Journal of Scientific Computing (2021) 88:4
https://doi.org/10.1007/s10915-021-01491-2

Efficient Spectral Methods for PDEs with Spectral Fractional Laplacian

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Received: 20 June 2020 / Revised: 9 December 2020 / Accepted: 11 April 2021

Abstract
We develop efficient spectral methods for the spectral fractional Laplacian equation and parabolic PDEs with spectral fractional Laplacian on rectangular domains. The key idea is to construct eigenfunctions of discrete Laplacian (also referred to Fourier-like basis) by using the Fourierization method. Under this basis, the non-local fractional Laplacian operator can be trivially evaluated, leading to very efficient algorithms for PDEs involving spectral fractional Laplacian. We provide a rigorous error analysis of the proposed methods for the case with homogeneous boundary conditions, as well as ample numerical results to show their effectiveness.

Keywords Spectral fractional Laplacian · Fourier-like basis function · Spectral method · Error analysis

Mathematics Subject Classification Primary: 26A33 · 35R11 · 35S15 · 41A58 · 65N35

1 Introduction

We consider in this paper numerical approximation of the spectral fractional Laplacian equation

$$(-\Delta)^{\frac{\alpha}{2}} v = f, \quad \forall \boldsymbol{x} \in \Omega, \tag{1.1}$$

The work of J. Shen is partially supported by NSF DMS-2012585 and AFOSR FA9550-20-1-0309.

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Published online: 21 May 2021

with suitable boundary conditions, and parabolic PDEs with spectral fractional Laplacian:

$$v_t + \epsilon(-\Delta)^{\frac{\alpha}{2}} v + \mathcal{N}(v, t) = 0, \quad \forall (\boldsymbol{x}, t) \in D := \Omega \times (0, T], \tag{1.2}$$

with suitable initial and boundary conditions. In the above, Ω is a rectangular domain and $(-\Delta)^{\frac{\alpha}{2}}$ is the spectral fractional Laplacian operator defined by the spectral decomposition

$$(-\Delta)^{\frac{\alpha}{2}} u(\boldsymbol{x}) = \sum_n \tilde{u}_n \lambda_n^{\frac{\alpha}{2}} \phi_n(\boldsymbol{x}), \quad \alpha \in (0, 2), \tag{1.3}$$

where $\{\lambda_n, \phi_n\}_{n\geq 0}$ are the eigenvalues and eigenfunctions of the Laplace operator $-\Delta$ with given zeros Dirichlet, Neumann or the mixed boundary conditions (cf. (2.1) below). We refer to [21] for more details on the definition of spectral fractional Laplacian.

Approximation of spectral fractional Laplacian (1.3) has been the subject of many investigations recently. For problems with periodic boundary conditions, it is natural and effective to use Fourier spectral methods e.g., [2,7,34]. For non-periodic boundary conditions, there are essentially four different approaches:

- Use space spanned by eigenfunctions of Laplacian operator as approximation space. An one-dimensional example is considered in [17]. However, since the eigenfunctions of Laplacian, $\sin c_k x$ or $\cos c_k x$, have very poor approximation properties for non-periodic functions, the convergence of such method is very slow, even if the solution is smooth.
- Use space spanned by eigenfunctions of discrete Laplacian operator as approximation space. In [30], a spectral-element method is used to construct discrete eigenfunctions which are then used to approximate the fractional Laplacian operator.
- Use the Caffarelli–Silvestre extension [9,31]. This approach was first considered in [22] using a finite-element method followed by extension to space-time parabolic fractional PDEs in [23] and improvements with tensor product finite elements and adaptivity in [3]. A spectral method for the extended problem is presented in [11]. A different approach using the Caffarelli–Silvestre extension is given in [1].
- Use the Dunford–Taylor formula. This formula can be viewed as a semi-analytic solution of the Caffarelli–Silvestre extension in which the extended direction is analytically represented by an integral formula. This approach was first adopted in [5] with a finite-element method in space.

For more detailed presentation on numerical methods for fractional Laplacians (in spectral form and integral form) and up-to-date references, we refer to two excellent recent review papers [4,21].

In this paper, we focus on the numerical approximation of spectral fractional Laplacian (1.3) with non-periodic boundary conditions. We adopt the second approach. More precisely, we use space spanned by eigenfunctions of **discrete** Laplacian operator where the discretization is done by a Legendre–Galerkin method [24].

It is well-known that only a portion of the discrete eigenpairs are good approximation of the corresponding exact eigenpairs, so a main question in using the second approach is whether to use all discrete eigenpairs or only those who are good approximations of the exact ones. Numerical results in [30] indicated that better results can be obtained by using all discrete eigenpairs, but no theoretical justification is available on why spurious discrete eigenpairs should be included in the approximation space. A main purpose of this paper is to provide a theoretical justification by carrying out a delicate error analysis.

A main bottleneck in using the second approach is that computing and storing all discrete eigenpairs in multidomains is usually prohibitively expensive. So another purpose of this paper is to develop a robust and efficient method to compute discrete eigenpairs. To this end, we restrict our attention in this paper to rectangular domains, with the expectation that the method developed here can be extend to more generally domain using the recently developed novel spectral methods for complex domains in [13]. Following the idea in [28], we construct eigenfunctions of **discrete** Laplacian operator in the Legendre–Galerkin formulation using the so called Fourier-like basis functions. The benefits of such choice are significant: the construction of Fourier-like basis functions only involves finding all the eigenpairs of a symmetric positive definite penta-diagonal matrix, and thanks to the orthonormality of the Fourier-like basis functions in the underlying inner products, the linear system for approximating (1.1) is diagonal so its solution can be obtained very efficiently.

We also construct a space-time spectral method for (1.2) by using the approximation based on the discrete eigenpairs in space and a dual-Petrov Galerkin method in time. For linear parabolic systems, this method leads to a sequence of one-dimensional tridiagonal systems that can be easily solved, and nonlinear parabolic systems can also be solved very efficiently with a preconditioned iterative procedure using a suitable linear parabolic system as a preconditioner.

We highlight below the main advantages of the proposed methods and main contributions of the paper:

- Accuracy: the space spanned by the eigenfunctions of **discrete** Laplacian operator with Legendre–Galerkin method has excellent approximation properties: it leads to exponential convergence for smooth functions, and can double the convergence rate of Fourier-spectral method or standard finite-element methods for problems with corner singularities which are present in problems with spectral fractional Laplacian.
- Efficiency: Unlike the approaches based on the Caffarelli–Silvestre extension and Dunford–Taylor formula which involve an extra-dimension, the presence of the nonlocal fractional Laplacian operator does not introduce any additional computational complexity so the cost of our proposed methods are essentially the same as the very efficient spectral-Galerkin method for the Poisson type Eq. [24] and dual-Petrov Galerkin method for parabolic type equations [28].
- We derived error bounds between the discrete fractional Laplacian $(-\Delta_M)^{\frac{\alpha}{2}}$ and the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, which in particular justifies the use of all discrete eigenfunctions.
- We established error analysis of our proposed methods for fractional Laplacian equation and linear fractional reaction–diffusion equation.

The rest of the paper is organized as follows. In the next section, we construct the Fourier-like basis functions as the discrete eigenfunctions of fractional Laplacian operator. In Sect. 3, we provide a complete error analysis of the proposed methods for fractional Poisson equation and fractional PDE. Numerical experiments for boundary value problem involving spectral fractional Laplacian are carried out in Sect. 4. In Sect. 5 we develop efficient methods for space fractional reaction–diffusion equations. Some concluding remarks are given in the last section.

2 Algorithms

We construct below the Fourier-like basis functions which are the eigenfunctions of discrete fractional Laplacian, and present spectral algorithms for fractional Laplacian equation and a space-time spectral method for linear parabolic PDEs with spectral fractional Laplacian on rectangular domains.

Let us first introduce some notations.

- Let $\mathbb{R}$ (resp. $\mathbb{N}$) be the set of all real numbers (resp. positive integers), and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- We use boldface lowercase letters to denote d-dimensional multi-indexes, vectors and multi-variables, e.g., $\boldsymbol{j} = (j_1, \ldots, j_d)$, $\boldsymbol{k} = (k_1, \ldots, k_d)$ and $\boldsymbol{x} = (x_1, \ldots, x_d)$. Also, let $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^d$, $e_i = (0, \ldots, 1, \ldots, 0)$ be the i-th unit vector in $\mathbb{R}^d$, and use the following conventions

$$\boldsymbol{\alpha} \geq \boldsymbol{k} \Leftrightarrow \forall_{1 \leq j \leq d},\ \alpha_j \geq k_j.$$

- Denote by $|\boldsymbol{\xi}|_1, |\boldsymbol{\xi}|_2, |\boldsymbol{\xi}|_\infty$ be the l^1, l^2, l^∞ norm of $\boldsymbol{\xi}$ in $\mathbb{R}^d$, respectively.
- Denote by $\mathcal{P}_M$ the set of all algebraic polynomials of degree $\leq M$.
- Let ω be a positive weight function in Λ, and denote by $(u, v)_{\Lambda,\omega} := \int_\Lambda uv\omega \, \mathrm{d}\Lambda$ the inner product of $L^2_\omega(\Lambda)$ whose norm is denoted by $\|\cdot\|_{\Lambda,\omega}$. In cases where no confusion arise, ω and Λ may be dropped from the notations whenever $\omega \equiv 1$. We use the expression $A \lesssim B$ to mean that $A \leq cB$.

2.1 Fourier-Like Basis Functions

We consider $\Omega := (-1, 1)^d$ with the following general homogeneous boundary conditions:

$$a_j^{\pm} v_j(\pm 1, t) + b_j^{\pm} \partial_{x_j} v_j(\pm 1, t) = 0, \quad t \geq 0, \quad 1 \leq j \leq d, \tag{2.1}$$

where $v_j(\pm 1, t) := v(x_1, \ldots, x_{j-1}, \pm 1, x_{j+1}, \ldots, x_d, t)$. To ensure the well-posedness, we assume that the constants $a_j^{\pm}$ and $b_j^{\pm}$ with $1 \leq j \leq d$ satisfy the following conditions

$$(i)\ a_j^{\pm} \geq 0; \quad (ii)\ (a_j^-)^2 + (b_j^-)^2 \neq 0,\ a_j^- b_j^- \leq 0; \quad (iii)\ (a_j^+)^2 + (b_j^+)^2 \neq 0,\ a_j^+ b_j^+ \geq 0. \tag{2.2}$$

We first define the one-dimensional spatial approximation space as

$$V_M = \{v \in \mathcal{P}_M(\Lambda) : a^{\pm} v(\pm 1) + b^{\pm} v_x(\pm 1) = 0\}, \tag{2.3}$$

and denote

$$h_k(x) = L_k(x) + a_k L_{k+1}(x) + b_k L_{k+2}(x), \quad 0 \leq k \leq M-2, \tag{2.4}$$

where $L_k(x)$ is the Legendre polynomial of degree k. It is shown in [25] that, for boundary conditions in form of (2.1), there exists a unique set $\{a_k, b_k\}$ such that $h_k \in V_{k+2}$.

We recall below the construction of the Fourier-like basis functions [28]. Denote by $\boldsymbol{M}$ (with entries $m_{pq} = (h_p, h_q)$) and $\boldsymbol{S}$ (with entries $s_{pq} = -(h_p'', h_q)$) be the mass matrix and stiffness matrix, respectively. Let $\boldsymbol{E} := (e_{pq})_{p,q=0,\ldots,M-2}$ be the matrix formed by the orthonormal eigenvectors of generalized eigenvalue problem of $\boldsymbol{M}$ and $\boldsymbol{S}$, and $\boldsymbol{\Lambda} = diag(\lambda_{M,i})$ be the diagonal matrix with main diagonal being the corresponding eigenvalues, i.e.,

$$\boldsymbol{SE} = \boldsymbol{ME\Lambda}, \quad \boldsymbol{E}^T\boldsymbol{SE} = \boldsymbol{\Lambda}, \quad \boldsymbol{E}^T\boldsymbol{ME} = \boldsymbol{I}_{M-1}. \tag{2.5}$$

Then, the Fourier-like basis is given by

$$\phi_{M,n}(x) = \sum_{j=0}^{M-2} e_{jn} h_j(x), \quad 0 \leq n \leq M-2. \tag{2.6}$$

Thank to (2.5), it is easy to verify that

$$(\phi_{M,p},\phi_{M,q})=\sum_{k,j=0}^{M-2}e_{kp}e_{jq}(h_k,h_j)=\sum_{k,j=0}^{N-2}e_{jq}m_{jk}e_{kp}=\big(\boldsymbol{E}^T\boldsymbol{M}\boldsymbol{E}\big)_{pq}=\delta_{pq},$$
$$(-\phi''_{M,p},\phi_{M,q})=-\sum_{k,j=0}^{N-2}e_{kp}e_{jq}(h''_k,h_j)=\sum_{k,j=0}^{M-2}e_{jq}s_{jk}e_{kp}=\big(\boldsymbol{E}^T\boldsymbol{S}\boldsymbol{E}\big)_{pq}=\lambda_{M,q}\delta_{pq}. \tag{2.7}$$

Let us define d-dimensional tensorial eigenfunctions and eigenvalues

$$\boldsymbol{\phi}_{M,\boldsymbol{n}}(\boldsymbol{x})=\prod_{j=1}^{d}\phi_{M,n_j}(x_j),\quad \boldsymbol{x}\in\Omega,\quad\text{and}\quad \boldsymbol{\lambda}_{M,\boldsymbol{n}}=(\lambda_{M,n_1},\ldots,\lambda_{M,n_d})^T,$$

and the d-dimensional approximation space

$$V_M^d:=\operatorname{span}\{\boldsymbol{\phi}_{M,\boldsymbol{n}}(\boldsymbol{x}):\boldsymbol{n}\in\Upsilon_M,\boldsymbol{x}\in\Omega\}, \tag{2.8}$$

where the index set

$$\Upsilon_M:=\big\{n=(n_1,\ldots,n_d)\,:\,0\le n_j\le M-2,\ 1\le j\le d\big\}. \tag{2.9}$$

One verifies by using (2.7) that

$$(\boldsymbol{\phi}_{M,\boldsymbol{n}},\boldsymbol{\phi}_{M,\boldsymbol{m}})=\delta_{\boldsymbol{mn}}\,,\quad (-\Delta\boldsymbol{\phi}_{M,\boldsymbol{n}},\boldsymbol{\phi}_{M,\boldsymbol{m}})=|\boldsymbol{\lambda}_{M,\boldsymbol{n}}|_1\,\delta_{\boldsymbol{mn}}, \tag{2.10}$$

where $\boldsymbol{m},\boldsymbol{n}\in\Upsilon_M$ and

$$\delta_{\boldsymbol{mn}}=\prod_{j=1}^{d}\delta_{m_jn_j},\quad |\boldsymbol{\lambda}_{M,\boldsymbol{n}}|_1=\lambda_{M,n_1}+\ldots+\lambda_{M,n_d}. \tag{2.11}$$

Indeed, the discrete Laplacian operator $-\Delta_M:V_M^d\to V_M^d$ can be interpreted as $-\Delta\big|_{V_M^d}$, which satisfies

$$\langle-\Delta_M\boldsymbol{\phi}_{M,\boldsymbol{n}},\boldsymbol{\phi}_{M,\boldsymbol{m}}\rangle=-(\Delta\boldsymbol{\phi}_{M,\boldsymbol{n}},\boldsymbol{\phi}_{M,\boldsymbol{m}})=|\boldsymbol{\lambda}_{M,\boldsymbol{n}}|_1\delta_{\boldsymbol{mn}},\quad\forall\boldsymbol{\phi}_{M,\boldsymbol{n}},\boldsymbol{\phi}_{M,\boldsymbol{m}}\in V_M^d.$$

Then, we arrive at following definition of discrete spectral fractional Laplacian.

Definition 2.1 Let $\{\boldsymbol{\lambda}_{M,\boldsymbol{n}},\boldsymbol{\phi}_{M,\boldsymbol{n}}\}_{\boldsymbol{n}\in\Upsilon_M}$ be the discrete eigenpairs of the Laplacian operator $-\Delta_M$ on V_M^d. For any $u\in\mathscr{D}(-\Delta):=H^2(\Omega)\cap H_0^1(\Omega)$, the discrete spectral fractional Laplacian is given by

$$(-\Delta_M)^{\frac{\alpha}{2}}\Pi_M u(\boldsymbol{x})=\sum_{\boldsymbol{n}\in\Upsilon_M}\widetilde{u}_{\boldsymbol{n}}|\boldsymbol{\lambda}_{M,\boldsymbol{n}}|_1^{\frac{\alpha}{2}}\boldsymbol{\phi}_{M,\boldsymbol{n}}(\boldsymbol{x}). \tag{2.12}$$

where $\Pi_M:\mathscr{D}(-\Delta)\to V_M^d$ denote the orthogonal projection. Moreover, for any $u_M\in V_M^d$, namely, $u_M(\boldsymbol{x})=\sum_{\boldsymbol{n}\in\Upsilon_M}\widetilde{u}_{\boldsymbol{n}}\boldsymbol{\phi}_{M,\boldsymbol{n}}(\boldsymbol{x})$, the spectral fractional Laplacian is given by

$$(-\Delta_M)^{\frac{\alpha}{2}}u_M(\boldsymbol{x})=\sum_{\boldsymbol{n}\in\Upsilon_M}\widetilde{u}_{\boldsymbol{n}}|\boldsymbol{\lambda}_{M,\boldsymbol{n}}|_1^{\frac{\alpha}{2}}\boldsymbol{\phi}_{M,\boldsymbol{n}}(\boldsymbol{x}). \tag{2.13}$$

2.2 Spectral Method for Fractional Laplacian Equation

We consider (1.1) with (2.1). Then, our spectral Galerkin method using the Fourier-like basis functions is as follows:

Find $u_M \in V_M^d$ such that

$$\left((-\Delta_M)^{\frac{\alpha}{2}} u_M, v\right)_\Omega = (f, v)_\Omega \quad \forall v \in V_M^d. \tag{2.14}$$

We write

$$u_M(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\Upsilon_M} \tilde{u}_{\boldsymbol{m}} \boldsymbol{\phi}_{M,\boldsymbol{m}}(\boldsymbol{x}). \tag{2.15}$$

Substituting (2.15) into (2.14) and taking $v = \boldsymbol{\phi}_{M,\boldsymbol{n}}(\boldsymbol{x})$, the scheme (2.14) becomes

$$\sum_{\boldsymbol{m}\in\Upsilon_M} \tilde{u}_{\boldsymbol{m}}((-\Delta_M)^{\frac{\alpha}{2}} \boldsymbol{\phi}_{M,\boldsymbol{m}}, \boldsymbol{\phi}_{M,\boldsymbol{n}})_\Omega = (f, \boldsymbol{\phi}_{M,\boldsymbol{n}})_\Omega. \tag{2.16}$$

By the orthogonality (2.10) and (2.13), we obtain

$$\tilde{u}_{\boldsymbol{n}} = |\lambda_{M,\boldsymbol{n}}|_1^{-\frac{\alpha}{2}} \tilde{f}_{\boldsymbol{n}}, \quad \forall \boldsymbol{n} \in \Upsilon_M,$$

where

$$\tilde{f}_{\boldsymbol{n}} = (f, \boldsymbol{\phi}_{M,\boldsymbol{n}})_\Omega.$$

Finally, the numerical solutions of (2.14) can be obtained from (2.15).

2.3 Space-Time Spectral Method

We shall start by considering a special case of (1.2) with $\mathcal{N}(v,t) = \beta v - g$:

$$v_t + \epsilon(-\Delta)^{\frac{\alpha}{2}} v + \beta v = g, \quad \forall(\boldsymbol{x}, t) \in D = \Omega \times (0, T], \tag{2.17}$$

with the initial condition $v(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \in \mathscr{D}((-\Delta)^{\frac{\alpha}{2}})$ (see (3.5) below) and boundary conditions (2.1). Here the constants $\epsilon, \beta > 0$ and g is a given function. We shall extend the algorithm to the general case of (1.2) in Sect. 5.

We propose below a space-time spectral method for (2.17) based on the Fourier-like basis in space and a dual-Petrov Legendre–Galerkin formulation in time.

We first decompose the solution $v(\boldsymbol{x}, t)$ into two parts as

$$v(\boldsymbol{x}, t) = u(\boldsymbol{x}, t) + u_0(\boldsymbol{x}), \tag{2.18}$$

with $u(\boldsymbol{x}, 0) = 0$. Hence, by (2.18) , the equation (2.17) is equivalent to the following equation

$$u_t + \epsilon(-\Delta)^{\frac{\alpha}{2}} u + \beta u = f, \quad \forall(\boldsymbol{x}, t) \in D, \tag{2.19}$$

where

$$f(\boldsymbol{x}, t) = g(\boldsymbol{x}, t) - \epsilon(-\Delta)^{\frac{\alpha}{2}} u_0(\boldsymbol{x}) - \beta u_0(\boldsymbol{x}),$$

with

$$u(\boldsymbol{x}, 0) = 0, \quad \forall \boldsymbol{x} \in \Omega, \tag{2.20}$$

and the boundary conditions (2.1).

Since (2.17) or (2.19) has first order derivative in time, it is suitable to use the dual-Petrov Legendre–Galerkin method in time direction [26,28]. To simplify the presentation, we first scale the time interval from $(0, T)$ to $(-1, 1)$, and define a pair of "dual" approximation spaces (in time):

$$S_N = \{u \in \mathcal{P}_N : u(-1) = 0\}, \quad S_N^* = \{u \in \mathcal{P}_N : u(1) = 0\}. \tag{2.21}$$

Then, after scaling the time interval to $(-1, 1)$, the space-time spectral approximation of (2.19) is: Find $u_L \in V_M^d \otimes S_N$ such that

$$(\partial_t u_L, v)_D, +\epsilon((-\Delta_M)^{\frac{\alpha}{2}} u_L, v)_D + \beta(u_L, v)_D = (f, v)_D, \quad \forall v \in V_M^d \otimes S_N^*. \tag{2.22}$$

We now describe the numerical implementation of (2.22). An obvious choice for V_M^d in space is the Fourier-like basis function $\{\boldsymbol{\phi}_{M,\boldsymbol{k}}\}_{\boldsymbol{k}\in\Upsilon_M}$. As for S_N and S_N^* in time, we set

$$\psi_n(t) = L_n(t) + L_{n+1}(t), \qquad \psi_q^*(t) = L_q(t) - L_{q+1}(t), \tag{2.23}$$

and write

$$u_L(\boldsymbol{x}, t) = \sum_{\boldsymbol{m}\in\Upsilon_M} \sum_{n=0}^{N-1} \tilde{u}_{n,\boldsymbol{m}} \boldsymbol{\phi}_{M,\boldsymbol{m}}(\boldsymbol{x}) \psi_n(t). \tag{2.24}$$

Substituting (2.24) into (2.22) and taking $v = \boldsymbol{\phi}_{M,\boldsymbol{p}}(\boldsymbol{x})\psi_q^*(t)$, the scheme (2.22) becomes

$$\sum_{\boldsymbol{m}\in\Upsilon_M} \sum_{n=0}^{N-1} \tilde{u}_{n\boldsymbol{m}} \Big\{ (\boldsymbol{\phi}_{M,\boldsymbol{m}}, \boldsymbol{\phi}_{M,\boldsymbol{p}})_\Omega\, (\partial_t \psi_n, \psi_q^*)_I + \epsilon((-\Delta_M)^{\frac{\alpha}{2}} \boldsymbol{\phi}_{M,\boldsymbol{m}}, \boldsymbol{\phi}_{M,\boldsymbol{p}})_\Omega\, (\psi_n, \psi_q^*)_I + \beta(\boldsymbol{\phi}_{M,\boldsymbol{m}}, \boldsymbol{\phi}_{M,\boldsymbol{p}})_\Omega\, (\psi_n, \psi_q^*)_I \Big\} = (f, \boldsymbol{\phi}_{M,\boldsymbol{p}} \psi_q^*)_D. \tag{2.25}$$

Denote

$$\begin{aligned} u_{\boldsymbol{m}} &= (\tilde{u}_{0,\boldsymbol{m}}, \tilde{u}_{1,\boldsymbol{m}}, \ldots, \tilde{u}_{N-1,\boldsymbol{m}})^T, \\ s_{qn}^t &= (\partial_t \psi_n, \psi_q^*)_I, \quad m_{qn}^t = (\psi_j, \psi_i^*)_I, \\ \boldsymbol{S}^t &= (s_{qn}^t)_{0\le q,n\le N-1}, \quad \boldsymbol{M}^t = (m_{qn}^t)_{0\le q,n\le N-1}, \\ \tilde{f}_{q,\boldsymbol{p}} &= (f, \boldsymbol{\phi}_{M,\boldsymbol{p}} \psi_q^*)_D, \quad \boldsymbol{f}_{\boldsymbol{m}} = (\tilde{f}_{0,\boldsymbol{m}}, \tilde{f}_{1,\boldsymbol{m}}, \ldots, \tilde{f}_{N-1,\boldsymbol{m}})^T. \end{aligned} \tag{2.26}$$

One may verifies that [28]

$$s_{qn}^t = (\partial_t \psi_n, \psi_q^*) = 2\delta_{qn}, \quad m_{qn}^t = (\psi_n, \psi_q^*) = 0, \text{ if } |q-n| > 1. \tag{2.27}$$

Then, from (2.26) and (2.27), we find that (2.22) is equivalent to the following

$$(2\boldsymbol{I} + (\epsilon|\lambda_{M,\boldsymbol{m}}|_1^{\frac{\alpha}{2}} + \beta)\boldsymbol{M}^t) u_{\boldsymbol{m}} = \boldsymbol{f}_{\boldsymbol{m}}, \qquad \boldsymbol{m} \in \Upsilon_M. \tag{2.28}$$

Since $\boldsymbol{M}^t$ is tri-diagonal, the above systems can be efficiently solved.

Finally, we obtain the numerical solutions of (2.22) by (2.24).

3 Error Analysis

The aim of this section is to perform error analysis for the two spectral algorithms (2.14) and (2.22) described in the previous section. For the sake of brevity, we shall restrict ourselves to the cases with homogeneous Dirichlet boundary conditions.

Throughout this section, we assume that $\alpha \in (1, 2)$ and $s = \frac{\alpha}{2}$.

3.1 Approximation of Fractional Laplacian

We first recall some results in [18], which play a very important role in the forthcoming analysis.

Definition 3.1 (see, e.g., [18, Appendix]) Let $(\mathcal{X}, \|\cdot\|_X)$ and $(\mathcal{Y}, \|\cdot\|_Y)$ be two Banach spaces. A linear operator $A : \mathscr{D}(A) \subset \mathcal{X} \to \mathcal{Y}$ with operator norm

$$\|A\|_{\mathscr{L}(\mathcal{X},\mathcal{Y})} = \sup_{0 \neq x \in \mathcal{X}} \frac{\|Ax\|_Y}{\|x\|_X}$$

is said to be of type (ω, K), if

1. A is densely defined and closed;
2. the resolvent set of $-A$ contains the sector $|\arg \lambda| < \pi - \omega$, $0 < \omega < \pi$, and $\lambda(\lambda + A)^{-1}$ is uniformly bounded in each smaller sector $|\arg \lambda| < \pi - \omega - \varepsilon$, $\varepsilon > 0$ with $\|\lambda(\lambda + A)^{-1}\|_{\mathscr{L}(\mathcal{Y},\mathcal{X})} \leq K$ for $\lambda > 0$.

The following Corollary can be found in [18, Appendix].

Corollary 3.1 *If A is of type $(\frac{\pi}{2}, K)$, then we have*

$$\|(\mu + A)^{-1} A u\|_X \leq (1 + K)\|u\|_X, \quad \|(\mu + A)^{-1}\|_{\mathscr{L}(\mathcal{Y},\mathcal{X})} \leq K\mu^{-1}, \quad \textit{for } \mu > 0. \tag{3.1}$$

For $u \in \mathscr{D}(-\Delta) := H^2(\Omega) \cap H_0^1(\Omega)$, the operator $-\Delta : H^2(\Omega) \to L^2(\Omega)$ can be defined via a bilinear form $\mathcal{A}(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, namely

$$\mathcal{A}(u, v) := \langle -\Delta u, v \rangle = (\nabla u, \nabla v), \quad \text{for} \quad u, v \in H_0^1(\Omega), \tag{3.2}$$

which satisfies continuity and coercivity.

For $u_M, v_M \in V_M^d$, the operator $-\Delta_M$ can be defined by the following bilinear form

$$\langle -\Delta_M u_M, v_M \rangle = (\nabla u_M, \nabla v_M), \quad u_M, v_M \in V_M^d, \tag{3.3}$$

which satisfies continuity and coercivity.

Because of the continuity and coercivity, we immediately get the following lemma.

Lemma 3.1 *The operator $-\Delta$ is of type $(\frac{\pi}{2}, 1)$. Moreover, the operator $-\Delta_M$ is also of type $(\frac{\pi}{2}, 1)$.*

Proof According to [18,33], the operator Δ is a closed and maximal accretive operator. On the other hand, we have the following equivalence (see, e.g., [18])

- the operator $-\Delta$ in Ω is a closed and maximal accretive operator;
- the operator $-\Delta$ is of type $(\frac{\pi}{2}, 1)$;

which will automatically lead to the desired results. Note that $-\Delta_M = -\Delta|_{V_M^d}$, so the operator $-\Delta_M$ is also of type $(\frac{\pi}{2}, 1)$ (see, e.g., [18]). □

We now turn to fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$. By density, the operator $(-\Delta)^{\frac{s}{2}}$ can be extended to the Hilbert space

$$\mathbb{H}^s(\Omega) = \left\{ u = \sum_{k=1}^{\infty} u_k \phi_k \in L^2(\Omega) : \|u\|_{\mathbb{H}^s(\Omega)}^2 = \sum_{k=1}^{\infty} |u_k|^2 \lambda_k^s < \infty \right\}. \tag{3.4}$$

The theory of Hilbert scales presented in [20, Chap. 1] shows that

$$\left[H_0^1(\Omega), L^2(\Omega)\right]_\theta = \mathscr{D}((-\Delta)^{\frac{s}{2}}), \quad \text{with} \quad \theta = 1 - s. \tag{3.5}$$

This implies the following characterization of the space $\mathbb{H}^s(\Omega)$

$$\mathbb{H}^s(\Omega) = \begin{cases} H^s(\Omega) = \left[H^1(\Omega), L^2(\Omega)\right]_{1-s}, \; s \in (0, \frac{1}{2}), \\ H_{00}^{\frac{1}{2}}(\Omega) = \left[H_0^1(\Omega), L^2(\Omega)\right]_{\frac{1}{2}}, \quad s = \frac{1}{2}, \\ H_0^s(\Omega) = \left[H_0^1(\Omega), L^2(\Omega)\right]_{1-s}, \; s \in (\frac{1}{2}, 1), \end{cases} \tag{3.6}$$

where $H^s(\Omega)$ and $H_0^s(\Omega)$, $s \neq \frac{1}{2}$, are the classical fractional Sobolev spaces, and $H_{00}^{\frac{1}{2}}$ denote the Lions-Magenes space, which can be characterized as

$$H_{00}^{\frac{1}{2}} = \left\{ u \in H^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{u^2(y)}{\mathrm{dist}(y, \partial\Omega)} \mathrm{d}y < \infty \right\}.$$

If the boundary of Ω is Lipschitz, we have $H_0^s(\Omega) = H^s(\Omega)$ for $s \in (0, \frac{1}{2}]$; and $H_0^s(\Omega)$ is strictly contained in $H^s(\Omega)$ for $s \in (\frac{1}{2}, 1)$. In particular, we have the strict inclusion $H_{00}^{\frac{1}{2}}(\Omega) \subsetneq H_0^{\frac{1}{2}}(\Omega) = H^{\frac{1}{2}}(\Omega)$.

Then, we write $\mathscr{D}((-\Delta)^{\frac{\alpha}{2}}) = \left\{u : (-\Delta)^{\frac{\alpha}{2}} u \in L^2(\Omega), u \in H_0^{\frac{\alpha}{2}}(\Omega)\right\}$ with $\alpha \in (1, 2)$. For $u \in \mathscr{D}((-\Delta)^{\frac{\alpha}{2}})$, the operator $(-\Delta)^{\frac{\alpha}{2}}$ can also be defined via a bilinear form

$$\mathcal{A}^{\frac{\alpha}{2}}(u, v) := \langle (-\Delta)^{\frac{\alpha}{2}} u, v \rangle = ((-\Delta)^{\frac{\alpha}{4}} u, (-\Delta)^{\frac{\alpha}{4}} v), \quad \text{for} \quad u, v \in H_0^{\frac{\alpha}{2}}(\Omega). \tag{3.7}$$

Lemma 3.2 *For $v \in H_0^{\frac{\alpha}{2}}(\Omega)$, $\alpha \neq 1$ we have*

$$|(-\Delta)^{\frac{\alpha}{4}} v| \cong |v|_{H^{\frac{\alpha}{2}}(\Omega)}. \tag{3.8}$$

Proof One derives immediately from (3.6) that the equality holds. □

Using the first equality of (3.8), we deduce the following:

$$\begin{aligned} |\mathcal{A}^{\frac{\alpha}{2}}(u, v)| &\lesssim \|u\|_{H^{\frac{\alpha}{2}}(\Omega)} \|v\|_{H^{\frac{\alpha}{2}}(\Omega)}, \quad \text{for} \quad u, v \in H_0^{\frac{\alpha}{2}}(\Omega), \\ |\mathcal{A}^{\frac{\alpha}{2}}(u, u)| &\gtrsim \|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)}, \qquad \text{for} \quad u \in H_0^{\frac{\alpha}{2}}(\Omega). \end{aligned} \tag{3.9}$$

From Definition 2.1, for any $u_M, v_M \in V_M^d$, the operator $(-\Delta_M)^{\frac{\alpha}{2}} : V_M^d \to V_M^d$ can be defined via following bilinear form

$$\langle (-\Delta_M)^{\frac{\alpha}{2}} u_M, v_M \rangle = ((-\Delta_M)^{\frac{\alpha}{4}} u_M, (-\Delta_M)^{\frac{\alpha}{4}} v_M). \tag{3.10}$$

The following Lemma can be found in [33, Lemma 4.1].

Lemma 3.3 *For $0 \le s \le \frac{1}{2}$, there is a constant C, such that for any $v_M \in V_M^d$,*

$$\|(-\Delta)^s v_M\| \le C \|(-\Delta_M)^s v_M\|. \tag{3.11}$$

For $0 \le s \le 1$, there is a constant C, such that for any $v_M \in V_M^d$,

$$\|(-\Delta_M)^s v_M\| \le C \|(-\Delta)^s v_M\|. \tag{3.12}$$

Therefore, we deduce from Lemma 3.2 and Lemma 3.3 that

$$\begin{aligned}
|((-\Delta_M)^{\frac{\alpha}{4}}u_M, (-\Delta_M)^{\frac{\alpha}{4}}v_M)| &\lesssim \|(-\Delta_M)^{\frac{\alpha}{4}}u_M\|_{L^2(\Omega)}\|(-\Delta_M)^{\frac{\alpha}{4}}v_M\|_{L^2(\Omega)} \\
&\lesssim \|(-\Delta)^{\frac{\alpha}{4}}u_M\|_{L^2(\Omega)}\|(-\Delta)^{\frac{\alpha}{4}}v_M\|_{L^2(\Omega)} \\
&\lesssim \|u_M\|_{H^{\frac{\alpha}{2}}(\Omega)}\|v_M\|_{H^{\frac{\alpha}{2}}(\Omega)}, \qquad \text{for} \ \ u_M, v_M \in V_M^d, \\
|((-\Delta_M)^{\frac{\alpha}{4}}u_M, (-\Delta_M)^{\frac{\alpha}{4}}u_M)| &\gtrsim \|(-\Delta_M)^{\frac{\alpha}{4}}u_M\|^2_{L^2(\Omega)} \\
&\gtrsim \|(-\Delta)^{\frac{\alpha}{4}}u_M\|^2_{L^2(\Omega)} \gtrsim \|u_M\|^2_{H^{\frac{\alpha}{2}}(\Omega)}, \quad \text{for} \ \ u_M \in V_M^d.
\end{aligned} \tag{3.13}$$

Next, we define negative norms by

$$\|u\|_{-s} = \sup\left\{\frac{(u,v)}{\|v\|_{H^s}}; v \in H^s\right\}, \quad \text{for} \quad s \ge 0. \tag{3.14}$$

The following Lemma plays a key role in the error analysis.

Lemma 3.4 *For any* $0 < s < 1$ *and* $u \in H_0^1(\Omega) \cap H^{2-s}(\Omega)$*, we have*

$$\|(-\Delta)^s u - (-\Delta_M)^s \Pi_M u\|_{H^{-s}(\Omega)} \le c_{u,s} \, \|(-\Delta)u - (-\Delta_M)\Pi_M u\|^s_{H^{-s}(\Omega)}.$$

where $c_{u,s} = c(s)\,(\|\Pi_M u\|_{H^{-s}(\Omega)} + \|u\|_{H^{-s}(\Omega)})^{1-s}$.

Proof By virtue of $-\Delta_M$ is of type $(\frac{\pi}{2}, 1)$ and $(-\Delta_M)^{-1}$ is bounded, we follow the procedure used in [18], the operator $(-\Delta_M)^s$ can be defined indirectly by

$$(-\Delta_M)^s \Pi_M u = \frac{\sin(\pi s)}{\pi}\int_0^\infty \mu^{s-1}\,(\mu + (-\Delta_M))^{-1}\,(-\Delta_M)\Pi_M u \, \mathrm{d}\mu. \tag{3.15}$$

Subtracting from (3.15) a similar expression with M replaced by M', $M \le M'$, we have

$$\begin{aligned}
&(-\Delta_M)^s\Pi_M u - (-\Delta_{M'})^s\Pi_{M'}u \\
&= \frac{\sin(\pi s)}{\pi}\Big[\int_0^\delta \mu^{s-1}(\mu + (-\Delta_M))^{-1}(-\Delta_M)\Pi_M u \, \mathrm{d}\mu \\
&\quad - \int_0^\delta \mu^{s-1}(\mu + (-\Delta_{M'}))^{-1}(-\Delta_{M'})\Pi_{M'}u \, \mathrm{d}\mu \\
&\quad + \int_\delta^\infty \mu^s(\mu + (-\Delta_M))^{-1}(\mu + (-\Delta_{M'}))^{-1}\Big((-\Delta_M)\Pi_M u - (-\Delta_{M'})\Pi_{M'}u\Big)\mathrm{d}\mu\Big].
\end{aligned} \tag{3.16}$$

Moreover, we can derive from (3.1) with $K = 1$ and the norm of linear operators (see, e.g., [6, p. 102]) upon H^{-s} that

$$\|(\mu + (-\Delta_M))^{-1}(-\Delta_M)\Pi_M u\|_{H^{-s}} \le 2\|\Pi_{M'}u\|_{H^{-s}(\Omega)}$$

and

$$\begin{aligned}
&\|(\mu + (-\Delta_M))^{-1}(\mu + (-\Delta_{M'}))^{-1}\,((-\Delta_M)\Pi_M u - (-\Delta_{M'})\Pi_{M'}u)\,\|_{H^{-s}(\Omega)} \\
&\le \|(\mu + (-\Delta_M))^{-1}(\mu + (-\Delta_{M'}))^{-1}\|_{\mathscr{L}(V_M^d, V_M^d)}\|(-\Delta_M)\Pi_M u - (-\Delta_{M'})\Pi_{M'}u\|_{H^{-s}(\Omega)} \\
&\le \mu^{-2}\|(-\Delta_M)\Pi_M u - (-\Delta_{M'})\Pi_{M'}u\|_{H^{-s}(\Omega)}.
\end{aligned}$$

This, along with equation (3.16), yields

$$\begin{aligned}
&\|(-\Delta_M)^s \Pi_M u - (-\Delta_{M'})^s \Pi_{M'} u\|_{H^{-s}(\Omega)} \\
&\le \frac{\sin(\pi s)}{\pi}\Big[2(\|\Pi_M u\|_{H^{-s}(\Omega)} + \|\Pi_{M'} u\|_{H^{-s}(\Omega)}) \int_0^\delta \mu^{s-1}\mathrm{d}\mu \\
&\quad + \|(-\Delta_M)\Pi_M u - (-\Delta_{M'})\Pi_{M'} u\|_{H^{-s}(\Omega)} \int_\delta^\infty \mu^{s-2}\mathrm{d}\mu\Big] \\
&= \frac{\sin(\pi s)}{\pi}\Big[2s^{-1}\delta^s(\|\Pi_M u\|_{H^{-s}(\Omega)} + \|\Pi_{M'} u\|_{H^{-s}(\Omega)}) \\
&\quad + (1-s)^{-1}\delta^{s-1}\|(-\Delta_M)\Pi_M u - (-\Delta_{M'})\Pi_{M'} u\|_{H^{-s}(\Omega)}\Big].
\end{aligned}$$

Taking $\delta = \|(-\Delta_M)\Pi_M u - (-\Delta_{M'})\Pi_{M'} u\|_{H^{-s}(\Omega)}/(\|\Pi_M u\|_{H^{-s}(\Omega)} + \|\Pi_{M'} u\|_{H^{-s}(\Omega)})$, we deduce that

$$\begin{aligned}
&\|(-\Delta_M)^s \Pi_M u - (-\Delta_{M'})^s \Pi_{M'} u\|_{H^{-s}(\Omega)} \\
&\quad \le c(s)(\|\Pi_M u\|_{H^{-s}(\Omega)} + \|\Pi_{M'} u\|_{H^{-s}(\Omega)})^{1-s}\|(-\Delta_M)\Pi_M u - (-\Delta_{M'})\Pi_{M'} u\|^s_{H^{-s}(\Omega)}.
\end{aligned}$$

Finally, letting $M' \to \infty$ leads to the desired results. □

Remark 3.1 The regularity assumption of the solution $u \in H_0^1(\Omega) \cap H^{2-\frac{\alpha}{2}}(\Omega)$ is necessary for the result in Lemma 3.4, which is essential for the error estimates in Theorems 3.1 and 3.1. Note that we can follow the same line as in [5] to obtain the following estimate:

$$\|(-\Delta)^{-\frac{\alpha}{2}} f - (-\Delta_M)^{-\frac{\alpha}{2}} \Pi_M f\|_{L^2(\Omega)} \le cM^{-\alpha}\|f\|_{L^2(\Omega)}, \tag{3.17}$$

which only requires $f \in L^2(\Omega)$ (see [12,16] for similar results on FEM), but it is difficult to derive a result similar to (3.17) for fractional parabolic PDEs (2.19). □

3.2 Error Bounds for (2.14)

To fix the idea, we restrict our attention to $\Omega = \Lambda^d$, $\Lambda = (-1, 1)$. Let us begin with the L^2-orthogonal projection $\widehat{\Pi}_M : L^2(\Lambda) \to \mathcal{P}_M$ such that

$$\big(\widehat{\Pi}_M u - u, v_M\big)_\Lambda = 0, \quad \forall v_M \in \mathcal{P}_M. \tag{3.18}$$

According to [14, Theorem 2.9]: If $u \in H^r(\Lambda)$, $r \ge 0$ and $\mu \le r$, then for $\mu \in [0, 1]$,

$$\|\widehat{\Pi}_M u - u\|_{H^\mu(\Lambda)} \le cM^{\frac{3}{2}\mu - r}\|u\|_{H^r(\Lambda)}. \tag{3.19}$$

Moreover, we define the H_0^1-orthogonal projection $\Pi_M^{1,0} : H_0^1(\Lambda) \to V_M$ such that

$$\big(\partial_x(\Pi_M^{1,0} u - u), \partial_x v_M\big)_\Lambda = 0, \quad \forall v_M \in V_M. \tag{3.20}$$

Lemma 3.5 *If $u \in H_0^1(\Lambda) \cap H^r(\Lambda)$, then for $0 \le \mu \le 1 \le r$,*

$$\|\Pi_M^{1,0} u - u\|_{H^\mu(\Lambda)} \le cM^{\mu - r}\|u\|_{H^r(\Lambda)}, \tag{3.21}$$

while for $1 \le \mu \le 2 \le r$,

$$\|\Pi_M^{1,0} u - u\|_{H^\mu(\Lambda)} \le cM^{\frac{3}{2}\mu - \frac{1}{2} - r}\|u\|_{H^r(\Lambda)}, \tag{3.22}$$

where c is a positive constant independent of M and u.

Proof The result (3.21) can be found in [14, Theorem 2.11]. Now, we prove (3.22) with $\mu = 2$. By using the triangle inequality and (3.19), we find that

$$\begin{aligned}\|\partial_x^2(\Pi_M^{1,0}u - u)\|_\Lambda &\le \|\partial_x(\partial_x\Pi_M^{1,0}u - \widehat{\Pi}_M\partial_x u)\|_\Lambda + \|\partial_x(\widehat{\Pi}_M\partial_x u - \partial_x u)\|_\Lambda \\ &\le cM^{\frac{5}{2}-r}\|u\|_{H^r(\Lambda)} + \|\partial_x(\partial_x\Pi_M^{1,0}u - \widehat{\Pi}_M\partial_x u)\|_\Lambda.\end{aligned} \tag{3.23}$$

Recall some properties of Legendre polynomials (cf. [27]):

$$\int_{-1}^{1} L_n(x)L_m(x)\,\mathrm{d}x = \frac{2}{2n+1}\delta_{mn}, \tag{3.24a}$$

$$(2n+3)L_{n+1}(x) = L'_{n+2}(x) - L'_n(x), \quad n \ge 0, \tag{3.24b}$$

$$L'_n(x) = \sum_{\substack{k=0\\ k+n\ \text{odd}}}^{n-1} (2k+1)L_k(x). \tag{3.24c}$$

For any $u \in H_0^1(\Lambda)$, we write

$$u(x) = \sum_{n=0}^{\infty} \tilde{u}_n(L_{n+2}(x) - L_n(x)).$$

Then, we obtain from (3.24b) that

$$\begin{aligned}\widehat{\Pi}_M\partial_x u &= \widehat{\Pi}_M\sum_{n=0}^{\infty}\tilde{u}_n(L'_{n+2}(x) - L'_n(x)) = \widehat{\Pi}_M\sum_{n=0}^{\infty}\tilde{u}_n(2n+3)L_{n+1}(x)\\ &= \sum_{n=0}^{M-1}\tilde{u}_n(2n+3)L_{n+1}(x),\\ \partial_x\Pi_M^{1,0}u &= \partial_x\Pi_M^{1,0}\sum_{n=0}^{\infty}\tilde{u}_n(L_{n+2}(x) - L_n(x)) = \sum_{n=0}^{M-2}\tilde{u}_n(L'_{n+2}(x) - L'_n(x))\\ &= \sum_{n=0}^{M-2}\tilde{u}_n(2n+3)L_{n+1}(x).\end{aligned}$$

Thereby

$$\widehat{\Pi}_M\partial_x u(x) - \partial_x\Pi_M^{1,0}u(x) = \tilde{u}_{M-1}(2M+1)L_M(x).$$

This together with (3.21), (3.24a) and (3.24c) leads to

$$\begin{aligned}\|\partial_x(\widehat{\Pi}_M\partial_x u - \partial_x\Pi_M^{1,0}u)\|_\Lambda^2 &= \tilde{u}_{M-1}^2(2M+1)^2\|\partial_x L_M\|_\Lambda^2\\ &= 2\tilde{u}_{M-1}^2(2M+1)^2\sum_{\substack{k=0\\ k+n\ \text{odd}}}^{M-1}(2k+1) \le 2\tilde{u}_{M-1}^2(2M+1)^2M^2\\ &\le 2M^2\|\partial_x(\Pi_M^{1,0}u - u)\|_\Lambda^2 \le cM^{2(2-r)}\|u\|_{H^r(\Lambda)}^2.\end{aligned} \tag{3.25}$$

A combination of (3.21) with $\mu = 1$, (3.23), (3.25) and the standard space interpolation leads to (3.22). This completes the proof. □

Now, we turn to the d-dimensional $H_0^1(\Omega)$-orthogonal projection $\Pi_M^d : H_0^1(\Omega) \to V_M^d$ satisfy

$$(\nabla(\Pi_M^d u - u), \nabla v)_\Omega = 0, \quad \forall v \in V_M^d. \tag{3.26}$$

Then, we can derive the following approximation property of Π_M^d.

Lemma 3.6 *If $u \in H_0^1(\Omega) \cap H^r(\Omega)$ with $r \geq d$ then for $\mu \in [0, 1]$,*

$$\|\Pi_M^d u - u\|_{H^\mu(\Omega)} \lesssim M^{\mu - r} \|u\|_{H^r(\Omega)}, \tag{3.27}$$

while for $\mu \in (1, 2]$,

$$\|\Pi_M^d u - u\|_{H^\mu(\Omega)} \lesssim M^{\frac{3}{2}\mu - \frac{1}{2} - r} \|u\|_{H^r(\Omega)}. \tag{3.28}$$

Proof The estimate (3.27) is a direct extension of [27, Theorem 8.2 & 8.3]. Thus we only need to focus on the estimate (3.28) with $\mu = 2$. In view of (3.26), we have

$$\Pi_M^d := \Pi_M^{(1)} \circ \ldots \circ \Pi_M^{(d)} \quad \text{with} \quad \Pi_M^{(j)} = \Pi_M^{1,0}, \ j = 1, \ldots, d. \tag{3.29}$$

For clarity, we only prove the results with $d = 3$, as it is straightforward to extend the results to the case with $d > 3$. Using integration by parts, we have

$$\begin{aligned} \|\Delta(\Pi_M^3 u - u)\|_\Omega^2 &= \sum_{j=1}^3 \|\partial_{x_j}^2 (\Pi_M^3 u - u)\|_\Omega^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^3 \int_{\Lambda^3} \partial_{x_i}^2 (\Pi_M^3 u - u) \partial_{x_j}^2 (\Pi_M^3 u - u) \mathrm{d}\boldsymbol{x} \\ &= \sum_{j=1}^3 \|\partial_{x_j}^2 (\Pi_M^3 u - u)\|_\Omega^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^3 \|\partial_{x_i} \partial_{x_j} (\Pi_M^3 u - u)\|_\Omega^2 := I_1 + I_2. \end{aligned} \tag{3.30}$$

By virtue of (3.22) with $\mu = 2$ and (3.29), we obtain that

$$\begin{aligned} I_1 &\leq 2\Big(\|\partial_{x_1}^2 (\Pi_M^{(1)} u - u)\|_\Omega^2 + \|\partial_{x_1}^2 \Pi_M^{(1)} \circ (\Pi_M^{(2)} \circ \Pi_M^{(3)} u - u)\|_\Omega^2 \\ &\quad + \|\partial_{x_2}^2 (\Pi_M^{(2)} u - u)\|_\Omega^2 + \|\partial_{x_2}^2 \Pi_M^{(2)} \circ (\Pi_M^{(1)} \circ \Pi_M^{(3)} u - u)\|_\Omega^2 \\ &\quad + \|\partial_{x_3}^2 (\Pi_M^{(3)} u - u)\|_\Omega^2 + \|\partial_{x_3}^2 \Pi_M^{(3)} \circ (\Pi_M^{(1)} \circ \Pi_M^{(2)} u - u)\|_\Omega^2 \Big) \\ &\leq c M^{5-2r} \|u\|_{H^r(\Omega)}^2 + 2M \|\Pi_M^{(2)} \circ \Pi_M^{(3)} (\partial_{x_1}^2 u) - (\partial_{x_1}^2 u)\|_\Omega^2 \\ &\quad + 2M \|\Pi_M^{(1)} \circ \Pi_M^{(3)} (\partial_{x_2}^2 u) - (\partial_{x_2}^2 u)\|_\Omega^2 + 2M \|\Pi_M^{(1)} \circ \Pi_M^{(2)} (\partial_{x_3}^2 u) - (\partial_{x_3}^2 u)\|_\Omega^2 \end{aligned} \tag{3.31}$$

Hence, it remains to estimate the last three terms in (3.31), and we only need to consider the last term as they are similar to each other. Then, we can derive from (3.31), (3.21) with $\mu = 0$ and (3.21) with $\mu = 0, r = 1$ that

$$\begin{aligned} &\|\Pi_M^{(1)} \circ \Pi_M^{(2)} (\partial_{x_3}^2 u) - (\partial_{x_3}^2 u)\|_\Omega \leq \|\Pi_M^{(2)} (\partial_{x_3}^2 u) - (\partial_{x_3}^2 u)\|_\Omega \\ &\quad + \|\Pi_M^{(1)} (\partial_{x_3}^2 u) - (\partial_{x_3}^2 u)\|_\Omega + \|(\mathcal{I} - \Pi_M^{(2)}) \big(\Pi_M^{(1)} (\partial_{x_3}^2 u) - (\partial_{x_3}^2 u)\big)\|_\Omega \\ &\leq c M^{2-r} \|u\|_{H^r(\Omega)} + c M^{-1} \big\| \Pi_M^{(1)} (\partial_{x_2} \partial_{x_3}^2 u) - (\partial_{x_2} \partial_{x_3}^2 u) \big\|_\Omega \leq c M^{2-r} \|u\|_{H^r(\Omega)}. \end{aligned}$$

Applying this argument repeatedly leads to

$$I_1 \leq c M^{5-2r} \|u\|_{H^r(\Omega)}^2. \tag{3.32}$$

Now, we turn to estimate I_2. Similarly, we only consider $i=1$, $j=2$ in I_2, since the other terms in I_2 can be derived in a similar fashion. Then, we obtain from (3.29) and the triangle inequality that

$$\begin{aligned}\|\partial_{x_1}\partial_{x_2}(\Pi_M^3 u-u)\|_\Omega &\le \|\partial_{x_1}\partial_{x_2}(\Pi_M^{(1)}\circ\Pi_M^{(2)}u-u)\|_\Omega+\|\partial_{x_1}\partial_{x_2}\Pi_M^{(1)}\circ\Pi_M^{(2)}(\Pi_M^{(3)}u-u)\|_\Omega\\ &\le cM^{2-r}\|u\|_{H^r(\Omega)}.\end{aligned}$$

Applying this argument repeatedly leads to

$$I_2 \le cM^{4-2r}\|u\|^2_{H^r(\Omega)}. \tag{3.33}$$

According to [27, Lemma 8.8], there holds

$$|\Pi_M^3 u-u|_{H^2(\Omega)} \le c\|\Delta(\Pi_M^3 u-u)\|_\Omega.$$

This, together with (3.32) and (3.33) leads to desired result (3.28) with $\mu=2$. A combination of the above facts and the standard space interpolation leads to (3.28) with $d=3$. It is straightforward to extend the above derivation to $d>3$. This completes the proof. □

Theorem 3.1 *Let $\alpha\in(1,2)$ and let u, u_M be respectively the solutions of (1.1) and (2.14). If $u\in H_0^{\frac{\alpha}{2}}(\Omega)$ and $u\in H^{2-\frac{\alpha}{2}}(\Omega)\cap H^r(\Omega)$ with $r\ge d$, then we have*

$$\|u-u_M\|_{H^{\frac{\alpha}{2}}(\Omega)} \lesssim M^{\frac{\alpha}{2}-r}\|u\|_{H^r(\Omega)}+c_{u,\frac{\alpha}{2}}M^{\frac{\alpha}{2}(\frac{5}{2}-\frac{3}{4}\alpha-r)}\|u\|^{\frac{\alpha}{2}}_{H^r(\Omega)}. \tag{3.34}$$

Proof Applying the first Strang Lemma (see, e.g., [32]) to (2.14), we can obtain

$$\begin{aligned}&\|u-u_M\|_{H^{\frac{\alpha}{2}}(\Omega)}\\ &\lesssim \inf_{w_M\in V_M^d}\left\{\|u-w_M\|_{H^{\frac{\alpha}{2}}(\Omega)}+\sup_{0\ne v_M\in V_M^d}\frac{\langle(-\Delta)^{\frac{\alpha}{2}}w_M,v_M\rangle-\langle(-\Delta_M)^{\frac{\alpha}{2}}w_M,v_M\rangle}{\|(-\Delta)^{\frac{\alpha}{4}}v_M\|_{L^2(\Omega)}}\right\}\\ &\lesssim \inf_{w_M\in V_M^d}\left\{\|u-w_M\|_{H^{\frac{\alpha}{2}}(\Omega)}+\sup_{0\ne v_M\in V_M^d}\frac{\langle(-\Delta)^{\frac{\alpha}{2}}w_M,v_M\rangle-\langle(-\Delta_M)^{\frac{\alpha}{2}}w_M,v_M\rangle}{\|v_M\|_{H^{\frac{\alpha}{2}}(\Omega)}}\right\}\\ &= \inf_{w_M\in V_M^d}\left\{\|u-w_M\|_{H^{\frac{\alpha}{2}}(\Omega)}+\|(-\Delta)^{\frac{\alpha}{2}}w_M-(-\Delta_M)^{\frac{\alpha}{2}}w_M\|_{H^{-\frac{\alpha}{2}}(\Omega)}\right\}.\end{aligned}$$

Take $w_M=\Pi_M^d u$ in the above, we derive from (3.27) with $\mu=\frac{\alpha}{2}$ and Lemma 3.4 that

$$\|u-u_M\|_{H^{\frac{\alpha}{2}}(\Omega)} \lesssim M^{\frac{\alpha}{2}-r}\|u\|_{H^r(\Omega)}+\|(-\Delta)^{\frac{\alpha}{2}}(\Pi_M^d u-u)+(-\Delta)^{\frac{\alpha}{2}}u-(-\Delta_M)^{\frac{\alpha}{2}}\Pi_M^d u\|_{H^{-\frac{\alpha}{2}}(\Omega)}.$$

Thus, it remains to estimate the last term of above equation. By (3.27), (3.28) with $\mu = 2-\frac{\alpha}{2}$, the triangle inequality and Lemma 3.4, we obtain

$$\begin{aligned}
&\|(-\Delta)^{\frac{\alpha}{2}}(\Pi_M^d u - u) + (-\Delta)^{\frac{\alpha}{2}} u - (-\Delta_M)^{\frac{\alpha}{2}}\Pi_M^d u\|_{H^{-\frac{\alpha}{2}}(\Omega)}\\
&\lesssim \|(-\Delta)^{\frac{\alpha}{2}}(\Pi_M^d u - u)\|_{H^{-\frac{\alpha}{2}}(\Omega)} + \|(-\Delta)^{\frac{\alpha}{2}} u - (-\Delta_M)^{\frac{\alpha}{2}}\Pi_M^d u\|_{H^{-\frac{\alpha}{2}}(\Omega)}\\
&\lesssim \|\Pi_M^d u - u\|_{H^{\frac{\alpha}{2}}(\Omega)} + c_{u,\frac{\alpha}{2}}\|(-\Delta)u - (-\Delta_M)\Pi_M^d u\|^{\frac{\alpha}{2}}_{H^{-\frac{\alpha}{2}}(\Omega)}\\
&\lesssim M^{\frac{\alpha}{2}-r}\|u\|_{H^r(\Omega)} + c_{u,\frac{\alpha}{2}}\|(-\Delta)u - (-\Delta)\Pi_M^d u\|^{\frac{\alpha}{2}}_{H^{-\frac{\alpha}{2}}(\Omega)}\\
&\lesssim M^{\frac{\alpha}{2}-r}\|u\|_{H^r(\Omega)} + c_{u,\frac{\alpha}{2}}\|u - \Pi_M^d u\|^{\frac{\alpha}{2}}_{H^{2-\frac{\alpha}{2}}(\Omega)}\\
&\lesssim M^{\frac{\alpha}{2}-r}\|u\|_{H^r(\Omega)} + c_{u,\frac{\alpha}{2}} M^{\frac{\alpha}{2}(\frac{5}{2}-\frac{3}{4}\alpha-r)}\|u\|^{\frac{\alpha}{2}}_{H^r(\Omega)}.
\end{aligned}$$

A combination of above facts lead to the desired result. □

3.3 Error Bounds for (2.22)

In the error analysis, we compare the numerical solution with a suitable orthogonal projection of the exact solution. The orthogonal projection in time $\pi_N^{0,-1} : L^2_{\omega^{0,-1}}(I) \to S_N$, is defined by

$$(\pi_N^{0,-1} v - v, \phi)_{I,\omega^{0,-1}} = 0, \quad \forall \phi \in S_N. \tag{3.35}$$

Defining

$$\widehat{H}^1(I) := \{u : u \in H^1(I) \cap L^2_{\omega^{0,-2}}(I)\},$$

one observes that for any $v \in \widehat{H}^1(I)$ and $\psi \in S_N^*$,

$$\big(\partial_t(\pi_N^{0,-1} v - v), \psi\big)_I = -\big(\pi_N^{0,-1} v - v, \omega^{0,1}\partial_t\psi\big)_{I,\omega^{0,-1}} = 0, \tag{3.36}$$

which follows from the fact $\omega^{0,1}\partial_t\phi \in S_N$ and the definition (3.35). According to Theorem 1.1 in [15], we have

Lemma 3.7 *If $v \in L^2_{\omega^{0,-1}}(I)$ and $\partial_x^k v \in L^2_{\omega^{k,k-1}}(I)$ for $1 \le k \le n$, then*

$$\|\partial_t^l(\pi_N^{0,-1} v - v)\|_{\omega^{l,l-1}} \lesssim N^{l-n}\|\partial_t^n v\|_{\omega^{n,n-1}}, \quad l \le n, \quad l = 0, 1. \tag{3.37}$$

For notational convenience, we denote by $A^r(D)$ (respectively $B^n(D)$) a function space consisting of measurable functions satisfying $\|u\|_{A^r(D)} < \infty$ (respectively $\|u\|_{B^n(D)} < \infty$), where for real number $r \ge d$ and integers $n \ge 0$

$$\begin{aligned}
\|u\|_{A^r(D)} &= \Big(\|\partial_t u\|^2_{H^r(\Omega; L^2_{\omega^{2,0}}(I))} + \|u\|^2_{H^r(\Omega; L^2_{\omega^{0,-1}}(I))}\Big)^{\frac{1}{2}},\\
\|u\|_{B^n(D)} &= \Big(\|\partial_t^n u\|^2_{H^{\frac{\alpha}{2}}(\Omega, L^2_{\omega^{n,n-1}}(I))} + \|\partial_t^n u\|^2_{L^2(\Omega, L^2_{\omega^{n,n-1}}(I))}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Theorem 3.2 *Let $\alpha \in (1,2)$, $\epsilon > 0$ and $\beta > 0$, and let u, u_L be respectively the solutions of (2.19) and (2.22). If $u \in L^2_{\omega^{1,-1}}(I; H_0^{\frac{\alpha}{2}}(\Omega)) \cap \widehat{H}^1(I; L^2(\Omega)) \cap A^r(D) \cap B^n(D)$ and*

$\partial_t^n u \in H^{2-\frac{\alpha}{2}}(\Omega, L^2_{\omega^{n,n-1}}(I))$ *with* $r \geq d$ *and integers* $n \geq 0$, *then we have*

$$\|u - u_L\|_{L^2(\Omega, L^2_{\omega^{0,-1}}(I))} + \|u - u_L\|_{H^{\frac{\alpha}{2}}(\Omega, L^2_{\omega^{1,-1}}(I))} \lesssim M^{\frac{\alpha}{2}-r}\|u\|_{A^r(D)} + N^{-n}\|u\|_{B^n(D)} + d_u^\alpha M^{\frac{\alpha}{2}(\frac{5}{2}-\frac{3}{4}\alpha-r)}\|u\|^{\frac{\alpha}{2}}_{H^r(\Omega; L^2_{\omega^{0,-1}}(I))} + d_u^\alpha N^{-\frac{\alpha}{2}n}\|\partial_t^n u\|^{\frac{\alpha}{2}}_{H^{2-\frac{\alpha}{2}}(\Omega, L^2_{\omega^{n,n-1}}(I))},$$

where $d_u^\alpha := c(\alpha)\big(\|\Pi_M^d u\|_{H^{-\frac{\alpha}{2}}(\Omega, L^2_{\omega^{1,-1}}(I))} + \|u\|_{H^{-\frac{\alpha}{2}}(\Omega, L^2_{\omega^{1,-1}}(I))}\big)^{1-\frac{\alpha}{2}}$.

Proof Let us denote $\widetilde{u}_L := \pi_N^{0,-1}\Pi_M^d u = \Pi_M^d \pi_N^{0,-1} u$ and $e_L := u_L - \widetilde{u}_L$. By virtue of (1.2) with $\mathcal{N}(u) = \beta u - f$ and (2.22), we have

$$\begin{aligned} a(u, v) &:= \big(\partial_t u, v\big)_D + \epsilon\big((-\Delta)^{\frac{\alpha}{4}} u, (-\Delta)^{\frac{\alpha}{4}} v\big)_D + \beta\big(u, v\big)_D = (f, v)_D, \\ a_M(u_L, v) &:= \big(\partial_t u_L, v\big)_D + \epsilon\langle(-\Delta_M)^{\frac{\alpha}{2}} u_L, v\rangle_D + \beta\big(u_L, v\big)_D = (f, v)_D, \quad v \in V_M^d \otimes S_N, \end{aligned} \tag{3.38}$$

which imply

$$a(u, v) = a_M(u_L, v), \quad v \in V_M^d \otimes S_N.$$

This, along with (3.38), yields

$$\begin{aligned} a_M(e_L, v) &= a(u - \widetilde{u}_L, v) + a(\widetilde{u}_L, v) - a_M(\widetilde{u}_L, v) \\ &= \big(\partial_t(u - \widetilde{u}_L), v\big)_D + \epsilon\big((-\Delta)^{\frac{\alpha}{4}}(u - \widetilde{u}_L), (-\Delta)^{\frac{\alpha}{4}} v\big)_D + \beta\big(u - \widetilde{u}_L, v\big)_D \\ &\quad + \epsilon\langle(-\Delta)^{\frac{\alpha}{2}}\widetilde{u}_L - (-\Delta_M)^{\frac{\alpha}{2}}\widetilde{u}_L, v\rangle_D, \end{aligned} \tag{3.39}$$

for all $v \in V_M^d \otimes S_N^*$. Due to (3.36), the above equation can be simplified to

$$\begin{aligned} a_M(e_L, v) =& \big(\partial_t(u - \Pi_M^d u), v\big)_D + \epsilon\big((-\Delta)^{\frac{\alpha}{4}}(u - \pi_N^{0,-1}\Pi_M^d u), (-\Delta)^{\frac{\alpha}{4}} v\big)_D \\ &+ \beta\big(u - \widetilde{u}_L, v\big)_D + \epsilon\langle(-\Delta)^{\frac{\alpha}{2}}\widetilde{u}_L - (-\Delta_M)^{\frac{\alpha}{2}}\widetilde{u}_L, v\rangle_D. \end{aligned} \tag{3.40}$$

Taking $v = \frac{1-t}{1+t} e_L (\in V_M^d \otimes S_N^*)$ in above equation, and using Lemma 3.2, we arrive at

$$\begin{aligned} &\|e_L\|_{L^2(\Omega, L^2_{\omega^{0,-2}}(I))} + \sqrt{\epsilon}\|e_L\|_{H^{\frac{\alpha}{2}}(\Omega, L^2_{\omega^{1,-1}}(I))} + \sqrt{\beta}\|e_L\|_{L^2(\Omega, L^2_{\omega^{1,-1}}(I))} \\ &\cong \|e_L\|_{L^2(\Omega, L^2_{\omega^{0,-2}}(I))} + \sqrt{\epsilon}\|(-\Delta)^{\frac{\alpha}{4}} e_L\|_{L^2(\Omega, L^2_{\omega^{1,-1}}(I))} + \sqrt{\beta}\|e_L\|_{L^2(\Omega, L^2_{\omega^{1,-1}}(I))} \\ &\lesssim \|\partial_t(u - \Pi_M^d u)\|_{L^2(\Omega, L^2_{\omega^{2,0}}(I))} + \|(-\Delta)^{\frac{\alpha}{4}}(u - \pi_N^{0,-1}\Pi_M^d u)\|_{L^2(\Omega, L^2_{\omega^{1,-1}}(I))} \\ &\quad + \|u - \widetilde{u}_L\|_{L^2(\Omega, L^2_{\omega^{1,-1}}(I))} + \|(-\Delta)^{\frac{\alpha}{2}}\widetilde{u}_L - (-\Delta_M)^{\frac{\alpha}{2}}\widetilde{u}_L\|_{H^{-\frac{\alpha}{2}}(\Omega, L^2_{\omega^{1,-1}}(I))}. \end{aligned} \tag{3.41}$$

The first two terms at the right-hand side can be bounded by using Lemma 3.6–3.7, (3.36), (3.37) and (3.27) as follows:

$$\begin{aligned} &\|\partial_t(u - \Pi_M^d u)\|_{L^2(\Omega, L^2_{\omega^{2,0}}(I))} \lesssim M^{-r}\|\partial_t u\|_{H^r(\Omega; L^2_{\omega^{2,0}}(I))}, \\ &\|(-\Delta)^{\frac{\alpha}{4}}(u - \pi_N^{0,-1}\Pi_M^d u)\|_{L^2(\Omega, L^2_{\omega^{1,-1}}(I))} \\ &\leq \|(-\Delta)^{\frac{\alpha}{4}}(u - \pi_N^{0,-1} u)\|_{L^2(\Omega, L^2_{\omega^{1,-1}}(I))} + \|\pi_N^{0,-1}(-\Delta)^{\frac{\alpha}{4}}(u - \Pi_M^d u)\|_{L^2(\Omega, L^2_{\omega^{1,-1}}(I))} \\ &\lesssim \|u - \pi_N^{0,-1} u\|_{H^{\frac{\alpha}{2}}(\Omega, L^2_{\omega^{1,-1}}(I))} + \|u - \Pi_M^d u\|_{H^{\frac{\alpha}{2}}(\Omega, L^2_{\omega^{0,-1}}(I))} \\ &\lesssim N^{-n}\|\partial_t^n u\|_{H^{\frac{\alpha}{2}}(\Omega, L^2_{\omega^{n,n-1}}(I))} + M^{\frac{\alpha}{2}-r}\|u\|_{H^r(\Omega; L^2_{\omega^{0,-1}}(I))}, \end{aligned}$$

and

$$\begin{aligned}\|\widetilde{u}_L - u\|_{L^2(\Omega,L^2_{\omega^{1,-1}}(I))} &\lesssim \|\pi_N^{0,-1}(u-\Pi_M^d u)\|_{L^2(\Omega,L^2_{\omega^{0,-1}}(I))} + \|u-\pi_N^{0,-1}u\|_{L^2(\Omega,L^2_{\omega^{0,-1}}(I))}\\ &\lesssim \|u-\Pi_M^d u\|_{L^2(\Omega,L^2_{\omega^{0,-1}}(I))} + \|u-\pi_N^{0,-1}u\|_{L^2(\Omega,L^2_{\omega^{0,-1}}(I))}\\ &\lesssim M^{-r}\|u\|_{H^r(\Omega;L^2_{\omega^{0,-1}}(I))} + N^{-n}\|\partial_t^n u\|_{L^2(\Omega,L^2_{\omega^{n,n-1}}(I))}.\end{aligned}$$

A combination of the above leads to

$$\begin{aligned}&\|e_L\|_{L^2(\Omega,L^2_{\omega^{0,-2}}(I))} + \sqrt{\epsilon}\|e_L\|_{H^{\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I)} \lesssim M^{-m}\|\partial_t u\|_{H^m(\Omega;L^2_{\omega^{2,0}}(I))}\\ &+ N^{-n}\|\partial_t^n u\|_{H^{\frac{\alpha}{2}}(\Omega,L^2_{\omega^{n,n-1}}(I))} + M^{\frac{\alpha}{2}-r}\|u\|_{H^r(\Omega;L^2_{\omega^{0,-1}}(I))}\\ &+ N^{-n}\|\partial_t^n u\|_{L^2(\Omega,L^2_{\omega^{n,n-1}}(I))} + \|(-\Delta)^{\frac{\alpha}{2}}\widetilde{u}_L - (-\Delta_M)^{\frac{\alpha}{2}}\widetilde{u}_L\|_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))}.\end{aligned} \tag{3.42}$$

We can estimate the last term by using an argument similar to that for the proof in Theorem 3.1,

$$\begin{aligned}&\|(-\Delta)^{\frac{\alpha}{2}}\widetilde{u}_L - (-\Delta_M)^{\frac{\alpha}{2}}\widetilde{u}_L\|_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))}\\ &\le \|(-\Delta)^{\frac{\alpha}{2}}(\widetilde{u}_L - u) + (-\Delta)^{\frac{\alpha}{2}}u - (-\Delta_M)^{\frac{\alpha}{2}}\widetilde{u}_L\|_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))}\\ &\le \|(-\Delta)^{\frac{\alpha}{2}}(\widetilde{u}_L - u)\|_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))} + \|(-\Delta)^{\frac{\alpha}{2}}u - (-\Delta_M)^{\frac{\alpha}{2}}\widetilde{u}_L\|_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))}.\end{aligned} \tag{3.43}$$

By (3.37) and (3.27), the bound of first term in (3.43) is given by

$$\begin{aligned}\|(-\Delta)^{\frac{\alpha}{2}}(\widetilde{u}_L - u)\|_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))} &\lesssim \|\widetilde{u}_L - u\|_{H^{\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))}\\ &\lesssim N^{-n}\|\partial_t^n u\|_{H^{\frac{\alpha}{2}}(\Omega,L^2_{\omega^{n,n-1}}(I))} + M^{\frac{\alpha}{2}-r}\|u\|_{H^r(\Omega;L^2_{\omega^{0,-1}}(I))}.\end{aligned} \tag{3.44}$$

Then, by Lemma 3.4 and (3.28), we estimate the last term of (3.43) by

$$\begin{aligned}&\|(-\Delta)^{\frac{\alpha}{2}}u - (-\Delta_M)^{\frac{\alpha}{2}}\widetilde{u}_L\|_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))} \lesssim d_u^{\alpha}\|(-\Delta)u - (-\Delta)\widetilde{u}_L\|^{\frac{\alpha}{2}}_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))}\\ &\lesssim d_u^{\alpha}\|\Delta(u-\pi_N^{0,-1}u)\|^{\frac{\alpha}{2}}_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))} + d_u^{\alpha}\|\pi_N^{0,-1}\Delta(u-\Pi_M^d u)\|^{\frac{\alpha}{2}}_{H^{-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))}\\ &\lesssim d_u^{\alpha}\|u-\pi_N^{0,-1}u\|^{\frac{\alpha}{2}}_{H^{2-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))} + d_u^{\alpha}\|u-\Pi_M^d u\|^{\frac{\alpha}{2}}_{H^{2-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{1,-1}}(I))}\\ &\lesssim d_u^{\alpha}N^{-\frac{\alpha}{2}n}\|\partial_t^n u\|^{\frac{\alpha}{2}}_{H^{2-\frac{\alpha}{2}}(\Omega,L^2_{\omega^{n,n-1}}(I))} + d_u^{\alpha}M^{\frac{\alpha}{2}(\frac{5}{2}-\frac{3}{4}\alpha-r)}\|u\|^{\frac{\alpha}{2}}_{H^r(\Omega;L^2_{\omega^{0,-1}}(I))}.\end{aligned} \tag{3.45}$$

On the other hand, we have $u - u_L = u - \widetilde{u}_L + e_L$. Then, using Lemma 3.7 and (3.28), again yields

$$\|u-\widetilde{u}_L\|_{L^2(\Omega,L^2_{\omega^{0,-1}}(I))} \lesssim M^{-r}\|u\|_{H^r(\Omega;L^2_{\omega^{0,-1}}(I))} + N^{-n}\|\partial_t^n u\|_{L^2(\Omega,L^2_{\omega^{n,n-1}}(I))}.$$

Consequently, the desired result follows from above estimates, the triangle inequality and (3.45). □

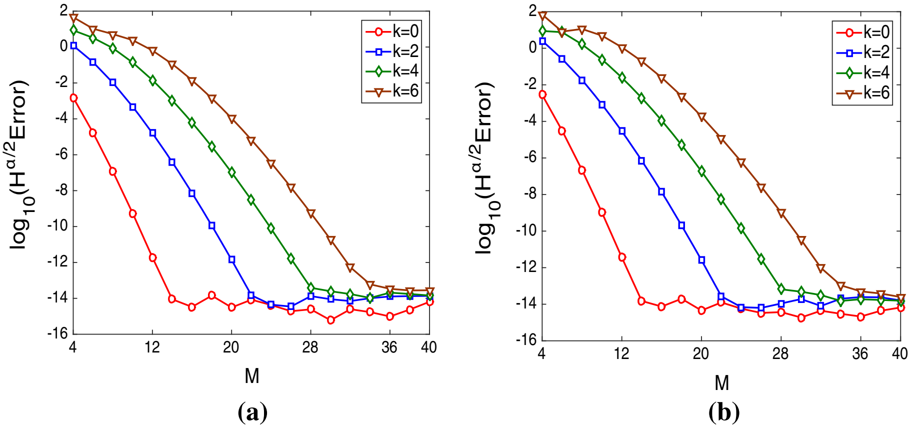


Fig. 1 **a** $H^{\frac{\alpha}{2}}$-error in semi-log scale of (1.1) with $u(x) = \varphi_k(x)$; **b** $H^{\frac{\alpha}{2}}$-error in semi-log scale of (1.1) with $u(x, y) = \varphi_k(x)\varphi_k(y)$

4 Numerical Results for Linear Fractional Equations

In this section, we present some numerical results obtained by the spectral method (2.14) for (1.1) and the space-time spectral method (2.22) for (2.19).

Example 4.1 (with known smooth exact solutions) We consider first the following smooth exact solutions for (1.1) with homogeneous Dirichlet boundary conditions:

$$u(x) = \varphi_k(x) \quad \text{and} \quad u(x, y) = \varphi_k(x)\varphi_{k'}(y), \quad k, k' = 0, 1, 2, \ldots, \tag{4.1}$$

where $\{\lambda_k, \varphi_k\}_{k\geq 0}$ denote the eigenpairs of spectral Laplacian operator on $(-1, 1)$, and have explicit formulas

$$\varphi_k(x) = \sin(\sqrt{\lambda_k}(x+1)), \quad \lambda_k = \left(\frac{(k+1)\pi}{2}\right)^2. \tag{4.2}$$

The corresponding source term can be calculated by using the definition of spectral fractional Laplacian

$$f(x) = \lambda_k^{\frac{\alpha}{2}} \varphi_k(x), \quad f(x, y) = (\lambda_k + \lambda_{k'})^{\frac{\alpha}{2}} \varphi_k(x)\varphi_{k'}(y).$$

In Fig. 1, we list the $H^{\frac{\alpha}{2}}$-errors in semi-log scale with $\alpha = 1.8$. The plots in Fig. 1 indicate the numerical errors decay exponentially. This is consistent with the theoretical result in Theorem 3.1 which predicts that for this smooth solution, the convergence rate is faster than any algebraic rate.

Example 4.2 (with a unknown weakly singular solution) We consider now the problem (1.1) with $f(x) = 1$ and homogeneous Dirichlet boundary conditions. Since the exact solution is unknown, we use the truncation of the exact solution

$$u(x) \approx \sum_{j=0}^{10000} \lambda_j^{-\frac{\alpha}{2}} \langle 1, \varphi_j \rangle \varphi_j(x), \tag{4.3}$$

as the reference solution, where $\{\lambda_j, \varphi_j\}_{j\geq 0}$ are given in (4.2). It is known that the exact solution has weak singularities at the boundary [3]. In the left of Fig. 2, we plot the $H^{\frac{\alpha}{2}}$-error

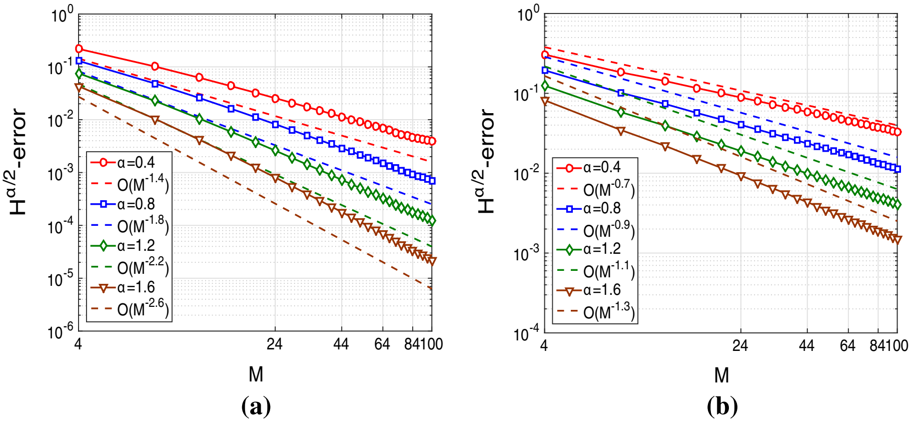


Fig. 2 **a** $H^{\frac{\alpha}{2}}$-error, in log–log scale, of our method for (1.1) with right hand side term $f(x) = 1$; **b** $H^{\frac{\alpha}{2}}$-error, in log–log scale, of Fourier spectral method in [7] for (1.1) with right hand side term $f(x) = 1$

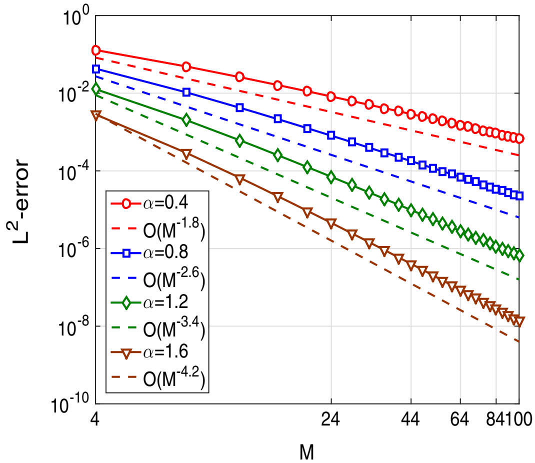


Fig. 3 L^2-error, in log–log scale, of our method for (1.1) with right hand side term $f(x) = 1$

of our Legendre spectral method in log-log scale. We observe that the convergence rate is clearly $O(M^{-1-\alpha})$. As a comparison, we also plot the $H^{\frac{\alpha}{2}}$-error of the Sine-spectral method [7] in the right of Fig. 2. We observe that the Sine-spectral method converge at $O(M^{-0.5-\alpha/2})$, which means the convergence rate of our method is twice that of Sine-spectral method under $H^{\frac{\alpha}{2}}$-norm.

We plot in Fig. 3 the L^2-error of our Legendre spectral method in log-log scale. While one can not use the usual duality argument to improve the error estimate in L^2-norm, we do observe that the L^2-error decays as $O(M^{-1-2\alpha})$, an improvement of order α over the $H^{\frac{\alpha}{2}}$-error.

Example 4.3 (a fractional diffusion equation) As the last example, we consider the following fractional diffusion equation

$$\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u = 0, \tag{4.4}$$

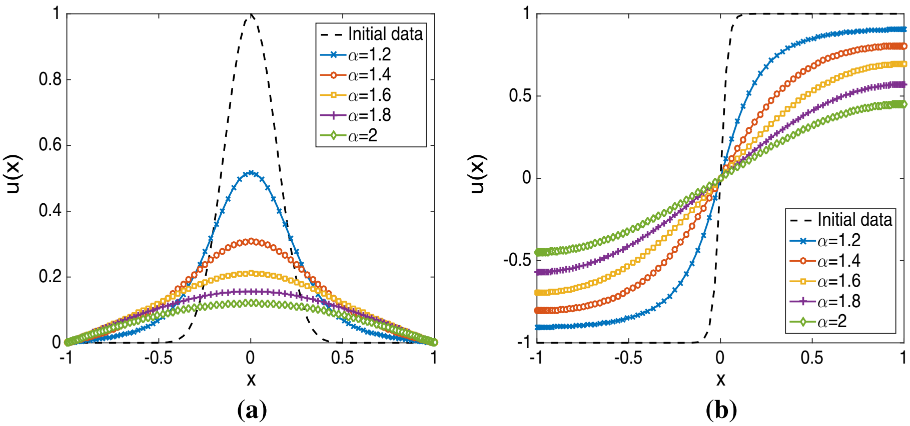


Fig. 4 Numerical solution of fractional heat equation (4.4) for different α: **a** initial data: $\mathrm{e}^{-25x^2/(1-x^2)}$; **b** initial data: $\tanh(25x/\sqrt{1-x^2})$

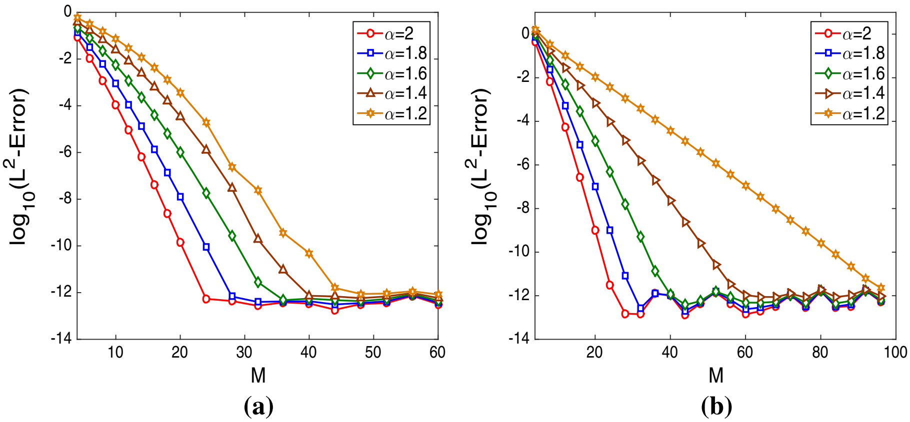


Fig. 5 L^2-error for (4.4) at $T = 0.1$ for different α; **a** initial data $u_0 = \mathrm{e}^{-25x^2/(1-x^2)}$; **b** initial data $u_0 = \tanh(25x/\sqrt{1-x^2})$

with the initial conditions: (a) a Gaussian like function $e^{-25x^2/(1-x^2)}$ (with homogeneous Dirichlet boundary condition); (b) a sigmoid exhibiting sharper gradients $\tanh(25x/\sqrt{1-x^2})$ (with Neumann boundary condition). The numerical solution computed with our space-time spectral method with $M = N = 100$ at $T = 0.1$ are plotted in Fig. 4, and the black dotted line is the initial conditions. We observe that, as expected, the diffusion rate decreases as the fractional power α decreases.

In Fig. 5, we list the L^2-errors of (4.4), compared with an "exact" solution computed with a refined mesh, in semi-log scale against various $M = N$ at $T = 0.1$. We observe an exponential convergence with respect to $M = N$ for our space-time spectral method, but as α decreases, the rate of convergence also decreases. This is due to the fact that, with the given initial conditions, the solution is essentially smooth in the time interval that we considered.

5 Application to Nonlinear Fractional Equations

The method presented in Sect. 2 can be used as an effective preconditioner in the Jacobian-free Newton–Krylov algorithm [19] to solve nonlinear fractional equations (1.2). More precisely, the linearized (about a function w) equation of (1.2) is:

$$\mathcal{L}_w v := \partial_t v + \epsilon^2(-\Delta)^{\frac{\alpha}{2}} v + \mathcal{N}'(w)v = 0. \tag{5.1}$$

Hence, with a suitable constant β, $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 v := \partial_t v + \epsilon^2(-\Delta)^{\frac{\alpha}{2}} v + \beta v$$

will be an effective preconditioner for $\mathcal{L}_w$. Since $v \to \mathcal{L}_w v$ and $v \to \mathcal{L}_0^{-1} v$ can be efficiently performed in the space-time approximation space described in Sect. 2, we can solve (1.2) efficiently with the the Jacobian-free Newton–Krylov algorithm. In addition, the following strategies are used:

- The convergence rate of the Newton–Krylov iteration depends on the quality of the initial guess. We use the following simple semi-implicit scheme to generate such an initial guess: Find $v_M^{n+1} \in V_M^d$ s.t.

$$\left(\frac{v_M^{n+1} - v_M^n}{t^{n+1} - t^n}, w_M\right) + (\epsilon^2(-\Delta)^{\frac{\alpha}{2}} v_M^{n+1}, w_M) + (N(v_M^n), w_M) = 0 \quad \forall w_M \in V_M^d, \tag{5.2}$$

  where t^k are the scaled Legendre–Gauss-Radau points. The above equation can be easily solved by using the Fourier-like basis of V_M^d.
- To integrate the nonlinear problems (1.2) for a large time interval $[0, T]$, we can first divide $[0, T]$ into a number of smaller intervals $[0, T] = \cup_{i=1}^K [T_{i-1} - T_i]$, and apply the space-time spectral method on each interval using the final solution at the interval $[T_{i-1} - T_i]$ as the initial condition for the interval $[T_i - T_{i+1}]$.

5.1 Fractional FitzHugh–Nagumo Model

The FitzHugh–Nagumo model is a system of reaction–diffusion equations describing wave propagation in an excitable medium. It takes the following form (with $\alpha = 2$):

$$\begin{aligned} \partial_t u &= -K_u(-\Delta)^{\frac{\alpha}{2}} u + u(1-u)(u-a) - v, \\ \partial_t v &= \epsilon(\beta u - \gamma v - \delta), \\ \partial_n u|_{\partial\Omega} &= \partial_n v|_{\partial\Omega} = 0, \end{aligned} \tag{5.3}$$

where u is a "fast" variable which describes membrane potential of a cell and v is a "slow" variable which connects with the medium conductivity by inverse ratio. Here, we also consider the fractional FitzHugh–Nagumo model, represented by the above system with $\alpha \in (1, 2)$, which takes into accounts non-local interactions [7].

Let $\Omega = (-1, 1)^2$, and set the parameters in (5.3) to be $a = 0.1$, $\epsilon = 0.01$, $\beta = 0.5$, $\gamma = 1$, $\delta = 0$. These parameters are considered in [7], and they lead to stable patterns in the system in the form of re-entrant spiral waves. In our simulations, the trivial state $(u, v) = (0, 0)$ was perturbed by setting the lower-left quarter of the domain to $u = 1$ and the upper half part to $v = 0.1$, which allows the initial condition to rotate clockwise to generate spiral patterns. We discretize the spatial domain using 256×256 points, and compute the solution up to $T = 2000$ so that the solution reaches the steady state.

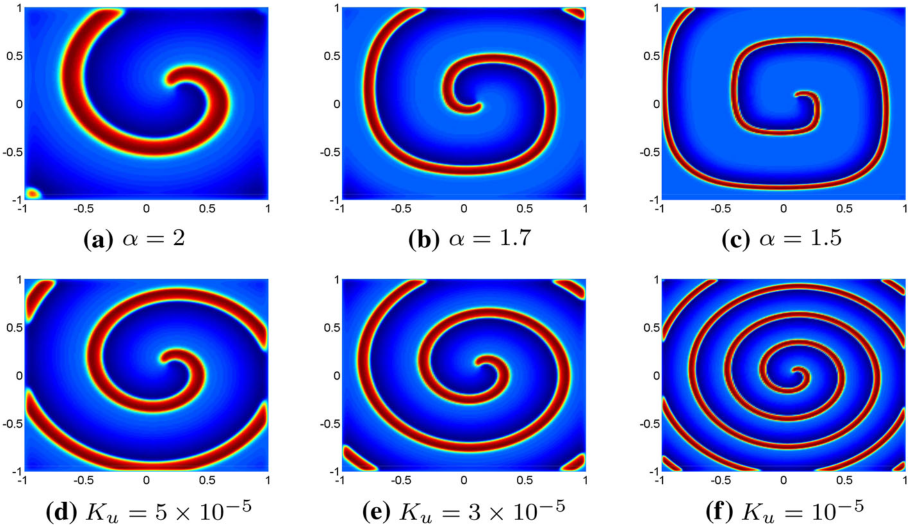


Fig. 6 Spiral wave in FitzHugh–Nagumo model for various α and K_u; top: $K_u = 10^{-4}$; bottom: $\alpha = 1.7$

The steady state rotating solutions at $t = 2000$, with different diffusion coefficient K and fractional order α, are presented in Fig. 6. we observe similar behaviors as reported in [7]. Namely, as α decreases, the width of the excitation wavefront decreases (cf. the top row of Fig. 6), which also happens as we decrease the diffusion coefficient (cf. the bottom row of Fig. 6). However, there is also a significant difference between our simulation with homogeneous Neumann boundary conditions and the simulation in [7] with a periodic boundary condition: in our simulations, the rotation angles are more aligned with the boundary which reflects the effect of the homogeneous Neumann boundary conditions, while in their simulations the curvatures are essentially uniform.

5.2 Fractional Allen–Cahn Equation

The Allen–Cahn equation is a reaction–diffusion equation describing the process of phase separation in crystalline solids. It takes the following form (with $\alpha = 2$):

$$\partial_t u + \epsilon^2 (-\Delta)^{\frac{\alpha}{2}} u + (u^2 - 1)u = 0. \tag{5.4}$$

The fractional Allen–Cahn equation (with $\alpha \in (1, 2)$) has received some attention recently [2,8,29]. In the following, we shall present some numerical results for both the regular and fractional Allen–Cahn equation. In all computations below, we set $\epsilon = 0.1$.

5.2.1 Width of 1-D Interfacial Layer

It is well-known that the parameter ϵ represents the interfacial width of the regular Allen–Cahn equation, and that the interfacial width deceases as α decreases in the fractional Allen–Cahn equation [2,8,29]. More precisely, it has been observed computationally [29] in and proved in [2] that the interfacial width behaves like $O(\epsilon^{1/\alpha})$. In this example, We take the initial

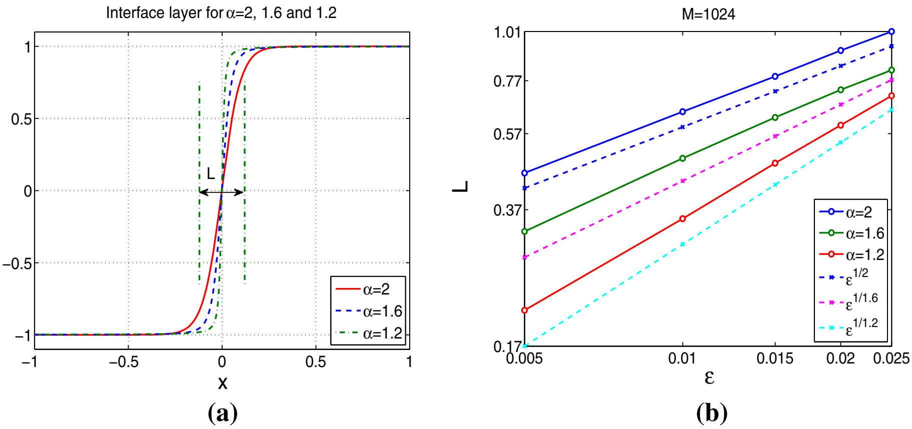


Fig. 7 **a** Interfacial layers with different α; **b** Layer width L against ϵ for different values of α

solution to be

$$u_0(x) = \begin{cases} 1, & 0 \le x \le 1, \\ -1, & -1 \le x < 0, \end{cases} \tag{5.5}$$

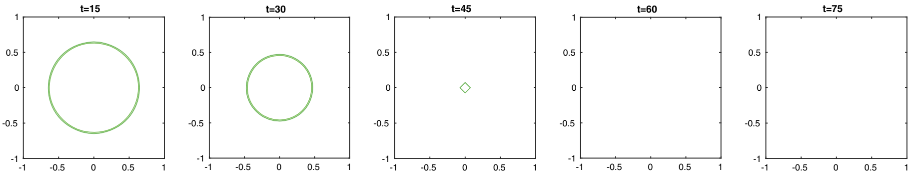
and compute the solution up to time=1 using $N = 64$, $M = 1024$ in our space-time spectral method. In the left of Fig. 7, we plot the interfacial layer for different α at time $T = 1$. And the thickness L is defined to be $\{x | -1 < x < 1 \text{ and } |u(x)| < 0.99\}$ (as is shown in Fig. 7 (a)). In the right of Fig. 7, we plot the interfacial width L with different α and ϵ, and we observe indeed that as the interfacial width decreases as α decreases, and behaves like $L = O(\epsilon^{1/\alpha})$ [2,29].

5.2.2 Velocity of 2-D Moving Interface

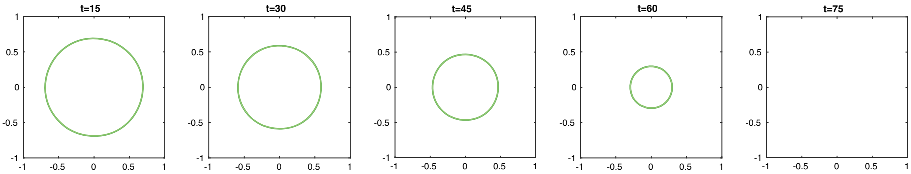
As suggested in [10], in order to predict quantitatively the kinetics of microstructural evolution, we have to calculate not only the accurate equilibrium profiles but also the accurate velocity of a moving interface. To compare the influence of fractional order α on the velocity of a moving interface, we consider the two dimensions Allen–Cahn equation with system size 128×128 in domain $[-1, 1]^2$. In Fig. 8 we plotted the contour near $u = 0$ which captures the movement of the interface. At time $t = 0$, there is a circular interface boundary with a radius of 74. The order parameter values inside the circle are assigned to be $+1$ and -1 outside. Such a circular interface is unstable and will shrink under the mean curvature. In Fig. 9 we compute the radius of the inner circle with respect to time t for different α. It is numerically observed that the radius of the circle shrinks as $R_0^2 - R^2 = O(t^{\alpha/2})$ where R_0 is the initial radius.

6 Conclusion

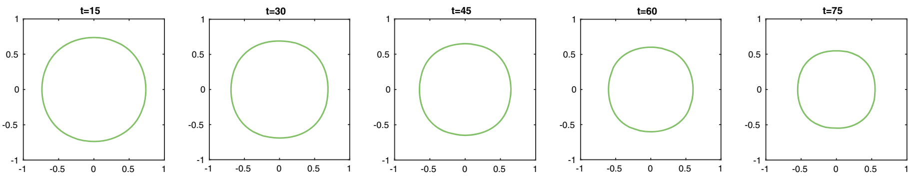
In this paper, we developed efficient spectral methods for solving PDEs with fractional Laplacians, and carried out rigorous error analysis for them. The methods are based on the Fourier-like basis functions which are the eigenfunctions of the discrete fractional Laplacian with non periodic boundary conditions. Therefore, the nonlocal fractional Laplacian operator



(a) $\alpha = 2.0$



(b) $\alpha = 1.8$



(c) $\alpha = 1.5$

Fig. 8 Temporal evolution of circular domain for various α

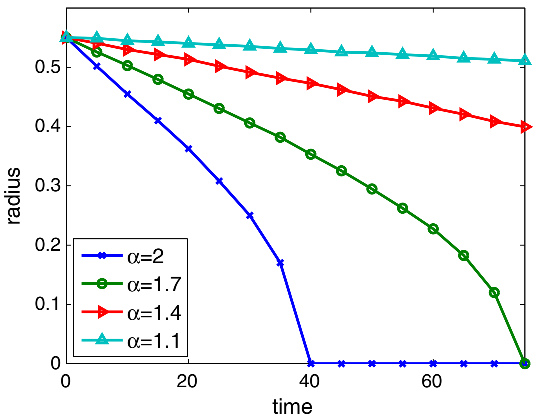


Fig. 9 Radius of the circular domain as a function of time obtained with different fractional order α using time step $\Delta t = 0.5$

can be naturally handled, leading to simple, efficient and accurate numerical algorithms. Our numerical experiments demonstrated that our algorithms are robust, efficient and accurate for a variety of linear and nonlinear PDEs with fractional Laplacian.

The analytical framework we introduced to estimate the errors between fractional Laplacian and discrete fractional Laplacian is not restricted to spectral methods and can be used to analyze numerical approximations of fractional Laplacian by using other Galerkin type approximations such as finite-element methods.

We only considered rectangular domains in this paper. However, by using a recently developed spectral method for general domains [13], we expect to be able to extend the approach proposed here to problems in more general domains.

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