_h^{n+1}, \vartheta) = (\nabla \cdot \hat{u}_h^{n+1}, \vartheta), \quad \forall \vartheta \in DG^k. \tag{6.4}$$

Step 4. Update u_h^{n+1} by

$$u_h^{n+1} = \hat{u}_h^{n+1} - \omega_h^{n+1}, \tag{6.5}$$

determine $s_h^{n+1} \in DG^k$ from

$$(s_h^{n+1}, v) = (s_h^n - \nabla \cdot \hat{u}_h^{n+1}, v) \quad \forall v \in DG^k, \tag{6.6}$$

and set

$$p_h^{n+1} = \mu s_h^{n+1} - \frac{1}{\tau}\phi_h^{n+1}. \tag{6.7}$$

We shall first establish a result similar to Lemma 2.

Lemma 5. *Assume that there exists two constants $c, C > 0$ such that $c \leq \rho_h^0(x) \leq C, \forall x \in \bar{\Omega}$. Then, the numerical density ρ_h^n determined from (6.1) satisfies:*

$$c \leq \rho_h^n(x) \leq C \quad \forall x \in \bar{\Omega},\ \forall n. \tag{6.8}$$

Proof. Let $P_h^{2k} : L^2 \to V_h^{2k}$ be the L^2 projection operator. Since $D_\tau \rho_h^{n+1} + P_h^{2k}(u_h^n \cdot \nabla \rho_h^{n+1}) \in V_h^{2k}$, we derive from (6.1) that

$$D_\tau \rho_h^{n+1} + P_h^{2k}(u_h^n \cdot \nabla \rho_h^{n+1}) = 0.$$

Then, for any element K in $\mathcal{T}_h$, by using exactly the same argument as in Lemma 2, we can show $c \leq \rho_h^n(x) \leq C \quad \forall x \in \bar{K},\ \forall n$, which implies (6.8). □

Next, we prove the following stability result for the above FEM scheme.

Theorem 3. *The Gauge–Uzawa Algorithm is unconditionally stable in the sense that, assuming that $\rho_h^n > 0$ then for all $\tau > 0$ and $1 \leq N \leq T/\tau$ the following a priori bounds hold:*

$$\|\rho_h^N\|^2 + \sum_{k=0}^{N-1} \|\rho_h^{k+1} - \rho_h^k\|^2 = \|\rho_h^0\|^2, \tag{6.9}$$

and

$$\|\sigma_h^N \hat{u}^N\|^2 + \mu\tau\|s_h^N\|^2 + \sum_{k=0}^{N-1}\left(\|\sigma_h^k(\hat{u}_h^{k+1} - u_h^k)\|^2 + \frac{\mu}{2}\tau\|\nabla\hat{u}_h^{k+1}\|^2\right)$$
$$\leq \|\sigma_h^0\hat{u}_h^0\|^2 + C\mu\tau\sum_{k=0}^{N-1}\|f^{k+1}\|_{-1}^2, \tag{6.10}$$

where $\sigma_h^n = \sqrt{\rho_h^n}$.

Proof. Thanks to (6.4) and (6.5), it follows that

$$\nabla\cdot u_h^n = 0, \quad \forall n. \tag{6.11}$$

Taking the inner product of (6.1) with $2\tau\rho_h^{n+1}$ to get

$$\|\rho_h^{n+1}\|^2 - \|\rho_h^n\|^2 + \|\rho_h^{n+1} - \rho_h^n\|^2 = 0,$$

where by (6.11) and $u_h^n\cdot\vec{n} = 0$ on $\partial\Omega$ we have used the fact

$$(u_h^n\cdot\nabla\rho_h^{n+1}, \rho_h^{n+1}) = 0. \tag{6.12}$$

Then summing up over n from 0 to $N-1$ leads to (6.9).

Next, taking the inner product of (6.2) with $2\tau\hat{u}_h^{n+1}$, we find

$$2(\rho_h^n(\hat{u}_h^{n+1} - u_h^n), \hat{u}_h^{n+1}) + 2\tau(\rho_h^{n+1}(u_h^n\cdot\nabla)\hat{u}_h^{n+1}, \hat{u}_h^{n+1}) + 2\tau(\mu\nabla s_h^n, \hat{u}_h^{n+1}) + 2\tau\mu\|\nabla\hat{u}_h^{n+1}\|^2$$
$$= 2\tau(f^{n+1}, \hat{u}_h^{n+1}). \tag{6.13}$$

Setting $\sigma_h^n = \sqrt{\rho_h^n}$, we can rewrite the first term in the above as

$$2(\rho_h^n(\hat{u}_h^{n+1} - u_h^n), \hat{u}_h^{n+1}) = \|\sigma_h^n\hat{u}_h^{n+1}\|^2 - \|\sigma_h^n u_h^n\|^2 + \|\sigma_h^n(\hat{u}_h^{n+1} - u_h^n)\|^2. \tag{6.14}$$

Since by (6.3), (6.5), (6.11) and the fact that $\mathring{\mathbf{V}}_h^k \subset \mathring{RT}^k$ [12,13] it holds that

$$\|\sigma_h^n u_h^n\|^2 = (\rho_h^n u_h^n, u_h^n) = (\rho_h^n\hat{u}_h^n - \rho_h^n\omega_h^n, u_h^n) = (\rho_h^n\hat{u}_h^n, u_h^n) = (\rho_h^n\hat{u}_h^n, \hat{u}_h^n - \omega_h^n)$$
$$= \|\sigma_h^n\hat{u}_h^n\|^2 - (\rho_h^n u_h^n + \rho_h^n\omega_h^n, \omega_h^n) = \|\sigma_h^n\hat{u}_h^n\|^2 - \|\sigma_h^n\omega_h^n\|^2, \tag{6.15}$$

we only need to derive a suitable relation between $\|\sigma_h^n\hat{u}_h^{n+1}\|^2$ and $\|\sigma_h^{n+1}\hat{u}_h^{n+1}\|^2$. To this end, taking inner product of (6.1) with $\tau\hat{u}_h^{n+1}\cdot\hat{u}_h^{n+1}$ to get

$$(\rho_h^{n+1} - \rho_h^n, \hat{u}_h^{n+1}\cdot\hat{u}_h^{n+1}) + \tau(u_h^n\cdot\nabla\rho_h^{n+1}, \hat{u}_h^{n+1}\cdot\hat{u}_h^{n+1})$$
$$= \|\sigma_h^{n+1}\hat{u}_h^{n+1}\|^2 - \|\sigma_h^n\hat{u}_h^{n+1}\|^2 - \tau((\nabla\cdot u_h^n)\rho_h^{n+1}, \hat{u}_h^{n+1}\cdot\hat{u}_h^{n+1}) - 2\tau(\rho_h^{n+1}(u_h^n\cdot\nabla)\hat{u}_h^{n+1}, \hat{u}_h^{n+1}) = 0. \tag{6.16}$$

Due to (6.11) it holds that

$$((\nabla\cdot u_h^n)\rho_h^{n+1}, \hat{u}_h^{n+1}\cdot\hat{u}_h^{n+1}) = 0,$$

which together with (6.16) implies that

$$\|\sigma_h^{n+1}\hat{u}_h^{n+1}\|^2 - \|\sigma_h^n\hat{u}_h^{n+1}\|^2 = 2\tau(\rho_h^{n+1}(u_h^n\cdot\nabla)\hat{u}_h^{n+1}, \hat{u}_h^{n+1}). \tag{6.17}$$

Combining (6.17) and (6.15) with (6.14), we arrive at

$$2(\rho_h^n(\hat{u}_h^{n+1} - u_h^n), \hat{u}_h^{n+1}) + 2\tau(\rho_h^{n+1}(u_h^n\cdot\nabla)\hat{u}_h^{n+1}, \hat{u}_h^{n+1})$$
$$= \|\sigma_h^{n+1}\hat{u}_h^{n+1}\|^2 - \|\sigma_h^n\hat{u}_h^n\|^2 + \|\sigma_h^n(\hat{u}_h^{n+1} - u_h^n)\|^2 + \|\sigma_h^n\omega_h^n\|^2. \tag{6.18}$$

Then substituting (6.18) into (6.13), we have

$$\|\sigma_h^{n+1}\hat{u}_h^{n+1}\|^2 - \|\sigma_h^n\hat{u}_h^n\|^2 + \|\sigma_h^n(\hat{u}_h^{n+1} - u_h^n)\|^2 + \|\sigma_h^n\omega_h^n\|^2 + 2\tau\mu\|\nabla\hat{u}_h^{n+1}\|^2$$
$$= 2\tau(f^{n+1}, \hat{u}_h^{n+1}) + 2\tau(\mu s_h^n, \nabla\cdot\hat{u}_h^{n+1}) := Y_1 + Y_2. \tag{6.19}$$

We can bound

$$Y_1 \leq C\tau\|f^{n+1}\|_{-1}^2 + \frac{\mu\tau}{2}\|\nabla\hat{u}_h^{n+1}\|^2.$$

Table 1
Error and convergence rate for convective GUM with $k = 1$.

τ	$\|\rho_h - \rho\|_{L^\infty(L^2)}$	Rate	$\|u_h - u\|_{L^2(\mathbf{L}^2)}$	Rate
1.25E−02	4.59E−04	0.92	1.30E−06	0.83
6.25E−03	2.34E−04	0.97	6.84E−07	0.93
3.12E−03	1.18E−04	0.98	3.52E−07	0.96

Table 2
Error and convergence rate for convective GUM with $k = 1$.

τ	$\|u_h - u\|_{L^2(\mathbf{H}^1)}$	Rate	$\|p_h - p\|_{L^2(L^2)}$	Rate
1.25E−02	2.18E−05	0.93	4.19E−05	0.82
6.25E−03	1.13E−05	0.94	2.24E−05	0.92
3.12E−03	5.71E−06	0.97	1.15E−05	0.96

Thanks to (3.2) and (6.6),

$$\begin{aligned} Y_2 &= -2\mu\tau(s_h^{n+1} - s_h^n, s_h^n) = -\mu\tau(\|s_h^{n+1}\|^2 - \|s_h^{n+1} - s_h^n\|^2 - \|s_h^n\|^2) \\ &= -\mu\tau(\|s_h^{n+1}\|^2 - \|s_h^n\|^2) + \mu\tau\|\nabla\cdot\hat{u}_h^{n+1}\|^2 \le -\mu\tau(\|s_h^{n+1}\|^2 - \|s_h^n\|^2) + \mu\tau\|\nabla\hat{u}_h^{n+1}\|^2, \end{aligned}$$

where from (6.6) we have used the fact that

$$-\nabla\cdot\hat{u}_h^{n+1} = s_h^{n+1} - s_h^n.$$

Inserting the above estimates for Y_1 and Y_2 into (6.19) yields

$$\begin{aligned} &\|\sigma_h^{n+1}\hat{u}_h^{n+1}\|^2 - \|\sigma_h^n\hat{u}_h^n\|^2 + \|\sigma_h^n(\hat{u}_h^{n+1} - u_h^n)\|^2 + \|\sigma_h^n\omega_h^n\|^2 + \mu\tau(\|s_h^{n+1}\|^2 - \|s_h^n\|^2) + \frac{\tau\mu}{2}\|\nabla\hat{u}_h^{n+1}\|^2 \\ &\quad \le C\tau\|f^{n+1}\|_{-1}^2. \end{aligned}$$

Summing the above over n from 0 to $N-1$ leads to (6.10). □

7. Numerical experiments

In this section, we present numerical examples to test the accuracy of the first-order algorithm proposed in Section 6. We solve problem (1.1)–(1.3) using an analytical solution defined on the disk with a radius of 0.1:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 0.01\}.$$

The exact solution is:

$$\begin{cases} \rho(x, y, t) = 2 + r\cos(\theta - sin(t)), \\ u_1(x, y, t) = -y\cos(t), \\ u_2(x, y, t) = x\cos(t), \\ p(x, y, t) = \sin(t)\sin(y)\sin(t), \end{cases}$$

where $r := \sqrt{x^2 + y^2}$ and $\theta := \arctan(y/x)$. Note that the above exact solutions satisfy the mass conservation (1.1). We set $\mu = 1$ so that the corresponding right-hand side in the momentum equation (1.2) is

$$f(x, y, t) = \begin{pmatrix} \big(y\sin(t) - x\cos^2(t)\big)\,\rho(x, y, t) + \cos(x)\sin(y)\sin(t) \\ -\big(x\sin(t) + y\cos^2(t)\big)\,\rho(x, y, t) + \sin(x)\cos(y)\sin(t) \end{pmatrix}.$$

Then we choose $k = 1$ in the algorithm of Section 6 which is related with the order of finite element space. Hence, the mass conservation equation (1.1) is discretized in space using P^2 continuous finite elements. To approximate the velocity and pressure, we both use linear elements. To solve the problem (6.3)–(6.4) the mixed element $(\mathring{RT}^1, DG^1)$ is used. We perform the accuracy tests over the time interval [0,1] with respect to τ on a uniformly triangulation mesh with mesh size $h = \tau/10$.

In Tables 1 and 2, we display the errors and convergence rates in time for all variables. We observe that the errors for all variables in the reported norms are of first-order which are consistent with our error estimates except that the first-order convergence rate for the velocity error in $L^2(\mathbf{H}^1)$ is faster than the half-order proved in Lemma 3.

Then the same problem is solved using the algorithm of Section 6 again but now $k = 2$, which means the mass conservation equation (1.1) is discretized in space using P^4 continuous finite elements and the other equations are solved using one order higher finite elements. We also perform the accuracy tests over the time interval [0,1] with respect to τ on a uniformly triangulation mesh with mesh size $\tau = 30h^2$. The results are shown in Tables 3 and 4, from which we observe that all the errors have the same order with respect to τ as in the first case.

Table 3
Error and convergence rate for convective GUM with $k = 2$.

τ	$\|\rho_h - \rho\|_{L_2(L_\infty)}$	Rate	$\|u_h - u\|_{L_2(L_2)}$	Rate
6.25E−02	2.29E−04	0.96	6.51E−07	0.89
3.12E−02	1.17E−04	0.97	3.43E−07	0.93
1.56E−02	5.89E−05	0.99	1.77E−07	0.96

Table 4
Error and convergence rate for convective GUM with $k = 2$.

τ	$\|u_h - u\|_{L_2(H_1)}$	Rate	$\|p_h - p\|_{L_2(L_2)}$	Rate
6.25E−02	2.13E−05	0.91	2.22E−05	0.87
3.12E−02	1.12E−05	0.92	1.29E−05	0.91
1.56E−02	5.71E−06	0.97	6.78E−06	0.94

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