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STABILITY AND CONVERGENCE ANALYSIS OF A FULLY DISCRETE SEMI-IMPLICIT SCHEME FOR STOCHASTIC ALLEN-CAHN EQUATIONS WITH MULTIPLICATIVE NOISE

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Abstract. We consider a fully discrete scheme for stochastic Allen-Cahn equation in a multi-dimensional setting. Our method uses a polynomial based spectral method in space, so it does not require the elliptic operator A and the covariance operator Q of noise in the equation commute, and thus successfully alleviates a restriction of Fourier spectral method for stochastic partial differential equations pointed out by Jentzen, Kloeden and Winkel [Ann. Appl. Probab. 21 (2011), pp. 908–950]. The discretization in time is a tamed semi-implicit scheme which treats the nonlinear term explicitly while being unconditionally stable. Under regular assumptions which are usually made for SPDEs, we establish strong convergence rates in the one spatial dimension for our fully discrete scheme. We also present numerical experiments which are consistent with our theoretical results.

1. Introduction

We consider numerical approximation of the following nonlinear stochastic PDE perturbed by multiplicative noise:

$$\begin{cases} du = Audt + F(u)dt + G(u)dW^Q(t), x \in \mathcal{O} \subset \mathbb{R}^d \ (d=1,2), \\ u(t,x) = 0, x \in \partial\mathcal{O}, \\ u(0,x) = u_0(x), x \in \mathcal{O}, \end{cases} \tag{1.1}$$

where A is the Laplacian operator on $\mathcal{O}$, F is the Nemytskii operator defined by $F(u)(\xi) = f(u(\xi)),\ \xi \in \mathcal{O}$, where $f(u) = u - u^3$. $G(u)(\xi) = g(u(\xi))$ is another Nemytskii operator, where $g(u)$ is a Lipschitz continuous function with linear growth satisfying Assumption 2.2, and $W^Q(t)$ is a Q-Wiener process on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ defined by (cf. [33])

$$W^Q(t) = \sum_{j=1}^{\infty} \sqrt{q}_j e_j \beta_j(t),$$

where $\beta_j(t)$ are independent standard Wiener processes, and $\{(q_j, e_j)\}_{j=1}^{\infty}$ are eigenpairs of a symmetric nonnegative operator Q. We emphasize that $\{e_j\}_{j=1}^{\infty}$ are not necessarily eigenfunctions of A in $\mathcal{O}$.

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It is well known that for $u_0 \in C(\mathcal{O})$, (1.1) admits a unique mild solution in $L^p(\Omega; C((0,T);H) \cap L^\infty(0,T;H))$ for arbitrary $p \geq 1$ that satisfy (cf. [9])

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(u(s))ds + \int_0^t e^{(t-s)A} g(u(s))dW^Q(s). \tag{1.2}$$

Moreover, under certain conditions to be specified later, $\sup_{t\in[0,T]} \mathbf{E}\|(-A)^{\frac{\gamma}{2}}u(t)\|^2 < \infty$ holds for some $\gamma > d/2$ (cf. Theorem 2.1).

Many mathematical models in physics, biology, chemistry etc. are formulated as SPDEs (cf. [10, 13, 29]), and various numerical methods have been proposed for solving SPDEs. We refer to [3, 21, 26, 40, 42] and references therein for an incomplete account of numerical approaches for SPDEs with global Lipschitz condition on f. In contrast, SPDEs with nonglobally Lipschitz condition on f are more difficult to deal with, we refer to [5, 6, 8, 11, 12, 17, 24, 25, 30, 31, 34] for some recent advances in this regard. Moreover, most of these works are concerned with additive noise (cf. [5, 8, 11, 15, 21–24, 34, 40]), while SPDEs with local Lipschitz condition driven by multiplicative noise have received much less attention. We would like to point out that in [17], the authors considered a finite element method (FEM) for stochastic Allen-Cahn equation driven by the gradient type multiplicative noise under sufficient spatial regularity assumptions, and in [30], the authors also investigated a FEM for the same equation perturbed by multiplicative noise of type $g(u)\beta(t)$, where $\beta(t)$ is a Brownian motion. In both cases, fully implicit time discretization schemes are used so that a nonlinear system has to be solved at each time step.

The main goal of this paper is to design and analyze a strongly convergent, linear and fully decoupled numerical method for stochastic Allen-Cahn equation in a multi-dimensional framework. To avoid using a fully implicit scheme for SPDEs with local Lipschitz condition, we construct a tamed semi-implicit scheme in time (cf. [18, 19, 39, 41]) and show that it is unconditionally stable under a quite general setting. On the other hand, we adopt a polynomial based spectral-Galerkin method as spatial discretization. Distinguished for their high resolution and relative low computational cost for a given accuracy threshold, spectral methods have become a major computational tool for solving PDEs. However, only limited attempts have been made for using spectral methods for SPDEs (cf. [5, 21, 22]), and most of these attempts are confined to Fourier spectral methods. Note that the use of Fourier spectral methods in these works is essential as Fourier basis functions are eigenfunctions of the elliptic operator $-A$. Since in our tamed semi-implicit scheme, the nonlinear term and the noise terms are treated explicitly, we shall employ a polynomial-based spectral method for spatial approximation to overcome the restriction mentioned above. A key ingredient is to use a set of specially constructed Fourier-like discrete eigenfunctions of A (cf. [37, Chapter 8]), which are mutually orthogonal in both $L^2(\mathcal{O})$ and $H^1(\mathcal{O})$.

Combining the above ingredients together, we develop an efficient and unconditionally stable fully discretized scheme based on a tamed semi-implicit approach. Moreover, we derive the following convergence rate in the one-dimensional case under regular assumptions (Assumption 2.1-Assumption 2.3) for general multiplicative noise

$$\mathbf{E}\|u(t_k) - u_N^k\|^2 \leq C(N^{-2\gamma} + \tau + \tau^{-1}N^{-4\gamma}), \tag{1.3}$$

where u_N^k is the full discretization of u at t_k, N is the number of points in each direction in our spatial approximation, τ is the time step size and γ is the index measuring the regularity of noise, which can be arbitrarily large provided that Assumptions 2.2 and 2.3 hold. It extends the results in [11,34] for stochastic Allen-Cahn equation with additive noise with finite-element approximation under the essential assumption $\|(-A)^{\frac{\gamma-1}{2}}Q^{\frac{1}{2}}\|_{L^2} < \infty$.

In summary, the main contributions of this paper include:

- We investigate the optimal spatial regularity of solution for (1.1), which lifts the previous results $\gamma \in (1,2]$ (cf. [11,21,22,26,28,34,40,41]) to possible arbitrarily large γ provided Assumption 2.1-Assumption 2.3 are fulfilled, and derive optimal spatial convergence rate for our fully-discretized scheme based on the improved regularity.
- Our tamed time discretization for (1.1) treats the nonlinear terms explicitly while is still unconditionally stable. Thus, it avoids solving nonlinear systems at each time step, which is in contrast to the popular backward Euler method (cf. [21,22,26,28,34,41]).
- We use the Legendre spectral method, instead of the usual Fourier spectral method, for spatial discretization which does not require the commutativity of operators A and Q, and circumvents a restriction of Fourier approximation for SPDE pointed out in [21]. Through a matrix diagonalization process, our method based on the Legendre approximation can also be efficiently implemented as with a Fourier approximation.
- We establish the strong convergence result (1.3) when $d = 1$.

The rest of this paper is organized as follows. In Section 2, some preliminaries including our main assumptions and optimal spatial regularity of solution of (1.1) are presented. Section 3 is devoted to spatial semi-discretization and its analysis. In Section 4, we present our semi-implicit tamed Euler full discretization for (1.1), and derive an unconditional stability result. In Section 5, we derive further stability estimates and strong convergence results for the scheme under regular assumptions. In Section 6, we present an efficient implementation of our scheme using the spectral-Galerkin method, and present numerical results for the stochastic Allen-Cahn equation to validate our main theoretical results.

2. Preliminaries

In this section, we first describe some notations and a few lemmas which will be used in our analysis, and then we present several general assumptions for the problem under consideration.

2.1. **Notations.** We begin with notations. Denote by H the standard $L^2(\mathcal{O})$ space. Let U and V be separable Hilbert spaces and let $\mathcal{L}(U,V)$ be the Banach space of all bounded linear operators $U \to V$ endowed with the uniform norm $\|\cdot\|_{\mathcal{L}}$. We denote the norm in $L^p(\Omega,\mathcal{F},\mathbf{P};U)$ by $\|\cdot\|_{L^p(\Omega;U)}$, that is,

$$\|Y\|_{L^p(\Omega;U)} = \big(\mathbf{E}\big[\|Y\|_U^p\big]\big)^{\frac{1}{p}}, Y \in L^p(\Omega,\mathcal{F},\mathbf{P};U).$$

Denote by $L_1(U,V)$ the nuclear operator space from U to V and for $T \in L_1(U,V)$, its norm is given by

$$\|T\|_{L_1} = \sum_{i=1}^{\infty} |(Te_i,e_i)_U| \text{ and } Tr(T) = \sum_{i=1}^{\infty}(Te_i,e_i)_U$$

for any orthonormal basis $\{e_i\}$ of U. In particular, if $T > 0$, then $\|T\|_{L^1} = Tr(T)$. In this work, we assume that $W^Q(t)$ is of trace class, i.e. $Tr(Q) < \infty$. Let $L_2(U, V)$ be the Hilbert-Schmidt space such that for any $T \in L_2(U, V)$

$$\|T\|_{L_2} = \left(\sum_{i=1}^{\infty} \|Te_i\|^2 \right)^{1/2} < \infty.$$

Moreover, if Q is of trace class, we introduce $L_2^0 = L_2(U_0, V)$ with norm

$$\|T\|_{L_2^0} = \|TQ^{1/2}\|_{L_2(U,V)},$$

where $U_0 = Q^{1/2}(U)$.

Finally, when no confusion arises, we will drop the spatial dependency from the notations, i.e., $u(t) = u(t, x)$.

2.2. **Some useful lemmas.** We shall start with the Burkholder-Davis-Gundy (BDG) inequality for a sequence of H-valued discrete martingales (cf. [20, 28])

Lemma 2.1. *Let $p \geq 2$ and $\{Z_m\}$ be a sequence of H-valued random variables with bounded p-moments such that $\mathbf{E}[Z_{m+1}|Z_0, \cdots, Z_m] = 0$ for all $1 \leq m \leq N-1$. Then there exists a constant $C = C(p)$ such that*

$$\left(\mathbf{E} \left\| \sum_{i=0}^{m} Z_i \right\|^p \right)^{\frac{1}{p}} \leq C \left(\sum_{i=0}^{m} (\mathbf{E}\|Z_i\|^p)^{\frac{2}{p}} \right)^{\frac{1}{2}}. \tag{2.1}$$

Recall the following generalized Gronwall's inequality:

Lemma 2.2 (Generalized Gronwall's lemma [16])**.** *Let $T > 0$ and $C_1, C_2 \geq 0$ and let ϕ be a nonnegative and continuous function. Let $\beta > 0$. If we have*

$$\phi(t) \leq C_1 + C_2 \int_0^t (t-s)^{\beta-1} \phi(s) ds,$$

then there exists a constant $C = C(C_2, T, \beta)$ such that

$$\phi(t) \leq CC_1.$$

2.3. **Assumptions and observations.** We describe below our main assumptions.

Assumption 2.1 (Operator A)**.** The linear operator $-A : dom(A) \subset H \to H$ is densely defined, self-adjoint and positive definite with compact inverse.

Under Assumption 2.1, the operator A generates an analytic semi-group $E(t) = e^{tA}, t \geq 0$ on H and the fractional powers of $(-A)$ and its domain $H^r := \mathrm{dom}((-A)^{r/2})$ for all $r \in \mathbb{R}$ equipped with inner product $(\cdot, \cdot)_r = ((-A)^{r/2}\cdot, (-A)^{r/2}\cdot)$ and the induced norm $\|\cdot\|_r = (\cdot, \cdot)_r^{1/2}$. In particular, we denote $\|\cdot\| = \|\cdot\|_0$. Let $L_{2,r}^0 = L_2(U_0, H^r)$ with norm $\|T\|_{L_{2,r}^0} = \|(-A)^{r/2}T\|_{L_2^0}$. Moreover, the following inequalities hold (cf. [32, Theorem 6.13], [26]).

(i) For any $\mu \geq 0,$ it holds that

$$(-A)^{\mu} E(t)v = E(t)(-A)^{\mu} v, \quad \text{for } v \in H^{2\mu},$$

and there exists a constant C such that

$$\|(-A)^{\mu} E(t)\| \leq Ct^{-\mu}, \quad t > 0; \tag{2.2}$$

(ii) For any $0 \leq \nu \leq 1,$ there exists a constant C such that

$$\|(-A)^{-\nu}(E(t) - I)\| \leq Ct^{\nu}, \ t > 0. \tag{2.3}$$

Assumption 2.2 (Linear growth and Lipschitz condition for g)**.** Given $\gamma > \frac{d}{2}$. The mapping $g(v)$ satisfies

$$\|g(u)\|_{L^0_{2,\nu}} \leq c\|u\|_{\nu}, \quad u \in H^{\nu}(\mathcal{O})$$

with $\nu = 0$ and $\nu = \gamma$, and

$$\|g(u) - g(v)\| \leq c\|u - v\|, \ u, v \in L^2(\mathcal{O}).$$

Observation 2.1 (Nonlinearity). f satisfies the following coercivity and one-sided Lipschitz condition

$$\begin{aligned} &(f(u), u) \leq -\theta\|u\|^4 + K\|u\|^2, \qquad \text{for some } \theta, K > 0; \\ &(f(u) - f(v), u - v) \leq L\|u - v\|^2, \quad L > 0, \ u, v \in L^{2P}(\mathcal{O}), \end{aligned} \tag{2.4}$$

for some $L > 0$.

Remark 2.1. The parameter γ essentially determines (see Theorem 2.1) spatial regularity. It is clear that linear functions satisfy the assumption which relaxes the sublinear growth condition of g to some extent (cf. [4]).

Assumption 2.3 (Initial condition)**.** Let $\gamma > \frac{d}{2}$ be the same as in Assumption 2.2. We assume that the initial condition u_0 is $\mathscr{F}_0/\mathscr{B}(H^{\gamma})$-measurable and

$$\mathbf{E}\|u_0\|_{\gamma}^p < \infty, \quad p \geq 2.$$

Under Assumptions 2.1-2.3, there exists a unique predictable process u (cf. [28]) such that for any $p \geq 1$, one has

$$\mathbf{E}\left(\sup_t \|u(t)\|_{\gamma}^p\right) < \infty, \quad \text{for } \gamma \in (1, 2). \tag{2.5}$$

Remark 2.2 (On the well-posedness of (1.1)).

- If both f and g are globally Lipschitz continuous with linear growth condition, then the well-posedness of (1.1) is standard and has been provided in, for instance, [26, Chap. 2].
- If $f(v)$ is a general polynomial of degree P, to guarantee the existence and uniqueness of solution for (1.1) for cylindrical white noise (cf. [9]), $g(u)$ is required to have the following restriction

$$\|g(u)\| \leq C(1 + \|u\|^{1/P}), \ u \in H.$$

2.4. **Spatial regularity of u.** We proceed to exploit the regularity of the solution (1.2) under these assumptions. We note that an optimal spatial regularity has been established for additive noise under the conditions $\|(-A)^{\frac{\gamma-1}{2}}Q^{\frac{1}{2}}\| < \infty$ (cf. [8,34]).

Theorem 2.1. *Under Assumption* 2.1*-Assumption* 2.3*, the unique mild solution* $u(t)$ *of* (1.1) *satisfies*

$$\sup_t \mathbf{E}\|u(t)\|_\gamma^p, \sup_t \mathbf{E}\|f(u(t))\|_\gamma^p < \infty, \quad \forall p \geq 2.$$

Proof. We start with (1.2). For any $t > 0$

$$\begin{aligned} \|u(t)\|_{L^p(\Omega;H^\gamma)} \leq \left\|(-A)^{\frac{\gamma}{2}}E(t)u_0\right\|_{L^p(\Omega;H)} + \left\|(-A)^{\frac{\gamma}{2}}\int_0^t E(t-\sigma)f(u(\sigma))d\sigma\right\|_{L^p(\Omega;H)} \\ + \left\|(-A)^{\frac{\gamma}{2}}\int_0^t E(t-\sigma)g(u)dW^Q(\sigma)\right\|_{L^p(\Omega;H)}. \end{aligned} \tag{2.6}$$

The assumption on $u_0 : \Omega \to H^\gamma$ implies the bound for the first term

$$\left\|(-A)^{\frac{\gamma}{2}}E(t)u_0\right\|_{L^p(\Omega;H)} \leq \|u_0\|_{L^p(\Omega;H^\gamma)} < C. \tag{2.7}$$

For the last term in (2.6), we use the Burkholder-Davis-Gundy inequality, Assumption 2.2 and generalized Gronwall inequality to obtain

$$\begin{aligned} &\left\|\int_0^t (-A)^{\frac{\gamma}{2}}E(t-\sigma)g(u(\sigma))dW^Q(\sigma)\right\|_{L^p(\Omega;H)} \\ &\leq C(p)\bigg(\int_0^t \left(\mathbf{E}\left\|(-A)^{\frac{\gamma}{2}}E(t-\sigma)g(u(\sigma))\right\|_{L_2^0}^p\right)^{\frac{2}{p}}d\sigma\bigg)^{\frac{1}{2}} \\ &\leq C(p)\bigg(\int_0^t \left(\mathbf{E}\left\|(-A)^{\frac{1-\epsilon}{2}}E(t-\sigma)(-A)^{\frac{\gamma-1+\epsilon}{2}}g(u(\sigma))\right\|_{L_2^0}^p\right)^{\frac{2}{p}}d\sigma\bigg)^{\frac{1}{2}} \\ &\leq C\bigg(\int_0^t (t-\sigma)^{\epsilon-1}\left(\mathbf{E}\|g(u(\sigma))\|_{L^0_{2,\gamma-1+\epsilon}}^p\right)^{\frac{2}{p}}d\sigma\bigg)^{\frac{1}{2}} \\ &\leq C\bigg(\int_0^t (t-\sigma)^{\epsilon-1}\left(\mathbf{E}\|(u(\sigma))\|_{L^0_{2,\gamma-1+\epsilon}}^p\right)^{\frac{2}{p}}d\sigma\bigg)^{\frac{1}{2}} \\ &\leq C\bigg(\int_0^t (t-\sigma)^{\epsilon-1}\|u(\sigma)\|^2_{L^p(\Omega;H^\gamma)}d\sigma\bigg)^{\frac{1}{2}}. \end{aligned} \tag{2.8}$$

It remains to bound the second term in (2.6). Towards this end, we consider γ in differently intervals separately as follows.

(i) Case $\gamma \in (\frac{d}{2}, 2)$:

The regularity result on u for $\gamma \in (1,2)$ has been proved in [28]. We only need to prove the case for $\gamma \in (\frac{1}{2}, 1]$ with $d = 1$.

$$\left\|(-A)^{\frac{\gamma}{2}}\int_0^t E(t-\sigma)f(u(\sigma))d\sigma\right\|_{L^p(\Omega;H)} \leq \int_0^t (t-\sigma)^{-\frac{\gamma}{2}}\|f(u(\sigma))\|_{L^p(\Omega;H)}d\sigma. \tag{2.9}$$

Therefore, $u(t) \in L^p(\Omega; H^\gamma)$ by the generalized Gronwall's inequality using (2.7), (2.9) and (2.8).

Since $\gamma > \frac{d}{2}, p \geq 2$, we have $W^{\gamma,p}(\mathcal{O})$ is a Banach algebra for $d = 1, 2$ (cf. [1, Page 106]). Hence, $\sup_t \mathbf{E}\|f(u(t))\|_\gamma^p < \infty$.

(ii) Case $\gamma = 2$:

From the previous case, one has $u(t) \in L^p(\Omega; H^\mu)$ for some $\mu \in (1,2)$. Hence,

$$\begin{aligned}
&\left\| \int_0^t E(t-\sigma) A f(u(\sigma)) d\sigma \right\|_{L^p(\Omega;H)} \\
&\quad = \left\| \int_0^t E(t-\sigma)(-A)^{1-\frac{\mu}{2}}(-A)^{\frac{\mu}{2}} f(u(\sigma)) d\sigma \right\|_{L^p(\Omega;H)} \\
&\quad \le \int_0^t (t-\sigma)^{\frac{\mu}{2}-1} \| f(u(\sigma)) \|_{L^p(\Omega;H^\mu)} d\sigma < C, \qquad (2.10)
\end{aligned}$$

by the result of case (i). Therefore, $u(t) \in L^p(\Omega; H^2)$ by the generalized Gronwall's inequality using (2.7), (2.10) and (2.8), and by the same reason as in the previous case, $\sup_t \mathbf{E}\| f(u(t))\|_2^p < \infty$.

(iii) Case $\gamma \in (2,4)$:

By virtue of the results of case (ii),

(2.11)
$$\left\| \int_0^t (-A)^{\frac{\gamma}{2}} E(t-\sigma) f(u(\sigma)) d\sigma \right\|_{L^p(\Omega;H)} \le \int_0^t (t-\sigma)^{1-\frac{\gamma}{2}} \| f(u(\sigma)\|_{L^p(\Omega;H^2)} d\sigma < C.$$

We repeat the above process for arbitrarily large γ as long as both (2.7) and (2.8) hold or Assumptions 2.2 and 2.3 hold.

The proof is completed.

□

Remark 2.3. Theorem 2.1 lifts an essential restriction on γ in [11, 24, 25, 27, 28, 34], and allows us to obtain higher-order convergence in space, as opposed to the low-order convergence rate of linear FEM approximation considered in [11, 24, 25, 27, 34].

3. Spatial semi-discretization

We describe below our spatial semi-discretization and carry out an error analysis. We assume $\mathcal{O} = (0,1)^d$,$(d = 1, 2)$.

3.1. **Spatial semi-discretization.** Let $\mathcal{P}_N$ be the space of polynomials on $\mathcal{O}$ with degree at most N in each direction and $V_N = \{v | v \in \mathcal{P}_N, v|_{\partial\mathcal{O}} = 0\}$. We define $P_N : H^{-1} \to V_N$ a generalized projection by (cf. [26]):

$$(P_N v, y_N) = (\nabla A^{-1} v, \nabla y_N),\ \forall v \in H^{-1},\ y_N \in V_N. \qquad (3.1)$$

It is clear that for $v \in L^2(\mathcal{O})$, we have

$$(P_N v, y_N) = (v, y_N),\ \forall y_N \in V_N,$$

from which we derive [7]

$$\| P_N v - v \| \le \inf_{v_N \in V_N} \| v_N - v \| \le C N^{-r} \| u \|_r, \quad \forall r > 0. \qquad (3.2)$$

We introduce a discrete operator $A_N : V_N \to V_N$ defined by

$$(A_N v_N, \chi_N) := -((-A)^{1/2} v_N, (-A)^{1/2} \chi_N),\ \ \forall v_N, \chi_N \in V_N.$$

Then the spectral Galerkin approximation of (1.1) yields

$$du_N = A_N u_N dt + P_N f(u_N) dt + P_N g(u_N) dW^Q(t), u_N(0) = P_N u_0. \qquad (3.3)$$

Similar as the continuous case, there exists a unique mild solution u_N to (3.3) which can be written as

(3.4)
$$u_N(t)=E_N(t)P_Nu_0+\int_0^t E_N(t-s)P_Nf(u_N(s))ds+\int_0^t E_N(t-s)P_Ng(u_N)dW^Q(s),$$

where $E_N(t)=e^{tA_N}$. Similar to [38, Lemma 3.9], one has the property

$$\|(-A_N)^\mu E_N v_N\| \le Ct^{-\mu}\|v_N\| \text{ for all } t>0,\ v_N\in\mathcal{P}_N, \tag{3.5}$$

and defines the operator

$$F_N(t) := E_N(t)P_N - E(t).$$

Lemma 3.1. *Let $0\le\nu\le\mu$. Then there exists a constant C such that*

$$\|F_N(t)u\| \le CN^{-\mu}t^{-\frac{\mu-\nu}{2}}\|u\|_\nu,\ \forall u\in H^\nu.$$

Proof. Thanks to (3.2), this result can be proved by using the same technique used for finite elements (cf. [38, Theorem 3.5]), so we omit the detail here. □

Lemma 3.2. *Let Assumptions 2.1-2.3 hold and u_N is given by (3.3). Then, for all $p\ge2$,*

$$\sup_t \mathbf{E}\|u_N(t)\|_\infty^p < C,$$

where C is independent of N.

Proof. The proof is the similar as that of full discretization scheme (see Theorem 5.1) and we omit it here. □

Theorem 3.1. *Let u and u_N be the solutions of (1.1) and (3.3). Then, under Assumptions 2.1-2.3, there exists a constant C independent of N such that*

$$\|u(t)-u_N(t)\|_{L^2(\Omega;H)} \le CN^{-\gamma},\quad t>0. \tag{3.6}$$

Proof. Let us introduce an auxiliary process $\widetilde{u}_N(t)$ defined by

(3.7)
$$d\widetilde{u}_N(t) = A_N\widetilde{u}_N(t)dt + P_Nf(u(t))dt + P_N[g(u(t))dW^Q(t)],\quad \widetilde{u}_N(0)=P_Nu_0.$$

We can easily obtain the following stability result by following the same proof for $u_N(t)$

$$\sup_t \mathbf{E}\|\widetilde{u}_N(t)\|_\infty^p \le C,\quad \forall p\ge2. \tag{3.8}$$

It is clear that

$$\begin{aligned}\|u(t)-\widetilde{u}_N(t)\|_{L^p(\Omega,H)} &\le \left\|F_N(t)u_0\right\|_{L^p(\Omega,H)} + \left\|\int_0^t F_N(t-s)f(u(s))ds\right\|_{L^p(\Omega,H)}\\ &\quad + \left\|\int_0^t F_N(t-s)g(u(s)dW^Q(s)\right\|_{L^p(\Omega,H)}\\ &:= I_1+I_2+I_3. \end{aligned}\tag{3.9}$$

By Lemma 3.1 and Theorem 2.1, we easily have

$$I_1 \le CN^{-\gamma}\|u_0\|_{L^p(\Omega,H^\gamma)},\quad I_2\le CN^{-\gamma}\int_0^t \|f(u(s))\|_{L^p(\Omega,H^\gamma)}ds, \tag{3.10}$$

and by the BDG inequality

$$
\begin{aligned}
I_3^2 &\le C\int_0^t (\mathbf{E}\|F_N(t-s)g(u(s))Tr(Q)\|^p)^{\frac{2}{p}}ds \\
&\le CN^{-2\gamma}\int_0^t (\mathbf{E}\|g(u(s))\|_\gamma^p)^{\frac{2}{p}}ds \\
&\le CN^{-2\gamma}\int_0^t (\mathbf{E}\|u(s)\|_\gamma^p)^{\frac{2}{p}}ds \le CN^{-2\gamma}. \qquad (3.11)
\end{aligned}
$$

Therefore,

$$
\|u(t)-\widetilde{u}_N(t)\|_{L^p(\Omega,H)} \le CN^{-\gamma}. \qquad (3.12)
$$

Let $\widetilde{e}_N(t)=\widetilde{u}_N(t)-u_N(t)$. Then, we have

$$
\begin{aligned}
\widetilde{e}_N(t) =& \int_0^t E_N(t-s)P_N[f(u(s))-f(u_N(s))]ds \\
&+\int_0^t E_N(t-s)P_N[g(u(s))-g(u_N(s))dW^Q(s)]. \qquad (3.13)
\end{aligned}
$$

Now, we apply Itô's formula for $\widetilde{e}_N(t)$ (cf. [14]) and obtain

$$
\begin{aligned}
&\mathbf{E}\|\widetilde{e}_N(t)\|^2+2\int_0^t \mathbf{E}\|\nabla\widetilde{e}_N(s)\|^2ds \\
=&2\int_0^t \mathbf{E}(\widetilde{e}_N(s),P_N(f(u(s))-f(u_N(s)))ds+\int_0^t \mathbf{E}\|P_N(g(u(s))-g(u_N(s)))\|_{L_2^0}^2ds \\
\le&2L\int_0^t \mathbf{E}\|\widetilde{e}_N(s)\|^2ds+2\int_0^t \mathbf{E}(\widetilde{e}_N(s),P_N(f(u(s))-f(\widetilde{u}_N(s)))ds \\
&+C\int_0^t \mathbf{E}\|u(s)-u_N(s)\|^2ds \\
\le&C\int_0^t \mathbf{E}\|\widetilde{e}_N(s)\|^2ds+\int_0^t \mathbf{E}\|f(u(s))-f(\widetilde{u}_N(s))\|^2ds+C\int_0^t \mathbf{E}\|u(s)-\widetilde{u}_N(s)\|^2ds \\
\le&C\int_0^t \mathbf{E}\|\widetilde{e}_N(s)\|^2ds+\int_0^t \mathbf{E}[\|u(s)-\widetilde{u}_N(s)\|^2(1+\|u(s)\|_\infty^4+\|\widetilde{u}_N(s)\|_\infty^4)]+CN^{-2\gamma} \\
\le&C\int_0^t \mathbf{E}\|\widetilde{e}_N(s)\|^2ds+\int_0^t \sqrt{\mathbf{E}\|u(s)-\widetilde{u}_N(s)\|^4}\sqrt{\mathbf{E}(1+\|u(s)\|_\infty^4+\|\widetilde{u}_N(s)\|_\infty^4)^2}ds \\
&+CN^{-2\gamma} \\
\le&C\int_0^t \mathbf{E}\|\widetilde{e}_N(s)\|^2ds+CN^{-2\gamma}, \qquad (3.14)
\end{aligned}
$$

where we have used (2.4), Theorem 2.1 and (3.8).

Therefore, the Gronwall's inequality implies

$$
\mathbf{E}\|\widetilde{e}_N(t)\|^2 \le CN^{-2\gamma}. \qquad (3.15)
$$

Combining (3.12) and (3.15) leads to the desired result. □

4. Full discretization and an unconditional stability result

In this section, we present our fully discrete scheme and establish an unconditional stability result. The time discretization is based on a tamed semi-implicit discretization which leads to a linear system at each time step and is unconditionally stable.

Let τ be the time step size and $M = T/\tau$. We start with a semi-discrete time splitting discretization scheme for (1.1):

$$u^{k+1} - u^k = \tau\Delta u^{k+1} + \frac{\tau f(u^k)}{1+\tau\|f(u^k)\|^2} + g(u^k)\Delta W^Q(t_k), 0 \le k \le M-1. \tag{4.1}$$

Combining with (3.3), we have its fully discretized version:

$$(u_N^{k+1} - u_N^k, \psi) = \tau(\Delta u_N^{k+1}, \psi) + \frac{\tau}{1+\tau\|f(u_N^k)\|^2}(f(u_N^k),\psi) + (g(u_N^k),\psi), \tag{4.2}$$
$$u_N^0 = P_N u^0,$$

or

$$u_N^{k+1} - u_N^k = \tau A_N u_N^{k+1} + \frac{\tau}{1+\tau\|f(u_N^k)\|^2} P_N f(u_N^k) + P_N[g(u_N^k)\Delta W^Q(t_k)], \tag{4.3}$$
$$0 \le k \le M-1.$$

A remarkable property of the above tamed scheme is that, despite treating the nonlinear term explicitly, it is still unconditionally stable as we show below.

Theorem 4.1. *The schemes* (4.1) *and* (4.2) *admit a unique solution* u^{k+1} *and* u_N^{k+1}*, and are unconditionally stable in the sense that for* $1 \le k \le M-1$*, we have*

$$\mathbf{E}\|u_N^{k+1}\|^q \le C(\mathbf{E}\|u_0\|, q, T), \quad \forall q \ge 2.$$

Proof. The proof for the semi-discrete and full-discrete cases are essentially the same so we shall only prove the result for the full-discrete case. It is clear that the scheme (4.2) admits a unique solution.

Choosing $\psi = u_N^{n+1}$ in (4.2) gives

$$\begin{aligned}
&\|u_N^{n+1}\|^2 + \|u_N^{n+1} - u_N^n\|^2 + 2\tau\|\nabla u_N^{n+1}\|^2 \\
&\le \|u_N^n\|^2 + \frac{2\tau}{1+\tau\|f(u_N^n)\|^2}(f(u_N^n), u_N^{n+1}) + 2(g(u_N^n)\Delta W^n, u_N^{n+1}) \\
&\le \|u_N^n\|^2 + \frac{2\tau}{1+\tau\|f(u_N^n)\|^2}(f(u_N^n), u_N^{n+1} - u_N^n) + 2\tau(f(u_N^n), u_N^n) \\
&\quad + 2(g(u_N^n)\Delta W^n, u_N^{n+1}) \\
&\le \|u_N^n\|^2 + 4\tau + \frac{1}{4}\|u_N^{n+1} - u_N^n\|^2 - 2\tau\|u_N^n\|_{L^4}^4 + 2\tau\|u_N^n\|^2 \\
&\quad + 2(g(u_N^n)\Delta W^n, u_N^{n+1} - u_N^n) + 2(g(u_N^n)\Delta W^n, u_N^n) \\
&\le (1+2\tau)\|u_N^n\|^2 + 4\tau + \frac{1}{2}\|u_N^{n+1} - u_N^n\|^2 - 2\tau\|u_N^n\|_{L^4}^4 \\
&\quad + 4\|g(u_N^n)\Delta W^n\|^2 + 2(g(u_N^n)\Delta W^n, u_N^n).
\end{aligned} \tag{4.4}$$

Denote by $A = (1+2\tau)$ and a simple computation implies

$$\lim_{M\to\infty} A^n = \lim_{M\to\infty}\left(1 + \frac{2T}{M}\right)^n \le e^{2T}, \ n \le M.$$

Hence, (4.4) leads to

$$\|u_N^{n+1}\|^2 + \frac{1}{2}\sum_{j=0}^n A^{n-j}\|u_N^{j+1}-u_N^j\|^2 + 2\tau\sum_{j=0}^n A^{n-j}\|\nabla u_N^{j+1}\|^2 + 2\tau\sum_{j=0}^n A^{n-j}\|u_N^j\|_{L^4}^4$$

(4.5)

$$\leq A^{n+1}\|u_N^0\|^2 + 4\tau\sum_{j=0}^n A^j + 4\sum_{j=0}^n A^{n-j}\|g(u_N^j)\Delta W^n\|^2 + 2\sum_{j=0}^n A^{n-j}(g(u_N^j)\Delta W^j, u_N^j).$$

Next, we take m-moment on (4.5) and obtain

(4.6)

$$\|u_N^{n+1}\|^{2m} \leq \begin{cases} C\|u_N^0\|^{2m} + CT^m + C\Big[\sum_{j=0}^n \|g(u_N^j)\Delta W^n\|^2\Big]^m + C\Big[\sum_{j=0}^n (g(u_N^j)\Delta W^j, u_N^j)\Big]^m, \\ \qquad\qquad \text{if } \sum_{j=0}^n (g(u_N^j)\Delta W^j, u_N^j) \geq 0; \\ C\|u_N^0\|^{2m} + CT^m + C\Big[\sum_{j=0}^n \|g(u_N^j)\Delta W^n\|^2\Big]^m, \qquad \text{otherwise.} \end{cases}$$

Since $(g(u_N^j)\Delta W^j, u_N^j)$ are martingales independent from each other, we derive from the BDG inequality (Lemma 2.1) and Assumption 2.2 that

$$\begin{aligned}
\mathbf{E}\|u_N^{n+1}\|^{2m} &\leq C\mathbf{E}\|u_N^0\|^{2m} + CT^m + Cn^{m-1}\sum_{j=0}^n \mathbf{E}\|g(u_N^j)\Delta W^n\|^{2m} \\
&\quad + C\mathbf{E}\sum_{j=0}^n (g(u_N^j)\Delta W^j, u_N^j)^m \\
&\leq C\mathbf{E}\|u_N^0\|^{2m} + CT^m + C\tau\sum_{j=0}^n \mathbf{E}\|g(u_N^j)\|^{2m} \\
&\quad + C\left(\sum_{j=0}^n [\mathbf{E}(|(g(u_N^j)\Delta W^j, u_N^j)|^m]^{\frac{2}{m}}\right)^{\frac{m}{2}} \\
&\leq C\mathbf{E}\|u_N^0\|^{2m} + CT^m + C\tau\sum_{j=0}^n \mathbf{E}\|g(u_N^j)\|^{2m} \\
&\quad + C\left(\sum_{j=0}^n \{\mathbf{E}[(\|g(u_N^j)\|^4 + \|u_N^j\|^4)\tau]^{\frac{m}{2}}\}^{\frac{2}{m}}\right)^{\frac{m}{2}} \\
&\leq C\mathbf{E}\|u_N^0\|^{2m} + CT^m + C\tau\sum_{j=0}^n \mathbf{E}\|u_N^j\|^{2m} + C\tau^{\frac{m}{2}}\left(\sum_{j=0}^n (\mathbf{E}\|u_N^j\|^{2m})^{\frac{2}{m}}\right)^{\frac{m}{2}} \\
&\leq C\mathbf{E}\|u_N^0\|^{2m} + CT^m + C\tau\sum_{j=0}^n \mathbf{E}\|u_N^j\|^{2m} + C\tau\sum_{j=0}^n (\mathbf{E}\|u_N^j\|^{2m})
\end{aligned}$$

(4.7)

$$\leq C\mathbf{E}\|u_N^0\|^{2m} + CT^m + C\tau\sum_{j=0}^n \mathbf{E}\|u_N^j\|^{2m}.$$

Hence, the discrete Gronwall's inequality implies

$$\mathbf{E}\|u_N^{n+1}\|^{2m} \le C(\|u^0\|, T, m). \tag{4.8}$$

Furthermore, we also have

(4.9)

$$\mathbf{E}\|u_N^{n+1}\|^{2m} + \mathbf{E}\Big(\sum_{j=0}^{n} \|\nabla u_N^{j+1}\|^2\tau\Big)^m + \mathbf{E}\Big(\sum_{j=0}^{n} \|u_N^j\|_{L^4}^4\tau\Big)^m \le C(\|u_0\|, T, m).$$

□

Remark 4.1. We note that the full discretization scheme (3.3) can be directly extended to rectangular domains in d dimension, and that the above stability result is also valid for any dimension.

5. Error analysis in one spatial dimension

We established an unconditional stability result for $d = 1, 2$ in the last section, but the result in Theorem 4.1 is not sufficient to derive a strong convergence result. Therefore, we confine ourselves mostly to the case $d = 1$ in this section. We first derive further stability results in L^∞ norm, and then use it to establish a strong convergence result.

5.1. **Further stability results.** We first establish a result which is valid in any dimension d, and will be used below.

Lemma 5.1. *There exist positive constants c_1 and c_2 independent of N and τ such that*

$$c_1\|(-\Delta_N)^\gamma v_N\| \le \|(-\Delta)^\gamma v_N\| \le c_2\|(-\Delta_N)^\gamma v_N\|, \quad -\frac{1}{2} \le \gamma \le \frac{1}{2}. \tag{5.1}$$

Proof. We follow a similar argument in [2] for a finite-element approximation.

We start with the first inequality in (5.1). By definition of Δ_N,

$$\begin{aligned}\|(-\Delta_N)^{\frac{1}{2}} P_N v\| &= \|\nabla P_N v\| = \|(-\Delta)^{\frac{1}{2}} P_N v\| = \|P_N v\|_1 \le C\|v\|_1 \\ &= C\|(-\Delta)^{\frac{1}{2}} v\|, \ v \in H^1.\end{aligned} \tag{5.2}$$

Moreover,

$$\begin{aligned}\|(-\Delta_N)^{-\frac{1}{2}} P_N v\| &= \sup_{\psi\in P_N} \frac{(v,\psi)}{\|(-\Delta_N)^{\frac{1}{2}}\psi\|} = \sup_{\psi\in P_N} \frac{(v,\psi)}{\|(-\Delta)^{\frac{1}{2}}\psi\|} \\ &\le \sup_{\psi\in H^1} \frac{(v,\psi)}{\|(-\Delta)^{\frac{1}{2}}\psi\|} = \|(-\Delta)^{-\frac{1}{2}} v\|.\end{aligned} \tag{5.3}$$

Hence, by interpolation, we have

$$\|(-\Delta_N)^\gamma P_N v\| \le C\|(-\Delta)^\gamma v\|, \ v \in H^\gamma, \ -\frac{1}{2} \le \gamma \le \frac{1}{2}. \tag{5.4}$$

The desired inequality is proved by choosing $v = v_N$.

Next, we prove the second inequality in (5.1). We note from (5.2) that

$$\|(-\Delta_N)^{\frac{1}{2}} P_N (-\Delta)^{-\frac{1}{2}}\|_{\mathcal{L}} \le C. \tag{5.5}$$

Hence,

$$\begin{aligned}\|(-\Delta)^{-\frac{1}{2}}(-\Delta_N)^{\frac{1}{2}}P_N\|_{\mathcal{L}} &= \|[(-\Delta)^{-\frac{1}{2}}(-\Delta_N)^{\frac{1}{2}}P_N]^*\|_{\mathcal{L}} \\ &= \|(-\Delta_N)^{\frac{1}{2}}P_N(-\Delta)^{-\frac{1}{2}}\|_{\mathcal{L}} \le C,\end{aligned} \tag{5.6}$$

so that

$$\|(-\Delta)^{-\frac{1}{2}}(-\Delta)_N^{\frac{1}{2}}P_N v\| \le C\|v\|$$

or

$$\|(-\Delta)^{-\frac{1}{2}}v_N\| \le C\|(-\Delta_N)^{-\frac{1}{2}}v_N\|, \quad v_N \in P_N. \tag{5.7}$$

In addition,

$$\|(-\Delta_N)^{\frac{1}{2}}v_N\| = \|\nabla v_N\| = \|(-\Delta)^{\frac{1}{2}}v_N\|,\ v_N \in P_N. \tag{5.8}$$

Again, by interpolation,

$$\|(-\Delta)^{\gamma}v_N\| \le C\|(-\Delta_N)^{\gamma}v_N\|,\ v_N \in P_N,\ -\frac{1}{2} \le \gamma \le \frac{1}{2}. \tag{5.9}$$

□

We denote

$$E^n = (I - \tau A_N)^{-n}, 1 \le n \le M.$$

To proceed, we need to prove a stability result for the operator E^m.

Lemma 5.2. *Let $d = 1$. We have for $0 \le \gamma \le 1, m \ge 1$,*

$$\|E^m P_N v\|_\infty \le C(m\tau)^{\frac{\gamma}{2}-\frac{1}{4}-\epsilon}\|v\|_\gamma, \quad v \in H^\gamma. \tag{5.10}$$

Proof. We note that the function

$$h(x) = \frac{x^{\frac{1}{2}+\epsilon-\gamma}}{(1+\tau x)^{2m}}, x > 0$$

attains its maximum at the point $x^* = \frac{1+2\epsilon-2\gamma}{[4m-(1+2\epsilon-2\gamma)]\tau}$ with the bound

$$|h(x)| \le h(x^*) \le C(m\tau)^{\gamma-\frac{1}{2}-\epsilon}.$$

Therefore, by Lemma 5.1 and the bound of h, we deduce that

$$\begin{aligned}\|E^m P_N v\|_\infty^2 &\le C\|E^m P_N v\|_{\frac{1}{2}+\epsilon}^2 = C\|(-\Delta)^{\frac{1}{4}+\frac{\epsilon}{2}}E^m P_N v\|^2 \\ &\le C\|(-\Delta_N)^{\frac{1}{4}+\frac{\epsilon}{2}}E^m P_N v\|^2 \\ &= C\sum_{j=1}^{N}\frac{(\lambda_N^j)^{\frac{1}{2}+\epsilon-\gamma}}{(1+\tau\lambda_N^j)^{2m}}(\lambda_N^j)^{\gamma}|(v, e_N^j)|^2 \\ &\le C(m\tau)^{\gamma-\frac{1}{2}-2\epsilon}\sum_{j=1}^{N}(\lambda_N^j)^{\gamma}|(v, e_N^j)|^2 \\ &\le C(m\tau)^{\gamma-\frac{1}{2}-2\epsilon}\|(-\Delta_N)^{\frac{\gamma}{2}}P_N v\|^2 \\ &\le C(m\tau)^{\gamma-\frac{1}{2}-2\epsilon}\|v\|_\gamma^2, \quad \forall\ 0 \le \gamma \le 1,\end{aligned} \tag{5.11}$$

which implies (5.10). □

Now we are ready to prove a new stability result needed for the error analysis below.

Theorem 5.1. *Let $T>0$, $q\geq 1$ and $d=1$. There exists a constant $C(T,q,\|u_0\|)$ such that*

$$\sup_n[\mathbf{E}\|u_N^n\|_\infty^q]\leq \begin{cases} C(T,q,\|u_0\|)t_n^{-\frac{q}{4}-\epsilon}, u_0\in L^2(\mathcal{O}),\\ C(T,q,\|u_0\|), \qquad u_0\in H^\gamma(\mathcal{O}),\gamma>\frac12.\end{cases}$$

Proof. With Theorem 4.1 in hand, we bound the following stochastic term first. Applying Lemma 2.1 and (3.5) and Assumption 2.2 successively, we have

$$\begin{aligned}\mathbf{E}\Big\|\sum_{j=0}^k E^{k+1-n}P_Ng(u_N^j)\Delta W^j\Big\|_\infty^q &\leq C\mathbf{E}\Big\|\sum_{j=0}^k E^{k+1-n}P_Ng(u_N^j)\Delta W^j\Big\|_{\frac12+\epsilon}^q\\ &\leq C\mathbf{E}\Big\|\sum_{j=0}^k (-A_N)^{\frac14+\frac{\epsilon}{2}}E^{k+1-n}P_Ng(u_N^j)\Delta W^j\Big\|^q\\ &\leq C\Big(\sum_{j=0}^k(\mathbf{E}\|(-A_N)^{\frac14+\frac{\epsilon}{2}}E^{k+1-n}P_Ng(u_N^j)\Delta W^j\|^q)^{\frac2q}\Big)^{\frac q2}\\ &\leq C\Big(\sum_{j=0}^k t_{k+1-n}^{-\frac12-\epsilon}(\mathbf{E}\|g(u_N^j)\Delta W^j\|^q)^{\frac2q}\Big)^{\frac q2}\\ &\leq C\Big(\sum_{j=0}^k t_{k+1-n}^{-\frac12-\epsilon}\tau(\mathbf{E}\|u_N^j\|^q)^{\frac2q}\Big)^{\frac q2}<\infty. \qquad (5.12)\end{aligned}$$

Now, we are ready to bound u_N^{k+1}. By (5.12) and the Gagliardo-Nirenberg inequality

$$\begin{aligned}\mathbf{E}\|u_N^{k+1}\|_\infty^q &= \mathbf{E}\Big\|E^{k+1}u_N^0+\sum_{j=0}^k\frac{\tau E^{k+1-j}P_N}{1+\tau\|f(u_N^j)\|^2}f(u_N^j)+\sum_{j=0}^kE^{k+1-j}P_Ng(u_N^j)\Delta W^Q(t_j)\Big\|_\infty^q\\ &\leq 3^{q-1}\mathbf{E}\|E^{k+1}u_N^0\|_\infty^q+3^{q-1}\mathbf{E}\Big(\sum_{j=0}^k\|E^{k+1-j}P_Nf(u_N^j)\|_\infty\tau\Big)^q\\ &\quad+3^{q-1}\mathbf{E}\Big\|\sum_{j=0}^kE^{k+1-n}P_Ng(u_N^j)\Delta W^Q(t_j)\Big\|_\infty^q\\ &\leq C\mathbf{E}\|E^{k+1}u_N^0\|_\infty^q+3^{q-1}\mathbf{E}\Big(\sum_{j=n}^k t_{k+1-j}^{-\frac14-\epsilon}\|f(u_N^j)\|\tau\Big)^q+C\\ &\leq C\mathbf{E}\|E^{k+1}u_N^0\|_\infty^q+C\mathbf{E}\Big(\sum_{i=0}^k t_{k+1-i}^{-\frac14-\epsilon}(1+\|u_N^i\|_{L^6}^3)\tau\Big)^q+C\\ &\leq C\mathbf{E}\|E^{k+1}P_Nu^0\|_\infty^q+C(q)\mathbf{E}\Big(\sum_{i=0}^k t_{k+1-i}^{-\frac14-\epsilon}(1+\|\nabla u_N^i\|^{\frac43}+\|u_N^i\|^8)\tau\Big)^q+C\\ &\leq C(q)\Big(\sum_{i=0}^k t_{k+1-i}^{-\frac34-3\epsilon}\tau\Big)^{\frac q3}\mathbf{E}\Big(\sum_{i=0}^k(1+\|\nabla u_N^i\|^2+\|u_N^i\|^{12})\tau\Big)^{\frac{2q}{3}}\\ &\quad+C+\begin{cases}Ct_{k+1}^{-\frac q4-\epsilon}\mathbf{E}\|u_N^0\|^q,\ u^0\in L^2(\mathcal{O})\\ C\mathbf{E}\|u^0\|_\gamma^q,\ u_0\in H^\gamma(\mathcal{O}),\gamma>\frac12\end{cases},\end{aligned}$$

where we have used Lemma 5.2 to bound u_N^0. Then, the desired result follows from the above and previous results on u_N^k in (4.9). □

5.2. **Convergence analysis.** Now, we carry out a convergence analysis for (4.2).

Lemma 5.3. *Under Assumption* 2.1*, we have*

$$\|(-A)^{\frac{\rho}{2}}(E(t_n)-E^n)v\| \le C\tau^{\frac{\beta}{2}}t_n^{-\frac{\beta-\gamma+\rho}{2}}\|(-A)^{\frac{\gamma}{2}}v\|,\ 0\le\gamma\le\beta+\rho,\gamma\ge 0,\beta\in[0,2];$$

(5.13)

$$\|(E(t)-E^nP_N)v\| \le C(N^{-\mu}+\tau^{\min\{\frac{\mu}{2},1\}})\|v\|_\mu,\quad v\in H^\mu.$$

Proof. The first inequality can be found in [24]. We only prove the second one.

$$\|(E(t)-E^nP_N)v\| \le \|(E^nP_N-E(t_n))v\|+\|(E(t_n)-E(t))v\|, t\in[t_{n-1},t_n].$$

It is clear that

$$\begin{aligned}\|(E(t_n)-E(t))v\| &= \|(-A)^{-\frac{\mu}{2}}(E(t_n-t))-I)E(t)(-A)^{\frac{\mu}{2}}v\|\\ &\le C(t_n-t)^{\frac{\mu}{2}}\|x\|_\mu \le C\tau^{\frac{\mu}{2}}\|v\|_\mu,\end{aligned}$$

where (2.3) is applied and therefore we require $0\le\mu<2$ for this estimate.

Furthermore, since v is smooth, we can follow the proof of [38, Theorem 7.8] to derive

$$\|(E^nP_N-E(t_n))v\| \le C(N^{-\mu}\|v\|_\mu+\tau\|v\|_2),\ t_n\ge 0.$$

□

Remark 5.1. For the above estimate, we only require $v\in\dot{H}^\mu$, where μ can be arbitrarily large. Thus, the spatial error can be made arbitrarily small provided v is sufficiently smooth whereas the temporal error is at most of order $\mathcal{O}(\tau)$ which cannot be improved.

We start by establishing some temporal properties of $u(s)$.

Lemma 5.4. *Under Assumptions* 2.1-2.3*, we have*

$$\|u(t)-u(s)\|_{L^p(\Omega;H)} \le C(t-s)^{\min\{\frac{\gamma}{2},\frac{1}{2}\}},\quad p\ge 2. \tag{5.14}$$

Proof. Suppose that $0\le s\le t\le T$. Using (1.2),

$$\begin{aligned}&\|u(t)-u(s)\|_{L^p(\Omega;H)}\\ &\quad\le \|(E(t)-E(s))u_0\|_{L^p(\Omega;H)}\\ &\qquad+\left\|\int_0^t E(t-\sigma)f(u(\sigma))d\sigma-\int_0^s E(s-\sigma)f(u(\sigma))d\sigma\right\|_{L^p(\Omega;H)}\\ &\qquad+\left\|\int_0^t E(t-\sigma)g(u(\sigma))dW^Q(\sigma)-\int_0^s E(s-\sigma)g(u(\sigma))dW^Q(\sigma)\right\|_{L^p(\Omega;H)}\\ &\quad:= H_1+H_2+H_3.\end{aligned} \tag{5.15}$$

Using (2.3),

$$\begin{aligned}H_1 &\le \|E(s)(-A)^{-\min\{\frac{\gamma}{2},1\}}(E(t-s)-I)(-A)^{\min\{\frac{\gamma}{2},1\}}u_0\|_{L^p(\Omega;H)}\\ &\le C(t-s)^{\min\{\frac{\gamma}{2},1\}}\|u_0\|_{L^p(\Omega;H^{\min\{\gamma,2\}})}.\end{aligned} \tag{5.16}$$

Similarly,

$$\begin{aligned}H_2 &\le \left\| \int_0^s \big(E(t-\sigma)-E(s-\sigma)\big) f(u(\sigma))) d\sigma \right\|_{L^p(\Omega;H)} \\ &\quad + \left\| \int_s^t E(t-\sigma) f(u(\sigma)) d\sigma \right\|_{L^p(\Omega;H)} \\ &\le C(t-s)^{\min\{\frac{\gamma}{2},1\}} \int_0^s \|f(u(\sigma))\|_{L^p(\Omega;H^{\min\{\gamma,2\}})} d\sigma \\ &\quad + \int_s^t \|E(t-\sigma) f(u(\sigma))\|_{L^p(\Omega;H)} ds \\ &\le C(t-s)^{\min\{\frac{\gamma}{2},1\}}. \end{aligned} \tag{5.17}$$

By the BDG inequality and Assumption 2.2, we have

$$\begin{aligned} H_3^2 &\le C\mathbf{E}\left\| \int_0^s \big(E(t-\sigma)-E(s-\sigma)\big) g(u(\sigma)) dW^Q(\sigma) \right\|^2_{L^p(\Omega;H)} \\ &\quad + C\mathbf{E}\left\| \int_s^t E(t-\sigma) g(u(\sigma)) dW^Q(\sigma) \right\|^2_{L^p(\Omega;H)} \\ &\le C\int_0^s (\mathbf{E}\|\big(E(t-\sigma)-E(s-\sigma)\big) g(u(\sigma))\|^p_{L_2^0})^{\frac{2}{p}} d\sigma \\ &\quad + C\int_s^t (\mathbf{E}\|E(t-\sigma) g(u(\sigma))\|^p_{L_2^0})^{\frac{2}{p}} d\sigma \\ &\le C\int_0^s (\mathbf{E}\|(-A)^{\frac{1-\epsilon}{2}} E(s-\sigma)(-A)^{-\min\{\frac{\gamma}{2},1\}}(E(t-s)-I)(-A)^{\min\{\frac{\gamma}{2},1\}-\frac{1-\epsilon}{2}} g(u(\sigma))\|^p_{L_2^0})^{\frac{2}{p}} d\sigma \\ &\quad + C(t-s) \\ &\le C(t-s)^{\min\{\gamma,2\}} \int_0^s (s-\sigma)^{\epsilon-1} (\mathbf{E}\|g(u)\|^p_{\gamma-1+\epsilon})^{\frac{2}{p}} d\sigma + C(t-s) \\ &\le C(t-s)^{\min\{\gamma,2\}} (\mathbf{E}\sup_\sigma \|u(\sigma)\|^p_{\gamma-1+\epsilon})^{\frac{2}{p}} + C(t-s). \end{aligned} \tag{5.18}$$

The result follows by combining estimates of H_1, H_2 and H_3. □

Theorem 5.2. *Let $d=1$, and $u(t)$ and u_N^m be solutions of* (1.2) *and* (4.2) *respectively. Then, under Assumptions* 2.1-2.3*, there exists a constant C independent of N and τ such that*

$$\|u(t_m)-u_N^m\|_{L_2(\Omega;H)} \le C(N^{-\gamma}+\tau^{\min\{\frac{\gamma}{2},\frac{1}{2}\}}+\tau^{-\frac{1}{2}}N^{-2\gamma}), \quad t>0. \tag{5.19}$$

Proof. Following the idea from [28] (see also [34, 35]), we introduce an auxiliary process

$$\tilde{u}_N^n - \tilde{u}_N^{n-1} = \tau A_N \tilde{u}^n + \frac{\tau P_N f(u(t_{n-1}))}{1+\tau\|f(u(t_n))\|^2} + P_N g(u(t_{n-1}))\Delta W^Q(t_n), \tag{5.20}$$

which can be rewritten as

$$\tilde{u}_N^n = E^n P_N u_0 + \tau \sum_{k=1}^n \frac{E^{n-k} P_N f(u(t_{k-1}))}{1+\tau\|f(u(t_{k-1}))\|^2} + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} E^{n-k} P_N g(u(t_{k-1})) dW^Q(s). \tag{5.21}$$

By the proof of Theorem 2.1, we easily infer that $\mathbf{E}\|\tilde{u}_N^n\|_\gamma^p<\infty$, $\mathbf{E}\|f(\tilde{u}_N^n)\|_\gamma^2<\infty$, for all $1\le n\le M$ (see also [34, 35]).

Note that (4.2) can also be written in closed form

$$u_N^n = E^n P_N u_0 + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{E^{n-k} P_N f(u_N^{k-1})}{1+\tau\|f(u_N^{k-1})\|^2} ds + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} E^{n-k} P_N g(u(t_{k-1})) dW^Q(s). \tag{5.22}$$

Next, we split the error $\|u(t_n) - u_N^n\|_{L^2(\Omega;H)}, 1 \le n \le M$ into two parts, and bound them individually.

$$\|u(t_n) - u_N^n\|_{L^2(\Omega;H)} \le \|u(t_n) - \tilde{u}_N^n\|_{L^2(\Omega;H)} + \|u_N^n - \tilde{u}_N^n\|_{L^2(\Omega;H)}. \tag{5.23}$$

Subtracting (5.21) from (1.2) and taking the associated norm gives

$$\begin{aligned} &\|u(t_n) - \tilde{u}_N^n\|_{L^p(\Omega;H)} \\ &\le \|(E(t_n) - E^n P_N) u_0\|_{L^p(\Omega;H)} \\ &\quad + \left\| \int_0^{t_n} E(t_n - s) f(u(s)) ds - \tau \sum_{k=1}^{n} \frac{E^{n-k} P_N f(u(t_{k-1}))}{1+\tau\|f(u(t_{k-1}))\|^2} \right\|_{L^p(\Omega;H)} \\ &\quad + \left\| \int_0^{t_n} E(t_n - s) g(u(s)) dW^Q(s) - \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} E^{n-k} P_N g(u(t_{k-1})) dW^Q(s) \right\|_{L^p(\Omega;H)} \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{5.24}$$

An application of (5.13) gives

$$I_1 \le C(N^{-\gamma} + \tau^{\min\{\frac{\gamma}{2},1\}}) \|u_0\|_{L^p(\Omega;H^\gamma)}. \tag{5.25}$$

I_2 can be decomposed in the following way:

$$\begin{aligned} I_2 &= \left\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left[E(t_n - s) f(u(s)) - \frac{E^{n-k} P_N f(u(t_{k-1}))}{1+\tau\|f(u(t_{k-1}))\|^2} \right] ds \right\|_{L^p(\Omega;H)} \\ &\le \left\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} E(t_n - s)[f(u(s)) - f(u(t_{k-1}))] ds \right\|_{L^p(\Omega;H)} \\ &\quad + \left\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} [E(t_n - s) - E^{n-k}] f(u(t_{k-1})) ds \right\|_{L^p(\Omega;H)} \\ &\quad + \left\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left[E^{n-k} f(u(t_{k-1})) - \frac{E^{n-k} P_N f(u(t_{k-1}))}{1+\tau\|f(u(t_{k-1}))\|^2} \right] ds \right\|_{L^p(\Omega;H)} \\ &:= I_{21} + I_{22} + I_{23}. \end{aligned} \tag{5.26}$$

2 By Lemma 5.4 and Theorem 2.1, we bound I_{21} as follows.

$$\begin{aligned}
&\|I_{21}\|_{L^p(\Omega:H)}\\
&= \Big\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} E(t_n - s)(f(u(s)) - f(u(t_{k-1})))ds \Big\|_{L^p(\Omega;H)}\\
&\le C \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \big(1 + \|u(s)\|^4_{L^{2p}(\Omega;L^\infty)} + \|u(t_{k-1})\|^4_{L^{2p}(\Omega;L^\infty)}\big) \|u(s) - u(t_{k-1})\|_{L^{2p}(\Omega;H)}\\
&\le C \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|u(s) - u(t_{k-1})\|_{L^{2p}(\Omega;H)} ds\\
&\le C\tau^{\min\{\frac{\gamma}{2},\frac{1}{2}\}}.
\end{aligned}$$

By (5.13),

$$\begin{aligned}
\|I_{22}\|_{L^p(\Omega;H)} &\le \Big\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} [E(t_n - s) - E^{n-k}P_N] f(u(t_{k-1}))ds \Big\|_{L^p(\Omega;H)}\\
&\quad + \Big\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} [E^{n-k}P_N - E^{n-k}] f(u(t_{k-1}))ds \Big\|_{L^p(\Omega;H)}\\
&\le C(N^{-\gamma} + \tau^{\min\{\frac{\gamma}{2},1\}}).
\end{aligned} \tag{5.27}$$

Similarly, using Theorem 2.1 and (5.13) gives

$$\begin{aligned}
&\|I_{23}\|_{L^p(\Omega;H)} = \Big\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big[E^{n-k} f(u(t_{k-1})) - \frac{E^{n-k}P_N f(u(t_{k-1}))}{1 + \tau\|f(u(t_{k-1}))\|^2} \Big] ds \Big\|_{L^p(\Omega;H)}\\
&\le \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big\| \frac{(E^{n-k} - E^{n-k}P_N) f(u(t_{k-1})) + \tau\|f(u(t_{k-1}))\|^2 E^{n-k} f(u(t_{k-1}))}{1 + \tau\|f(u(t_{k-1}))\|^2} \Big\|_{L^p(\Omega;H)} ds\\
&\le C(N^{-\gamma} + \tau^{\min\{\frac{\gamma}{2},1\}} \|f(u)\|_{L^p(\Omega;H^\gamma)} + C\tau(\|f(u(t_{k-1}))\|^4_{L^{4p}(\Omega;H)}\\
&\qquad + \|f(u(t_{k-1}))\|^2_{L^{2p}(\Omega;H)})\\
&\le C(N^{-\gamma} + \tau^{\min\{\frac{\gamma}{2},1\}}).
\end{aligned} \tag{5.28}$$

Hence,

$$\|I_2\|_{L^p(\Omega;H)} \le C(N^{-\gamma} + \tau^{\min\{\frac{\gamma}{2},\frac{1}{2}\}}). \tag{5.29}$$

I_3 can be bounded by using the Burkholder-Davis-Gundy inequality, Assumption 2.2, Lemma 5.4, and (5.13). Note that $Tr(Q) < \infty$.

$$\begin{aligned}
\|I_3\|^2_{L^p(\Omega;H)} &= \Big\| \sum_{k=1}^n \int_{t_{k-1}}^{t_k} [E(t_n - s)g(u(s)) - E^{n-k} P_N g(u(t_{k-1})] dW^Q(s) \Big\|^2_{L^p(\Omega;H)} \\
&\le C \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|E(t_n - s)g(u(s)) - E^{n-k} P_N g(u(t_{k-1})\|^2_{L^p(\Omega;L_2^0)} ds \\
&\le C \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|E(t_n - s)(g(u(s)) - g(u(t_{k-1})))\|^2_{L^p(\Omega;L_2)} ds \\
&\quad + C \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|(E(t_n - s) - E^{n-k} P_N) g(u(t_{k-1}))\|^2_{L^p(\Omega;L_2)} ds \\
&\le C \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|u(s) - u(t_{k-1})\|^2_{L^p(\Omega;L_2)} ds + C(N^{-2\gamma} + \tau^{\min\{\gamma,2\}}) \\
&\le C(N^{-2\gamma} + \tau^{\min\{\gamma,1\}}). \qquad (5.30)
\end{aligned}$$

Thus, combining estimates of I_1, I_2 and I_3, we can obtain

$$\|u(t_n) - \tilde{u}_N^n\|_{L^p(\Omega;H)} \le C(N^{-\gamma} + \tau^{\min\{\frac{\gamma}{2},\frac{1}{2}\}}). \qquad (5.31)$$

Next, we estimate $\|\tilde{u}_N^n - u_N^n\|_{L^p(\Omega;H)}$. Denote $\tilde{e}_n := u_N^n - \tilde{u}_N^n$. It is clear that $\tilde{e}_n$ satisfies the equation

$$\begin{aligned}
\tilde{e}^n - \tilde{e}^{n-1} &= \tau A_N \tilde{e}^n + \frac{\tau f(u_N^{n-1})}{1 + \tau\|f(u_N^{n-1})\|^2} - \frac{\tau P_N f(u(t_{n-1})}{1 + \tau\|f(u(t_{n-1}))\|^2} \\
&\quad + P_N(g(u_N^{n-1}) - g(u(t_{n-1})))\Delta W^Q(t_n), \qquad (5.32)\\
\tilde{e}_0 &= 0.
\end{aligned}$$

Multiplying both sides by $\tilde{e}^n$ gives

$$\begin{aligned}
&\frac{1}{2}\|\tilde{e}^n\|^2 - \frac{1}{2}\|\tilde{e}^{n-1}\|^2 + \frac{1}{2}\|\tilde{e}^n - \tilde{e}^{n-1}\|^2 + \tau\|\nabla\tilde{e}^n\|^2 \\
&\quad = \left(\frac{\tau f(u_N^{n-1})}{1 + \tau\|f(u_N^{n-1})\|^2} - \frac{\tau f(u(t_{n-1})}{1 + \tau\|f(u(t_{n-1}))\|^2}, \tilde{e}^n \right) \\
&\qquad + (g(u_N^{n-1}) - (g(u(t_{n-1})))\Delta W^Q(t_n), \tilde{e}^n) \\
&\quad := J + K. \qquad (5.33)
\end{aligned}$$

A careful computation gives

$$\begin{aligned}
J &= \frac{\tau\Big(f(u_N^{n-1}) - f(u(t_{n-1})), \tilde{e}^n\Big) + \tau^2\Big(\|f(u(t_{n-1}))\|^2 f(u(t_N^{n-1}) - \|f(u_N^{n-1}))\|^2 f(u(t_{n-1})), \tilde{e}^n\Big)}{(1+\tau\|f(u(t_{n-1}))\|^2)(1+\tau\|f(u_N^{n-1})\|^2)} \\
&= \frac{\tau^2(\|f(u(t_{n-1}))\|^2 - \|f(u_N^{n-1})\|^2)(f(u(t_{n-1})), \tilde{e}^n)}{(1+\tau\|f(u(t_{n-1}))\|^2)(1+\tau\|f(u_N^{n-1})\|^2)} + \frac{\tau(f(u_N^{n-1}) - f(u(t_{n-1}), \tilde{e}^n)}{1+\tau\|f(u_N^{n-1})\|^2} \qquad (5.34)\\
&:= J_1 + J_2.
\end{aligned}$$

By Theorem 5.1, we have

$$\begin{aligned}\mathbf{E}J_1 &\leq C\tau^3\mathbf{E}[\|f(u_N^{n-1})\|^4\|f(u(t_{n-1}))\|^2+\|(f(u(t_{n-1}))\|^6]+\tau\mathbf{E}\|\tilde{e}^n\|^2\\ &\leq C\tau^2+\tau\mathbf{E}\|\tilde{e}^n\|^2. \end{aligned}\tag{5.35}$$

From Observation 2.1 of f,

$$\begin{aligned}J_2 &= \frac{\tau(f(u(t_{n-1})-f(u_N^{n-1}),\tilde{e}^n)}{1+\tau\|f(u_N^{n-1})\|^2}\\ &= \frac{\tau(f(u(t_{n-1})-f(\tilde{u}^{n-1}),\tilde{e}^n\rangle+\tau\langle f(\tilde{u}^{n-1})-f(u_N^{n-1}),\tilde{e}^{n-1})+\tau(f(\tilde{u}^{n-1})-f(u_N^{n-1}),\tilde{e}^n-\tilde{e}^{n-1})}{1+\tau\|f(u_N^{n-1})\|^2}\\ &\leq \tau\left[\|f(u(t_{n-1})-f(\tilde{u}^{n-1})\|^2+\frac{1}{4}\|\tilde{e}^n\|^2\right]+L\tau\|\tilde{e}^{n-1}\|^2+\tau^2\|f(\tilde{u}^{n-1})-f(u_N^{n-1})\|^2\\ &\quad+\frac{1}{4}\|\tilde{e}^n-\tilde{e}^{n-1}\|^2. \end{aligned}\tag{5.36}$$

Based upon Theorem 2.1 and (5.31), we deduce that

$$\begin{aligned}&\tau\mathbf{E}\|f(u(t_{n-1})-f(\tilde{u}^{n-1})\|^2\\ &\leq C\tau\mathbf{E}\left[(1+\|u(t_{n-1})\|_\infty^2+\|\tilde{u}^{n-1}\|_\infty^2)^2\|u(t_{n-1})-\tilde{u}^{n-1}\|^2\right]\\ &\leq C\tau^2\mathbf{E}(1+\|u(t_{n-1})\|_\infty^4+\|\tilde{u}^{n-1}\|_\infty^4)+C\mathbf{E}\|u(t_{n-1})-\tilde{u}^{n-1}\|^4\\ &\leq C\tau^2+C(N^{-4\gamma}+\tau^{\min\{2\gamma,2\}}) \end{aligned}\tag{5.37}$$

and $\tau^2\mathbf{E}\|f(\tilde{u}^{n-1})-f(u_N^{n-1})\|^2\leq C\tau^2$. Hence,

$$\mathbf{E}J\leq C(\tau^{\min\{2\gamma,2\}}+N^{-4\gamma})+C\tau\mathbf{E}\|\tilde{e}^n\|^2+C\tau\mathbf{E}\|\tilde{e}^{n-1}\|^2+\frac{1}{4}\mathbf{E}\|\tilde{e}^n-\tilde{e}^{n-1}\|^2.\tag{5.38}$$

Now it remains to bound K. By Assumption 2.2 and (5.31),

$$\begin{aligned}\mathbf{E}K &= \mathbf{E}(g(u(t_{n-1}))-g(u_N^{n-1}))\Delta W^Q(t_n),\tilde{e}^n-\tilde{e}^{n-1})\\ &\leq \mathbf{E}\|g(u(t_{n-1}))-g(u_N^{n-1}))\Delta W^Q(t_n)\|^2+\frac{1}{4}\mathbf{E}\|\tilde{e}^n-\tilde{e}^{n-1}\|^2\\ &\leq C\tau Tr(Q)\mathbf{E}\|g(u(t_{n-1}))-g(u_N^{n-1}))\|^2+\frac{1}{4}\mathbf{E}\|\tilde{e}^n-\tilde{e}^{n-1}\|^2\\ &\leq C\tau\mathbf{E}\|u(t_{n-1})-u_N^{n-1}\|^2+\frac{1}{4}\mathbf{E}\|\tilde{e}^n-\tilde{e}^{n-1}\|^2\\ &\leq C\tau\mathbf{E}\|u(t_{n-1})-\tilde{u}_N^{n-1}\|^2+C\tau\mathbf{E}\|\tilde{e}^{n-1}\|^2+\frac{1}{4}\mathbf{E}\|\tilde{e}^n-\tilde{e}^{n-1}\|^2\\ &\leq C\tau(N^{-2\gamma}+\tau^{\min\{\gamma,1\}})+C\tau\mathbf{E}\|\tilde{e}^{n-1}\|^2+\frac{1}{4}\mathbf{E}\|\tilde{e}^n-\tilde{e}^{n-1}\|^2. \end{aligned}\tag{5.39}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\mathbf{E}\|\tilde{e}^n\|^2-\frac{1}{2}\mathbf{E}\|\tilde{e}^{n-1}\|^2+\tau\mathbf{E}\|\nabla\tilde{e}^n\|^2\\ &\leq C_1\tau\mathbf{E}\|\tilde{e}^n\|^2+C_2\tau\mathbf{E}\|\tilde{e}^{n-1}\|^2+C(\tau^{\min\{2\gamma,2\}}+N^{-4\gamma})+C\tau(N^{-2\gamma}+\tau^{\min\{\gamma,1\}}). \end{aligned}\tag{5.40}$$

Hence, substituting the bounds of J and K into (5.33) and taking expectation, we have for τ sufficiently small

$$\mathbf{E}\|\tilde{e}^n\|^2 \le A(\tau)\mathbf{E}\|\tilde{e}^{n-1}\|^2 + C\tau(N^{-2\gamma} + \tau^{\min\{\gamma,1\}}) + C(\tau^2 + N^{-4\gamma}), \tag{5.41}$$

where

$$A(\tau) = \frac{1+2C_2\tau}{1-2C_1\tau}.$$

By a simple calculation,

$$\lim_{n\to\infty} A(\tau)^n = e^{(2C_2+2C_1)T}.$$

Therefore,

$$\begin{aligned}\mathbf{E}\|\tilde{e}^n\|^2 &\le A(\tau)^n\mathbf{E}\|\tilde{e}^0\|^2 + C\big[\tau(N^{-2\gamma} + \tau^{\min\{\gamma,1\}}) + \tau^2 + N^{-4\gamma}\big]\sum_{k=1}^{n-1} A(\tau)^k \\ &\le C(N^{-2\gamma} + \tau + \tau^{-1}N^{-4\gamma}). \end{aligned} \tag{5.42}$$

The result follows by a combination of (5.31) and (5.42). □

6. Efficient implementation and numerical experiments

In this section, we first present an efficient implementation of our scheme with the spectral-Galerkin method, and then present some numerical experiments.

6.1. Efficient implementation with spectral-Galerkin method. We present below an efficient implementation by using the spectral-Galerkin method [36] which will greatly simplify the implementation and increase the efficiency. To fix the idea, we take $\mathcal{O} = (0,1)^2$ and $A = \Delta$ as an example.

Our spectral semi-discretization (3.3) is equivalent to finding $u_N \in V_N$ such that

$$(du_N, \chi_N) = (A_N u_N, \chi_N)dt + (f(u_N), \chi_N)dt + (g(u_N)dW^Q(t), \chi_N), \quad \chi_N \in V_N, \tag{6.1}$$

where $W^Q(t) \approx \sum_{j_1,j_2=1}^{J} \sqrt{q_{j_1j_2}} e_{j_1j_2}(x,y)\beta_{j_1j_2}(t)$.

Let $\{\phi_m(\cdot)\}_{m=1}^N$ be the basis functions of V_N in 1-D so that $\{\phi_m(x)\phi_j(y)\}_{m,j=1}^N$ forms a basic for V_N in 2-D.

$$u_N(t) = \sum_{m,n=0}^{N-2} c_{mn}(t)\phi_m(x)\phi_n(y), \mathbf{C}(t) = (c_{mn}(t))_{m,n=0,1,\cdots,N-2};$$

$$a_{mn} = \int_0^1 \phi_m'(x)\phi_n'(x)dx, \mathbf{A} = (a_{mn})_{m,n=0,1,\cdots,N-2};$$

$$b_{mn} = \int_0^1 \phi_m(x)\phi_n(x)dx, \mathbf{B} = (b_{mn})_{m,n=0,1,\cdots,N-2};$$

$$f_{mn} = \int_{\mathcal{O}} f(u_N(t))\phi_m(x)\phi_n(y)dxdy, \mathbf{F}(t) = (f_{mn})_{m,n=0,1,\cdots,N-2};$$

$$g_{mn}^{j_1j_2} = \int_{\mathcal{O}} g(u_N(t))e_{j_1j_2}(x,y)\phi_m(x)\phi_n(y)dxdy, \mathbf{G}_{j_1j_2}(t) = (g_{m,n}^{j_1j_2})_{m,n=0,1,\cdots,N-2}.$$

Then, (6.1) can be transformed into

(6.2)

$$\mathbf{B}(d\mathbf{C}(t))\mathbf{B} = -[\mathbf{A}\mathbf{C}(t)\mathbf{B} + \mathbf{B}\mathbf{C}(t)\mathbf{A}]dt + \mathbf{F}(t)dt + \sum_{j_1,j_2=1}^{J} \sqrt{q_{j_1j_2}}\mathbf{G}_{j_1j_2}(t)d\beta_{j_1j_2}(t).$$

We now perform a matrix diagonalization technique (cf. [37, Chap 8]) to the above system. Let $(\lambda_i, \bar{h}_i)$ $(i = 0, 1, \cdots, N-2)$ be the generalized eigenpairs such that $\mathbf{B}\bar{h}_i = \lambda_i \mathbf{A}\bar{h}_i$, and set

$$\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_{N-2}), \mathbf{H} = (\bar{h}_0, \bar{h}_1, \cdots, \bar{h}_{N-2}). \tag{6.3}$$

Then, we have $\mathbf{BH} = \mathbf{AH\Lambda}$. Note that since $\mathbf{A}$ and $\mathbf{B}$ are symmetric, we have $\mathbf{H}^{-1} = \mathbf{H}^T$.

Writing $\mathbf{C}(t) = \mathbf{H}\mathbf{V}(t)\mathbf{H}^T$ in (6.2), we arrive at

$$\begin{aligned}\mathbf{H}\boldsymbol{\Lambda}d\mathbf{V}(t)\boldsymbol{\Lambda}\mathbf{H}^T = -[\mathbf{H}\mathbf{V}(t)\boldsymbol{\Lambda}\mathbf{H}^T + \mathbf{H}\boldsymbol{\Lambda}\mathbf{V}(t)\mathbf{H}^T]dt + \mathbf{A}^{-1}(\mathbf{F}(t)dt \\ + \sum_{j_1,j_2=1}^{J} \sqrt{q_{j_1j_2}}\mathbf{G}_{j_1j_2}(t)d\beta_{j_1j_2}(t))\mathbf{A}^{-1}.\end{aligned}$$

Multiplying the left (resp. right) of the above equation by $\mathbf{H}^T$ (resp. $\mathbf{H}$), we arrive at

$$\begin{aligned}\boldsymbol{\Lambda}d\mathbf{V}(t)\boldsymbol{\Lambda} = -[\mathbf{V}(t)\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\mathbf{V}(t)]dt + \mathbf{H}^T\mathbf{A}^{-1}(\mathbf{F}(t)dt \\ + \sum_{j_1,j_2=1}^{J} \sqrt{q_{j_1j_2}}\mathbf{G}_{j_1j_2}(t)d\beta_{j_1j_2}(t))\mathbf{A}^{-1}\mathbf{H},\end{aligned}$$

which can be rewritten componentwise as a system of nonlinear SDEs with decoupled linear parts:

$$\begin{aligned}\lambda_m\lambda_n dV_{mn}(t) = -[\lambda_m + \lambda_n]V_{mn}(t)dt + (\mathbf{H}^T\mathbf{A}^{-1}\mathbf{F}(t)\mathbf{A}^{-1}\mathbf{H})_{mn}dt \\ + \sum_{j_1,j_2=1}^{J} \sqrt{q_{j_1j_2}}(\mathbf{H}^T\mathbf{A}^{-1}\mathbf{G}_{j_1j_2}(t)\mathbf{A}^{-1}\mathbf{H})_{mn}d\beta_{j_1j_2}(t), \\ 0 \le m, n \le N-2.\end{aligned} \tag{6.4}$$

Now, we are in a position to discretize the above SDE. Writing $u_N^k = \sum_{m,n=0}^{N-2} c_{mn}^k \phi_m(x)\phi_n(y)$ in (4.2), setting $\mathbf{C}^k = (c_{mn}^k) = \mathbf{H}\mathbf{V}^k\mathbf{H}^T$ with $\mathbf{V}^k = (V_{mn}^k)_{m,n=0,1,\cdots,N-2}$, we derive

$$\begin{aligned}\lambda_m\lambda_n \frac{V_{mn}^k - V_{mn}^{k-1}}{\tau} + (\lambda_m + \lambda_n)V_{mn}^k = \left(\frac{1}{1+\tau\|f(u_N^{k-1})\|^2}\right)(\mathbf{H}^T\mathbf{F}^{k-1}\mathbf{H})_{mn} \\ + \sum_{j_1,j_2=1}^{J} \sqrt{q_{j_1j_2}}(\mathbf{H}^T\mathbf{G}_{j_1j_2}^{k-1}\mathbf{H})_{mn}\Delta\beta_{j_1j_2}(t_{k-1}).\end{aligned} \tag{6.5}$$

Here $\Delta\beta_{j_1j_2}(t_{k-1})$ are i.i.d. random variables following $N(0,\tau)$-distribution and

$$\begin{aligned}\mathbf{F}^{k-1} = (f_{mn}^{k-1}), f_{mn}^{k-1} = \int_{\mathcal{O}} f(u_N^{k-1})\phi_m(x)\phi_n(y)dxdy; \\ \mathbf{G}_{j_1j_2}^{k-1} = (g_{mn}^{j_1j_2,k-1}), g_{mn}^{j_1j_2,k-1} = \int_{\mathcal{O}} g(u_N^{k-1})e_{j_1j_2}(x,y)\phi_m(x)\phi_n(y)dxdy.\end{aligned} \tag{6.6}$$

Hence, we can determine V_{mn}^k explicitly from (6.5).

Note that in general $\mathbf{F}^{k-1}$ and $\mathbf{G}_{j_1j_2}^{k-1}$ cannot be computed exactly. In practice, the following pseudo-spectral approach is used to approximately compute $\mathbf{F}^{k-1}$ and $\mathbf{G}_{j_1j_2}^{k-1}$. Let $\{x_i, y_i\}_{i=0,N}$ be the Legendre-Gauss Lobatto points, and $\mathbf{P}_N$ be the set of polynomials with degree less than or equal to N in each direction. We define an interpolation operator $I_N : C(\bar{\mathcal{O}}) \to P_N$ such that $I_N u(x_i, y_j) = u(x_i, y_j), i, j = 01, 1, \cdots, N$. Then, we approximate $\mathbf{F}^{k-1}$ and $\mathbf{G}_{j_1j_2}^{k-1}$ as follows:

$$f_{mn}^{k-1} \approx \int_{\mathcal{O}} I_N(f(u_N^{k-1}))\phi_m(x)\phi_n(y)dxdy;$$

$$g_{mn}^{j_1j_2,k-1} \approx \int_{\mathcal{O}} I_N(g(u_N^{k-1})e_{j_1j_2}(x,y))\phi_m(x)\phi_n(y)dxdy. \tag{6.7}$$

Since $I_N(f(u_N^{k-1})) \in \mathbf{P}_N$, we can determine h_{mn}^{k-1} such that $I_N(f(u_N^{k-1})) = \sum_{m,n=0}^N h_{mn}^{k-1} L_n(x)L_m(y)$ where $\{L_j(\cdot)\}$ are the shifted Legendre polynomials. Hence, f_{mn}^{k-1} can be easily obtained using the orthogonality of Legendre polynomials. The total cost of computing $\mathbf{H}^T\mathbf{F}^{k-1}\mathbf{H}$ being $O(N^{d+1})$ for the d-dimensional problem. One can compute $g_{mn}^{j_1j_2,k-1}$ in a similar way with the total cost of computing $\mathbf{H}^T\mathbf{G}_{j_1j_2}^{k-1}\mathbf{H}$ is J^dN^{d+1} for the d-dimensional problem.

In summary, our algorithm can be described as follows:

(1) Compute the eigenvalues and eigenvectors of the generalized eigenvalue problem $\mathbf{BH} = \mathbf{H\Lambda}$;
(2) Find $\mathbf{C}^0$ by projecting u_0 onto $\mathcal{P}_N \otimes \mathcal{P}_N$;
(3) At time step t_{k-1}, compute $\mathbf{F}^{k-1}$, $\mathbf{G}_{j_1j_2}^{k-1}$ and generate a random matrix $\Delta\beta_{j_1j_2}(t_k)$;
(4) Use (6.5) to obtain $\mathbf{V}^k$, set $\mathbf{C}^k = \mathbf{HV}^k\mathbf{H}^T$ and $u_N^k = \sum_{m,n=0}^{N-2} c_{mn}^k \phi_m(x)\phi_n(y)$;
(5) Go to the next step.

Remark 6.1.

- In principle, one can solve the above system of nonlinear SDEs using any standard SDE solver.
- The above procedure is also applicable to a separable operator A in the form $Au = \partial_x(a(x)\partial_x u) + \partial_y(b(y)\partial_y u)$, and can be extended in a straightforward fashion to three dimensions.
- In the special case of $A = \Delta$ considered above, we can use

$$\phi_m(x) = \frac{1}{2\sqrt{4m+6}}(L_m(x) - L_{m+2}(x)), \quad m \geq 0, \tag{6.8}$$

where $L_m(x)$ is the shifted Legendre polynomials on $[0, 1]$ such that $\phi_m(0) = \phi_m(1) = 0$ and $(\phi_m', \phi_n') = \delta_{mn}$ [36]. Hence, $\mathbf{A}$ is the identity matrix, and the entries of $\mathbf{B}$ have the explicit form [36]

$$b_{mn} = b_{nm} = \begin{cases} \frac{1}{4(4m+6)}\left(\frac{1}{2m+1} + \frac{1}{2m+5}\right), & m = n, \\ -\frac{1}{4\sqrt{(4m+6)(4m-2)}}\frac{1}{2m+1}, & m = n+2, \\ 0, & \text{otherwise.} \end{cases} \tag{6.9}$$

6.2. **Numerical experiments.** In this section, two numerical experiments are provided to illustrate the theoretical results claimed in the previous sections.

Example 6.1. Consider the following 1-d stochastic Allen-Cahn equation on the time domain $0 \leq t \leq 1$:

$$
(6.10) \qquad \begin{cases} du = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}dt + (u - u^3)dt + g(u)dW^Q(t), x \in I = (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,\cdot) = \sin \pi x \end{cases}
$$

and we take

$$
W^Q(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\sin(j\pi x)\beta_j(t).
$$

Here, $L_j(x)$ is the shifted Legendre polynomials on $[0,1]$ with q_j to be specified below.

Obviously, eigenfunctions of $A = \frac{\partial^2}{\partial x^2}$ with homogeneous Dirichlet boundary condition on I are $\{\sin j\pi x\}_{j=1}^{\infty}$, and A and Q commute for this case. To measure the spatial error, we run $K = 200$ independent realizations for each spatial expansion term with $N = 12, 14, 16, 18, 20$ and temporal steps $\tau = 1E-5$ and truncate the first 100 terms in $W^Q(t)$. Since the true solution is unknown, we take the numerical solution with $\tau = 1E-5$ and $N = 100$ as a surrogate. The error $E\|U_N^k - u(t_k,\cdot)\|$ is approximated by

$$
(6.11) \qquad E\|u(t_k,\cdot) - u_N^k\| \approx \sqrt{\frac{1}{K}\sum_{i=1}^{K}\|u_N^k(\omega_i) - u(t_k,\omega_i)\|^2}.
$$

First, we consider additive noise and take $g(u) = I$. Hence, we examine the condition $\|A^{\frac{\gamma-1}{2}}Q^{\frac{1}{2}}\|_{L^2} < \infty$ associated with q_j and γ, and consider the following two cases:

(1) $q_j = j^{-1.001}$, associated with $\gamma = 1$;
(2) $q_j = j^{-5.001}$, associated with $\gamma = 3$.

One observes from Figure 6.1 that the spatial error decays at a rate of $\mathcal{O}(N^{-\gamma})$ for both cases as Theorem 5.2 predicts, and the restriction $\gamma < 2$ is lifted in contrast to [11, 24, 26, 34, 40].

Similarly, in order to find the temporal error convergence rate, we freeze $N = 100$ and split the time interval $[0,1]$ into 96, 144, 192, 256, 384 subintervals for (1) and 256, 384, 768, 1152, 1536 for (2), and truncate the first 100 terms in $W^Q(t)$. A surrogate of true solution is obtained using $N = 100$ and $M = 9216$. Figure 6.2 demonstrates that the temporal error decays at a rate of $\mathcal{O}(\tau^{\min\{\frac{\gamma}{2},1\}})$.

Secondly, in order to demonstrate the prediction in Theorem 5.2, we also choose $g(u) = \frac{1-u^2}{1+u^2}$ and $q_j = j^{-5.001}$ in $W^Q(t)$ and repeat the process above. From Figure 6.3, it is evident that the convergence rate is $\mathcal{O}(N^{-3} + \tau^{1/2})$, which is consistent with Theorem 5.2.

Example 6.2. Consider the following 2-d stochastic Allen-Cahn equation:

$$
\begin{cases} du = \frac{1}{2}\Delta u dt + (u - u^3)dt + g(u)dW^Q, (x,y) \in (0,1)^2, \\ u_0(x,y,0) = \sin(\pi x)\sin(\pi y), \end{cases}
$$

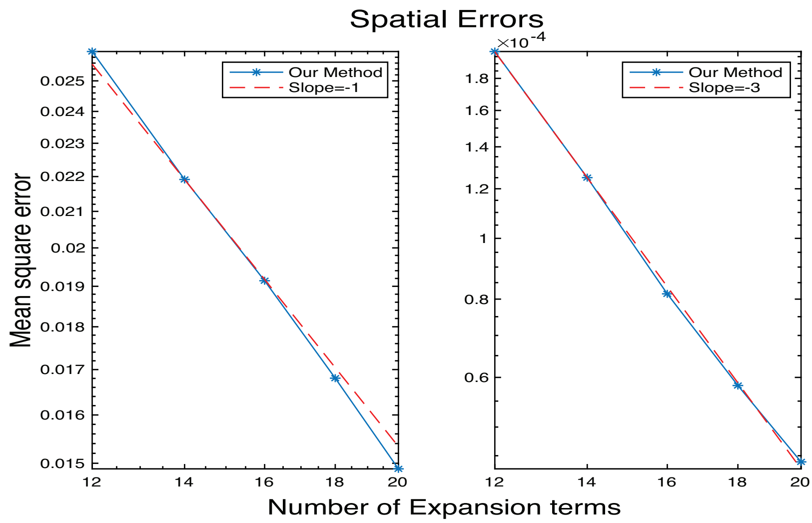

FIGURE 6.1. Spatial errors of 1-d stochastic Allen-Cahn equation with $g(u) = I$: (Left) $q_j = j^{-1.001}$ and (right) $q_j = j^{-5.001}$

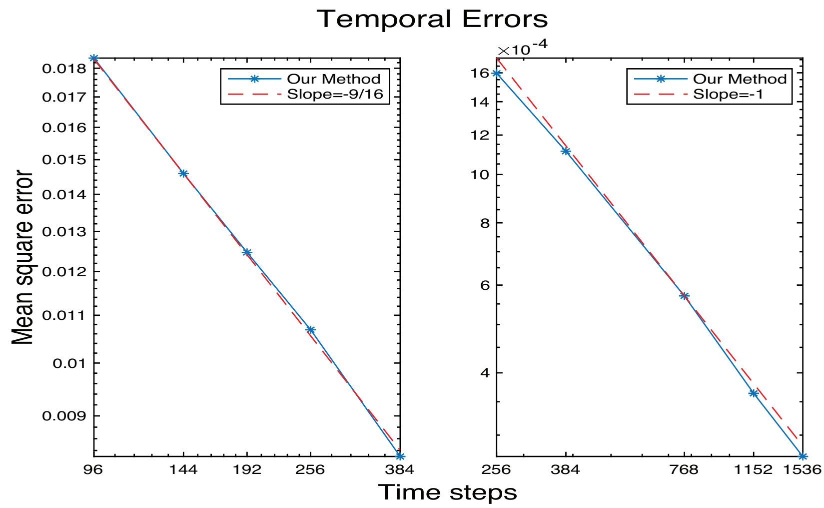

FIGURE 6.2. Temporal errors of 1-d stochastic Allen-Cahn equation with $g(u) = I$: (Left) $q_j = j^{-1.001}$ and (right) $q_j = j^{-5.001}$

where $g(u) = \sin(u)$ and

$$W^Q(t) = \sum_{j_1,j_2=1}^{\infty} 1/\sqrt{(j_1^2 + j_2^2)^3}(\sin(j_1\pi x + \phi_{j_1}(x)))(\sin(j_2\pi y) + \phi_{j_2}(y))\beta_{j_1,j_2}(t).$$

Here, $\phi(x)$ is defined in (6.8).

In the experiment, we choose $K = 200$ in (6.11) to measure the error again. To balance the CPU runtime and accuracy, we truncate the first 10 terms in each direction of $W^Q(t)$. In order to find the spatial convergence rate, we use the numerical

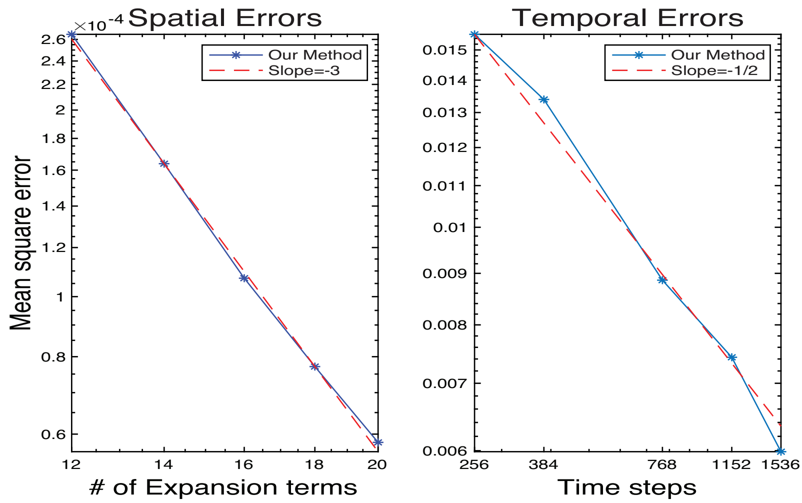

FIGURE 6.3. Numerical errors of 1-d Allen-Cahn equation with $g(u) = (1-u^2)/(1+u^2)$

solution with $N = 100$ and $M = 1000$ for $T = 0.1$ as a surrogate of true solution. From Figure 6.4, we can clearly observe a spatial convergence rate of approximately $\mathcal{O}(N^{-3/2})$ for $N = 16, 17, 19, 21$ and 22.

Similarly, in order to find temporal convergence rate, we use the numerical solution with $N = 60$ and $M = 2304$ for $T = 0.5$ as a surrogate of true solution. It is clear that temporal convergence rate $\mathcal{O}(\tau^{1/2})$ for $\tau = 1/64, 1/72, 1/96, 1/128, 1/144$ can be observed from Figure 6.4. These numerical evidences offer strong indication that the results in Theorem 5.2 may still hold in two spatial dimensions.

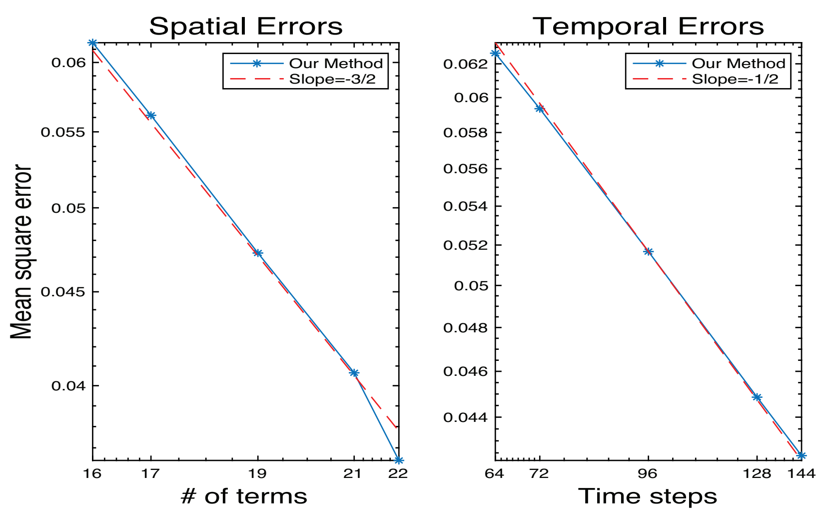

FIGURE 6.4. Numerical errors of 2d stochastic A-C equation

7. Concluding remarks

We developed a fully discrete scheme for stochastic Allen-Cahn equation driven by multiplicative noise in a multi-dimensional setting. The space discretization is a Legendre spectral method, so it does not require the elliptic operator A and the covariance operator Q of noise in the equation commute, while can still be efficiently implemented as with a Fourier method. The time discretization is a tamed semi-implicit scheme which treats the nonlinear term explicitly while being unconditionally stable, and it avoids solving nonlinear systems at each time step. Under reasonable regularity assumptions, we established strong convergence results in one spatial dimension for our fully discrete scheme. We also presented several numerical experiments to validate our theoretical results.

Although we only proved strong convergence results in one spatial dimension, our numerical results indicate that the convergence results in Theorem 5.2 still hold in two spatial dimensions. However, how to extend the analysis to the two dimensional case is still an open problem.

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