ERROR ESTIMATES FOR A FULLY DISCRETIZED SCHEME TO A CAHN-HILLIARD PHASE-FIELD MODEL FOR TWO-PHASE INCOMPRESSIBLE FLOWS

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ABSTRACT. In this paper, we carry out a rigorous error analysis for a finite-element discretization of the linear, weakly coupled energy stable scheme introduced by Shen and Yang for a Cahn-Hilliard phase-field model of two-phase incompressible flows with matching density.

1. INTRODUCTION

Phase-field approaches for multi-phase incompressible flows have attracted considerable interest in recent years (cf. [1, 15, 16, 18, 20, 22, 27] and the references therein). For two-phase incompressible flows, the phase-field models consist of either a Navier-Stokes-Cahn-Hilliard (NSCH) system or a Navier-Stokes-Allen-Cahn (NSAC) system. How to design efficient and accurate numerical methods for these coupled nonlinear systems brings great challenge to the scientific computing community.

In this paper, we focus on the following Cahn-Hilliard phase-field model with matching density,

$$(1.1) \qquad \phi_t + u \cdot \nabla\phi - \gamma\Delta w = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^d,$$

$$(1.2) \qquad w = -\Delta\phi + f(\phi), \quad \text{in} \quad \Omega \subset \mathbb{R}^d,$$

$$(1.3) \qquad \rho_0(u_t + (u \cdot \nabla)u) - \mu_0\Delta u + \nabla p - \lambda w\nabla\phi = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^d,$$

$$(1.4) \qquad \nabla \cdot u = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^d,$$

$$(1.5) \qquad u|_{\partial\Omega} = 0, \frac{\partial\phi}{\partial\mathbf{n}}|_{\partial\Omega} = 0, \frac{\partial w}{\partial\mathbf{n}}|_{\partial\Omega} = 0,$$

with given initial data $u(0) = u_0$, $\phi(0) = \phi_0$. In the above, $d = 2$ or 3, ϕ is the phase function, where $\phi \approx \pm 1$ corresponds to two different fluids, w is the chemical potential, u is the velocity field and p is the pressure. ρ_0 is the density of both fluids; γ is a relaxation constant; λ is the mixing energy density, $f(\phi) = F'(\phi)$, where $F(\phi) = \frac{(1-\phi^2)^2}{4\varepsilon^2}$, and the parameter $\varepsilon > 0$ represents the interfacial thickness.

The above system satisfies the following energy law:

$$(1.6) \qquad \frac{d}{dt}\int_\Omega(\frac{\rho_0}{2}|u|^2 + \frac{\lambda}{2}|\nabla\phi|^2 + \lambda F(\phi))dx = -\int_\Omega(\mu_0|\nabla u|^2 + \lambda\gamma|\nabla w|^2)dx.$$

While various convergence results and error estimates have been derived for the Navier-Stokes equations [8, 13, 25], there are only a few error estimates available for phase-field models of multi-phase flows. In [4], Feng proved convergence of discrete finite-element solutions for a Cahn-Hilliard phase-field model with matching density, and in [5] the authors established similar convergence results for an Allen-Cahn phase-field model with matching density. Most recently, Grün [9] proved convergence results for a Cahn-Hilliard phase-field model with variable densities. Later, Grün et al. [10] established convergence of a convex splitting scheme for a Cahn-Hilliard phase-field model with general mass densities, and Diegel et al. [3] obtained error estimates of a fully discretized convex splitting method for a Cahn-Hilliard-Darcy-Stokes model. However, the schemes considered in these papers are fully coupled (velocity and pressure) and/or nonlinear. From a computational point of view, it is more efficient to use a decoupled or weakly coupled linear scheme. In [23, 24], some weakly coupled linear, energy stable schemes are constructed, where the phase equations are discretized by the stabilized scheme [21, 26] and the Navier-Stokes (NS) equations are discretized by a projection scheme [13]. These schemes lead to, at each time step, a weakly coupled elliptic equation for the phase function and velocity, and a decoupled pressure Poisson equation for the pressure. Hence, they are very efficient and easy to implement.

Though various error estimates are available for projection type methods to the Navier-Stokes equations [13] and for the Cahn-Hilliard/Allen-Cahn equations [6, 7, 17, 21], it is highly nontrivial to deal with the systems which couple Navier-Stokes and Cahn-Hilliard/Allen-Cahn, since the splitting error in the projection step affects the whole system. The major difficulties arise from the projection step to deal with the incompressibility constraint and from the coupling between the phase function and the velocity. To the best of our knowledge, error estimates for such schemes in fully discrete form are not yet available. In a recent work [2], we carried out an error analysis for the schemes presented [23, 24] in semi-discrete-in-time form. However, the analysis in [2] cannot be easily extended to the fully discrete case, as the full discretization adds another level of difficulty, particularly in obtaining optimal error estimates for the pressure due to the splitting error in the scheme. The main purpose of this paper is to provide a rigorous error analysis for the energy stable scheme in [24] for the Navier-Stokes-Cahn-Hilliard system in fully discrete form. To make our analysis applicable to more general settings, we make only standard assumptions on the finite-element discretization.

The rest of the paper is organized as follows. In section 2, we recall some preliminary results regarding the Navier-Stokes and the Cahn-Hilliard equations and for the finite-element approximation. In section 3, we introduce the fully discretized scheme for the Cahn-Hilliard phase-field model based on the scheme introduced in [24]. Section 4 is devoted to the error analysis, where we prove the error estimates for phase functions, velocity field and the pressure, under the assumption that the exact solution is sufficiently smooth. In section 5, we show that the numerical solution converges to the weak solution of the continuous problem without assuming extra regularity of the exact solution. We conclude with a few remarks in section 6.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a connected, bounded, open domain with $C^{1,1}$ boundary $\partial\Omega$ (or such that the Stokes problem has H^2 regularity, see (2.9) below). The following Sobolev inequalities hold (cf. for instance [19]):

$$(2.1) \qquad \begin{aligned} \|g\|_{L^4(\Omega)}^2 &\leq C(2, \Omega)\|g\|_{L^2(\Omega)}\|g\|_{H^1(\Omega)}, & d &= 2, \\ \|g\|_{L^3(\Omega)}^2 &\leq C(3, \Omega)\|g\|_{L^2(\Omega)}\|g\|_{H^1(\Omega)}, & d &= 3. \end{aligned}$$

In particular, $\|g\|_{L^6(\Omega)} \leq C\|g\|_{H^1(\Omega)}$.

Let $t^n = n\delta t$ $(0 \leq n \leq N = [T/\delta t])$, where δt is the time step. For a sequence of functions $\varphi^0, \varphi^1, \ldots, \varphi^N$ in some Hilbert space E, we denote the sequence by $\varphi_{\delta t}$ and define the following discrete norms for $\varphi_{\delta t}$,

$$(2.2) \qquad \|\varphi_{\delta t}\|_{l^2(E)} = \left(\delta t \sum_{n=0}^{N} \|\varphi^n\|_E^2\right)^{1/2}, \qquad \|\varphi_{\delta t}\|_{l^\infty(E)} = \max_{0 \leq n \leq N}\left(\|\varphi^n\|_E\right).$$

Let $\|\cdot\|_k$ denote the usual $H^k(\Omega)$ $(H^k(\Omega)^d)$ norm, and $\|\cdot\|_{k,p}$ denote the $W^{k,p}$ norm. In particular, $\|\cdot\|$ and $(\cdot, \cdot)$ are the $L^2(\Omega)$ $(L^2(\Omega)^d)$ norm and inner product, respectively.

We also denote

$$(2.3) \qquad X = H_0^1(\Omega)^d, \qquad M = L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q = 0\right\},$$

and the following spaces of incompressible vector fields,

$$(2.4) \qquad \begin{aligned} H &= \left\{v \in L^2(\Omega)^d; \nabla \cdot v = 0; v \cdot \mathbf{n}|_{\partial\Omega} = 0\right\}, \\ V &= \left\{v \in H^1(\Omega)^d; \nabla \cdot v = 0; v|_{\partial\Omega} = 0\right\}. \end{aligned}$$

It holds that [25]

$$(2.5) \qquad L^2(\Omega)^d = H \oplus \nabla(H^1(\Omega)),$$

where $\nabla(H^1(\Omega)) = \{\nabla g | g \in H^1(\Omega)\}$. Define P_H as the L^2-orthogonal projector in H, i.e.,

$$(2.6) \qquad (u - P_H u, v) = 0 \qquad \forall u \in L^2(\Omega)^d, \quad v \in H.$$

We also define the inverse Stokes operator $\mathcal{S} : H^{-1}(\Omega)^d \to V$ as follows. For all $v \in H^{-1}(\Omega)^d$, $(\mathcal{S}(v), r) \in V \times L_0^2(\Omega)$ is the solution to the following Stokes problem,

$$(2.7) \qquad \begin{cases} (\nabla\mathcal{S}(v), \nabla w) - (r, \nabla \cdot w) = \langle v, w \rangle & \forall w \in H_0^1(\Omega)^d, \\ (2.8) \qquad (q, \nabla \cdot \mathcal{S}(v)) = 0 & \forall q \in L_0^2(\Omega), \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denote the pairing between $H^{-1}(\Omega)^d$ and $H_0^1(\Omega)^d$ and r is the pressure.

By the assumption on Ω, we have the following H^2 regularity results [25]:

$$(2.9) \qquad \|\mathcal{S}(v)\|_2 + \|\nabla r\| \leq c\|v\| \quad \forall v \in L^2(\Omega)^d.$$

The following properties of $\mathcal{S}$ are shown in [12].

Lemma 2.1. *For all v in $H_0^1(\Omega)^d$ and for all $0 < \theta < 1$, we have*

$$(\nabla\mathcal{S}(v), \nabla v) \geq (1 - \theta)\|v\|^2 - c(\theta)\|v - v^*\|^2 \quad \forall v^* \in H.$$

In particular,

$$(\nabla\mathcal{S}(v), \nabla v) = \|v\|^2 \quad \forall v \in V.$$

Lemma 2.2. *The bilinear form* $H^{-1}(\Omega)^d \times H^{-1}(\Omega)^d \ni (v, w) \longmapsto \langle \mathcal{S}(v), w \rangle \in \mathbb{R}$ *induces a semi-norm on* $H^{-1}(\Omega)^d$ *that we denote by* $|\cdot|_*$, *and*

$$|v|_* = \|\mathcal{S}(v)\|_1 \le c_* \|v\|_{-1} \quad \forall v \in H^{-1}(\Omega)^d.$$

2.1. **Variational formulation.** Since we are interested in the values of the phase variable ϕ in the range of $[-1, 1]$, it is a common practice to replace $F(\phi)$ by

$$(2.10) \qquad F(\phi) = \begin{cases} \frac{1}{\varepsilon^2}(\phi - 1)^2, & \phi > 1, \\ \frac{1}{4\varepsilon^2}(\phi^2 - 1)^2, & \phi \in [-1, 1], \\ \frac{1}{\varepsilon^2}(\phi + 1)^2, & \phi < -1. \end{cases}$$

It can be checked that $f(\phi) = F'(\phi)$ has a bounded first order derivative $|f'(\phi)| \le 2/\varepsilon^2$, $f'(\phi)$ is Lipschitz and $|f(\phi)| \le \frac{2}{\varepsilon^2}(|\phi| + 1)$. Hereafter, we shall assume that $F(\phi)$ takes the above modified form.

To write the variational formulation, we adopt similar notations as those in [11, 13]. Define the linear, continuous operator $A : X \to X' = H^{-1}(\Omega)^d$ (resp. bilinear form $a : X \times X \to \mathbb{R}$) such that for all $(u, v) \in X \times X$:

$$(2.11) \qquad \langle Au, v \rangle = a(u, v) := (\nabla u, \nabla v) = \sum_{i,j=1}^{d} \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right).$$

$\tilde{A} : H^1(\Omega) \to H^{-1}(\Omega)$ (resp. bilinear form $\tilde{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$) so that for all $(\phi, \varphi) \in H^1(\Omega) \times H^1(\Omega)$,

$$(2.12) \qquad \langle \tilde{A}\phi, \varphi \rangle = \tilde{a}(\phi, \varphi) := (\nabla\phi, \nabla\varphi).$$

Introduce the operator $B : X \to M$ and its transpose $B^T : M \to X'$ (resp. linear form $b : X \times M \to \mathbb{R}$) so that for all $v \in X$ and $q \in M$,

$$(2.13) \qquad (Bv, q) = b(v, q) = -(\nabla \cdot v, q).$$

Define the bilinear operator $D : X^2 \to X'$ (resp. trilinear form $d : X \times X \times X \to \mathbb{R}$) so that for $(u, v, w) \in X \times X \times X$,

$$(2.14) \qquad \langle D(u, v), w \rangle = d(u, v, w) := ((u \cdot \nabla)v, w) + \frac{1}{2}(\nabla \cdot u, v \cdot w),$$

where $u \cdot v$ denote the Euclidean scalar product. It is easy to check that for any $u \in X$, $d(u, \cdot, \cdot)$ is a skew-symmetric bilinear form and thus $d(u, v, v) = 0$.

Then, the weak formulation for the continuous problem (1.1)-(1.5) is: Find

$$\phi \in L^\infty(0, T; H^1(\Omega)), \quad \phi_t \in L^2(0, T; H^{-1}(\Omega)) \quad \text{for all} \quad T > 0,$$
$$w \in L^2(0, T; H^1(\Omega)) \quad \text{for all} \quad T > 0,$$
$$u \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; H^1(\Omega)^d), \quad u_t \in L^2(0, T; (H^{-1})^d),$$
$$p \in L^2(0, T; M) \quad \text{for all} \quad T > 0,$$

such that

$$(2.15) \qquad \begin{cases} \phi_t + u \cdot \nabla\phi + \gamma\tilde{A}w = 0, \\ (2.16) \qquad w = \tilde{A}\phi + f(\phi), \\ (2.17) \qquad \rho_0(u_t + D(u, u)) + \mu_0 Au + B^T p - \lambda w \nabla\phi = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \\ (2.18) \qquad Bu = 0, \end{cases}$$

with $\phi(t = 0) = \phi_0$ and $u(t = 0) = u_0$. Initial values of p and w can be determined from the equations by ϕ_0 and u_0.

2.2. **FEM discretization.** Let $\mathcal{T}_h$ be a regular, quasi-uniform triangulation of Ω of mesh size $0 < h < 1$ and $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$. For a nonnegative integer r, denote $P_r(K)$ as the space of polynomials of degree less than or equal to r on K.

Let $X_h \subset X$, $M_h \subset M$ be a pair of inf-sup compatible (see (2.25) below) mixed finite-element spaces based on the triangulation $\mathcal{T}_h$, and let Ψ_h be a finite-dimensional subspace of $H^1(\Omega)$. Let Y_h be a finite-dimensional subspace of $L^2(\Omega)^d$, and we assume that either Y_h is conformal in

$$(2.19) \qquad H_0^{\text{div}}(\Omega) = \left\{ v \in L^2(\Omega)^d, \quad \nabla \cdot v \in L^2(\Omega), \quad v \cdot \mathbf{n}|_{\partial\Omega} = 0 \right\}$$

or M_h is conformal in $H^1(\Omega)$. In particular, we assume $X_h \subseteq Y_h$ and $h(x) \equiv 1 \in \Psi_h$.

We assume that the finite-element spaces satisfy the following approximation properties: There exists $l \geq 1$ and $c > 0$ such that for all $0 \leq r \leq l$,

$$(2.20) \qquad \inf_{v_h \in X_h} \left\{ \|v - v_h\| + h\|v - v_h\|_1 \right\} \leq ch^{r+1}\|v\|_{r+1} \qquad \forall v \in H^{r+1}(\Omega)^d \cap X,$$

$$\inf_{v_h \in X_h} \|v - v_h\|_{1,p} \leq ch^r\|v\|_{r+1,p}, \quad 2 \leq p \leq \infty \quad \forall v \in W^{r+1,p}(\Omega)^d \cap X$$

and

$$(2.21)$$

$$\inf_{\psi_h \in \Psi_h} \left\{ \|\psi - \psi_h\| + h\|\psi - \psi_h\|_1 \right\} \leq ch^{r+1}\|\psi\|_{r+1} \qquad \forall \psi \in H^{r+1}(\Omega) \cap H^1(\Omega),$$

$$\inf_{\psi_h \in \Psi_h} \|\psi - \psi_h\|_{1,p} \leq ch^r\|\psi\|_{r+1,p}, \quad 2 \leq p \leq \infty \quad \forall \psi \in W^{r+1,p}(\Omega) \cap H^1(\Omega).$$

There exists some constant $c > 0$ such that for all $0 \leq r \leq l$,

$$(2.22) \qquad \inf_{q_h \in M_h} \|q - q_h\| \leq ch^r\|q\|_r \quad \forall q \in H^r(\Omega) \cap M.$$

In addition, for the choice of $M_h \subset H^1(\Omega)$, we assume that there exists some constant $c > 0$, such that for all $0 \leq r \leq l$,

$$(2.23) \qquad \inf_{q_h \in M_h} \left(\|q - q_h\| + h\|q - q_h\|_1 \right) \leq ch^r\|q\|_r \quad \forall q \in H^r(\Omega) \cap M,$$

and the following inverse inequality holds:

$$(2.24) \qquad \|v_h\|_{n,p} \leq ch^{m-n+\frac{d}{p}-\frac{d}{q}}\|v_h\|_{m,q}$$
$$\forall v_h \in X_h \text{ or } \Psi_h \quad 0 \leq m \leq n \leq 1, \quad 1 \leq q \leq p \leq \infty.$$

In addition, we also assume the inverse inequality (2.24) holds for Y_h if $Y_h \subset H_0^{\text{div}}$. One possible choice is the following [5]:

$$X_h = Y_h = \{ v_h \in [C^0(\overline{\Omega})]^d \cap [H_0^1(\Omega)]^d; v_h|_K \in [P_2(K)]^d \},$$

$$V_h = \{ v_h \in X_h; (\nabla \cdot v_h, q_h) = 0 \quad \forall q_h \in M_h \},$$

$$M_h = \{ q_h \in L_0^2(\Omega); q_h|_K \in P_0(K) \}, \quad \Psi_h = \{ \psi_h \in C^0(\overline{\Omega}); \psi_h|_K \in P_2(K) \}.$$

Define the L^2-orthogonal projections: $P_{X_h} : L^2(\Omega)^d \to X_h$, $P_{\Psi_h} : L^2(\Omega) \to \Psi_h$ and $P_{M_h} : L^2(\Omega) \to M_h$ such that

$$(v - P_{X_h}v, v_h) = 0 \qquad \forall v \in L^2(\Omega)^d, \quad v_h \in X_h,$$

$$(\psi - P_{\Psi_h}\psi, \psi_h) = 0 \qquad \forall \psi \in L^2(\Omega), \quad \psi_h \in \Psi_h,$$

$$(q - P_{M_h}q, q_h) = 0 \qquad \forall q \in L^2(\Omega), \quad q_h \in M_h.$$

Define the discrete version of the divergence operator $B_h : X_h \to M_h$ and its transpose $B_h^T : M_h \to X_h$ so that for every couple $(v_h, q_h) \in X_h \times M_h$ there holds $(B_h v_h, q_h) = b(v_h, q_h)$ and $(v_h, B_h^T q_h) = b(v_h, q_h)$. We also assume B_h is surjective, i.e., the following inf-sup condition holds:

$$(2.25) \qquad \exists \beta > 0, \qquad \inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{(B_h v_h, q_h)}{\|q_h\| \|v_h\|_1} \geq \beta.$$

We introduce the following discrete divergence operator on Y_h: Let $C_h : Y_h \to M_h$ such that for every couple $(v_h, q_h) \in Y_h \times M_h$, either $(C_h v_h, q_h) = -(\nabla \cdot v_h, q_h)$ if $Y_h \subset H_0^{\mathrm{div}}$ or $(C_h v_h, q_h) = (v_h, \nabla q_h)$ if $M_h \subset H^1(\Omega)$.

Let $A_h : X_h \to X_h$ be the linear, continuous operator such that $(A_h u_h, v_h) = a(u_h, v_h)$ for all $(u_h, v_h) \in X_h \times X_h$. Define $D_h : X_h \times X_h \to X_h'$ such that $(D_h(u_h, v_h), w_h) = d(u_h, v_h, w_h)$ for $(u_h, v_h, w_h) \in X_h \times X_h \times X_h$. Then it holds that $d(u_h, v_h, v_h) = 0$.

Define the linear, continuous operator $\tilde{A}_h : \Psi_h \to \Psi_h \cap L_0^2(\Omega)$ by

$$(2.26) \qquad \left(\tilde{A}_h \phi_h, \psi_h \right) = (\nabla \phi_h, \nabla \psi_h) \qquad \forall \phi_h, \psi_h \in \Psi_h.$$

Let $i_h : X_h \to Y_h$ be the continuous injection of X_h into Y_h and let i_h^T be the transpose of i_h, i.e., the L^2-orthogonal projection onto X_h.

Proposition 2.1 (cf. [11]). *C_h is an extension of B_h and $B_h = C_h i_h$, $i_h^T C_h^T = B_h^T$; we have the commutative diagrams below:*
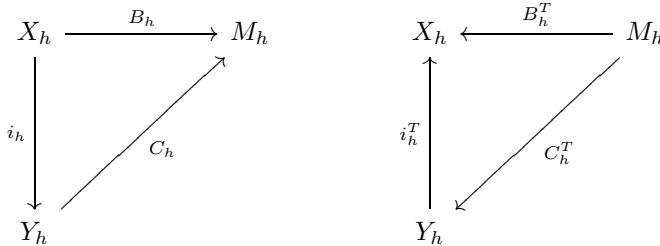
Set $H_h = \ker(B_h)$, we then have the L^2-orthogonal decomposition of X_h as $X_h = H_h \oplus B_h^T(M_h)$ [11]. Similarly, set $\tilde{H}_h = \ker(C_h)$, $Y_h = \tilde{H}_h \oplus C_h^T(M_h)$ [11].

3. Full discretization and its stability

In this section, we will present a full finite-element Galerkin approximation based on the stabilized semi-discrete schemes introduced in [24], and show that it is unconditionally stable. Let $J_{\delta t} = \{t^n\}_{n=0}^N$ be a quasi-uniform partition of $[0, T]$ of mesh size $\delta t := T/N$. A fully discrete finite-element approximation based on the stabilized semi-discrete scheme is defined as: Given suitable approximations $(\phi_h^0, w_h^0, u_h^0, p_h^0)$ of initial data (ϕ_0, w_0, u_0, p_0) (w_0 and p_0 can be determined from (1.2) and (1.3), respectively), find $\{(\phi_h^{n+1}, w_h^{n+1}, u_h^{n+1}, \tilde{u}_h^{n+1}, p_h^{n+1})\}_{n=0}^{N-1} \in$

$\Psi_h \times \Psi_h \times Y_h \times X_h \times M_h$ such that

(3.1) $\left\{ \begin{array}{l} \dfrac{\phi_h^{n+1} - \phi_h^n}{\delta t} + P_{\Phi_h}\left((\tilde{u}_h^{n+1} \cdot \nabla)\phi_h^n\right) + \gamma \tilde{A}_h w_h^{n+1} = 0, \end{array} \right.$

(3.2) $w_h^{n+1} - \dfrac{S}{\varepsilon^2}(\phi_h^{n+1} - \phi_h^n) = \tilde{A}_h \phi_h^{n+1} + P_{\Phi_h} f(\phi_h^n),$

(3.3) $\dfrac{\rho_0}{\delta t}\left(\tilde{u}_h^{n+1} - i_h^T u_h^n\right) + \rho_0 P_{X_h}\left(D_h(\tilde{u}_h^n, \tilde{u}_h^{n+1})\right) + \mu_0 A_h \tilde{u}_h^{n+1}$
$\qquad\qquad\qquad\qquad\qquad + B_h^T p_h^n - \lambda P_{X_h}\left(w_h^{n+1}\nabla\phi_h^n\right) = 0,$

(3.4) $\dfrac{\rho_0}{\delta t}(u_h^{n+1} - i_h\tilde{u}_h^{n+1}) + C_h^T(p_h^{n+1} - p_h^n) = 0,$

(3.5) $C_h u_h^{n+1} = 0,$

where $S \geq \frac{1}{2}$ is a stabilizing parameter. Initially, we set $u_h^0 = \tilde{u}_h^0 \in X_h \subset Y_h$.

Before discussing the numerical procedure, we show below that the above scheme is unconditionally stable. For a sequence $\{v^n\}_{n=0}^N$, we denote

(3.6) $$\delta_t v^n = v^{n+1} - v^n.$$

Theorem 3.1. *The fully discrete scheme* (3.1)-(3.5) *is unconditionally stable and satisfies the following discrete energy law:*

(3.7)
$$\left[\frac{\rho_0}{2}\|u_h^{n+1}\|^2 + \frac{\lambda}{2}\|\nabla\phi_h^{n+1}\|^2 + \lambda(F(\phi_h^{n+1}), 1) \right]$$
$$+ \frac{\delta t^2}{2\rho_0}\|C_h^T p_h^{n+1}\|^2 + \mu_0\delta t\|\nabla\tilde{u}_h^{n+1}\|^2 + \lambda\gamma\delta t\|\nabla w_h^{n+1}\|^2$$
$$\leq \left[\frac{\rho_0}{2}\|u_h^n\|^2 + \frac{\lambda}{2}\|\nabla\phi_h^n\|^2 + \lambda(F(\phi_h^n), 1) \right] + \frac{\delta t^2}{2\rho_0}\|C_h^T p_h^n\|^2, \quad n \geq 0.$$

Proof. Testing (3.1) with $\lambda\delta t w_h^{n+1} \in \Psi_h$, (3.2) with $-\lambda(\phi_h^{n+1} - \phi_h^n) \in \Psi_h$, (3.3) with $\delta t \tilde{u}_h^{n+1} \in X_h$, we obtain

$$\lambda\left(\phi_h^{n+1} - \phi_h^n, w_h^{n+1}\right) + \lambda\delta t\left((\tilde{u}_h^{n+1} \cdot \nabla)\phi_h^n, w_h^{n+1}\right) + \lambda\gamma\delta t\|\nabla w_h^{n+1}\|^2 = 0,$$

$$-\lambda\left(\phi_h^{n+1} - \phi_h^n, w_h^{n+1}\right) + \frac{\lambda S}{\varepsilon^2}\|\phi_h^{n+1} - \phi_h^n\|^2$$
$$= -\frac{\lambda}{2}\left(\|\nabla\phi_h^{n+1}\|^2 + \|\nabla\delta_t\phi_h^n\|^2 - \|\nabla\phi_h^n\|^2\right) - \lambda\left(f(\phi_h^n), \phi_h^{n+1} - \phi_h^n\right),$$

$$\frac{\rho_0}{2}\left(\|\tilde{u}_h^{n+1}\|^2 + \|\tilde{u}_h^{n+1} - i_h^T u_h^n\|^2 - \|i_h^T u_h^n\|^2\right) + \mu_0\delta t\|\nabla\tilde{u}_h^{n+1}\|^2 - \delta t\left(B_h^T p_h^n, \tilde{u}_h^{n+1}\right)$$
$$- \lambda\delta t\left(w_h^{n+1}\nabla\phi_h^n, \tilde{u}_h^{n+1}\right) = 0.$$

Summing up the above identities, we find

(3.8)
$$\frac{\lambda}{2}\left(\|\nabla\phi_h^{n+1}\|^2 + \|\nabla\delta_t\phi_h^n\|^2 - \|\nabla\phi_h^n\|^2\right)$$
$$+ \lambda\left(f(\phi_h^n), \phi_h^{n+1} - \phi_h^n\right) + \frac{S\lambda}{\varepsilon^2}\|\phi_h^{n+1} - \phi_h^n\|^2 + \lambda\gamma\delta t\|\nabla w_h^{n+1}\|^2$$
$$+ \frac{\rho_0}{2}\left(\|\tilde{u}_h^{n+1}\|^2 + \|\tilde{u}_h^{n+1} - i_h^T u_h^n\|^2 - \|i_h^T u_h^n\|^2\right)$$
$$+ \mu_0\delta t\|\nabla\tilde{u}_h^{n+1}\|^2 - \delta t\left(B_h^T p_h^n, \tilde{u}_h^{n+1}\right) = 0.$$

Testing (3.4) with $C_h^T p_h^n \in Y_h$, in view of (3.5), we obtain

$$(3.9) \quad (i_h \tilde{u}_h^{n+1}, C_h^T p_h^n) = (u_h^{n+1}, C_h^T p_h^n) - \frac{\delta t}{\rho_0} \left(C_h^T p_h^n, C_h^T (p_h^{n+1} - p_h^n) \right)$$

$$= (C_h u_h^{n+1}, p_h^n) - \frac{\delta t}{2\rho_0} \left(\|C_h^T p_h^{n+1}\|^2 - \|C_h^T p_h^n\|^2 - \|C_h^T (p_h^{n+1} - p_h^n)\|^2 \right)$$

$$= -\frac{\delta t}{2\rho_0} \left(\|C_h^T p_h^{n+1}\|^2 - \|C_h^T p_h^n\|^2 - \|C_h^T (p_h^{n+1} - p_h^n)\|^2 \right).$$

Using $i_h^T C_h^T = B_h^T$ leads to

$$(3.10) \qquad (i_h \tilde{u}_h^{n+1}, C_h^T p_h^n) = (\tilde{u}_h^{n+1}, i_h^T C_h^T p_h^n) = (\tilde{u}_h^{n+1}, B_h^T p_h^n).$$

Moreover, we rewrite (3.4) as

$$(3.11) \qquad i_h \tilde{u}_h^{n+1} = u_h^{n+1} + \frac{\delta t}{\rho_0} C_h^T (p_h^{n+1} - p_h^n),$$

with both sides belonging to $Y_h \subseteq L^2$; taking the L^2-norm of both sides and using (3.5), we get

$$(3.12) \qquad \|i_h \tilde{u}_h^{n+1}\|^2 = \|u_h^{n+1}\|^2 + \frac{\delta t^2}{\rho_0^2} \|C_h^T (p_h^{n+1} - p_h^n)\|^2.$$

Combining (3.8), (3.9), (3.10), and (3.12) together and noticing that $\|i_h \tilde{u}_h^{n+1}\| = \|\tilde{u}_h^{n+1}\|$, we find

$$(3.13) \qquad \frac{\lambda}{2} \left(\|\nabla \phi_h^{n+1}\|^2 + \|\nabla \delta_t \phi_h^n\|^2 - \|\nabla \phi_h^n\|^2 \right) + \lambda \left(f(\phi_h^n), \phi_h^{n+1} - \phi_h^n \right)$$

$$+ \frac{S\lambda}{\varepsilon^2} \|\phi_h^{n+1} - \phi_h^n\|^2 + \lambda \gamma \delta t \|\nabla w_h^{n+1}\|^2$$

$$+ \frac{\rho_0}{2} \left(\|u_h^{n+1}\|^2 + \|\tilde{u}_h^{n+1} - i_h^T u_h^n\|^2 - \|i_h^T u_h^n\|^2 \right)$$

$$+ \frac{\delta t^2}{2\rho_0} \left(\|C_h^T p_h^{n+1}\|^2 - \|C_h^T p_h^n\|^2 \right) + \mu_0 \delta t \|\nabla \tilde{u}_h^{n+1}\|^2$$

$$= 0.$$

In addition, Taylor expansion and (2.10) imply that for $S \geq 1/2$,

$$F(\phi_h^{n+1}) - F(\phi_h^n) = f(\phi_h^n)(\phi_h^{n+1} - \phi_h^n) + \frac{f'(\xi)}{2} |\phi_h^{n+1} - \phi_h^n|^2$$

$$\leq f(\phi_h^n)(\phi_h^{n+1} - \phi_h^n) + \frac{S}{\varepsilon^2} |\phi_h^{n+1} - \phi_h^n|^2.$$

Hence, we draw the conclusion via substituting the above inequality into (3.13) and noticing the fact that $i_h^T : Y_h \to X_h$ is an L^2-orthogonal projection. $\qquad \square$

Remark 3.1. Based on the above stability analysis, if we rewrite (3.1)-(3.3) as a coupled linear system for the unknown $(w_h^{n+1}, \phi_h^{n+1} - \phi_h^n, \tilde{u}_h^{n+1})^T$, the matrix of the linear system is then positive definite (but not symmetric). Therefore, the coupled system (3.1)-(3.3) can be solved efficiently by an iterative method such as BICGSTAB (cf. [22] and references therein). For the projection step (3.4)-(3.5), the following equation is solved in practice by applying C_h to (3.4),

$$(3.14) \qquad C_h C_h^T (p_h^{n+1} - p_h^n) = \frac{\rho_0}{\delta t} B_h \tilde{u}_h^{n+1}.$$

Noticing the inf-sup condition (2.25), the coefficient matrix of (3.14) is positive definite and (3.14) admits a unique solution.

From Remark 3.1, the practical procedure for implementing scheme (3.1)-(3.5) can be described as:

Algorithm 1: Implementation of (3.1)-(3.5)

Initialize $(\phi_h^0, w_h^0, \tilde{u}_h^0, p_h^0)$ from initial data (ϕ_0, w_0, u_0, p_0), where ϕ_0 and u_0 are given, w_0 and p_0 are determined from (1.2) and (1.3), respectively. Set $u_h^0 = \tilde{u}_h^0$ and $n = 0$;

while $n \leq N - 1$ **do**

$\quad$ Solve the weakly coupled linear system (3.1)-(3.3) for

$\quad$ $(w_h^{n+1}, \phi_h^{n+1} - \phi_h^n, \tilde{u}_h^{n+1})^T$, by an iterative method such as BICGSTAB;

$\quad$ Solve the pressure equation (3.14) for p_h^{n+1} and update u_h^{n+1};

end

---

4. ERROR ESTIMATES

4.1. Preparations. Let $(\phi(t), w(t), u(t), p(t))$ be the exact solution. As usual, we will compare the numerical solution with the interpolates defined below. Let $(u_h(t), p_h(t)) \in X_h \times M_h$ be the solution of the following discrete Stokes problem,

(4.1)
$$\begin{cases} \mu_0 \left(\nabla u_h(t), \nabla v_h \right) + \left(B_h^T p_h(t), v_h \right) = \mu_0 \left(\nabla u(t), \nabla v_h \right) - (p(t), \nabla \cdot v_h) & \forall v_h \in X_h, \\ (B_h u_h(t), r_h) = - \left(\nabla \cdot u(t), r_h \right) & \forall r_h \in M_h, \end{cases}$$

and define $(\phi_h(t), w_h(t)) \in \Psi_h \times \Psi_h$ as the solution of the following discrete elliptic problem

(4.2)
$$\begin{cases} (\nabla \phi_h(t), \nabla \psi_h) = (\nabla \phi(t), \nabla \psi_h) & \forall \psi_h \in \Psi_h, \\ (\phi_h(t), 1) = (\psi(t), 1), \\ (\nabla w_h(t), \nabla \varphi_h) = (\nabla w(t), \nabla \varphi_h) & \forall \varphi_h \in \Psi_h, \\ (w_h(t), 1) = (w(t), 1). \end{cases}$$

Using the H^2-regularity of the Stokes operator in regular domains with classical duality argument, we have the following results concerning interpolates $(u_h(t), p_h(t))$ [11]:

Lemma 4.1. *Under the assumptions $u(t) \in L^s([0,T]; H^{l+1}(\Omega)^d \cap V)$, $p(t) \in L^s([0,T]; H^l(\Omega) \cap M)$ for $1 \leq s \leq \infty$, there exists $c > 0$ such that*

(4.3)
$$\|u - u_h\|_{L^s([0,T];L^2(\Omega)^d)} + h \left[\|u - u_h\|_{L^s([0,T];H^1(\Omega)^d)} + \|p - p_h\|_{L^s([0,T];L^2(\Omega))} \right]$$
$$\leq ch^{l+1} \left[\|u\|_{L^s([0,T];H^{l+1}(\Omega)^d)} + \|p\|_{L^s([0,T];H^l(\Omega))} \right].$$

Lemma 4.2. *Under the assumptions $u(t) \in L^s([0,T]; H^2(\Omega)^d \cap V)$, $p(t) \in L^s([0,T]; H^1(\Omega) \cap M)$ for $1 \leq s \leq \infty$, there exists $c > 0$ such that*

(4.4)
$$\|u_h\|_{L^s([0,T];W^{0,\infty}(\Omega)^d \cap W^{1,3}(\Omega)^d)} \leq c \left(\|u\|_{L^s([0,T];H^2(\Omega)^d)} + \|p\|_{L^s([0,T];H^1(\Omega))} \right).$$

Lemma 4.3. *Under the assumptions $u(t) \in L^s([0,T]; H^2(\Omega)^d \cap H_0^1(\Omega)^d)$, $p(t) \in L^s([0,T]; H^1(\Omega) \cap M)$ for $1 \leq s \leq \infty$, there exists $c > 0$ such that*

$$(4.5) \qquad \|C_h^T p_h\|_{L^s([0,T]; L^2(\Omega)^d)} \leq c \left(\|u\|_{L^s([0,T]; H^2(\Omega)^d)} + \|p\|_{L^s([0,T]; H^1(\Omega))} \right).$$

For $(\phi_h(t), w_h(t)) \in \Psi_h \times \Psi_h$, we have the following.

Lemma 4.4. *Under the assumptions $\phi(t), w(t) \in L^s([0,T]; H^{l+1}(\Omega))$ for $1 \leq s \leq \infty$, there exists $c > 0$ such that*

$$(4.6)$$
$$\|\phi - \phi_h\|_{L^s([0,T]; L^2(\Omega))} + h\|\phi - \phi_h\|_{L^s([0,T]; H^1(\Omega))} \leq ch^{l+1} \|\phi\|_{L^s([0,T]; H^{l+1}(\Omega))},$$
$$\|w - w_h\|_{L^s([0,T]; L^2(\Omega))} + h\|w - w_h\|_{L^s([0,T]; H^1(\Omega))} \leq ch^{l+1} \|w\|_{L^s([0,T]; H^{l+1}(\Omega))}.$$

Lemma 4.5. *Under the assumptions $\phi(t), w(t) \in L^s([0,T]; H^2(\Omega))$ for $1 \leq s \leq \infty$, there exists $c > 0$ such that*

$$(4.7) \qquad \begin{aligned} \|\phi_h\|_{L^s([0,T]; W^{0,\infty}(\Omega) \cap W^{1,3}(\Omega))} &\leq c\|\phi\|_{L^s([0,T]; H^2(\Omega))}, \\ \|w_h\|_{L^s([0,T]; W^{0,\infty}(\Omega) \cap W^{1,3}(\Omega))} &\leq c\|w\|_{L^s([0,T]; H^2(\Omega))}. \end{aligned}$$

Discrete inverse Laplace operator with Neumann boundary condition.
Define the discrete inverse Laplace operator $L_h : H^{-1}(\Omega) \to \Psi_h$. For all $v \in H^{-1}(\Omega)$, $L_h(v) \in \Psi_h \cap L_0^2(\Omega)$ is the solution to the following problem,

$$(4.8) \qquad (\nabla L_h(v), \nabla \varphi_h) = (v - v_0, \varphi_h) \qquad \forall \varphi_h \in X_h,$$

where $v_0 = \frac{1}{|\Omega|}(v, 1)$.

L_h can be viewed as elliptic projection of the usual inverse Laplace operator $L : H^{-1}(\Omega) \to H^1(\Omega) \cap L_0^2(\Omega)$, where

$$(4.9) \qquad (\nabla L(\psi), \nabla \varphi) = (\psi - \psi_0, \varphi) \quad \forall \varphi \in H^1(\Omega),$$

with $\psi_0 = \frac{1}{|\Omega|}(\psi, 1)$.

In particular, using the H^2-regularity, we have

$$(4.10)$$
$$\|L(v) - L_h(v)\|_{L^2(\Omega)} + h\|L(v) - L_h(v)\|_{H^1(\Omega)} \leq c_1 h^2 \|L(v)\|_{H^2(\Omega)} \leq c_2 h^2 \|v\|_{L^2(\Omega)}.$$

Discrete inverse Stokes operator. Define the discrete inverse Stokes operator $\mathcal{S}_h : H^{-1}(\Omega)^d \to X_h$. For all $v \in H^{-1}(\Omega)^d$, $(\mathcal{S}_h(v), r_h) \in X_h \times M_h$ is the solution to the following problem:

$$(4.11) \qquad \begin{cases} (A_h(\mathcal{S}_h(v)), v_h) + \left(B_h^T r_h, v_h\right) = (v, v_h) & \forall v_h \in X_h, \\ (B_h \mathcal{S}_h(v), w_h) = 0 & \forall w_h \in M_h. \end{cases}$$

$(\mathcal{S}_h(v), r_h)$ can be viewed as an interpolate of $(\mathcal{S}(v), r)$ similar to (4.1).

In what follows, the following inequalities, which are valid for $d = 2, 3$, will be frequently used:

$$(4.12) \qquad |((u \cdot \nabla)v, w)| \leq c\|\nabla v\|_{L^2} \|u\|_{H^1} \|w\|_{L^2}^{1/2} \|w\|_{H^1}^{1/2} \quad \forall u, v, w \in H^1(\Omega)^d,$$

$$(4.13) \qquad |((u \cdot \nabla)v, w)| \leq \|u\|_{L^2} \|\nabla v\|_{L^3} \|w\|_{H^1} \quad \forall u, v, w \in H^1(\Omega)^d,$$

$$(4.14) \qquad |d(u, v, w)| \leq c(\|v\|_{L^\infty} + \|v\|_{W^{1,3}})\|w\|_{L^2} \|u\|_{H^1}$$
$$\forall v \in H^2(\Omega)^d, \ u \in H^1(\Omega)^d, \ w \in L^2(\Omega)^d.$$

We note that (4.12) and (4.13) hold for $u \in H^1(\Omega)^d$ with $v, w \in H^1(\Omega)$. Thanks to integration by parts, we also have

$$(4.15) \qquad |d(u, v, w)| \leq c(\|u\|_{L^\infty} + \|u\|_{W^{1,3}})\|v\|_{L^2}\|w\|_{H^1}$$
$$\forall u \in H^2(\Omega)^d \cap H_0^1(\Omega)^d, \ v, w \in H^1(\Omega)^d,$$

and

$$(4.16) \qquad |((u \cdot \nabla)v, w)| \leq c(\|u\|_{L^\infty} + \|u\|_{W^{1,3}})\|v\|_{L^2}\|w\|_{H^1}$$
$$\forall u \in H^2(\Omega)^d \cap H_0^1(\Omega)^d, \ v, w \in H^1(\Omega).$$

Obviously, we have

$$(4.17) \qquad \|u\|_{L^\infty} + \|u\|_{W^{1,3}} \leq c\|u\|_{H^2}.$$

4.2. Error analysis. Let $(\phi_h^n, w_h^n, u_h^n, \tilde{u}_h^n, p_h^n) \in \Psi_h \times \Psi_h \times Y_h \times X_h \times M_h$ be the numerical solutions and intermediate velocity obtained from scheme (3.1)-(3.5). We define the error functions for $n = 0, 1, 2, \ldots, N$ as

$$(4.18) \qquad \begin{aligned} e_\phi^n &= \phi_h(t^n) - \phi_h^n, & e_w^n &= w_h(t^n) - w_h^n, & e_u^n &= u_h(t^n) - u_h^n, \\ \tilde{e}_u^n &= u_h(t^n) - \tilde{u}_h^n, & e_p^n &= p_h(t^n) - p_h^n, \end{aligned}$$

and we denote by $\tilde{e}_{u,\delta t}, e_{w,\delta t}, e_{u,\delta t}, e_{\phi,\delta t}, e_{p,\delta t}$ the corresponding sequence of error functions.

Assumption A. *We assume that the solution $(\phi(t), w(t), u(t), p(t))$ of the Cahn-Hilliard phase-field model (1.1)-(1.5) is sufficiently smooth such that for some $l \geq 1$:*

$$\phi, w, \partial_t \phi \in L^\infty([0, T]; H^{l+1}), \quad u, \partial_t u \in L^\infty([0, T]; (H^{l+1})^d),$$
$$p, \partial_t p \in L^\infty([0, T]; H^l), \quad \partial_{tt}\phi \in L^\infty([0, T]; H^1),$$
$$\partial_{tt}u \in L^\infty([0, T]; (H^1)^d)), \quad \partial_{tt}p \in L^\infty([0, T]; L^2).$$

In addition, we assume the scheme is initialized such that

$$(4.19) \qquad \|e_u^0\| + \delta t^{1/2}\|\nabla e_u^0\| + \|e_\phi^0\| + \|\nabla e_\phi^0\| + \delta t\|C_h^T e_p^0\| \lesssim \delta t + h^l,$$

e.g., choose $u_h^0 = \tilde{u}_h^0 = P_{X_h}u_0$, $\phi_h^0 = P_{\Psi_h}\phi_0$, $p_h^0 = P_{M_h}p_0$. Then, the following error estimates hold.

Theorem 4.1. *Under Assumption* A, *for sufficiently small δt, the finite-element approximate solution to the scheme (3.1)-(3.5) satisfies*

$$\|\phi - \phi_{h,\delta t}\|_{l^\infty(L^2(\Omega))} + \|u - u_{h,\delta t}\|_{l^\infty(L^2(\Omega)^d)} + \|u - \tilde{u}_{h,\delta t}\|_{l^\infty(L^2(\Omega)^d)} \lesssim \delta t + h^l,$$
$$\|\phi - \phi_{h,\delta t}\|_{l^\infty(H^1(\Omega))} + \|u - \tilde{u}_{h,\delta t}\|_{l^2(H^1(\Omega)^d)} + \|w - w_{h,\delta t}\|_{l^2(H^1(\Omega))} \lesssim \delta t + h^l,$$
$$\|p - p_{h,\delta t}\|_{l^2(L^2(\Omega))} \lesssim (1 + h/\sqrt{\delta t})(\delta t + h^l).$$

If $X_h = Y_h$, the error on pressure becomes

$$\|p - p_{h,\delta t}\|_{l^2(L^2(\Omega))} \lesssim \delta t + h^l.$$

The proof of the above results will require a sequence of intermediate results that we establish below.

By definition, we have the following equations for the interpolates $(u_h(t), p_h(t), \phi_h(t), w_h(t))$:

$$(4.20) \quad \begin{cases} \dfrac{\phi_h(t^{n+1}) - \phi_h(t^n)}{\delta t} + \gamma \tilde{A}_h w_h(t^{n+1}) = P_{\Psi_h}\left[\tilde{R}_\phi^{n+1} - (u(t^{n+1}) \cdot \nabla)\phi(t^n) \right], \\[2mm] (4.21) \quad w_h(t^{n+1}) - \dfrac{S}{\varepsilon^2}(\phi_h(t^{n+1}) - \phi_h(t^n)) \\[2mm] \qquad\qquad\qquad\qquad = \tilde{A}_h \phi_h(t^{n+1}) + P_{\Psi_h}\left[\tilde{R}_w^{n+1} + f(\phi(t^n)) \right], \\[2mm] (4.22) \quad \dfrac{\rho_0}{\delta t}\left(u_h(t^{n+1}) - u_h(t^n) \right) + \mu_0 A_h u_h(t^{n+1}) + B_h^T p_h(t^{n+1}) \\[2mm] \qquad\qquad = P_{X_h}\left[\tilde{R}_u^{n+1} - \rho_0 D(u(t^n), u(t^{n+1})) + \lambda w(t^{n+1})\nabla\phi(t^n) \right], \\[2mm] (4.23) \quad B_h u_h(t^{n+1}) = 0, \end{cases}$$

where

$$(4.24)$$
$$\tilde{R}_\phi^{n+1} = \frac{\delta_t \phi_h(t^n)}{\delta t} - \partial_t \phi(t^{n+1}) - (u(t^{n+1}) \cdot \nabla)\phi(t^{n+1}) + (u(t^{n+1}) \cdot \nabla)\phi(t^n),$$
$$\tilde{R}_w^{n+1} = -\frac{S}{\varepsilon^2}(\phi_h(t^{n+1}) - \phi_h(t^n)) + f(\phi(t^{n+1})) - f(\phi(t^n)) + w(t^{n+1}) - w_h(t^{n+1}),$$
$$\tilde{R}_u^{n+1} = \rho_0 \frac{\delta_t u_h(t^n)}{\delta t} - \rho_0 \partial_t u(t^{n+1}) - \rho_0 \left(u(t^{n+1}) \cdot \nabla \right) u(t^{n+1}) + \lambda w(t^{n+1})\nabla\phi(t^{n+1})$$
$$\qquad + \rho_0 D(u(t^n), u(t^{n+1})) - \lambda w(t^{n+1})\nabla\phi(t^n).$$

Using the properties for the interpolates defined in (4.1) and (4.2), we have the following results.

Lemma 4.6. *Under Assumption* A, *we have*
$$(4.25)$$
$$\|\tilde{R}_\phi^{n+1}\| + \|\tilde{R}_u^{n+1}\| + \|\tilde{R}_w^{n+1}\| \lesssim \delta t + h^{l+1}, \quad \|\tilde{R}_w^{n+1}\|_1 \lesssim \delta t + h^l \quad \forall 0 \le n \le \frac{T}{\delta t} - 1.$$

Proof. We will only prove the bound for $\|\tilde{R}_\phi^{n+1}\|$, and omit the others as the arguments are similar. Noticing that

$$\tilde{R}_\phi^{n+1} = \partial_t \phi_h(t^{n+1}) - \partial_t \phi(t^{n+1}) - \left(u(t^{n+1}) \cdot \nabla \right)\left(\phi(t^{n+1}) - \phi(t^n) \right)$$
$$\qquad - \delta t \int_0^1 \int_0^\theta \partial_{tt}\phi_h(t_{n+1} - s\delta t)\, ds d\theta,$$

applying the properties of interpolates as well as the regularity assumption, we have

$$\|\tilde{R}_\phi^{n+1}\| \lesssim \|\partial_t \phi_h(t^{n+1}) - \partial_t \phi(t^{n+1})\| + \delta t\|u(t^{n+1})\|_{L^\infty}\|\partial_t \phi\|_{L^\infty(0,T;H^1)}$$
$$\qquad + \delta t\|\partial_{tt}\phi\|_{L^\infty(0,T;H^1)}$$
$$\qquad \lesssim h^{l+1}\|\partial_t \phi(t^{n+1})\|_{L^\infty(H^{l+1})} + \delta t \lesssim \delta t + h^{l+1},$$

where we have differentiated (4.2) in time t once and used approximation property in Lemma 4.4 to control $\|\partial_t \phi_h(t^{n+1}) - \partial_t \phi(t^{n+1})\|$. $\qquad\qquad\square$

Subtracting (3.1)-(3.3) from (4.20)-(4.22), respectively, we obtain the following error equations governing the growth of $\tilde{e}_u^{n+1}$, e_ϕ^{n+1} and e_w^{n+1}:

$$
\begin{cases}
(4.26) & \dfrac{e_\phi^{n+1} - e_\phi^n}{\delta t} + \gamma \tilde{A}_h e_w^{n+1} = P_{\Psi_h}\left[\tilde{R}_\phi^{n+1} + R_\phi^{n+1}\right], \\[2ex]
(4.27) & e_w^{n+1} - \dfrac{S}{\varepsilon^2}(e_\phi^{n+1} - e_\phi^n) = \tilde{A}_h e_\phi^{n+1} + P_{\Psi_h}\left[\tilde{R}_w^{n+1} + R_w^{n+1}\right], \\[2ex]
(4.28) & \dfrac{\rho_0}{\delta t}\left(\tilde{e}_u^{n+1} - i_h^T e_u^n\right) + \mu_0 A_h \tilde{e}_u^{n+1} + B_h^T \tilde{e}_p^n = P_{X_h}\left[\tilde{R}_u^{n+1} + R_{u,u}^{n+1} + R_{u,\phi}^{n+1}\right],
\end{cases}
$$

where

$$
\begin{aligned}
& R_\phi^{n+1} = -(u(t^{n+1}) \cdot \nabla)\phi(t^n) + (\tilde{u}_h^{n+1} \cdot \nabla)\phi_h^n, \\
& R_w^{n+1} = f(\phi(t^n)) - f(\phi_h^n) = (\phi(t^n) - \phi_h^n)\int_0^1 f'(s\phi(t^n) + (1-s)\phi_h^n)\,ds, \\
(4.29) \quad & \tilde{e}_p^n = p_h(t^{n+1}) - p_h^n = \delta_t p_h(t^n) + e_p^n, \\
& R_{u,u}^{n+1} = -\rho_0 D(u(t^n), u(t^{n+1})) + \rho_0 D_h(\tilde{u}_h^n, \tilde{u}_h^{n+1}), \\
& R_{u,\phi}^{n+1} = \lambda w(t^{n+1})\nabla\phi(t^n) - \lambda w_h^{n+1}\nabla\phi_h^n.
\end{aligned}
$$

In addition, since $u_h(t^{n+1}) \in X_h$, $B_h u_h(t^{n+1}) = 0$ and C_h is an extension of B_h, we have the error equation

$$
(4.30) \quad
\begin{cases}
\dfrac{\rho_0}{\delta t}(e_u^{n+1} - i_h \tilde{e}_u^{n+1}) + C_h^T(e_p^{n+1} - \tilde{e}_p^n) = 0, \\[2ex]
C_h e_u^{n+1} = 0.
\end{cases}
$$

It is convenient to write R_ϕ^{n+1} and $R_{u,\phi}^{n+1}$ as

$$
\begin{aligned}
(4.31) \quad R_\phi^{n+1} =& -\left((u(t^{n+1}) - u_h(t^{n+1})) \cdot \nabla\right)\phi(t^n) - (\tilde{e}_u^{n+1} \cdot \nabla)\phi_h^n \\
& - (u_h(t^{n+1}) \cdot \nabla)e_\phi^n - (u_h(t^{n+1}) \cdot \nabla)(\phi(t^n) - \phi_h(t^n)), \\
(4.32) \quad R_{u,\phi}^{n+1} =& \lambda(w(t^{n+1}) - w_h(t^{n+1}))\nabla\phi(t^n) + \lambda e_w^{n+1}\nabla\phi_h^n + \lambda w_h(t^{n+1})\nabla e_\phi^n \\
& + \lambda w_h(t^{n+1})\nabla(\phi(t^n) - \phi_h(t^n)).
\end{aligned}
$$

Taking the inner product of (4.26) with $\lambda e_\phi^{n+1} \in \Psi_h$, $\lambda e_w^{n+1} \in \Psi_h$, $L_h(e_\phi^{n+1} - e_\phi^n) \in \Psi_h \cap L_0^2(\Omega)$ and $1 \in \Psi_h$, respectively, we obtain

(4.33)
$$\frac{\lambda}{2\delta t}(\|e_\phi^{n+1}\|^2 + \|e_\phi^{n+1} - e_\phi^n\|^2 - \|e_\phi^n\|^2) + \lambda\gamma(\nabla e_w^{n+1}, \nabla e_\phi^{n+1})$$
$$= \lambda(\tilde{R}_\phi^{n+1} + R_\phi^{n+1}, e_\phi^{n+1}),$$

(4.34)
$$\frac{\lambda}{\delta t}\left(e_\phi^{n+1} - e_\phi^n, e_w^{n+1}\right) + \lambda\gamma(\nabla e_w^{n+1}, \nabla e_w^{n+1}) = \lambda(\tilde{R}_\phi^{n+1}, e_w^{n+1}) + \lambda(R_\phi^{n+1}, e_w^{n+1}),$$

(4.35)
$$\frac{\|\nabla L_h(\delta_t e_\phi^n)\|^2}{\delta t} + \gamma(e_w^{n+1}, \delta_t e_\phi^n) - \frac{\gamma}{|\Omega|}(\delta_t e_\phi^n, 1)(e_w^{n+1}, 1)$$
$$= (\tilde{R}_\phi^{n+1} + R_\phi^{n+1}, L_h(\delta_t e_\phi^n)),$$

(4.36)
$$(\delta_t e_\phi^n, 1) = \int_\Omega (e_\phi^{n+1} - e_\phi^n) = \delta t \int_\Omega (\tilde{R}_\phi^{n+1} + R_\phi^{n+1}).$$

Testing (4.27) with $e_w^{n+1} \in \Psi_h$ and $e_\phi^{n+1} - e_\phi^n \in \Psi_h$, respectively, we find

(4.37) $\quad \|e_w^{n+1}\|^2 - \dfrac{S}{\varepsilon^2}(e_\phi^{n+1} - e_\phi^n, e_w^{n+1}) = (\nabla e_\phi^{n+1}, \nabla e_w^{n+1}) + (R_w^{n+1} + \tilde{R}_w^{n+1}, e_w^{n+1}),$

(4.38) $\quad \dfrac{1}{2}(\|\nabla e_\phi^{n+1}\|^2 + \|\nabla \delta_t e_\phi^n\|^2 - \|\nabla e_\phi^n\|^2) = (e_w^{n+1}, \delta_t e_\phi^n) - \dfrac{S}{\varepsilon^2}\|\delta_t e_\phi^n\|^2$
$$- (R_w^{n+1} + \tilde{R}_w^{n+1}, \delta_t e_\phi^n).$$

Then taking the inner product of (4.28) with $\delta t \tilde{e}_u^{n+1} \in X_h$, we get

(4.39)
$$\frac{\rho_0}{2}(\|\tilde{e}_u^{n+1}\|^2 - \|i_h^T e_u^n\|^2 + \|\tilde{e}_u^{n+1} - i_h^T e_u^n\|^2) + \mu_0 \delta t\|\nabla \tilde{e}_u^{n+1}\|^2 + \delta t(B_h^T \tilde{e}_p^n, \tilde{e}_u^{n+1})$$
$$= \delta t(\tilde{R}_u^{n+1} + R_{u,\phi}^{n+1} + R_{u,u}^{n+1}, \tilde{e}_u^{n+1}).$$

Combining $\delta t \cdot (4.33) + \delta t \cdot (4.34) + (4.35) + \delta t\lambda\gamma \cdot (4.37) + (\lambda + \gamma) \cdot (4.38) + (4.39)$, we arrive at

(4.40)
$$\frac{\lambda}{2}(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2 + \|\delta_t e_\phi^n\|^2) + \frac{\rho_0}{2}(\|\tilde{e}_u^{n+1}\|^2 - \|i_h^T e_u^n\|^2 + \|\tilde{e}_u^{n+1} - i_h^T e_u^n\|^2)$$
$$+ \frac{\lambda + \gamma}{2}(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2 + \|\nabla \delta_t e_\phi^n\|^2) + \frac{1}{\delta t}\|\nabla L_h(\delta_t e_\phi^n)\|^2$$
$$+ \frac{S(\lambda + \gamma)}{\varepsilon^2}\|\delta_t e_\phi^n\|^2 + \mu_0 \delta t\|\nabla \tilde{e}_u^{n+1}\|^2 + \lambda\gamma\delta t\|e_w^{n+1}\|^2 + \lambda\gamma\delta t\|\nabla e_w^{n+1}\|^2$$
$$+ \delta t(B_h^T \tilde{e}_p^n, \tilde{e}_u^{n+1})$$
$$= \frac{S\delta t\lambda\gamma}{\varepsilon^2}(\delta_t e_\phi^n, e_w^{n+1}) + \lambda\delta t(R_\phi^{n+1}, e_w^{n+1}) + \delta t(R_{u,\phi}^{n+1}, \tilde{e}_u^{n+1})$$
$$+ \delta t\lambda\gamma(\tilde{R}_\phi^{n+1}, e_w^{n+1}) + \delta t\lambda\gamma(R_\phi^{n+1}, e_w^{n+1}) + \lambda\delta t(\tilde{R}_\phi^{n+1}, e_w^{n+1}) + \delta t(\tilde{R}_u^{n+1}, \tilde{e}_u^{n+1})$$
$$+ \delta t(R_{u,u}^{n+1}, \tilde{e}_u^{n+1}) + \lambda\delta t(R_\phi^{n+1}, e_\phi^{n+1}) + \lambda\delta t(\tilde{R}_\phi^{n+1}, e_\phi^{n+1}) + \gamma(\delta_t e_\phi^n, 1)(e_w^{n+1}, 1)$$
$$+ (\tilde{R}_\phi^{n+1} + R_\phi^{n+1}, L_h(\delta_t e_\phi^n)) - (\lambda + \gamma)(\tilde{R}_w^{n+1}, \delta_t e_\phi^n) - (\lambda + \gamma)(R_w^{n+1}, \delta_t e_\phi^n).$$

In order to control the right-hand side of (4.40), we establish below bounds for those nontrivial terms. As the first step, we provide the estimates for terms involving R_ϕ^{n+1} including $(R_\phi^{n+1}, e_w^{n+1})$, $(R_\phi^{n+1}, e_\phi^{n+1})$ and $(R_\phi^{n+1}, L_h(\delta_t e_\phi^n))$ as follows.

Lemma 4.7. *Under Assumption* A, *for any* $\varphi_h \in \Psi_h$ *and* $\kappa, \kappa' > 0$, *there holds*

$$(4.41) \quad (R_\phi^{n+1}, \varphi_h) \le c_{\kappa,\kappa'}(h^{2(l+1)} + \|e_\phi^n\|_1^2 + \|\varphi_h\|^2) + \kappa\|\nabla\varphi_h\|^2 + \kappa'\|\nabla\tilde{e}_u^{n+1}\|^2$$

and

$$(4.42) \quad (R_\phi^{n+1}, \varphi_h) \le c_{\kappa,\kappa'}(h^{2(l+1)} + \|e_\phi^n\|_1^2 + \|\tilde{e}_u^{n+1}\|^2) + \kappa\|\varphi_h\|_1^2 + \kappa'\|\nabla\tilde{e}_u^{n+1}\|^2.$$

In particular, we have

$$(4.43) \qquad\qquad |(R_\phi^{n+1}, 1)| \le c(h^{l+1} + \|\tilde{e}_u^{n+1}\| + \|\nabla e_\phi^n\|).$$

Proof. For any $\varphi_h \in \Psi_h$, in view of (4.31), we have

$$(R_\phi^{n+1}, \varphi_h) = -\big(((u(t^{n+1}) - u_h(t^{n+1})) \cdot \nabla)\phi(t^n), \varphi_h\big) - ((\tilde{e}_u^{n+1} \cdot \nabla)\phi_h^n, \varphi_h)$$
$$- ((u_h(t^{n+1}) \cdot \nabla)(\phi(t^n) - \phi_h(t^n)), \varphi_h) - ((u_h(t^{n+1}) \cdot \nabla)e_\phi^n, \varphi_h).$$

We estimate each term on the RHS as follows:

$$- \big(((u(t^{n+1}) - u_h(t^{n+1})) \cdot \nabla)\phi(t^n), \varphi_h\big)$$
$$\le c\|u(t^{n+1}) - u_h(t^{n+1})\|\|\phi(t^n)\|_{W^{1,3}}\|\varphi_h\|_1$$
$$\le ch^{l+1}(\|u\|_{L^\infty(H^{l+1})} + \|p\|_{L^\infty(H^l)})\|\varphi_h\|_1$$
$$\le \frac{c}{\kappa}h^{2(l+1)} + \frac{\kappa}{3}\|\varphi_h\|_1^2.$$

Using the a priori bound for $\|\nabla\phi_h^n\|$ in Theorem 3.1, we derive

$$-((\tilde{e}_u^{n+1} \cdot \nabla)\phi_h^n, \varphi_h) \le c\|\nabla\phi_h^n\| \|\tilde{e}_u^{n+1}\|_1\|\varphi_h\|^{1/2}\|\varphi_h\|_1^{1/2}$$
$$\le c_{\kappa,\kappa'}\|\varphi_h\|^2 + \frac{\kappa}{3}\|\nabla\varphi_h\|^2 + \kappa'\|\nabla\tilde{e}_u^{n+1}\|^2,$$

$$- ((u_h(t^{n+1}) \cdot \nabla)(\phi(t^n) - \phi_h(t^n)), \varphi_h)$$
$$\le c\|\phi(t^n) - \phi_h(t^n)\| \big(\|u_h(t^{n+1})\|_{L^\infty} + \|u_h(t^{n+1})\|_{W^{1,3}}\big) \|\varphi_h\|_1$$
$$\le ch^{l+1}(\|\phi\|_{L^\infty(H^{l+1})} + \|w\|_{L^\infty(H^{l+1})})(\|u\|_{L^\infty(H^2)} + \|p\|_{L^\infty(H^1)})\|\varphi_h\|_1$$
$$\le \frac{c}{\kappa}h^{2(l+1)} + \frac{\kappa}{3}\|\varphi_h\|_1^2,$$

and

$$-((u_h(t^{n+1}) \cdot \nabla)e_\phi^n, \varphi_h) \le \|u_h(t^{n+1})\|_{L^\infty}\|\nabla e_\phi^n\|\|\varphi_h\| \le c\|\nabla e_\phi^n\|^2 + \|\varphi_h\|^2.$$

Combining the above inequalities, we obtain (4.41). The other desired results in Lemma 4.7 can be derived in the same manner. $\square$

Lemma 4.8. *Under Assumption* A, *we have*

$$(4.44) \quad |(R_{u,u}^{n+1}, \tilde{e}_u^{n+1})| \le c(h^{2(l+1)} + \|\tilde{e}_u^{n+1}\|^2) + \frac{\mu_0}{8}\|\nabla\tilde{e}_u^{n+1}\|^2 + \frac{\mu_0}{16}\|\nabla\tilde{e}_u^n\|^2.$$

Proof. Using the skew-symmetric property of $d(u_h^n, \cdot, \cdot)$, we obtain

$$(R_{u,u}^{n+1}, \tilde{e}_u^{n+1}) = - \rho_0 \big[d(u(t^n), u(t^{n+1}), \tilde{e}_u^{n+1}) - d(\tilde{u}_h^n, \tilde{u}_h^{n+1}, \tilde{e}_u^{n+1})\big]$$
$$= - \rho_0 d(u(t^n) - u_h(t^n), u(t^{n+1}), \tilde{e}_u^{n+1}) - \rho_0 d(\tilde{e}_u^n, u_h(t^{n+1}), \tilde{e}_u^{n+1})$$
$$- \rho_0 d(u_h(t^n), u(t^{n+1}) - u_h(t^{n+1}), \tilde{e}_u^{n+1}).$$

Using the Cauchy-Schwarz inequality, the Sobolev inequalities, $\|\tilde{e}_u^{n+1}\|_1 \le c\|\nabla \tilde{e}_u^{n+1}\|$, and properties of the interpolates, we estimate each term on the RHS as follows:

$$
\begin{aligned}
|\rho_0 d(u(t^n) - u_h(t^n), u(t^{n+1}), \tilde{e}_u^{n+1})| &\le c\|u(t^{n+1})\|_2 \|u(t^n) - u_h(t^n)\| \|\tilde{e}_u^{n+1}\|_1 \\
&\le ch^{l+1}(\|u\|_{L^\infty(H^{l+1})} + \|p\|_{L^\infty(H^l)})\|\tilde{e}_u^{n+1}\|_1 \\
&\le ch^{2(l+1)} + \frac{\mu_0}{16}\|\nabla \tilde{e}_u^{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
|\rho_0 d(\tilde{e}_u^n, u_h(t^{n+1}), \tilde{e}_u^{n+1})| &\le c\left(\|u_h(t^{n+1})\|_{L^\infty} + \|u_h(t^{n+1})\|_{W^{1,3}}\right) \|\tilde{e}_u^n\|_1 \|\tilde{e}_u^{n+1}\| \\
&\le c\left(\|u\|_{L^\infty(H^2)} + \|p\|_{L^\infty(H^1)}\right) \|\tilde{e}_u^n\|_1 \|\tilde{e}_u^{n+1}\| \\
&\le c\|\tilde{e}_u^{n+1}\|^2 + \frac{\mu_0}{16}\|\nabla \tilde{e}_u^n\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|\rho_0 d(u_h(t^n), &u(t^{n+1}) - u_h(t^{n+1}), \tilde{e}_u^{n+1})| \\
&\le c\|u(t^{n+1}) - u_h(t^{n+1})\|(\|u_h(t^n)\|_{W^{1,3}} + \|u_h(t^n)\|_{L^\infty})\|\tilde{e}_u^{n+1}\|_1 \\
&\le c\|u(t^{n+1}) - u_h(t^{n+1})\| \|\tilde{e}_u^{n+1}\|_1 (\|u\|_{L^\infty(H^2)} + \|p\|_{L^\infty(H^1)}) \\
&\le ch^{l+1}(\|u\|_{L^\infty(H^{l+1})} + \|p\|_{L^\infty(H^l)})\|\tilde{e}_u^{n+1}\|_1 \\
&\le ch^{2(l+1)} + \frac{\mu_0}{16}\|\nabla \tilde{e}_u^{n+1}\|^2.
\end{aligned}
$$

Then the conclusion follows. $\qquad\square$

Lemma 4.9. *Under Assumption* A*, we have*

$$
\begin{aligned}
(R_{u,\phi}^{n+1}, \tilde{e}_u^{n+1}) \le{}& c(h^{2(l+1)} + \|\tilde{e}_u^{n+1}\|^2 + \|\nabla e_\phi^n\|^2) + \frac{\mu_0}{4}\|\nabla \tilde{e}_u^{n+1}\|^2 \\
&+ \frac{\lambda\gamma}{8}(\|e_w^{n+1}\|^2 + \|\nabla e_w^{n+1}\|^2).
\end{aligned}
$$

Proof. Noticing (4.32), we have

$$
\begin{aligned}
(R_{u,\phi}^{n+1}, \tilde{e}_u^{n+1}) ={}& \lambda\left((w(t^{n+1}) - w_h(t^{n+1}))\nabla\phi(t^n), \tilde{e}_u^{n+1}\right) + \lambda\left(e_w^{n+1}\nabla\phi_h^n, \tilde{e}_u^{n+1}\right) \\
&+ \lambda\left(w_h(t^{n+1})\nabla(\phi(t^n) - \phi_h(t^n)), \tilde{e}_u^{n+1}\right) + \lambda\left(w_h(t^{n+1})\nabla e_\phi^n, \tilde{e}_u^{n+1}\right).
\end{aligned}
$$

Using $\|\tilde{e}_u^{n+1}\|_1 \le c\|\nabla \tilde{e}_u^{n+1}\|$, we bound each term on the RHS as:

$$
\begin{aligned}
\lambda((w(t^{n+1}) - w_h(t^{n+1}))\nabla\phi(t^n), \tilde{e}_u^{n+1}) &\le \|w(t^{n+1}) - w_h(t^{n+1})\| \|\nabla\phi(t^n)\|_{L^3} \|\tilde{e}_u^{n+1}\|_1 \\
&\le ch^{l+1}\|w\|_{L^\infty(H^{l+1})}\|\phi(t^n)\|_{H^2}\|\tilde{e}_u^{n+1}\|_1 \\
&\le ch^{2(l+1)} + \frac{\mu_0}{16}\|\nabla \tilde{e}_u^{n+1}\|^2.
\end{aligned}
$$

Recalling the stability of the scheme, which implies the a priori bound for $\|\nabla\phi_h^n\|$,

we have

$$\lambda\left(e_w^{n+1}\nabla\phi_h^n, \tilde{e}_u^{n+1}\right) \leq c\|e_w^{n+1}\|_1\|\nabla\phi_h^n\|\|\tilde{e}_u^{n+1}\|^{1/2}\|\tilde{e}_u^{n+1}\|_1^{1/2}$$
$$\leq c\|\tilde{e}_u^{n+1}\|^2 + \frac{\mu_0}{16}\|\nabla\tilde{e}_u^{n+1}\|^2 + \frac{\lambda\gamma}{8}\left(\|e_w^{n+1}\|^2 + \|\nabla e_w^{n+1}\|^2\right),$$

$$\lambda(w_h(t^{n+1})\nabla(\phi(t^n) - \phi_h(t^n)), \tilde{e}_u^{n+1})$$
$$= -\lambda(w_h(t^{n+1})(\phi(t^n) - \phi_h(t^n)), \nabla\cdot\tilde{e}_u^{n+1})$$
$$- \lambda((\tilde{e}_u^{n+1}\cdot\nabla)w_h(t^{n+1}), \phi(t^n) - \phi_h(t^n))$$
$$\leq c\|w_h(t^{n+1})\|_{L^\infty}\|\phi(t^n) - \phi_h(t^n)\|\|\nabla\tilde{e}_u^{n+1}\|$$
$$+ c\|w_h(t^{n+1})\|_{W^{1,3}}\|\phi(t^n) - \phi_h(t^n)\|\|\tilde{e}_u^{n+1}\|_1$$
$$\leq ch^{l+1}\|\phi\|_{L^\infty(H^{l+1})}\|w\|_{L^\infty(H^2)}\|\tilde{e}_u^{n+1}\|_1$$
$$\leq ch^{2(l+1)} + \frac{\mu_0}{16}\|\nabla\tilde{e}_u^{n+1}\|^2$$

and

$$\lambda(w_h(t^{n+1})\nabla e_\phi^n, \tilde{e}_u^{n+1}) \leq c\|\nabla e_\phi^n\|\|w_h(t^{n+1})\|_1\|\tilde{e}_u^{n+1}\|_1$$
$$\leq c\|\nabla e_\phi^n\|\|\tilde{e}_u^{n+1}\|_1\|w(t^{n+1})\|_1$$
$$\leq c\|\nabla e_\phi^n\|^2 + \frac{\mu_0}{16}\|\nabla\tilde{e}_u^{n+1}\|^2.$$

This completes the proof. □

Lemma 4.10. *Under Assumption* A*, we have*

$$(4.45)\qquad |(e_w^{n+1}, 1)| \leq c(\delta t + h^{l+1} + \|\delta_t e_\phi^n\| + \|e_\phi^n\|).$$

Proof. We have from (4.27) that

$$(e_w^{n+1}, 1) = \frac{S}{\varepsilon^2}(\delta_t e_\phi^n, 1) + (\tilde{R}_w^{n+1}, 1) + (R_w^{n+1}, 1).$$

The term $(R_w^{n+1}, 1)$ can be bounded as

$$|(R_w^{n+1}, 1)| \leq \|f'\|_{L^\infty}(|\phi(t^n) - \phi_h^n|, 1) \leq c(\|\phi(t^n) - \phi_h(t^n)\| + \|e_\phi^n\|)$$
$$\leq ch^{l+1}\|\phi\|_{L^\infty(H^{l+1})} + c\|e_\phi^n\| \leq c(h^{l+1} + \|e_\phi^n\|).$$

Using Lemma 4.6, we obtain the desired inequality. □

We can now establish the following convergence results for $\tilde{u}_h^n$, u_h^n and ϕ_h^n, w_h^n.

Lemma 4.11. *Under Assumption* A*, for sufficiently small* δt*, the finite-element approximate solution to the scheme* (3.1)-(3.5) *satisfies*

$$\|\phi - \phi_{h,\delta t}\|_{l^\infty(L^2(\Omega))} + \|u - u_{h,\delta t}\|_{l^\infty(L^2(\Omega)^d)} + \|u - \tilde{u}_{h,\delta t}\|_{l^\infty(L^2(\Omega)^d)} \lesssim \delta t + h^l,$$
$$\|\phi - \phi_{h,\delta t}\|_{l^\infty(H^1(\Omega))} + \|w - w_{h,\delta t}\|_{l^2(H^1(\Omega))} + \|u - \tilde{u}_{h,\delta t}\|_{l^2(H^1(\Omega)^d)} \lesssim \delta t + h^l.$$

Proof. We start from (4.40).

(a) First, we bound the terms on the RHS of (4.40), except the last two terms, as follows:

$$\frac{S\delta t\lambda\gamma}{\varepsilon^2}(\delta_t e_\phi^n, e_w^{n+1}) = c\delta t\|\delta_t e_\phi^n\|\,\|e_w^{n+1}\| \leq c\delta t\|\delta_t e_\phi^n\|^2 + \frac{\delta t\lambda\gamma}{8}\|e_w^{n+1}\|^2.$$

From Lemmas 4.7 and 4.9 (choosing $\varphi_h = e_w^{n+1}$ in (4.42)), we get

$$\lambda \delta t (R_\phi^{n+1}, e_w^{n+1}) + \delta t (R_{u,\phi}^{n+1}, \tilde{e}_u^{n+1})$$
$$\leq c\delta t (h^{2(l+1)} + \|e_\phi^n\|_1^2 + \|\tilde{e}_u^{n+1}\|) + \frac{3\mu_0 \delta t}{8} \|\nabla \tilde{e}_u^{n+1}\|^2$$
$$+ \frac{3\lambda\gamma\delta t}{8} (\|e_w^{n+1}\|^2 + \|\nabla e_w^{n+1}\|^2).$$

Applying Lemma 4.6, we have

$$\delta t \lambda \gamma (\tilde{R}_w^{n+1}, e_w^{n+1}) \leq \lambda\gamma\delta t \|\tilde{R}_w^{n+1}\| \|e_w^{n+1}\| \leq c\delta t (\delta t^2 + h^{2l+2}) + \frac{\lambda\gamma\delta t}{8} \|e_w^{n+1}\|^2.$$

Using the expression of R_w^{n+1} and the assumption on $f(\cdot)$, we get

$$\delta t \lambda \gamma (R_w^{n+1}, e_w^{n+1}) \leq \delta t \lambda \gamma \|R_w^{n+1}\| \|e_w^{n+1}\| \leq c\delta t (\|\phi(t^n) - \phi_h(t^n)\| + \|e_\phi^n\|) \|e_w^{n+1}\|$$
$$\leq c\delta t (h^{l+1} (\|\phi\|_{L^\infty(H^{l+1})} + \|w\|_{L^\infty(H^{l+1})}) + \|e_\phi^n\|) \|e_w^{n+1}\|$$
$$\leq c\delta t (h^{2(l+1)} + \|e_\phi^n\|^2) + \frac{\lambda\gamma\delta t}{8} \|e_w^{n+1}\|^2.$$

Lemma 4.6 leads to

$$\lambda \delta t (\tilde{R}_\phi^{n+1}, e_w^{n+1}) \leq \lambda\delta t \|\tilde{R}_\phi^{n+1}\| \|e_w^{n+1}\| \leq c\delta t (\delta t + h^{l+1})^2 + \frac{\lambda\gamma\delta t}{8} \|e_w^{n+1}\|^2$$

and

$$\delta t (\tilde{R}_u^{n+1}, \tilde{e}_u^{n+1}) \leq \delta t \|\tilde{R}_u^{n+1}\| \|\tilde{e}_u^{n+1}\| \leq c\delta t (\delta t + h^{l+1})^2 + \delta t \|\tilde{e}_u^{n+1}\|^2.$$

Lemma 4.8 implies

$$|\delta t (R_{u,u}^{n+1}, \tilde{e}_u^{n+1})| \leq c\delta t (h^{2(l+1)} + \|\tilde{e}_u^{n+1}\|^2) + \frac{\mu_0\delta t}{8} \|\nabla \tilde{e}_u^{n+1}\|^2 + \frac{\mu_0\delta t}{16} \|\nabla \tilde{e}_u^n\|^2.$$

By Lemma 4.7, choosing $\varphi_h = e_\phi^{n+1}$ in (4.41), we have

$$\lambda \delta t (R_\phi^{n+1}, e_\phi^{n+1}) \leq c\delta t (h^{2(l+1)} + \|e_\phi^n\|_1^2 + \|e_\phi^{n+1}\|_1^2) + \frac{\mu_0}{16} \|\nabla \tilde{e}_u^{n+1}\|^2.$$

Based on Lemma 4.6, the following term can be bounded as

$$\lambda \delta t (\tilde{R}_\phi^{n+1}, e_\phi^{n+1}) \leq \lambda\delta t \|\tilde{R}_\phi^{n+1}\| \|e_\phi^{n+1}\| \leq c\delta t^3 + \delta t \|e_\phi^{n+1}\|^2.$$

Using (4.36), we have

$$(4.46) \qquad (\delta_t e_\phi^n, 1) = \delta t (\tilde{R}_\phi^{n+1}, 1) + \delta t (R_\phi^{n+1}, 1),$$

and we derive from Lemmas 4.7, 4.10, and 4.6 that

$$- \gamma (\delta_t e_\phi^n, 1)(e_w^{n+1}, 1)$$
$$\leq c\delta t (\|\tilde{R}_\phi^{n+1}\| + h^{l+1} + \|\tilde{e}_u^{n+1}\| + \|\nabla e_\phi^n\|)(\delta t + h^{l+1} + \|\delta_t e_\phi^n\| + \|e_\phi^n\|)$$
$$\leq c\delta t (h^{2(l+1)} + \delta t^2 + \|\delta_t e_\phi^n\|^2 + \|e_\phi^n\|^2 + \|\nabla e_\phi^n\|^2 + \|\tilde{e}_u^{n+1}\|^2).$$

Next, we deal with the first term in the last line of equation (4.40). By the definition of L_h, we have

$$\|L_h(\delta_t e_\phi^n)\| \leq c\|\nabla(L_h(\delta_t e_\phi^n))\| \leq c\|\delta_t e_\phi^n - (\delta_t e_\phi^n, 1)/|\Omega|\| \leq c\|\delta_t e_\phi^n\|.$$

Applying Lemma 4.7 and the Sobolev inequality (choose $\varphi_h = \frac{1}{\delta t} L_h(\delta_t e_\phi^n)$ in (4.42)), we derive

$$
\left(\tilde{R}_\phi^{n+1} + R_\phi^{n+1}, L_h(\delta_t e_\phi^n) \right)
$$

$$
\leq \|\tilde{R}_\phi^{n+1}\| \|L_h(\delta_t e_\phi^n)\| + \delta t (R_\phi^{n+1}, \frac{1}{\delta t} L_h(\delta_t e_\phi^n))
$$

$$
\leq \left[c\delta t (h^{2(l+1)} + \|\tilde{e}_u^{n+1}\|^2 + \|e_\phi^n\|_1^2) + \frac{1}{10\delta t} \|\nabla L_h(\delta_t e_\phi^n)\|^2 + \frac{\mu_0 \delta t}{16} \|\nabla \tilde{e}_u^{n+1}\|^2 \right]
$$

$$
\quad + c \|\tilde{R}_\phi^{n+1}\| \|\nabla L_h(\delta_t e_\phi^n)\|
$$

$$
\leq c\delta t (\delta t^2 + h^{2(l+1)} + \|\tilde{e}_u^{n+1}\|^2 + \|e_\phi^n\|_1^2) + \frac{1}{5\delta t} \|\nabla L_h(\delta_t e_\phi^n)\|^2 + \frac{\mu_0 \delta t}{16} \|\nabla \tilde{e}_u^{n+1}\|^2.
$$

(b) It remains to estimate the last two terms on the RHS of (4.40). The H^1-stability of P_{Ψ_h} is implied by the inverse inequality. Recalling that

$$
-(\lambda + \gamma)(\tilde{R}_w^{n+1}, \delta_t e_\phi^n)
$$

$$
= -(\lambda + \gamma)(\nabla(P_{\Psi_h} \tilde{R}_w^{n+1}), \nabla L_h(\delta_t e_\phi^n)) - (\lambda + \gamma)(\tilde{R}_w^{n+1}, 1)(\delta_t e_\phi^n, 1),
$$

by using (4.46), we derive

$$
-(\lambda + \gamma)(\tilde{R}_w^{n+1}, \delta_t e_\phi^n)
$$

$$
= -(\lambda + \gamma)(\nabla(P_{\Psi_h} \tilde{R}_w^{n+1}), \nabla L_h(\delta_t e_\phi^n)) - (\lambda + \gamma)(\tilde{R}_w^{n+1}, 1)(\delta_t e_\phi^n, 1)
$$

$$
\leq c\|\tilde{R}_w^{n+1}\|_1 \|\nabla L_h(\delta_t e_\phi^n)\| + c\delta t (\|\tilde{R}_\phi^{n+1}\| + h^{l+1} + \|\tilde{e}_u^{n+1}\| + \|\nabla e_\phi^n\|) \|\tilde{R}_w^{n+1}\|
$$

$$
\leq c\|\tilde{R}_w^{n+1}\|_1 \|\nabla L_h(\delta_t e_\phi^n)\| + c\delta t (\delta t + h^{l+1} + \|\tilde{e}_u^{n+1}\| + \|\nabla e_\phi^n\|) \|\tilde{R}_w^{n+1}\|
$$

$$
\leq \delta t \left(h^{2l} + \delta t^2 + \|\tilde{e}_u^{n+1}\|^2 + \|\nabla e_\phi^n\|^2 \right) + \frac{\|\nabla L_h(\delta_t e_\phi^n)\|^2}{5\delta t}.
$$

To deal with the last term in (4.40), we split R_w^{n+1} as

$$
R_w^{n+1} = R_{w,a}^{n+1} + R_{w,b}^{n+1},
$$

where

$$
R_{w,a}^{n+1} = f(\phi(t^n)) - f(\phi_h(t^n)), \quad R_{w,b}^{n+1} = f(\phi_h(t^n)) - f(\phi_h^n).
$$

It is obvious that, given the properties of f, interpolates $\phi_h(t^n)$, and exact solution ϕ, we have $|R_{w,a}^{n+1}| \leq c|\phi(t^n) - \phi_h(t^n)|$, and $|R_{w,b}^{n+1}| \leq c|e_\phi^n|$, $|\nabla R_{w,b}^{n+1}| \leq c(|\nabla e_\phi^n| + |\nabla \phi_h(t^n)| |e_\phi^n|)$. In addition, as f' is bounded and Lipschitz, we have

$$
|\nabla R_{w,a}^{n+1}| \leq c(|\nabla(\phi(t^n) - \phi_h(t^n))| + |\phi(t^n) - \phi_h(t^n)| |\nabla \phi(t^n)|).
$$

We can then estimate $(R_{w,a}^{n+1}, \delta_t e_\phi^n)$ by using the discrete inverse Laplacian L_h as follows:

$$
\begin{aligned}
-(\lambda + \gamma)&(R_{w,a}^{n+1}, \delta_t e_\phi^n) \\
&= -(\lambda + \gamma)(\nabla(P_{\Psi_h} R_{w,a}^{n+1}), \nabla L_h(\delta_t e_\phi^n)) - (\lambda + \gamma)(R_{w,a}^{n+1}, 1)(\delta_t e_\phi^n, 1) \\
&\leq c\|R_{w,a}^{n+1}\|_1 \|\nabla L_h(\delta_t e_\phi^n)\| - (\lambda + \gamma)(R_{w,a}^{n+1}, 1)(\delta_t e_\phi^n, 1) \\
&\leq c\left(\|\phi(t^n) - \phi_h(t^n)\|_1 + \|\phi(t^n) - \phi_h(t^n)\|_1 \|\nabla \phi(t^n)\|_{L^3}\right)\|\nabla L_h(\delta_t e_\phi^n)\| \\
&\quad + c\delta t(\delta t + h^{l+1} + \|\tilde{e}_u^{n+1}\| + \|\nabla e_\phi^n\|)\|\phi(t^n) - \phi_h(t^n)\| \\
&\leq ch^l\|\phi\|_{L^\infty(H^{l+1})}(1 + \|\phi(t^n)\|_2)\|\nabla L_h(\delta_t e_\phi^n)\| \\
&\quad + c\delta t h^{l+1}\|\phi\|_{L^\infty(H^{l+1})}(\delta t + h^{l+1} + \|\tilde{e}_u^{n+1}\| + \|\nabla e_\phi^n\|) \\
&\leq c\delta t(\delta t^2 + h^{2l} + \|\tilde{e}_u^{n+1}\|^2 + \|\nabla e_\phi^n\|^2) + \frac{1}{5\delta t}\|\nabla L_h(\delta_t e_\phi)\|^2.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
-(\lambda + \gamma)&(R_{w,b}^{n+1}, \delta_t e_\phi^n) \\
&= -(\lambda + \gamma)(\nabla(P_{\Psi_h} R_{w,b}^{n+1}), \nabla L_h(\delta_t e_\phi^n)) - (\lambda + \gamma)(R_{w,b}^{n+1}, 1)(\delta_t e_\phi^n, 1) \\
&\leq c\|R_{w,b}^{n+1}\|_1 \|\nabla L_h(\delta_t e_\phi^n)\| + c\delta t(\delta t + h^{l+1} + \|\nabla \tilde{e}_u^{n+1}\| + \|\nabla e_\phi^n\|)\|R_{w,b}^{n+1}\| \\
&\leq c(\|e_\phi^n\|_1 + \|\phi_h(t^n)\|_{W^{1,3}}\|e_\phi^n\|_1)\|\nabla L_h(\delta_t e_\phi^n)\| \\
&\quad + c\delta t(\delta t + h^{l+1} + \|\nabla \tilde{e}_u^{n+1}\| + \|\nabla e_\phi^n\|)\|e_\phi^n\| \\
&\leq \frac{1}{5\delta t}\|\nabla L_h(\delta_t e_\phi^n)\|^2 + \frac{\mu_0}{16}\|\nabla \tilde{e}_u^{n+1}\|^2 \\
&\quad + c\delta t(\delta t^2 + h^{2(l+1)} + \|e_\phi^n\|^2 + \|\nabla e_\phi^n\|^2 + \|\tilde{e}_u^{n+1}\|^2).
\end{aligned}
$$

Thus, combining all previous estimates, (4.40) implies that

(4.47)
$$
\begin{aligned}
\frac{\lambda}{2}&(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2) + \frac{\lambda}{8}\|\delta_t e_\phi^n\|^2 + \frac{\rho_0}{2}(\|\tilde{e}_u^{n+1}\|^2 - \|i_h^T e_u^n\|^2 + \|\tilde{e}_u^{n+1} - i_h^T e_u^n\|^2) \\
&+ \frac{\lambda + \gamma}{2}(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2) + (\lambda + \gamma)\|\nabla \delta_t e_\phi^n\|^2 + \frac{1}{5\delta t}\|\nabla L_h(\delta_t e_\phi^n)\|^2 \\
&+ \frac{S(\lambda + \gamma)}{\varepsilon^2}\|\delta_t e_\phi^n\|^2 + \frac{5\mu_0 \delta t}{16}\|\nabla \tilde{e}_u^{n+1}\|^2 + \frac{\lambda \gamma \delta t}{8}\|e_w^{n+1}\|^2 + \frac{5\lambda \gamma \delta t}{8}\|\nabla e_w^{n+1}\|^2 \\
&+ \delta t(B_h^T \tilde{e}_p^n, \tilde{e}_u^{n+1}) \\
&\quad \leq c\delta t\left[(\delta t + h^l)^2 + \|e_\phi^n\|_1^2 + \|e_\phi^{n+1}\|_1^2 + \|\delta_t e_\phi^n\|^2 + \|\tilde{e}_u^{n+1}\|^2\right] + \frac{\mu_0 \delta t}{16}\|\nabla \tilde{e}_u^n\|^2.
\end{aligned}
$$

(c) Now, we want to bound $\delta t(B_h^T \tilde{e}_p^n, \tilde{e}_u^{n+1})$ on the LHS of (4.47) to complete the estimates. Testing (4.30) with $\delta t^2 C_h^T \tilde{e}_p^n \in Y_h$, we have

$$
\begin{aligned}
-\delta t \rho_0 (B_h^T \tilde{e}_p^n, \tilde{e}_u^{n+1}) + \frac{\delta t^2}{2}\|C_h^T e_p^{n+1}\|^2 &= \frac{\delta t^2}{2}\|C_h^T(e_p^{n+1} - \tilde{e}_p^n)\|^2 + \frac{\delta t^2}{2}\|C_h^T \tilde{e}_p^n\|^2 \\
&= \frac{\rho_0^2}{2}\|e_u^{n+1} - i_h \tilde{e}_u^{n+1}\|^2 + \frac{\delta t^2}{2}\|C_h^T(\delta_t p_h(t^n) + e_p^n)\|^2.
\end{aligned}
$$

Applying Lemma 4.3, we obtain

$$-\delta t\rho_0(B_h^T e_p^n, \tilde{e}_u^{n+1}) + \frac{\delta t^2}{2}\|C_h^T e_p^{n+1}\|^2 - \frac{\rho_0^2}{2}\|e_u^{n+1} - i_h\tilde{e}_u^{n+1}\|^2 - \frac{\delta t^2}{2}\|C_h^T e_p^n\|^2$$
$$\leq c\delta t^2(\|C_h^T e_p^n\| + \|C_h^T \delta_t p_h(t^n)\|)\|C_h^T \delta_t p_h(t^n)\|$$
$$\leq c\delta t^2(\|C_h^T e_p^n\| + \|C_h^T \delta_t p_h(t^n)\|)\delta t(\|u_t\|_{L^\infty(H^2)} + \|p_t\|_{L^\infty(H^1)})$$

$$(4.48) \qquad \leq \delta t^3\|C_h^T e_p^n\|^2 + c\delta t^4.$$

Testing (4.30) with $\delta t e_u^{n+1} \in Y_h$, we get

$$(4.49) \qquad \frac{\rho_0}{2}\|e_u^{n+1}\|^2 + \frac{\rho_0}{2}\|e_u^{n+1} - i_h\tilde{e}_u^{n+1}\|^2 - \frac{\rho_0}{2}\|\tilde{e}_u^{n+1}\|^2 = 0.$$

Summing up $(4.47) + \frac{1}{\rho_0}(4.48) + (4.49)$, we obtain

(4.50)

$$\frac{\lambda}{2}(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2) + \frac{\lambda}{8}\|\delta_t e_\phi^n\|^2 + \frac{\rho_0}{2}(\|e_u^{n+1}\|^2 - \|i_h^T e_u^n\|^2 + \|\tilde{e}_u^{n+1} - i_h^T e_u^n\|^2)$$
$$+ \frac{\lambda+\gamma}{2}(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2) + (\lambda+\gamma)\|\nabla\delta_t e_\phi^n\|^2$$
$$+ \frac{\delta t^2}{2\rho_0}(\|C_h^T e_p^{n+1}\|^2 - \|C_h^T e_p^n\|^2) + \frac{1}{5\delta t}\|\nabla L_h(\delta_t e_\phi^n)\|^2 + \frac{S(\lambda+\gamma)}{\varepsilon^2}\|\delta_t e_\phi^n\|^2$$
$$+ \frac{5\mu_0\delta t}{16}\|\nabla\tilde{e}_u^{n+1}\|^2 + \frac{\lambda\gamma\delta t}{8}\|e_w^{n+1}\|^2 + \frac{5\lambda\gamma\delta t}{8}\|\nabla e_w^{n+1}\|^2$$
$$\leq c\delta t\left[(\delta t + h^l)^2 + \|e_\phi^n\|_1^2 + \|e_\phi^{n+1}\|_1^2 + \|\delta_t e_\phi^n\|^2 + \|\tilde{e}_u^{n+1}\|^2 + \delta t^2\|C_h^T q^n\|^2\right]$$
$$+ \frac{\mu_0\delta t}{16}\|\nabla\tilde{e}_u^n\|^2.$$

Summing up (4.50) for $n = 0, \ldots, m$, and noticing $\|\tilde{e}_u^{n+1}\| \leq \|\tilde{e}_u^{n+1} - i_h^T e_u^n\| + \|i_h^T e_u^n\|$ and $\|e_u^{n+1}\|^2 = \|e_u^{n+1} - i_h i_h^T e_u^{n+1}\|^2 + \|i_h^T e_u^{n+1}\|^2$, we obtain

(4.51)

$$\frac{\lambda}{2}\|e_\phi^{m+1}\|^2 + \frac{\rho_0}{2}\|i_h^T e_u^{m+1}\|^2 + \frac{\lambda+\gamma}{2}\|\nabla e_\phi^{m+1}\|^2 + \frac{\delta t^2}{2\rho_0}\|C_h^T e_p^{m+1}\|^2$$
$$+ \sum_{n=0}^m \left[\frac{\lambda}{8}\|\delta_t e_\phi^n\|^2 + \frac{\rho_0}{2}\|e_u^{n+1} - i_h i_h^T e_u^{n+1}\| + \frac{\rho_0}{2}\|\tilde{e}_u^{n+1} - i_h^T e_u^n\|\right.$$
$$+ (\lambda+\gamma)\|\nabla\delta_t e_\phi^n\|^2 + \frac{1}{5\delta t}\|\nabla L_h(\delta_t e_\phi^n)\|^2$$
$$\left.+ \frac{S(\lambda+\gamma)}{\varepsilon^2}\|\delta_t e_\phi^n\|^2 + \frac{\mu_0\delta t}{4}\|\nabla\tilde{e}_u^{n+1}\|^2 + \frac{\lambda\gamma\delta t}{8}\|e_w^{n+1}\|^2 + \frac{5\lambda\gamma\delta t}{8}\|\nabla e_w^{n+1}\|^2\right]$$
$$\leq c\delta t\sum_{n=0}^m(\|\tilde{e}_u^{n+1} - i_h^T e_u^n\|^2 + \|\delta_t e_\phi^n\|^2)$$
$$+ c(\|e_\phi^0\|_1^2 + \|e_u^0\|^2 + \delta t\|\nabla e_u^0\|^2 + \delta t^2\|C_h^T e_p^0\|^2)$$
$$+ \delta t\sum_{n=0}^m\left(\|e_\phi^n\|^2 + \|\nabla e_\phi^n\|^2 + \|i_h^T e_u^n\|^2 + \delta t^2\|C_h^T e_p^n\|^2\right) + c\,m\,\delta t(\delta t + h^l)^2.$$

Denote

$$I^n = \frac{\lambda}{2}\|e_\phi^{n+1}\|^2 + \frac{\rho_0}{2}\|i_h^T e_u^{n+1}\|^2 + \frac{\lambda+\gamma}{2}\|\nabla e_\phi^{n+1}\|^2 + \frac{\delta t^2}{2\rho_0}\|C_h^T e_p^{n+1}\|^2,$$

$$S^n = \frac{\lambda}{8}\|\delta_t e_\phi^n\|^2 + \frac{S(\lambda+\gamma)}{\varepsilon^2}\|\delta_t e_\phi^n\|^2 + \frac{\rho_0}{2}\|\tilde{e}_u^{n+1} - i_h^T e_u^n\|^2,$$

$$T^n = \frac{1}{5\delta t}\|\nabla L_h(\delta_t e_\phi^n)\|^2 + \frac{\rho_0}{2}\|e_u^{n+1} - i_h i_h^T e_u^{n+1}\|^2 + \frac{\mu_0\delta t}{4}\|\nabla\tilde{e}_u^{n+1}\|^2 + \frac{\lambda\gamma\delta t}{8}\|e_w^{n+1}\|^2$$
$$+ \frac{5\lambda\gamma\delta t}{8}\|\nabla e_w^{n+1}\|^2 + (\lambda+\gamma)\|\nabla\delta_t e_\phi^n\|^2.$$

Then, we have

$$I^m + \sum_{n=0}^m (S^n + T^n) \leq c_1\delta t\sum_{n=0}^m (I^n + S^n) + c_2(\delta t + h^l)^2$$
$$+ c_3(\|e_\phi^0\|_1^2 + \|e_u^0\|^2 + \delta t\|\nabla e_u^0\|^2 + \delta t^2\|C_h^T e_p^0\|^2).$$

Since the initial errors at $t = 0$ satisfy

$$\|e_\phi^0\|^2 + \|\nabla e_\phi^0\|^2 + \|e_u^0\|^2 + \delta t\|\nabla e_u^0\|^2 + \delta t^2\|C_h^T e_p^0\|^2 \lesssim h^{2l} + \delta t^2,$$

by applying the discrete Gronwall inequality, we obtain the following estimates for sufficiently small δt:

$$(4.52) \qquad I^n + \sum_{k=0}^n (S^k + T^k) \lesssim (\delta t + h^l)^2.$$

The desired results then follow from the properties of the interpolates. $\qquad\square$

Remark 4.1. In the proof of Lemma 4.11, we notice that the projection step implies that

$$(4.53) \qquad \rho_0\frac{e_u^{n+1} - e_u^n}{\delta t} + C_h^T(e_p^{n+1} - \tilde{e}_p^n) = \rho_0\frac{i_h\tilde{e}_u^{n+1} - e_u^n}{\delta t}.$$

Taking L^2 norm of both sides, using $C_h e_u^{n+1} = C_h e_u^n = 0$ and the properties of i_h, we find

$$(4.54) \qquad \|e_u^{n+1} - e_u^n\|^2 \leq \|i_h\tilde{e}_u^{n+1} - e_u^n\|^2 = \|\tilde{e}_u^{n+1} - i_h^T e_u^n\|^2 + \|e_u^n - i_h i_h^T e_u^n\|^2,$$

from which we derive that for $n \leq [T/\delta t] - 1$,

$$\|\delta_t e_u^n\| \lesssim \delta t + h^l, \qquad \left(\delta t\sum_{k=0}^{[T/\delta t]-1}\|\delta_t e_u^k\|^2\right)^{1/2} \lesssim \delta t^{1/2}(\delta t + h^l).$$

4.3. **Error on pressure.** As in the case of the pressure-correction scheme for Navier-Stokes equations [14], the key idea to obtain the optimal error estimate on the pressure is to use the inf-sup condition/inverse Stokes operator and improved estimates of the time increments for the errors.

In order to obtain error estimates on $\|e_p^{n+1}\|$, we introduce the following semi-norm for $v \in H^{-1}(\Omega)^d$:

$$(4.55) \qquad |v|_{*,h} = \sup_{v_h\in X_h}\frac{\langle v, v_h\rangle}{\|v_h\|_1}.$$

Denote $H_{\int=0}^1(\Omega) = H^1(\Omega) \cap M$, and we have the L^2-orthogonal decomposition

$$(4.56) \qquad L^2(\Omega)^d = H \oplus \nabla(H_{\int=0}^1).$$

Lemma 4.12. *For $v \in Y_h$ with $C_h v = 0$, there exists constant $c > 0$ independent of h such that*

$$(4.57) \qquad |v|_{*,h} = |i_h^T v|_{*,h} \leq c(\|\nabla \mathcal{S}_h(v)\| + h\|v\|).$$

In particular, if $X_h = Y_h$, for $v \in Y_h$ with $C_h v = 0$, there holds

$$(4.58) \qquad |v|_{*,h} \leq c\|\nabla \mathcal{S}_h(v)\|.$$

Proof. By the definition of $\mathcal{S}_h$ and $\mathcal{S}$, we have for some $p_h \in M_h, p \in M$, $B_h \mathcal{S}_h(v) = 0$ and the following holds:

$$(4.59) \qquad \begin{aligned} (\nabla \mathcal{S}_h(v), \nabla v_h) + (B_h^T p_h, v_h) &= \langle v, v_h \rangle \quad \forall v_h \in X_h, \\ (\nabla \mathcal{S}(v), \nabla \tilde{v}) + (p, \nabla \cdot \tilde{v}) &= \langle v, \tilde{v} \rangle \quad \forall \tilde{v} \in H_0^1(\Omega)^d. \end{aligned}$$

For $v_h \in X_h$, we have $(B_h^T p_h, v_h) = (p_h, \nabla \cdot v_h)$. Using the L^2-decomposition, we set $v_h = P_H v_h + \nabla R_h$ with $R_h \in H_{\int=0}^1$.

(i) If $Y_h \subset H_0^{\text{div}}$. We have

$$\langle v, v_h \rangle = \langle v, P_H v_h \rangle + \langle v, \nabla R_h \rangle = \langle v, P_H v_h \rangle + \langle \nabla \cdot v, R_h - r_h \rangle \quad \forall r_h \in M_h.$$

Making use of the inverse inequality, the H^1-stability of P_H [25], and approximation properties of M_h and properties of S_h and S, we deduce that $\|R_h\|_2 \leq c\|v_h\|_1$, and

$$\begin{aligned} \langle v, v_h \rangle =& \langle v, P_H v_h \rangle + \langle \nabla \cdot v, R_h - r_h \rangle \\ =& (\nabla \mathcal{S}(v), \nabla P_H v_h) + \langle \nabla \cdot v, R_h - r_h \rangle \\ \leq& c(\|\nabla \mathcal{S}_h(v)\| + \|\nabla \mathcal{S}_h(v) - \nabla \mathcal{S}(v)\|)\|\nabla v_h\| + ch^{-1}\|v\|h^2\|R_h\|_2 \\ \leq& c(\|\nabla \mathcal{S}_h(v)\| + h\|\mathcal{S}(v)\|_2)\|\nabla v_h\| + ch^{-1}\|v\|h^2\|R_h\|_2 \\ \leq& c(\|\nabla \mathcal{S}_h(v)\| + h\|v\|)\|\nabla v_h\| + ch^{-1}\|v\|h^2\|R_h\|_2 \\ \leq& c(\|\nabla \mathcal{S}_h(v)\| + h\|v\|)\|v_h\|_1. \end{aligned}$$

(ii) If $M_h \subset H^1$, we have

$$\langle v, v_h \rangle = \langle v, P_H v_h \rangle + \langle v, \nabla R_h \rangle = \langle v, P_H v_h \rangle + \langle v, \nabla(R_h - r_h) \rangle \quad \forall r_h \in M_h.$$

Using the same arguments as above, we derive

$$\begin{aligned} \langle v, v_h \rangle \leq& c\|\nabla \mathcal{S}(v)\|\|\nabla v_h\| + \|v\| \inf_{r_h \in M_h} \|R_h - r_h\|_1 \\ \leq& c(\|\nabla \mathcal{S}_h(v)\| + h\|v\|)\|\nabla v_h\| + c\|v\|h\|R_h\|_2 \\ \leq& c(\|\nabla \mathcal{S}_h(v)\| + h\|v\|)\|v_h\|_1. \end{aligned}$$

Hence, we obtain (4.57).

If $X_h = Y_h$, testing (4.59) with $v \in X_h$, making use of $C_h v = 0$ and the inverse inequality, we have

$$\|v\|^2 = (\nabla \mathcal{S}_h(v), \nabla v) \leq \|\nabla \mathcal{S}_h(v)\| \, \|\nabla v\| \leq c/h\|\nabla \mathcal{S}_h(v)\| \, \|v\|,$$

and so $h\|v\| \leq c\|\nabla \mathcal{S}_h(v)\|$, which would imply the conclusion (4.58) in light of (4.57). □

Lemma 4.13. *Under Assumption* A*, we have the following estimates:*

$$(4.60) \quad \|\nabla \mathcal{S}_h(\delta_t e_u^n)\| \lesssim \delta t^{1/2}(\delta t + h^l), \qquad \left(\delta t \sum_{n=0}^N \|\nabla \mathcal{S}_h(\delta_t e_u^n)\|^2\right)^{1/2} \lesssim \delta t(\delta t + h^l).$$

Proof. Summing i_h^T(4.30) + (4.28), we have

$$(4.61)$$
$$\frac{\rho_0}{\delta t}\left(i_h^T e_u^{n+1} - i_h^T e_u^n\right) + \mu_0 A_h \tilde{e}_u^{n+1} + B_h^T e_p^{n+1} = P_{X_h}\left[\tilde{R}_u^{n+1} + R_{u,u}^{n+1} + R_{u,\phi}^{n+1}\right],$$
$$(4.62)$$
$$C_h e_u^{n+1} = 0.$$

Testing (4.61) with $\mathcal{S}_h(i_h^T e_u^{n+1} - i_h^T e_u^n) = \mathcal{S}_h(e_u^{n+1} - e_u^n) \in X_h$, noticing that $B_h(\mathcal{S}_h(\delta_t e_u^n)) = 0$, we get

$$\rho_0 \frac{\|\nabla \mathcal{S}_h(\delta_t e_u^n)\|^2}{\delta t} + \mu_0(\nabla \tilde{e}_u^{n+1}, \nabla \mathcal{S}_h(\delta_t e_u^n)) = \left(\tilde{R}_u^{n+1} + R_{u,u}^{n+1} + R_{u,\phi}^{n+1}, \mathcal{S}_h(\delta_t e_u^n)\right).$$

Using the same arguments as those in the proof of Lemma 4.11, the RHS can be easily bounded as

$$\left(\tilde{R}_u^{n+1} + R_{u,u}^{n+1} + R_{u,\phi}^{n+1}, \mathcal{S}_h(\delta_t e_u^n)\right) \leq \frac{\rho_0}{4\delta t}\|\nabla \mathcal{S}_h(\delta_t e_u^n)\|^2$$
$$+ c\delta t\left[(\delta t + h^{l+1})^2 + \|e_w^{n+1}\|^2 + \|e_\phi^n\|_1^2 + \|e_w^{n+1}\|_1^2 + \|\tilde{e}_u^n\|_1^2\right].$$

We also have

$$-\mu_0(\nabla \tilde{e}_u^{n+1}, \nabla \mathcal{S}_h(\delta_t e_u^n)) \leq \frac{\rho_0}{4\delta t}\|\nabla \mathcal{S}_h(\delta_t e_u^n)\|^2 + c\delta t\|\tilde{e}_u^{n+1}\|_1^2.$$

Thus, we get

$$\frac{\rho_0\|\nabla \mathcal{S}_h(\delta_t e_u^n)\|^2}{2\delta t} \leq c\delta t(\|e_w^{n+1}\|^2 + \|e_\phi^n\|_1^2 + \|e_w^{n+1}\|_1^2 + \|\tilde{e}_u^n\|_1^2 + \|\tilde{e}_u^{n+1}\|_1^2) + c\delta t(\delta t + h^l)^2.$$

Applying Lemma 4.11, we draw the conclusion. $\qquad\square$

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We only need to bound the error on pressure. By inf-sup condition and (4.61), there holds

$$\|e_p^{n+1}\| \leq c \sup_{v_h \in X_h} \frac{\langle B_h^T e_p^{n+1}, v_h\rangle}{\|v_h\|_1}$$
$$\leq c\left(\frac{1}{\delta t}|i_h^T \delta_t e_u^n|_{*,h} + |A_h \tilde{e}_u^{n+1}|_{*,h} + |\tilde{R}_u^{n+1} + R_{u,u}^{n+1} + R_{u,\phi}^{n+1}|_{*,h}\right).$$

From Remark 4.1, Lemmas 4.12 and 4.13, we obtain

$$\frac{1}{\delta t}|i_h^T \delta_t e_u^n|_{*,h} = \frac{1}{\delta t}|\delta_t e_u^n|_{*,h} \leq c\left(\frac{1}{\delta t}\|\nabla \mathcal{S}_h(\delta_t e_u^n)\| + \frac{h}{\delta t}\|\delta_t e_u^n\|\right) \lesssim \frac{\delta t^{1/2} + h}{\delta t}(\delta t + h^l),$$
$$\left(\delta t \sum_{n=0}^{N-1} \frac{1}{\delta t^2}|i_h^T \delta_t e_u^n|_{*,h}^2\right)^{1/2} \lesssim (1 + h/\sqrt{\delta t})(\delta t + h^l), \quad N = [T/\delta t].$$

From the proof of Lemma 4.11, it can be easily seen that

$$|A_h \tilde{e}_u^{n+1}|_{*,h} + |\tilde{R}_u^{n+1} + R_{u,u}^{n+1} + R_{u,\phi}^{n+1}|_{*,h} \lesssim \frac{1}{\delta t^{1/2}} (\delta t + h^l),$$

$$\left(\delta t \sum_{n=0}^{N-1} (|A_h \tilde{e}_u^{n+1}|_{*,h}^2 + |\tilde{R}_u^{n+1} + R_{u,u}^{n+1} + R_{u,\phi}^{n+1}|_{*,h}^2) \right)^{1/2} \lesssim \delta t + h^l.$$

Then it follows that

$$\|e_{p,\delta t}\|_{l^2(L^2(\Omega))} \lesssim \left(\delta t + h^l \right) \left(1 + h/\sqrt{\delta t} \right).$$

Combining with the estimates for the interpolate and under the condition $h^2 \lesssim \delta t$, we obtain

$$\|p - p_{h,\delta t}\|_{l^2(L^2(\Omega))} \lesssim \delta t + h^l.$$

The proof is complete. $\qquad \qquad \square$

Remark 4.2. In this approach, since there is no particular assumption on the relation between finite-element spaces X_h and Y_h except $X_h \subset Y_h$, the use of the inverse Stokes operator will result in the type of estimates in Lemma 4.12. It is noticed that if $X_h = Y_h$, then $h^2 \lesssim \delta t$ is unnecessary. It is possible to study the time increment $\delta_t e_u^n$ and establish higher order convergence for $\delta_t e_u^n$ as in [11], instead of using the discrete inverse Stokes operator.

The same proof works for the Allen-Cahn phase-field equations, but we will have L^2-norm of e_w^{n+1} instead of the H^1-norm of e_w^{n+1}.

5. CONVERGENCE ANALYSIS

We derived error estimates of the fully discrete scheme (3.1)-(3.5) under appropriate regularity assumptions of the exact solution. Now we show below that the numerical solution of (3.1)-(3.5) converges to the exact solution of NSCH system (1.1)-(1.5) without such assumptions, when $h, \delta t \to 0$. There are several challenges towards such an analysis. In particular, the choices of the finite-element spaces are rather general, where $X_h \subset Y_h$ may not be identical, and this brings significant difficulties when analyzing the velocity splitting scheme considered here. In the following arguments, when $X_h \neq Y_h$, we will assume the technical condition $h^2 \lesssim \delta t$.

From Remark 3.1 and the stability result Theorem 3.1, we could obtain the following a priori bounds on the numerical solution

$$\{(\phi_h^{n+1}, w_h^{n+1}, u_h^{n+1}, \tilde{u}_h^{n+1}, p_h^{n+1})\}_{n=0}^{N-1}.$$

Lemma 5.1. *Given the initialization* $(\phi_h^0, w_h^0, u_h^0, p_h^0)^T$ *and* $\tilde{u}_h^0 = u_h^0$, *there exists a unique solution* $\{(\phi_h^{n+1}, w_h^{n+1}, u_h^{n+1}, \tilde{u}_h^{n+1}, p_h^{n+1})\}_{n=0}^{N-1} \in \Psi_h \times \Psi_h \times Y_h \times X_h \times M_h$ *to the fully discrete scheme* (3.1)-(3.5). *Assume* $(\phi_h^0, w_h^0, u_h^0, p_h^0)^T$ *is initialized such*

that $\frac{\rho_0}{2}\|u_h^0\|^2 + \frac{\lambda}{2}\|\phi_h^0\|^2 + \lambda(F(\phi_h^0),1) + \frac{\delta t^2}{\rho_0}\|C_h^T p_h^0\|^2 \leq c_0$, *the numerical approximation satisfies the following estimates:*

$$(5.1) \qquad \max_{0\leq n\leq N}\left\{\|u_h^n\|^2 + \|\nabla\phi_h^n\|^2 + (F(\phi_h^n),1) + \delta t^2\|C_h^T p_h^n\|^2\right\} \leq c,$$

$$(5.2)$$
$$\sum_{n=0}^{N-1}\left(\|\delta_t\phi_h^n\|_1^2 + \|\tilde{u}_h^{n+1} - i_h^T u_h^n\|^2 + \|u_h^{n+1} - i_h i_h^T u_h^{n+1}\|^2 + \|\delta_t u_h^n\|^2 + \|\delta_t\tilde{u}_h^n\|^2\right) \leq c,$$

$$(5.3) \qquad \delta t\sum_{n=0}^{N}\left(\|\nabla w_h^n\|^2 + \|\nabla\tilde{u}_h^n\|^2\right) \leq c,$$

$$(5.4) \qquad \max_{0\leq n\leq N}\left\{\|\phi_h^n\|^2 + \|\tilde{u}_h^n\|^2\right\} + \delta t\sum_{n=0}^{N}\|w_h^n\|^2 \leq c,$$

$$(5.5) \qquad \delta t\sum_{n=0}^{N-1}\left\|\frac{1}{\delta t}\delta_t i_h^T u_h^n\right\|_{-1}^{\frac{12}{6+d}} \leq c\left(1 + \left(\frac{h^2}{\delta t}\right)^{\frac{6}{d}}\right),$$

$$(5.6) \qquad \delta t\sum_{n=0}^{N}\|p_h^n\|^{\frac{12}{6+d}} \leq c\left(1 + \left(\frac{h^2}{\delta t}\right)^{\frac{6}{d}}\right),$$

$$(5.7) \qquad \delta t\sum_{n=0}^{N-1}\left\|\frac{1}{\delta t}\delta_t\tilde{u}_h^n\right\|_{-1}^{\frac{12}{6+d}} \leq c\left(1 + \left(\frac{h^2}{\delta t}\right)^{\frac{6}{d}}\right),$$

$$(5.8) \qquad \delta t\sum_{n=0}^{N-1}\left\|\frac{1}{\delta t}\delta_t\phi_h^n\right\|_{-1}^2 \leq c,$$

where the constant c depends on c_0, ρ_0, λ and γ. In particular, if Y_h is chosen as $Y_h = X_h$, then the estimates (5.5)-(5.7) become

$$(5.9) \qquad \frac{1}{\delta t}\sum_{n=0}^{N-1}\|\delta_t i_h^T u_h^n\|_{-1}^{\frac{12}{6+d}} + \frac{1}{\delta t}\sum_{n=0}^{N-1}\|\delta_t\tilde{u}_h^n\|_{-1}^{\frac{12}{6+d}} + \delta t\sum_{n=0}^{N}\|p_h^n\|^{\frac{12}{6+d}} \leq c.$$

Proof. (5.1)-(5.3) are direct consequences of the proof in Theorem 3.1 except those estimates on $\delta_t u_h^n$ and $\delta_t\tilde{u}_h^n$. Indeed, using the arguments in Remark 4.1, we find from (3.4) that

$$\frac{\rho_0}{\delta t}(u_h^{n+1} - u_h^n) + C_h^T(p_h^{n+1} - p_h^n) = \frac{\rho_0}{\delta t}(i_h\tilde{u}_h^{n+1} - u_h^n).$$

Since $C_h(u_h^{n+1} - u_h^n) = 0$, taking the L^2-norm on both sides, using triangle inequality, and the fact that i_h^T is an L^2-orthogonal projection, we get

$$\|u_h^{n+1} - u_h^n\| \leq \|i_h\tilde{u}_h^{n+1} - u_h^n\| \leq \|u_h^n - i_h i_h^T u_h^n\| + \|\tilde{u}_h^{n+1} - i_h^T u_h^n\|, \quad 0 \leq n \leq N-1,$$

and the estimates of $\|\delta_t u_h^n\|$ are implied. For $\delta_t\tilde{u}_h^n$, $\delta_t\tilde{u}_h^0 = \tilde{u}_h^1 - i_h^T u_h^0$ ($\tilde{u}_h^0 = i_h^T u_h^0$), for $n \geq 1$, we get

$$\|\delta_t\tilde{u}_h^n\| \leq \|i_h^T\delta_t u_h^{n-1}\| + \|\tilde{u}_h^n - i_h^T u_h^{n-1}\| + \|\tilde{u}_h^{n+1} - i_h^T u_h^n\|.$$

Hence, we derive the estimates on $\|\delta_t\tilde{u}_h^n\|$ in (5.2).

For (5.4), the potential term $F(\cdot)$ (2.10) implies that $s^2 \leq (1 + 2\varepsilon^2)(F(s) + 1)$ ($s \in \mathbb{R}$) and

$$\|\phi_h^n\|^2 \leq c\left((F(\phi_h^n),1) + |\Omega|\right),$$

where $|\Omega|$ denotes the total volume of Ω. The estimates on $\|\phi_h^n\|$ hold. Testing (3.1) with $\delta t \phi_h^{n+1} \in \Psi_h$ and using previous bounds (5.1)-(5.3), we arrive at

$$\frac{1}{2}\left(\|\phi_h^{n+1}\|^2 + \|\phi_h^{n+1} - \phi_h^n\|^2\right) - \frac{1}{2}\|\phi_h^n\|^2$$
$$= \delta t\left((\tilde{u}_h^{n+1} \cdot \nabla)\phi_h^n, \phi_h^{n+1}\right) - \gamma(\nabla w_h^{n+1}, \nabla \phi_h^{n+1})$$
$$\le c\delta t\left(\|\tilde{u}_h^{n+1}\|_{L^6}\|\nabla\phi_h^n\|\|\phi_h^{n+1}\|_1 + \|\nabla w_h^{n+1}\|\|\nabla\phi_h^{n+1}\|\right)$$
$$\le c\delta t\left(\|\nabla\tilde{u}_h^{n+1}\|^2 + \|\phi_h^{n+1}\|_1^2 + \|\nabla w_h^{n+1}\|^2\right),$$

which leads to the estimates for $\|\delta_t\phi_h^n\|$.

Testing (3.2) with $w_h \in \Psi_h$ and applying the Cauchy inequality, we obtain

$$\|w_h^{n+1}\|^2 = \frac{S}{\varepsilon^2}\left(\delta_t\phi_h^n, w_h^{n+1}\right) + \left(\nabla\phi_h^{n+1}, \nabla w_h^{n+1}\right) + (f(\phi_h^n), w_h^{n+1})$$
$$\le \frac{1}{2}\|w_h^{n+1}\|^2 + c\left(\|\delta_t\phi_h^n\|^2 + \|f(\phi_h^n)\|^2\right) + \|\nabla\phi_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2$$
$$\le \frac{1}{2}\|w_h^{n+1}\|^2 + c\left(\|\delta_t\phi_h^n\|^2 + \|\phi_h^n\|^2 + 1\right) + \|\nabla\phi_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2,$$

and the estimates of w_h^n in (5.4) are true by using (5.1)-(5.3).

Next, we prove (5.5). For $i_h^T\delta_t u_h^n \in X_h$, using Lemma 4.12 and H^1-stability of P_{X_h}, we have

$$\|i_h^T\delta_t u_h^n\|_{-1} = \sup_{0 \ne v \in H_0^1(\Omega)^d} \frac{\langle i_h^T\delta_t u_h^n, v\rangle}{\|v\|_1} = \sup_{0 \ne v \in H_0^1(\Omega)^d} \frac{(i_h^T\delta_t u_h^n, P_{X_h}v)}{\|P_{X_h}v\|_1} \cdot \frac{\|P_{X_h}v\|_1}{\|v\|_1}$$
$$(5.10) \qquad\qquad \le c|i_h^T\delta_t u_h^n|_{*,h} \le c(\|\nabla\mathcal{S}_h(i_h^T\delta_t u_h^n)\| + h\|i_h^T\delta_t u_h^n\|),$$

where the estimates of $\|i_h^T\delta_t u_h^n\|$ can be found in (5.1)-(5.2) as

$$(5.11) \qquad\qquad \sum_{n=0}^{N-1}\|i_h^T\delta_t u_h^n\|^2 \le \sum_{n=0}^{N-1}\|\delta_t u_h^n\|^2 \le c.$$

We only need to estimate $\|\nabla\mathcal{S}_h(i_h^T\delta_t u_h^n)\|$. Applying i_h^T to (3.4) and adding it to (3.3), we obtain
(5.12)
$$\frac{\rho_0}{\delta t}i_h^T\delta_t u_h^n + \rho_0 P_{X_h}\left(D_h(\tilde{u}_h^n, \tilde{u}_h^{n+1})\right) + \mu_0 A_h\tilde{u}_h^{n+1} + B_h^T p_h^{n+1} - \lambda P_{X_h}\left(w_h^{n+1}\nabla\phi_h^n\right) = 0.$$

Testing (5.12) with $v_h = \mathcal{S}_h(\delta_t u_h^n) = \mathcal{S}_h(i_h^T\delta_t u_h^n) \in X_h$, noticing (5.1)-(5.4), we derive

$$\frac{1}{\delta t}\|\nabla v_h\|^2 = -\frac{\mu_0}{\rho_0}\left(\nabla\tilde{u}_h^{n+1}, \nabla v_h\right) - d(\tilde{u}_h^n, \tilde{u}_h^{n+1}, v_h) + \left(w_h^{n+1}\nabla\phi_h^n, v_h\right)$$
$$\le \frac{\mu_0}{\rho_0}\|\nabla\tilde{u}_h^{n+1}\|\,\|\nabla v_h\| + \left(\|\tilde{u}_h^n\|_{L^3}\|\nabla\tilde{u}_h^{n+1}\| + \frac{1}{2}\|\nabla\tilde{u}_h^n\|\|\tilde{u}_h^{n+1}\|_{L^3}\right)\|v_h\|_{L^6}$$
$$+ \|w_h^n\|_{L^3}\|\nabla\phi_h^n\|\|v_h\|_{L^6}$$
$$\le c\left(\|\nabla\tilde{u}_h^{n+1}\| + \|w_h^n\|_1\right)\|\nabla v_h\| + c\|\tilde{u}_h^n\|^{\frac{6-d}{6}}\|\nabla\tilde{u}_h^n\|^{\frac{d}{6}}\|\nabla\tilde{u}_h^{n+1}\|\|\nabla v_h\|$$
$$+ c\|\tilde{u}_h^{n+1}\|^{\frac{6-d}{6}}\|\nabla\tilde{u}_h^{n+1}\|^{\frac{d}{6}}\|\nabla\tilde{u}_h^n\|\|\nabla v_h\|$$

and

$$\frac{1}{\delta t}\|\nabla v_h\| \le c\left(\|\nabla\tilde{u}_h^{n+1}\| + \|w_h^n\|_1 + \|\nabla\tilde{u}_h^n\|^{\frac{d}{6}}\|\nabla\tilde{u}_h^{n+1}\| + \|\nabla\tilde{u}_h^{n+1}\|^{\frac{d}{6}}\|\nabla\tilde{u}_h^n\|\right).$$

Combining the above estimates with Young's inequality, (5.1)-(5.3), (5.11) and (5.10), we have

$$\delta t \sum_{n=0}^{N-1} \left\| \frac{1}{\delta t} i_h^T \delta_t u_h^n \right\|_{-1}^{\frac{12}{6+d}} \leq c + \delta t \sum_{n=0}^{N-1} \left\| \frac{h}{\delta t} i_h^T \delta_t u_h^n \right\|^{\frac{12}{6+d}} \leq c \left(1 + \left(\frac{h^2}{\delta t} \right)^{\frac{6}{d}} \right).$$

Next, we prove the estimates on the pressure p_h^n. Using inf-sup condition (2.25) and similar arguments above as well as those in subsection 4.3 for the pressure error estimates, from (5.12), we have for $n = 0, \ldots, N-1$,

$$\|p_h^{n+1}\| \lesssim \left\| \frac{1}{\delta t} i_h^T \delta_t u_h^n \right\|_{-1}$$
$$+ \left(\|\nabla \tilde{u}_h^{n+1}\| + \|w_h^n\|_1 + \|\nabla \tilde{u}_h^n\|^{\frac{d}{6}} \|\nabla \tilde{u}_h^{n+1}\| + \|\nabla \tilde{u}_h^{n+1}\|^{\frac{d}{6}} \|\nabla \tilde{u}_h^n\| \right),$$

and (5.6) follows.

In a similar way, we can prove (5.7) for $\delta_t \tilde{u}_h^n$. Noticing that $\tilde{u}_h^0 = u_h^0$ by initialization and for $n \geq 1$, applying i_h^T to (3.4) with time step $n-1$ and adding it to (3.3) to eliminate $i_h^T u_h^n$, we have

$$(5.13) \qquad \frac{\rho_0}{\delta t} \delta_t \tilde{u}_h^n + \rho_0 P_{X_h} \left(D_h(\tilde{u}_h^n, \tilde{u}_h^{n+1}) \right) + \mu_0 A_h \tilde{u}_h^{n+1} + B_h^T (2p_h^n - p_h^{n-1})$$
$$- \lambda P_{X_h} \left(w_h^{n+1} \nabla \phi_h^n \right) = 0.$$

We then control

$$\left\| \frac{1}{\delta t} \delta_t \tilde{u}_h^n \right\|_{-1}$$
$$\lesssim \left(\|\nabla \tilde{u}_h^{n+1}\| + \|w_h^n\|_1 + \|\nabla \tilde{u}_h^n\|^{\frac{d}{6}} \|\nabla \tilde{u}_h^{n+1}\| + \|\nabla \tilde{u}_h^{n+1}\|^{\frac{d}{6}} \|\nabla \tilde{u}_h^n\| + \|2p_h^n - p_h^{n-1}\| \right),$$

and the estimates on $\|\delta_t \tilde{u}_h^n\|_{-1}$ can be derived.

Last, for any $\varphi \in H_0^1$, testing (3.1) with $\varphi_h = P_{\Psi_h} \varphi \in \Psi_h$, we get

$$\frac{1}{\delta t} \left(\delta_t \phi_h^n, \varphi \right) = - \left((\tilde{u}_h^{n+1} \cdot \nabla) \phi_h^n, \varphi_h \right) + \gamma \left(\nabla w_h^{n+1}, \nabla \varphi_h \right)$$
$$\leq c \left(\|\tilde{u}_h^{n+1}\|_{L^3} \|\nabla \phi_h^n\| \|\varphi_h\|_{L^6} + \|\nabla w_h^{n+1}\| \|\nabla \varphi_h\| \right)$$
$$\leq c \left(\|\nabla \tilde{u}_h^{n+1}\| + \|\nabla w_h^{n+1}\| \right) \|\nabla \varphi_h\|,$$

and we can derive (5.8) since P_{Ψ_h} is H^1 stable. $\qquad \square$

For the fully discrete numerical solution $\{(\phi_h^n, w_h^n, u_h^n, \tilde{u}_h^n, p_h^n)\}_{n=0}^N \in \Psi_h \times \Psi_h \times Y_h \times X_h \times M_h$, define the linear interpolation $(U_{h,\delta t}(x,t), \Phi_{h,\delta t}(x,t))$ of $\{(\tilde{u}_h^n, \phi_h^n)\}_{n=0}^N$ in time as

$$(5.14)$$
$$U_{h,\delta t}(\cdot, t) = \tilde{u}_h^n + \frac{t - t_n}{\delta t}(\tilde{u}_h^{n+1} - \tilde{u}_h^n), \quad \Phi_{h,\delta t}(\cdot, t) = \phi_h^n + \frac{t - t_n}{\delta t}(\phi_h^{n+1} - \phi_h^n),$$

for $t \in [t_n, t_{n+1}]$, $n = 0, 1, \ldots, N-1$. We also denote $\overline{P}_{h,\delta t}(x,t)$, $\overline{U}_{h,\delta t}(x,t)$, $\overline{\Phi}_{h,\delta t}(x,t)$, $\overline{W}_{h,\delta t}(x,t)$, and $\widetilde{U}_{h,\delta t}(x,t)$ as the piecewise constant extensions of

$\{p_h^n\}_{n=0}^N$, $\{u_h^n\}_{n=0}^N$, $\{\phi_h^n\}_{n=0}^N$, $\{w_h^n\}_{n=0}^N$ and $\{\tilde{u}_h^n\}_{n=0}^N$, respectively, i.e.,

$$(5.15) \qquad \overline{P}_{h,\delta t}(\cdot, t) := p_h^n, \quad t \in [t_n, t_{n+1}); \quad \overline{U}_{h,\delta t}(\cdot, t) := u_h^n, \quad t \in [t_n, t_{n+1});$$

$$(5.16) \qquad \overline{\Phi}_{h,\delta t}(\cdot, t) := \phi_h^n, \quad t \in [t_n, t_{n+1}); \quad \overline{W}_{h,\delta t}(\cdot, t) := w_h^n, \quad t \in [t_n, t_{n+1});$$

$$(5.17) \qquad \widetilde{U}_{h,\delta t}(\cdot, t) := \tilde{u}_h^n, \quad t \in [t_n, t_{n+1}), \quad n = 0, 1, \ldots, N-1.$$

Theorem 5.1. *Assume initial data $(\phi_0, u_0) \in H^1(\Omega) \times V$ and $(\phi_h^0, w_h^0, u_h^0, p_h^0)^T$ is initialized such that $\frac{\rho_0}{2}\|u_h^0\|^2 + \frac{\lambda}{2}\|\phi_h^0\|^2 + \lambda(F(\phi_h^0), 1) + \frac{\delta t^2}{\rho_0}\|C_h^T p_h^0\|^2 \le c_0$. Furthermore, we require $h^2 \lesssim \delta t$ if $Y_h \ne X_h$. Let (u, ϕ, w, p) denote the unique solution of NSCH system (1.1)-(1.5), and let $\left(\overline{\Phi}_{h,\delta t}, \overline{W}_{h,\delta t}, \overline{U}_{h,\delta t}, \widetilde{U}_{h,\delta t}, \overline{P}_{h,\delta t}\right)$ be defined as above. Then there holds*

$$(5.18) \quad \lim_{h,\delta t \to 0} \left(\|\overline{U}_{h,\delta t} - u\|_{L^2(L^2)} + \|\widetilde{U}_{h,\delta t} - u\|_{L^2(L^2)} + \|\overline{\Phi}_{h,\delta t} - \phi\|_{L^2(L^2)} \right) = 0,$$

$$(5.19) \qquad\qquad \overline{W}_{h,\delta t} \to w, \quad \text{weakly} \quad \text{in} \quad L^2([0.T]; H^1(\Omega)),$$

$$(5.20) \qquad\qquad \overline{P}_{h,\delta t} \to p, \quad \text{weakly} \quad \text{in} \quad L^{\frac{12}{6+d}}([0,T]; L^2(\Omega)).$$

Proof. We only sketch the proof below as the procedure is similar to that in [4]. Moreover, we work with the case $X_h \ne Y_h$, as the arguments are almost the same for the $X_h = Y_h$ case but without the condition $h^2 \lesssim \delta t$.

Step 1. First, we collect some a priori estimates and extract convergent subsequences. From Lemma 5.1, we know

$$\|\Phi_{h,\delta t}\|_{L^\infty(H^1)} + \|\partial_t \Phi_{h,\delta t}\|_{L^2(H^{-1})} + \|\overline{W}_{h,\delta t}\|_{L^2(H^1)} + \|F(\overline{\Phi}_{h,\delta t})\|_{L^\infty(L^1)} \le C,$$

$$\|\partial_t U_{h,\delta t}\|_{L^{\frac{12}{6+d}}(H^{-1})} + \|U_{h,\delta t}\|_{L^\infty(L^2)} + \|U_{h,\delta t}\|_{L^2(H^1)} \le C\left(1 + \left(\frac{h^2}{\delta t}\right)^{\frac{6+d}{2d}}\right),$$

$$\|\overline{P}_{h,\delta t}\|_{L^{\frac{12}{6+d}}(L^2)} \le C\left(1 + \left(\frac{h^2}{\delta t}\right)^{\frac{6+d}{2d}}\right).$$

We can extract convergent subsequences $\left\{(\Phi_{h,\delta t}, W_{h,\delta t}, \tilde{U}_{h,\delta t}, \overline{P}_{h,\delta t})\right\}$ (denoted as the original one for simplicity) and find a quadruple (ϕ_*, w_*, u_*, p_*) such that

$$\phi_* \in L^\infty([0,T]; H^1(\Omega)) \cap H^1([0,T]; H^{-1}(\Omega)), \qquad w_* \in L^2([0,T]; H^1(\Omega)),$$

$$u_* \in L^\infty([0,T]; L^2(\Omega)^d) \cap L^2([0,T]; H^1(\Omega)^d) \cap W^{1,12/6+d}([0,T]; H^{-1}(\Omega)^d),$$

$$p_* \in L^{12/6+d}([0,T]; L^2(\Omega)),$$

and

$$(5.21) \qquad\qquad \Phi_{h,\delta t} \overset{h,\delta t \searrow 0}{\longrightarrow} \phi_*, \quad \text{weakly} \star \text{ in } L^\infty([0,T]; H^1(\Omega)),$$

$$\text{weakly in } H^1([0,T]; H^{-1}(\Omega)),$$

$$\text{strongly in } L^2([0,T]; L^2(\Omega)),$$

$$(5.22) \qquad\qquad \overline{W}_{h,\delta t} \overset{h,\delta t \searrow 0}{\longrightarrow} w_*, \quad \text{weakly in } L^2([0,T]; H^1(\Omega)),$$

$$(5.23) \qquad U_{h,\delta t} \overset{h,\delta t \searrow 0}{\longrightarrow} u_*, \quad \text{weakly} \star \text{ in } L^\infty([0,T]; L^2(\Omega)^d),$$
$$\text{weakly in } L^2([0,T]; H^1(\Omega)^d),$$
$$\text{weakly in } W^{1,12/6+d}([0,T]; H^{-1}(\Omega)^d),$$
$$\text{strongly in } L^2([0,T]; L^2(\Omega)^d),$$

$$(5.24) \qquad \overline{P}_{h,\delta t} \overset{h,\delta t \searrow 0}{\longrightarrow} p_*, \quad \text{weakly in } L^{12/6+d}([0,T]; L^2(\Omega)).$$

Furthermore, recalling Lemma 5.1, we have

$$\|\Phi_{h,\delta t} - \overline{\Phi}_{h,\delta t}\|^2_{L^2(H^1)} = \sum_{n=1}^N \|\phi_h^n - \phi_h^{n-1}\|^2_{H^1} \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^2}{\delta t} \, dt$$
$$= \frac{\delta t}{3} \sum_{n=1}^N \|\phi_h^n - \phi_h^{n-1}\|^2_{H^1} \overset{h,\delta t \searrow 0}{\longrightarrow} 0,$$

$$\|U_{h,\delta t} - \widetilde{U}_{h,\delta t}\|^2_{L^2(L^2)} = \frac{\delta t}{3} \sum_{n=1}^N \|\tilde{u}_h^n - \tilde{u}_h^{n-1}\|^2_{L^2} \overset{h,\delta t \searrow 0}{\longrightarrow} 0,$$

$$\|\overline{U}_{h,\delta t} - \widetilde{U}_{h,\delta t}\|^2_{L^2(L^2)} = \delta t \sum_{n=1}^N \|u_h^{n-1} - \tilde{u}_h^{n-1}\|^2_{L^2} \overset{h,\delta t \searrow 0}{\longrightarrow} 0.$$

We then conclude from the above computations and the estimates in Lemma 5.1 that $\overline{\Phi}_{h,\delta t}$ converge to ϕ_* strongly in $L^2([0,T]; L^2(\Omega))$, and $\overline{\Phi}_{h,\delta t}$ converges to ϕ_* weakly $\star$ in $L^\infty([0,T]; H^1(\Omega))$; $\overline{U}_{h,\delta t}$ and $\widetilde{U}_{h,\delta t}$ converge to u_* strongly in $L^2([0,T]; L^2(\Omega)^d)$, and $\widetilde{U}_{h,\delta t}$ converges to u_* weakly in $L^2([0,T]; H^1(\Omega)^d)$.

Step 2. Now, we want to pass to limits as $h, \delta t \to 0$ in (3.1)-(3.5) and show that (ϕ_*, w_*, u_*, p_*) is a weak solution of (2.15)-(2.16). Applying i_h^T to (3.4) with time step $n-1$, adding it to (3.3) and denoting $p_h^{-1} = p_h^0$, we have

$$\frac{\rho_0}{\delta t} \delta_t \tilde{u}_h^n + \rho_0 P_{X_h}\left(D_h(\tilde{u}_h^n, \tilde{u}_h^{n+1})\right) + \mu_0 A_h \tilde{u}_h^{n+1} + B_h^T(2p_h^n - p_h^{n-1})$$
$$- \lambda P_{X_h}\left(w_h^{n+1}\nabla\phi_h^n\right) = 0.$$

Recalling (3.14), (3.1)-(3.5) can be written equivalently for $t \in [t_n, t_{n+1})$ as

(5.25)
$$(\partial_t \Phi_{h,\delta t}(t), \psi_h) + \left(\widetilde{U}_{h,\delta t}(t+\delta t) \cdot \nabla\overline{\Phi}_{h,\delta t}(t), \psi_h\right) + \gamma\left(\nabla\overline{W}_{h,\delta t}(t+\delta t), \nabla\psi_h\right) = 0,$$

(5.26)
$$\left(\overline{W}_{h,\delta t}(t+\delta t), \varphi_h\right) - \frac{S}{\varepsilon^2}\left(\overline{\Phi}_{h,\delta t}(t+\delta t) - \overline{\Phi}_{h,\delta t}(t), \varphi_h\right) - \left(\nabla\overline{\Phi}_{h,\delta t}(t), \nabla\varphi_h\right)$$
$$= \left(f(\overline{\Phi}_{h,\delta t}(t)), \varphi_h\right),$$

(5.27)
$$\rho_0\left(\partial_t U_{h,\delta t}(t), v_h\right) + \rho_0 d(\widetilde{U}_{h,\delta t}(t), \widetilde{U}_{h,\delta t}(t+\delta t), v_h) + \mu_0\left(\nabla\widetilde{U}_{h,\delta t}(t+\delta t), \nabla v_h\right)$$
$$+ \left((2\overline{P}_{h,\delta t}(t) - \overline{P}_{h,\delta t}(t-\delta t)), \nabla v_h\right) - \lambda\left(\overline{W}_{h,\delta t}(t+\delta t)\nabla\overline{\Phi}_{h,\delta t}(t), v_h\right) = 0,$$

with

$$(5.28) \quad (\nabla \cdot U_{h,\delta t}, q_h) = -\frac{1}{\rho_0} \left(\overline{P}^C_{h,\delta t}(t + \delta t) - \overline{P}^C_{h,\delta t}(t), \nabla q_h \right), \quad \text{if } M_h \subset H^1(\Omega),$$

$$(5.29) \quad (\nabla \cdot U_{h,\delta t}, q_h) = \frac{1}{\rho_0} \left(\nabla \cdot (\overline{P}^C_{h,\delta t}(t + \delta t) - \overline{P}^C_{h,\delta t}(t)), q_h \right), \quad \text{if } Y_h \subset H_0^{\text{div}}(\Omega),$$

for all $(\psi_h, \varphi_h, v_h, q_h) \in \Psi_h \times \Psi_h \times X_h \times M_h$, where $\overline{P}^C_{h,\delta t}$ denotes the piecewise extension of $\delta t C_h^T p_h^n$. From Lemma 5.1, it is easy to see there exists $p_*^c \in L^2([0,T]; L^2(\Omega)^d)$ such that a subsequence $\overline{P}^C_{h,\delta t} \to p_*^c$ weakly in $L^2([0,T]; L^2(\Omega)^d)$ (denoted as the original sequence for simplicity) as $h, \delta t \to 0$.

For any $\eta(t) \in C[0,T]$, multiplying (5.25)-(5.29) by $\eta(t)$ and integrate from 0 to T, we have

$$\int_0^T \left[(\partial_t \Phi_{h,\delta t}(t), \psi_h) + (\widetilde{U}_{h,\delta t}(t+\delta t) \cdot \nabla \overline{\Phi}_{h,\delta t}(t), \psi_h) + \gamma (\nabla \overline{W}_{h,\delta t}(t+\delta t), \nabla \psi_h) \right] \eta(t) \, dt$$
$$= 0,$$

$$\int_0^T \left[(\overline{W}_{h,\delta t}(t+\delta t), \varphi_h) - \frac{S}{\varepsilon^2} \left(\overline{\Phi}_{h,\delta t}(t+\delta t) - \overline{\Phi}_{h,\delta t}(t), \varphi_h \right) - \left(\nabla \overline{\Phi}_{h,\delta t}(t), \nabla \varphi_h \right) \right] \eta(t) \, dt$$
$$= \int_0^T \left(f(\overline{\Phi}_{h,\delta t}(t)), \varphi_h \right) \eta(t) \, dt,$$

$$\int_0^T \left[\rho_0 \left(\partial_t U_{h,\delta t}(t), v_h \right) + \rho_0 d(\widetilde{U}_{h,\delta t}(t), \widetilde{U}_{h,\delta t}(t+\delta t), v_h) + \mu_0 \left(\nabla \widetilde{U}_{h,\delta t}(t+\delta t), \nabla v_h \right) \right] \eta(t) \, dt$$
$$+ \int_0^T \left[\left((2\overline{P}_{h,\delta t}(t) - \overline{P}_{h,\delta t}(t - \delta t)), \nabla v_h \right) - \lambda \left(\overline{W}_{h,\delta t}(t+\delta t) \nabla \overline{\Phi}_{h,\delta t}(t), v_h \right) \right] \eta(t) \, dt$$
$$= 0,$$

and if $M_h \subset H^1(\Omega)$,

$$(5.30)$$
$$\int_0^T \left(\nabla \cdot U_{h,\delta t}, q_h \right) \eta(t) \, dt = -\frac{1}{\rho_0} \int_0^T \left(\overline{P}^C_{h,\delta t}(t + \delta t) - \overline{P}^C_{h,\delta t}(t), \nabla q_h \right) \eta(t) \, dt,$$

or if $Y_h \subset H_0^{\text{div}}(\Omega)$,

$$(5.31)$$
$$\int_0^T \left(\nabla \cdot U_{h,\delta t}, q_h \right) \eta(t) \, dt = \frac{1}{\rho_0} \int_0^T \left(\nabla \cdot (\overline{P}^C_{h,\delta t}(t + \delta t) - \overline{P}^C_{h,\delta t}(t)), q_h \right) \eta(t) \, dt.$$

For any $(\psi, \varphi, v) \in H^1(\Omega) \times H^1(\Omega) \times X$, choose $(\psi_h, \varphi_h, v_h) \in \Psi_h \times \Psi_h \times X_h$ such that when $h \to 0$,

$$(5.32) \quad \psi_h \to \psi \quad \text{strongly} \quad \text{in} \quad H^1(\Omega); \quad \varphi_h \to \varphi \quad \text{strongly} \quad \text{in} \quad H^1(\Omega);$$

$$(5.33) \quad v_h \to v \quad \text{strongly} \quad \text{in} \quad X = H_0^1(\Omega)^d.$$

Since $U_{h,\delta t} \to u_*$ strongly in $L^2([0,T]; L^p(\Omega)^d)$ $(1 < p < \frac{2d}{d-2})$ and $\Phi_{h,\delta t} \to \phi_*$ strongly in $C([0,T]; L^p(\Omega))$ $(1 < p < \frac{2d}{d-2})$, combing with the results in *Step* 1 and

following [4], we can conclude that $u_*(t=0) = u_0$, $\phi_*(t=0) = \phi_0$,

$$\int_0^T \left[\langle \partial_t \phi_*, \psi \rangle + (u_* \cdot \nabla \phi_*, \psi) - \gamma (\nabla w_*, \nabla \psi) \right] \eta(t)\, dt = 0,$$

$$\int_0^T \left[(w_*, \varphi) - (\nabla \phi_*, \nabla \varphi) \right] \eta(t)\, dt = \int_0^T (f(\phi_*), \varphi)\, \eta(t)\, dt,$$

$$\int_0^T \left[\rho_0 \langle \partial_t u_*, v \rangle + \rho_0 d(u_*, u_*, v) + \mu_0 (\nabla u_*, \nabla v) + (p_*, \nabla v) - \lambda (w_* \nabla \phi_*, v) \right] \eta(t)\, dt = 0.$$

Thus (ϕ_*, w_*, u_*, p_*) satisfies (2.15)-(2.17) except the divergence free equation (2.18) in view of the fact that $C[0, T]$ is dense in $L^2[0, T]$. Next, we prove that u_* actually satisfies (2.18) and separates the proof into two cases.

(1) When $M_h \subset H^1(\Omega)$, for any $q \in H^1(\Omega) \cap L_0^2(\Omega)$, we can choose $q_h \in M_h$ such that $q_h \to q$ strongly in H^1. Then passing to the limit in (5.30) as $h, \delta t \to 0$, we have $\overline{P}_{h,\delta t}^C(t + \delta t) - \overline{P}_{h,\delta t}^C(t) \longrightarrow 0$ weakly in $L^2([0, T]; L^2(\Omega)^d)$ and

$$(5.34) \qquad \int_0^T \left[(\nabla \cdot u_*, q) \right] \eta(t)\, dt = 0.$$

Since $H^1(\Omega) \cap L_0^2(\Omega)$ is dense in $L_0^2(\Omega)$ and $C([0, T])$ is dense in $L^2[0, T]$, we find u_* fulfills (2.18).

(2) When $Y_h \subset H_0^{\mathrm{div}}(\Omega)$, for any $q \in L_0^2(\Omega)$, we can choose $q_h \in M_h$ such that $q_h \to q$ strongly in L^2. On the other hand, using (3.14) and the estimates in Lemma 5.1, we have $\nabla \cdot (\overline{P}_{h,\delta t}^C(t + \delta t) - \overline{P}_{h,\delta t}^C(t))$ is uniformly bounded in $L^2([0, T]; L^2)$, and we can extract a subsequence (denoted as the original one for simplicity) such that for some $p_*^\nabla \in L^2([0, T]; L^2)$, $\nabla \cdot (\overline{P}_{h,\delta t}^C(t + \delta t) - \overline{P}_{h,\delta t}^C(t)) \to p_*^\nabla$ weakly in $L^2([0, T]; L^2)$ as $h, \delta t \to 0$. Since we know $\overline{P}_{h,\delta t}^C(t + \delta t) - \overline{P}_{h,\delta t}^C(t) \longrightarrow 0$ weakly in $L^2([0, T]; L^2(\Omega)^d)$ as $h, \delta t \to 0$, we identify $p_*^\nabla = 0$. Thus, passing to the limit in (5.31) as $h, \delta t \to 0$, we get

$$(5.35) \qquad \int_0^T \left[(\nabla \cdot u_*, q) \right] \eta(t)\, dt = 0,$$

and then u_* fulfills (2.18).

Now, we have proved that (ϕ_*, w_*, u_*, p_*) is a weak solution of (2.15)-(2.18).

Step 3. Under the assumption that the system (1.1)-(1.5) admits a unique solution, the convergence results in Theorem 5.1 can be obtained by the same arguments in [4] and the detail is omitted here. $\qquad \square$

6. Concluding remarks

We derived rigorously in this paper error estimates for a fully discretized energy stable scheme of a Cahn-Hilliard phase-field model for two-phase incompressible flow. The full discretization is based on a finite-element discretization to the weakly coupled, linear, energy stable scheme introduced in [24]. The main difficulties for the error analysis were introduced by the splitting error in the projection step and the nonlinear coupling between the phase function and velocity. We derived optimal convergence rates for both phase functions and velocity in the H^1-norm and pressure in the L^2-norm, and established qualitative convergence of the numerical solution towards the weak solution of the continuous problem under minimal regularity

assumption. To the best of our knowledge, this is the first rigorous error analysis for a fully discrete scheme involving a projection step for a phase-field model of two phase flows.

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