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Stability and convergence analysis of rotational velocity correction methods for the Navier–Stokes equations

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Abstract

The velocity correction method has shown to be an effective approach for solving incompressible Navier–Stokes equations. It does not require the initial pressure and the inf-sup condition may not be needed. However, stability and convergence analyses have not been established for the nonlinear case. The challenge arises from the splitting associated with the nonlinear term and rotational term. In this paper, we carry out stability and convergence analysis of the first-order method in the nonlinear case. Our technique is a new Gauge–Uzawa formulation, which brings forth a telescoping symmetry into the rotational form. We also provide a stability proof for the second-order method in the linear case. Numerical results are provided for both first- and second-order methods.

Keywords Incompressible flow · Navier–Stokes · Projection correction · Stability · Convergence · Velocity correction · Rotational · Gauge–Uzawa

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1 Introduction

We consider in this paper a kind of time discretization of the unsteady incompressible Navier–Stokes equations in primitive variables. Below is the setup of the continuous

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system. Given a body force $\boldsymbol{f}(\boldsymbol{x}, t)$ and a solenoidal initial condition $\boldsymbol{u}_0$, we look for the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and the pressure $p(\boldsymbol{x}, t)$ such that:

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, &\quad \text{in } \Omega \times (0, T], \\ \nabla \cdot \boldsymbol{u} = 0, &\quad \text{in } \Omega \times (0, T], \\ \boldsymbol{u} = \boldsymbol{0}, &\quad \text{on } \Gamma, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, &\quad \text{in } \Omega, \end{aligned}$$

where T is the terminal time, Ω is an open, connected, and bounded subset of $\mathbb{R}^d$ ($d = 2$ or 3), Γ is the smooth boundary of Ω, and $\nu = Re^{-1}$ is the reciprocal of the Reynolds number Re.

The projection method was firstly introduced in [2] and [24]. The method has been popularized over the decades for discretizing the Navier–Stokes equations in time. As an improvement for the projection method, the pressure correction method first appeared in [8, 26]. It consists of two substeps at each time step: (i) treating the pressure explicitly and solving for the velocity in the momentum equation, and (ii) projecting the solved velocity to a divergence-free space. In the second substep, a Poisson-type equation is solved. The pressure correction method has been widely implemented (cf. [1, 22, 25, 26]). And the analysis has been done for both the standard form (cf. [7, 19, 20, 23]) and the rotational form (cf. [9, 10]).

The focus of this paper is another kind of projection scheme, the velocity correction method proposed in [14, 17]. Like the pressure correction method, it also consists of two substeps. Unlike the pressure correction method, it solves the pressure from the momentum equation at the first substep and corrects the velocity at the second substep. In other words, the two substeps have been swapped within an operator splitting framework. This type of scheme possesses the following advantages. First, the method does not require the initialization of the pressure. Secondly, the inf-sup condition is not needed in practice (cf. [5, 12]). Thirdly, it has a potential for deriving schemes that are more accurate and consistent. For example, the authors of [11] proposed a rotational form of the velocity correction method, which improved the order of convergence for both the velocity and pressure. A rigorous error analysis has been carried out for the linear case (cf. [11]). Later, as a further improvement, the fully nonlinear case was addressed by an unconditionally stable rotational velocity correction method (cf. [5]). The effectiveness of the method in [5] was validated through various experiments in fluid dynamics (cf. [3, 4, 6, 27]). Regarding the analysis, the stability of the standard form has been done in [5]. To the best of the authors' knowledge, currently there is no rigorous proof of the stability and convergence for its rotational form in the fully nonlinear case. The essential difficulty lies in the intertwine of the nonlinear term and the rotational term. The symmetry that is available in the standard form was lost in the rotational form, because the Laplacian operator was replaced by the curl-curl operator in the second substep. So, it becomes part of compromise on a consequence of the asymmetric correction equation to the divergence-free condition.

To resolve this issue, we introduce a Gauge–Uzawa formulation for the rotational form in the nonlinear case. The idea is related to earlier work on the Gauge method for projection methods (cf. [15, 16, 18]). We rewrite $\nabla \cdot \boldsymbol{u}$, a term supposing to be almost 0 on the discrete level, into a difference of Gauge series (see Eq. 3.2). This treatment will make the correction step symmetric. In all theorems we give in this paper, the above idea is manifested in the step where we take symmetric inner products and obtain telescoping terms.

The paper is organized as follows. In the next section, we describe basic notations and assumptions used throughout the content. In Section 3, we describe the existing formulations of the velocity correction method and introduce the new Gauge–Uzawa formulation. The stability results are established in Section 4. In Section 5, we prove the the $O(\Delta t)$ convergence in both $l^\infty(\boldsymbol{L}^2)$- and $l^2(\boldsymbol{H}^1)$-norms for the velocity. In Section 6, we provide the numerical experiments. We end the paper with a short conclusion in the last section.

2 Notations and preliminaries

In this section, we introduce notations and background materials that will be used. Throughout the paper, c denotes a generic constant that is independent of the time step size but may depend on the data and the regularity of the exact solution.

We denote $\|\cdot\|_m$ as the standard norms on Sobolev spaces H^m ($m \in \mathbb{Z}$) defined on Ω. And $\|\cdot\|$ denotes the L^2 norm. Bold fonts are used for vector fields, e.g., $\boldsymbol{u} \in \boldsymbol{H}^2$. Next, we define:

$$\boldsymbol{H}_0^1 \triangleq \{\boldsymbol{u} \in \boldsymbol{H}^1 : \boldsymbol{u}|_\Gamma = \boldsymbol{0}\}.$$

Also, define the following function space:

$$\boldsymbol{H} \triangleq \{\boldsymbol{u} \in \boldsymbol{L}^2 : \nabla \cdot \boldsymbol{u} = 0, \boldsymbol{u} \cdot \boldsymbol{n}|_\Gamma = 0\}.$$

We define the trilinear form $b(\cdot,\cdot,\cdot)$ as follows:

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{w}, \boldsymbol{u} \cdot \nabla \boldsymbol{v}).$$

By using Hölder inequality and Sobolev imbedding, it is easy to establish the following inequalities which are valid for $d \leq 4$:

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \leqslant \begin{cases} c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, \\ c\|\boldsymbol{u}\|\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_1, \\ c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|, \\ c\|\boldsymbol{u}\|\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_2, \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|. \end{cases} \tag{2.1}$$

For all $\boldsymbol{u} \in \boldsymbol{H}$, we have the following identities:

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1, \tag{2.2}$$

$$(\boldsymbol{u}, \nabla p) = 0, \quad \forall p \in H^1, \tag{2.3}$$

and for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1$, we have:

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}).$$

We assume that the Navier–Stokes Eq. 1 possesses a unique strong solution $\boldsymbol{u} \in \boldsymbol{H}_0^1 \cap \boldsymbol{H}^2$ and $p \in H^1 \backslash \mathbb{R}$. Furthermore, we assume that the exact solution (u, p) is sufficiently smooth; more precisely, we assume:

$$\begin{array}{l}\|\boldsymbol{u}(t)\|_{L^\infty(\boldsymbol{H}^m)} = \max_{0\leqslant t\leqslant T} \|\boldsymbol{u}(t)\|_m \leqslant c, \quad m = 0, 1, 2,\\ \|\boldsymbol{u}_t(t)\|_{L^2(\boldsymbol{H}^m)} = \left(\int_0^T \|\boldsymbol{u}_t(t)\|_m^2 \mathrm{d}t\right)^{1/2} \leqslant c, \quad m = 0, 1, 2,\\ \|\boldsymbol{u}_{tt}(t)\|_{L^2(\boldsymbol{H}^m)} = \left(\int_0^T \|\boldsymbol{u}_{tt}(t)\|_m^2 \mathrm{d}t\right)^{1/2} \leqslant c, \quad m = 0, 1, 2,\\ \|p(t)\|_{L^\infty(\boldsymbol{H}^m)} = \max_{0\leqslant t\leqslant T} \|q(t)\|_m \leqslant c, \quad m = 1, 2,\\ \|\boldsymbol{f}(t)\|_{L^\infty(\boldsymbol{H}^m)} = \max_{0\leqslant t\leqslant T} \|\boldsymbol{f}(t)\|_m \leqslant c, \quad m = 0, 1, 2.\end{array} \tag{2.4}$$

Now, we introduce the discrete norms. Let $\{\phi^0, \phi^1, \cdots, \phi^n\}$ be a sequence of functions in a Hilbert space W, and $\Delta t = T/n$. We introduce the following discrete norms:

$$\|\phi\|_{l^2(W)}^2 = \Delta t \sum_{i=0}^{n} \|\phi^i\|_W^2, \quad \|\phi\|_{l^\infty(W)}^2 = \max_{0\leqslant i\leqslant n} \|\phi^i\|_W^2.$$

The following notations are used for finite differences:

$$\begin{aligned}\delta\phi^{i+1} &= \phi^{i+1} - \phi^i,\\ \delta^2\phi^{i+1} &= \phi^{i+1} - 2\phi^i + \phi^{i-1},\\ \mathcal{D}^2\phi^{i+1} &= 3\phi^{i+1} - 4\phi^i + \phi^{i-1}.\end{aligned}$$

We recall the following version of discrete Gronwall's Lemma (cf. [13, 20]):

Lemma 2.1 (Discrete Gronwall's Lemma) Let y^n, h^n, g^n, and f^n be nonnegative series such that:

$$y^m + \Delta t \sum_{n=0}^{m} h^n \leqslant B + \Delta t \sum_{n=0}^{m} (g^n y^n + f^n), \quad \Delta t \sum_{n=0}^{M} g^n \leqslant K, \quad 0 \leqslant m \leqslant M = [\frac{T}{\Delta t}],$$

where B is a given constant typically related to the initial condition. In addition, assume that $g^n \Delta t < 1$ for every n. Define $\rho = \max_{0\leqslant n\leqslant M}(1 - g^n\Delta t)^{-1}$. Then:

$$y^m + \Delta t \sum_{n=0}^{m} h^n \leqslant e^{\rho K}(B + \Delta t \sum_{n=0}^{m} f^n), \quad 0 \leqslant m \leqslant M.$$

Finally, the following three identities will be frequently used in the analysis:

$$(a - b, 2a) = (a, a) - (b, b) + (a - b, a - b), \tag{2.5}$$

$$\begin{aligned}(3a - 4b + c, 2a) = {} & (a, a) + (2a - b, 2a - b) - (b, b) - (2b - c, 2b - c)\\ & + (a - 2b + c, a - 2b + c),\end{aligned} \tag{2.6}$$

$$\begin{aligned}(3a - 4b + c, 2(a - b)) = {} & (a - b, a - b) - (b - c, b - c)\\ & + (a - 2b + c, a - 2b + c) + 4(a - b, a - b).\end{aligned} \tag{2.7}$$

3 Rotational velocity correction schemes

We describe in this section the rotational velocity correction methods for the Navier–Stokes equations. The treatment of the nonlinear term in the schemes presented below are slightly different from the rotational velocity correction scheme (3-4) in [5], but consistent with the standard velocity correction scheme (17-18).

3.1 Rotational velocity correction schemes

First, we consider the first-order rotational velocity correction scheme for Eq. 1. Assume that at each time step, $\{\boldsymbol{u}^k, \tilde{\boldsymbol{u}}^k, p^k\}$ are given and one seeks $\{\boldsymbol{u}^{k+1}, \tilde{\boldsymbol{u}}^{k+1}, p^{k+1}\}$. In the first substep, we solve for $(\boldsymbol{u}^{k+1}, p^{k+1})$ from:

$$\begin{cases} \frac{\boldsymbol{u}^{k+1}-\tilde{\boldsymbol{u}}^k}{\Delta t} + \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k + \nu \nabla \times \nabla \times \tilde{\boldsymbol{u}}^k + \nabla p^{k+1} = \boldsymbol{f}^{k+1}, \\ \nabla \cdot \boldsymbol{u}^{k+1} = 0, \\ \boldsymbol{u}^{k+1} \cdot \boldsymbol{n}|_\Gamma = 0. \end{cases} \tag{3.1}$$

In the second substep, we correct $\boldsymbol{u}^{k+1}$ by solving $\tilde{\boldsymbol{u}}^{k+1}$ from

$$\begin{cases} \frac{\tilde{\boldsymbol{u}}^{k+1}-\boldsymbol{u}^{k+1}}{\Delta t} + \boldsymbol{u}^{k+1} \cdot \nabla \tilde{\boldsymbol{u}}^{k+1} - \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k - \nu \Delta \tilde{\boldsymbol{u}}^{k+1} - \nu \nabla \times \nabla \times \tilde{\boldsymbol{u}}^k = \boldsymbol{0}, \\ \tilde{\boldsymbol{u}}^{k+1}|_\Gamma = \boldsymbol{0}. \end{cases} \tag{3.2}$$

The above schemes can be easily extended to second-order as follows (cf. [5]). In the first substep, we solve for $(\boldsymbol{u}^{k+1}, p^{k+1})$ from:

$$\begin{cases} \frac{3\boldsymbol{u}^{k+1}-4\tilde{\boldsymbol{u}}^k+\tilde{\boldsymbol{u}}^{k-1}}{2\Delta t} + \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k + \nu \nabla \times \nabla \times \tilde{\boldsymbol{u}}^k + \nabla p^{k+1} = \boldsymbol{f}^{k+1}, \\ \nabla \cdot \boldsymbol{u}^{k+1} = 0, \\ \boldsymbol{u}^{k+1} \cdot \boldsymbol{n}|_\Gamma = 0. \end{cases} \tag{3.3}$$

In the second substep, we correct $\boldsymbol{u}^{k+1}$ by solving $\tilde{\boldsymbol{u}}^{k+1}$ from:

$$\begin{cases} \frac{3\tilde{\boldsymbol{u}}^{k+1}-3\boldsymbol{u}^{k+1}}{2\Delta t} + \boldsymbol{u}^{k+1} \cdot \nabla \tilde{\boldsymbol{u}}^{k+1} - \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k - \nu \Delta \tilde{\boldsymbol{u}}^{k+1} - \nu \nabla \times \nabla \times \tilde{\boldsymbol{u}}^k = \boldsymbol{0}, \\ \tilde{\boldsymbol{u}}^{k+1}|_\Gamma = \boldsymbol{0}. \end{cases} \tag{3.4}$$

3.2 The Gauge–Uzawa reformulation

A key step in establishing the stability result is to reformulate the rotational velocity correction schemes with a Gauge–Uzawa formulation. More precisely, we introduce a Gauge variable, $\{q^k\}$, and an axillary variable, $\{\boldsymbol{w}^k\}$, defined by:

$$\begin{aligned} &q^0 = 0; \quad q^{k+1} = \nabla \cdot \tilde{\boldsymbol{u}}^{k+1} + q^k, \quad k \geq 0, \\ &\boldsymbol{w}^k = \nu \nabla \times \nabla \times \tilde{\boldsymbol{u}}^k + \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k - \nu \nabla q^k. \end{aligned} \tag{3.5a}$$

Note that Eq. 3.5a is reminiscent of the Uzawa algorithm for the Stokes problem. Then, Eq. 3.2 can be reformulated as:

$$\begin{cases} \tilde{\boldsymbol{u}}^{k+1} + \Delta t \boldsymbol{w}^{k+1} = \boldsymbol{u}^{k+1} + \Delta t \boldsymbol{w}^k, \\ \tilde{\boldsymbol{u}}^{k+1}|_\Gamma = \boldsymbol{0}. \end{cases} \tag{3.6}$$

Similarly, Eq. 3.4 can also be reformulated as:

$$\begin{cases} 3\tilde{\boldsymbol{u}}^{k+1} + 2\Delta t \boldsymbol{w}^{k+1} = 3\boldsymbol{u}^{k+1} + 2\Delta t \boldsymbol{w}^{k}, \\ \tilde{\boldsymbol{u}}^{k+1}|_{\Gamma} = \boldsymbol{0}, \end{cases} \tag{3.7}$$

where $\boldsymbol{w}^k$ is again defined in Eq. 3.2. As we will show in the following section, this reformulation allows us to derive a stability result on $(\boldsymbol{u}^{k+1}, p^{k+1}, \tilde{\boldsymbol{u}}^{k+1})$ directly, without resorting to their differences as in [11].

4 Stability analysis

In this section, we provide the stability analysis for the reformulated schemes. Without loss of generality, we assume $\boldsymbol{f} \equiv 0$.

4.1 First-order scheme

Theorem 4.1 The scheme Eq. 3.1 and Eq. 3.2 with $\boldsymbol{f} \equiv 0$ is unconditionally energy stable in the sense that, for all $0 \le k \le T/\Delta t - 1$, we have:

$$\mathcal{E}_{k+1}^{(1)} - \mathcal{E}_{k}^{(1)} \le -\nu \Delta t \|\nabla \tilde{\boldsymbol{u}}^{k+1}\|^2,$$

where:

$$\mathcal{E}_{k}^{(1)} = \|\tilde{\boldsymbol{u}}^k\|^2 + \Delta t^2 \|\boldsymbol{w}^k\|^2 + \nu \Delta t \|q^k\|^2$$

is the modified energy at time step k.

Proof Take the inner product of Eq. 3.1 with $2\Delta t \boldsymbol{u}^{k+1}$ and use Eq. 2.3, to have:

$$\|\boldsymbol{u}^{k+1}\|^2 - \|\tilde{\boldsymbol{u}}^k\|^2 + \|\boldsymbol{u}^{k+1} - \tilde{\boldsymbol{u}}^k\|^2 + 2\Delta t(\boldsymbol{u}^{k+1}, \nu \nabla \times \nabla \times \tilde{\boldsymbol{u}}^k + \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k) = 0. \tag{4.1}$$

On each side of the Eq. 3.6, taking the inner product of that side with itself, we find:

$$\|\tilde{\boldsymbol{u}}^{k+1}\|^2 + \Delta t^2 \|\boldsymbol{w}^{k+1}\|^2 + 2\Delta t(\tilde{\boldsymbol{u}}^{k+1}, \boldsymbol{w}^{k+1}) = \|\boldsymbol{u}^{k+1}\|^2 + \Delta t^2 \|\boldsymbol{w}^{k}\|^2 + 2\Delta t(\boldsymbol{u}^{k+1}, \boldsymbol{w}^{k}). \tag{4.2}$$

Using Eq. 2.2 and the definition of q^k, we obtain:

$$\begin{aligned} (\tilde{\boldsymbol{u}}^{k+1}, \boldsymbol{w}^{k+1}) &= (\tilde{\boldsymbol{u}}^{k+1}, \nu \nabla \times \nabla \times \tilde{\boldsymbol{u}}^{k+1} - \nu \nabla q^{k+1}) \\ &= \nu \|\nabla \times \tilde{\boldsymbol{u}}^{k+1}\|^2 + \nu(\nabla \cdot \tilde{\boldsymbol{u}}^{k+1}, q^{k+1}), \\ &= \nu \|\nabla \times \tilde{\boldsymbol{u}}^{k+1}\|^2 + \nu(q^{k+1} - q^k, q^{k+1}), \\ &= \nu \|\nabla \times \tilde{\boldsymbol{u}}^{k+1}\|^2 + \tfrac{\nu}{2}(\|q^{k+1}\|^2 - \|q^k\|^2 + \|q^{k+1} - q^k\|^2). \end{aligned} \tag{4.3}$$

We derive from Eq. 2.3 that:

$$(\boldsymbol{u}^{k+1}, \boldsymbol{w}^k) = (\boldsymbol{u}^{k+1}, \nu \nabla \times \nabla \times \boldsymbol{u}^k + \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k). \tag{4.4}$$

Plugging Eq. 4.3 and Eq. 4.4 into Eq. 4.2, and summing up the result with Eq. 4.1, we obtain:

$$\begin{aligned} &\|\tilde{\boldsymbol{u}}^{k+1}\|^2 - \|\tilde{\boldsymbol{u}}^k\|^2 + \|\boldsymbol{u}^{k+1} - \tilde{\boldsymbol{u}}^k\|^2 + 2\nu \Delta t \|\nabla \times \tilde{\boldsymbol{u}}^{k+1}\|^2 + \Delta t^2(\|\boldsymbol{w}^{k+1}\|^2 - \|\boldsymbol{w}^k\|^2) \\ &\quad + \nu \Delta t(\|q^{k+1}\|^2 - \|q^k\|^2 + \|q^{k+1} - q^k\|^2) = 0. \end{aligned} \tag{4.5}$$

We can then conclude from the above, the fact that $\|q^{k+1}-q^k\|^2 = \|\nabla\cdot\tilde{\boldsymbol{u}}^{k+1}\|^2$, and the identity:

$$\|\nabla\times\boldsymbol{v}\|^2+\|\nabla\cdot\boldsymbol{v}\|^2=\|\nabla\boldsymbol{v}\|^2,\quad \forall\boldsymbol{v}\in\boldsymbol{H}_0^1. \tag{4.6}$$

□

Note that the Gauge-Uzawa formulation Eq. 3.6 plays a critical role in the above proof.

4.2 Stability of the second-order scheme in the linear case

The stability proof of the second-order scheme is much more delicate due to the special treatment required to deal with the second-order BDF formula. So, we shall only prove a stability result without the nonlinear term and with $\boldsymbol{f}\equiv 0$. Namely, we consider the scheme:

$$\begin{cases}\frac{3\boldsymbol{u}^{k+1}-4\tilde{\boldsymbol{u}}^k+\tilde{\boldsymbol{u}}^{k-1}}{2\Delta t}+\nu\nabla\times\nabla\times\tilde{\boldsymbol{u}}^k+\nabla p^{k+1}=0,\\ \nabla\cdot\boldsymbol{u}^{k+1}=0,\\ \boldsymbol{u}^{k+1}\cdot\boldsymbol{n}|_\Gamma=0;\end{cases} \tag{4.6}$$

and

$$\begin{cases}\frac{3\tilde{\boldsymbol{u}}^{k+1}-3\boldsymbol{u}^{k+1}}{2\Delta t}-\nu\Delta\tilde{\boldsymbol{u}}^{k+1}-\nu\nabla\times\nabla\times\tilde{\boldsymbol{u}}^k=\boldsymbol{0},\\ \tilde{\boldsymbol{u}}^{k+1}|_\Gamma=\boldsymbol{0}.\end{cases} \tag{4.7}$$

With the Gauge variable q^k and an auxiliary variable $\boldsymbol{w}^k$ defined by:

$$\begin{aligned} q^0=0;\quad q^{k+1}=\nabla\cdot\tilde{\boldsymbol{u}}^{k+1}+q^k,\quad k\geq 0,\\ \boldsymbol{w}^k=\nu\nabla\times\nabla\times\tilde{\boldsymbol{u}}^k-\nu\nabla q^k,\end{aligned} \tag{4.8}$$

Equation 4.7 can also be reformulated as Eq. 3.7.

Theorem 4.2 The scheme Eq. 4.6–Eq. 4.7 is unconditionally energy stable in the sense that, for all $0\leq k\leq T/\Delta t-1$, we have:

$$\mathcal{E}_{k+1}^{(2)}-\mathcal{E}_k^{(2)}\leq -2\nu\Delta t\|\nabla\tilde{\boldsymbol{u}}^{k+1}\|^2, \tag{4.9}$$

where:

$$\begin{aligned}\mathcal{E}_k^{(2)}=&\|\tilde{\boldsymbol{u}}^k\|^2+\|2\tilde{\boldsymbol{u}}^k-\tilde{\boldsymbol{u}}^{k-1}\|^2+\tfrac{2\Delta t\nu}{3}\|\nabla\times\delta\tilde{\boldsymbol{u}}^k\|^2+\tfrac{2\Delta t\nu}{3}\|\delta q^k\|^2\\ &+\tfrac{2\Delta t\nu}{3}\|2\delta q^k-\delta q^{k-1}\|^2+\tfrac{4\Delta t^2}{3}\|\boldsymbol{w}^k\|^2+2\nu\Delta t\|q^k\|^2\end{aligned}$$

is the modified energy at time step k.

Proof Taking the inner product of Eq. 4.6 with $4\Delta t\boldsymbol{u}^{k+1}$, we obtain:

$$I+4\nu\Delta t(\boldsymbol{u}^{k+1},\nabla\times\nabla\times\tilde{\boldsymbol{u}}^k)=0, \tag{4.10}$$

where we have denoted:

$$I=(3\boldsymbol{u}^{k+1}-4\tilde{\boldsymbol{u}}^k+\tilde{\boldsymbol{u}}^{k-1},2\boldsymbol{u}^{k+1}).$$

We will use a similar treatment as in [10] to deal with this term. More precisely, we rewrite the term I as:

$$I = 2(\mathcal{D}^2\tilde{\boldsymbol{u}}^{k+1}, \tilde{\boldsymbol{u}}^{k+1}) + 2(\mathcal{D}^2\tilde{\boldsymbol{u}}^{k+1}, \boldsymbol{u}^{k+1} - \tilde{\boldsymbol{u}}^{k+1}) + 6(\boldsymbol{u}^{k+1} - \tilde{\boldsymbol{u}}^{k+1}, \boldsymbol{u}^{k+1}) := I_1 + I_2 + I_3.$$

Using identity Eq. 2.6, we can rewrite I_1 as

$$I_1 = \|\tilde{\boldsymbol{u}}^{k+1}\|^2 + \|2\tilde{\boldsymbol{u}}^{k+1} - \tilde{\boldsymbol{u}}^k\|^2 - \|\tilde{\boldsymbol{u}}^k\|^2 - \|2\tilde{\boldsymbol{u}}^k - \tilde{\boldsymbol{u}}^{k-1}\|^2 + \|\delta^2\tilde{\boldsymbol{u}}^{k+1}\|^2. \tag{4.11}$$

Thanks to Eq. 3.7 and Eq. 4.8, we can rewrite I_2 as:

$$\begin{aligned} I_2 &= \tfrac{4\Delta t}{3}(\mathcal{D}^2\tilde{\boldsymbol{u}}^{k+1}, \boldsymbol{w}^{k+1} - \boldsymbol{w}^k) \\ &= \tfrac{4\Delta t\nu}{3}(\mathcal{D}^2\tilde{\boldsymbol{u}}^{k+1}, \nabla\times\nabla\times(\tilde{\boldsymbol{u}}^{k+1} - \tilde{\boldsymbol{u}}^k)) + \tfrac{4\Delta t\nu}{3}(\mathcal{D}^2\delta q^{k+1}, \delta q^{k+1}) := I_{2,1} + I_{2,2}. \end{aligned} \tag{4.12}$$

Using identity Eq. 2.7, we find:

$$I_{2,1} = \frac{2\Delta t\nu}{3}\Big(\|\nabla\times\delta\tilde{\boldsymbol{u}}^{k+1}\|^2 - \|\nabla\times\delta\tilde{\boldsymbol{u}}^k\|^2 + \|\nabla\times\delta^2\tilde{\boldsymbol{u}}^{k+1}\|^2 + 4\|\nabla\times\delta\tilde{\boldsymbol{u}}^{k+1}\|^2\Big).$$

By integration by parts, the definition of q^k, and Eq. 2.6, we have:

$$I_{2,2} = \frac{2\Delta t\nu}{3}(\|\delta q^{k+1}\|^2 - \|\delta q^k\|^2 + \|2\delta q^{k+1} - \delta q^k\|^2 - \|2\delta q^k - \delta q^{k-1}\|^2 + \|\delta^2\delta q^{k+1}\|^2).$$

We derive from identity Eq. 2.5 that:

$$I_3 = 3\Big(\|\boldsymbol{u}^{k+1}\|^2 - \|\tilde{\boldsymbol{u}}^{k+1}\|^2 + \|\boldsymbol{u}^{k+1} - \tilde{\boldsymbol{u}}^{k+1}\|^2\Big). \tag{4.13}$$

Next, taking the inner product of Eq. 3.7 with itself on both sides, we obtain:

$$\begin{aligned} &3(\|\tilde{\boldsymbol{u}}^{k+1}\|^2 - \|\boldsymbol{u}^{k+1}\|^2) + \tfrac{4\Delta t^2}{3}(\|\boldsymbol{w}^{k+1}\|^2 - \|\boldsymbol{w}^k\|^2) + 4\Delta t(\tilde{\boldsymbol{u}}^{k+1}, \boldsymbol{w}^{k+1}) \\ &= 4\Delta t(\boldsymbol{u}^{k+1}, \boldsymbol{w}^k) = 4\Delta t(\boldsymbol{u}^{k+1}, \nabla\times\nabla\times\boldsymbol{u}^k), \end{aligned}$$

where we applied the definition of $\boldsymbol{w}^k$ for the second equality. Note that Eq. 4.3 is still valid for $(\tilde{\boldsymbol{u}}^{k+1}, \boldsymbol{w}^{k+1})$ so that we can rewrite the above as:

$$\begin{aligned} &3(\|\tilde{\boldsymbol{u}}^{k+1}\|^2 - \|\boldsymbol{u}^{k+1}\|^2 + \|\tilde{\boldsymbol{u}}^{k+1} - \boldsymbol{u}^{k+1}\|^2) + \tfrac{4\Delta t^2}{3}(\|\boldsymbol{w}^{k+1}\|^2 - \|\boldsymbol{w}^k\|^2) \\ &+4\nu\Delta t\|\nabla\times\tilde{\boldsymbol{u}}^{k+1}\|^2 + 2\nu\Delta t(\|q^{k+1}\|^2 - \|q^k\|^2 + \|\delta q^{k+1}\|^2) \\ &= 4\Delta t(\boldsymbol{u}^{k+1}, \nabla\times\nabla\times\boldsymbol{u}^k). \end{aligned} \tag{4.14}$$

Summing up Eq. 4.10 and Eq. 4.14, and taking into account Eq. 4.11, Eq. 4.12, and Eq. 4.13, we obtain:

$$\begin{aligned} &\|\tilde{\boldsymbol{u}}^{k+1}\|^2 + \|2\tilde{\boldsymbol{u}}^{k+1} - \tilde{\boldsymbol{u}}^k\|^2 - \|\tilde{\boldsymbol{u}}^k\|^2 - \|2\tilde{\boldsymbol{u}}^k - \tilde{\boldsymbol{u}}^{k-1}\|^2 + \tfrac{2\Delta t\nu}{3}\Big(\|\nabla\times\delta\tilde{\boldsymbol{u}}^{k+1}\|^2 - \|\nabla\times\delta\tilde{\boldsymbol{u}}^k\|^2\Big) \\ &\quad + \tfrac{2\Delta t\nu}{3}(\|\delta q^{k+1}\|^2 - \|\delta q^k\|^2 + \|2\delta q^{k+1} - \delta q^k\|^2 - \|2\delta q^k - \delta q^{k-1}\|^2) + \\ &\quad + \tfrac{4\Delta t^2}{3}(\|\boldsymbol{w}^{k+1}\|^2 - \|\boldsymbol{w}^k\|^2) + 2\nu\Delta t(\|q^{k+1}\|^2 - \|q^k\|^2) \\ &\quad = -\Big(\|\delta^2\tilde{\boldsymbol{u}}^{k+1}\|^2 + \tfrac{2\Delta t\nu}{3}(\|\nabla\times\delta^2\tilde{\boldsymbol{u}}^{k+1}\|^2 + 4\|\nabla\times\delta\tilde{\boldsymbol{u}}^{k+1}\|^2) \\ &\qquad + \tfrac{2\Delta t\nu}{3}\|\delta^3 q^{k+1}\|^2 + 3\|\boldsymbol{u}^{k+1} - \tilde{\boldsymbol{u}}^{k+1}\|^2 + 4\nu\Delta t\|\nabla\times\tilde{\boldsymbol{u}}^{k+1}\|^2 + 2\nu\Delta t\|\delta q^{k+1}\|^2\Big), \end{aligned}$$

which leads to Eq. 4.9 with the fact that $\|\delta q^{k+1}\|^2 = \|\nabla\cdot\tilde{\boldsymbol{u}}^{k+1}\|^2$ and identity Eq. 4.6. □

5 Error analysis

We carry out a convergence analysis for the Navier–Stokes equations in this section. Since we only proved the stability of the first-order scheme for the Navier–Stokes equations, we shall only consider the first-order scheme.

5.1 Error equations

Define the following error terms:

$$\boldsymbol{e}^k = \boldsymbol{u}(\cdot, t^k) - \boldsymbol{u}^k, \quad \tilde{\boldsymbol{e}}^k = \boldsymbol{u}(\cdot, t^k) - \tilde{\boldsymbol{u}}^k, \quad h^k = p(\cdot, t^k) - p^k,$$

and

$$\tilde{\boldsymbol{d}}^k = \boldsymbol{u}(\cdot, t^k) \cdot \nabla \boldsymbol{u}(\cdot, t^k) - \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k = \boldsymbol{e}^k \cdot \nabla \boldsymbol{u}(\cdot, t^k) + \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{e}}^k.$$

First, we approximate the original governing system Eq. 1 at t^{k+1} as follows:

$$\frac{\boldsymbol{u}(\cdot, t^{k+1}) - \boldsymbol{u}(\cdot, t^k)}{\Delta t} + \boldsymbol{u}(\cdot, t^{k+1}) \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1}) - \nu \Delta \boldsymbol{u}(\cdot, t^{k+1}) + \nabla p(\cdot, t^{k+1}) = \boldsymbol{f}(\cdot, t^{k+1}) + \boldsymbol{R}^{k+1}, \tag{5.1}$$

where the error remainder term is:

$$\boldsymbol{R}^{k+1} = \frac{\boldsymbol{u}(\cdot, t^{k+1}) - \boldsymbol{u}(\cdot, t^k)}{\Delta t} - \boldsymbol{u}_t(\cdot, t^{k+1}).$$

We subtract Eq. 3.1 from Eq. 5.1 to obtain:

$$\frac{\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k}{\Delta t} + (\boldsymbol{u}(\cdot, t^{k+1}) \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1}) - \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k) + \nu \nabla \times \nabla \times (\boldsymbol{u}(\cdot, t^{k+1}) - \tilde{\boldsymbol{u}}^k) + \nabla h^{k+1} = \boldsymbol{R}^{k+1}. \tag{5.2}$$

Regarding the second step Eq. 3.6, we have:

$$\tilde{\boldsymbol{e}}^{k+1} + \Delta t \boldsymbol{s}^{k+1} = \boldsymbol{e}^{k+1} + \Delta t \boldsymbol{z}^k, \tag{5.3}$$

where $\boldsymbol{s}^{k+1}$ and $\boldsymbol{z}^k$ are obtained from subtracting both sides of Eq. 3.6 from:

$$\boldsymbol{u}(\cdot, t^{k+1}) + \Delta t \boldsymbol{u}(\cdot, t^{k+1}) \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1}) + \nu \Delta t \nabla \times \nabla \times \boldsymbol{u}(\cdot, t^{k+1}).$$

Namely,

$$\begin{aligned} \boldsymbol{s}^k &= \nu \nabla \times \nabla \times \tilde{\boldsymbol{e}}^k + \tilde{\boldsymbol{d}}^k + \nu \nabla q^k, \\ \boldsymbol{z}^k &= (\boldsymbol{u}(\cdot, t^{k+1}) \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1}) - \boldsymbol{u}^k \cdot \nabla \tilde{\boldsymbol{u}}^k) + \nu \nabla \times \nabla \times (\boldsymbol{u}(\cdot, t^{k+1}) - \tilde{\boldsymbol{u}}^k) + \nu \nabla q^k. \end{aligned}$$

By assuming that $\|\boldsymbol{u}_t\|_{L^2(\boldsymbol{H}^2)} \le c$ (the second line of Eq. 2.4), we have:

$$\begin{aligned} \|\boldsymbol{z}^k - \boldsymbol{s}^k\| &= \|\nu \nabla \times \nabla \times (\boldsymbol{u}(\cdot, t^{k+1}) - \boldsymbol{u}(\cdot, t^k)) - (\boldsymbol{u}(\cdot, t^{k+1}) \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1}) \\ &\quad - \boldsymbol{u}(\cdot, t^k) \cdot \nabla \boldsymbol{u}(\cdot, t^k))\| \\ &= \|\nu \nabla \times \nabla \times \delta \boldsymbol{u}(\cdot, t^{k+1}) - \delta\Big(\boldsymbol{u}(\cdot, t^{k+1}) \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1})\Big)\| \le c \Delta t, \end{aligned} \tag{5.4}$$

where c is a constant independent of k.

We can also rewrite Eq. 5.2 as:

$$\frac{\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k}{\Delta t} + (\boldsymbol{z}^k - \nu \nabla q^k) + \nabla h^{k+1} = \boldsymbol{R}^{k+1}. \tag{5.5}$$

5.2 Error estimates for the velocity

In this subsection, we establish error estimates for the velocity.

Theorem 5.1 Assuming that the exact solution (u, p) satisfies the regularity assumption in Eq. 2.4, we have the following error estimates for the scheme Eq. 3.1 and Eq. 3.2: for all, $0 \le m \le T/\Delta t - 1$, we have:

$$\|\tilde{\boldsymbol{e}}^{m+1}\|^2 + \|\boldsymbol{e}^{m+1}\|^2 + \Delta t^2 \|\boldsymbol{s}^{m+1}\|^2 + \frac{1}{2}\sum_{k=0}^{m}(\nu \Delta t \|\nabla \tilde{\boldsymbol{e}}^{k+1}\|^2 + \|\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k\|^2) \le c\Delta t^2,$$

where c is a constant independent of k.

Proof Take inner product of both sides of Eq. 5.5 with $2\Delta t \boldsymbol{e}^{k+1}$, to have:

$$\|\boldsymbol{e}^{k+1}\|^2 - \|\tilde{\boldsymbol{e}}^k\|^2 + \|\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k\|^2 + 2\Delta t(\boldsymbol{z}^k, \boldsymbol{e}^{k+1}) = 2\Delta t(\boldsymbol{R}^{k+1}, \boldsymbol{e}^{k+1}).$$

Take inner product of each side of Eq. 5.3 with that side itself, to have:

$$\|\tilde{\boldsymbol{e}}^{k+1}\|^2 + \Delta t^2 \|\boldsymbol{s}^{k+1}\|^2 + 2\Delta t(\tilde{\boldsymbol{e}}^{k+1}, \boldsymbol{s}^{k+1}) = \|\boldsymbol{e}^{k+1}\|^2 + \Delta t^2 \|\boldsymbol{z}^k\|^2 + 2\Delta t(\boldsymbol{e}^{k+1}, \boldsymbol{z}^k).$$

By using Eq. 5.4, we can estimate $\|\boldsymbol{z}^k\|^2$ as follows:

$$\begin{aligned}
\|\boldsymbol{z}^k\|^2 &\leqslant \|\boldsymbol{s}^k + (\boldsymbol{z}^k - \boldsymbol{s}^k)\|^2 \\
&\leqslant (\|\boldsymbol{s}^k\| + \|\boldsymbol{z}^k - \boldsymbol{s}^k)\|)^2 = \|\boldsymbol{s}^k\|^2 + 2\|\boldsymbol{s}^k\|\|\boldsymbol{z}^k - \boldsymbol{s}^k\| + \|\boldsymbol{z}^k - \boldsymbol{s}^k\|^2 \\
&\leqslant \|\boldsymbol{s}^k\|^2 + c\Delta t\|\boldsymbol{s}^k\| + c\Delta t^2 = \|\boldsymbol{s}^k\|^2 + c\Delta t^{\frac{1}{2}}\Delta t^{\frac{1}{2}}\|\boldsymbol{s}^k\| + c\Delta t^2 \\
&\leqslant \|\boldsymbol{s}^k\|^2 + c\Delta t + c\Delta t\|\boldsymbol{s}^k\|^2 + c\Delta t^2 \\
&\leqslant \|\boldsymbol{s}^k\|^2 + c\Delta t + c\Delta t\|\boldsymbol{s}^k\|^2,
\end{aligned}$$

where we used the same c to denote the absorbing constant. For $(\tilde{\boldsymbol{e}}^{k+1}, \boldsymbol{s}^{k+1})$ on the left-hand side, we apply the definition of $\boldsymbol{s}^{k+1}$:

$$(\tilde{\boldsymbol{e}}^{k+1}, \boldsymbol{s}^{k+1}) = (\tilde{\boldsymbol{e}}^{k+1}, \nu\nabla \times \nabla \times \tilde{\boldsymbol{e}}^{k+1} + \tilde{\boldsymbol{d}}^{k+1} + \nu\nabla q^{k+1}).$$

We break the above into three parts and estimate each of them below. First,

$$(\tilde{\boldsymbol{e}}^{k+1}, \nu\nabla \times \nabla \times \tilde{\boldsymbol{e}}^{k+1}) = \nu\|\nabla \times \tilde{\boldsymbol{e}}^{k+1}\|^2. \tag{5.6}$$

Secondly, since $\nabla \cdot \boldsymbol{u}^{k+1} = 0$, we derive by using the second branch of Eq. 2.1, the Cauchy–Schwarz inequality, and assuming that $\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H}^1)} \leqslant c$ (the first line of Eq. 2.4) that:

$$\begin{aligned}
(\tilde{\boldsymbol{e}}^{k+1}, \tilde{\boldsymbol{d}}^{k+1}) &= (\tilde{\boldsymbol{e}}^{k+1}, \boldsymbol{e}^{k+1} \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1})) + (\tilde{\boldsymbol{e}}^{k+1}, \boldsymbol{u}^{k+1} \cdot \nabla \tilde{\boldsymbol{e}}^{k+1}) \\
&= (\tilde{\boldsymbol{e}}^{k+1}, \boldsymbol{e}^{k+1} \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1})) \\
&= (\tilde{\boldsymbol{e}}^{k+1}, (\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k) \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1})) + (\tilde{\boldsymbol{e}}^{k+1}, \tilde{\boldsymbol{e}}^k \cdot \nabla \boldsymbol{u}(\cdot, t^{k+1})) \\
&\leqslant \tfrac{\nu}{4}\|\nabla \tilde{\boldsymbol{e}}^{k+1}\|^2 + c\|\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k\|^2 + c\|\tilde{\boldsymbol{e}}^k\|^2.
\end{aligned}$$

For the third term, noticing $\nabla \cdot \tilde{\boldsymbol{e}}^{k+1} = -(q^{k+1} - q^k)$, we have:

$$(\tilde{\boldsymbol{e}}^{k+1}, \nu\nabla q^{k+1}) = \frac{\nu}{2}(\|q^{k+1}\|^2 - \|q^k\|^2 + \|q^{k+1} - q^k\|^2).$$

The last term of the right hand of the above equation combines with Eq. 5.6 to give an account of the $\boldsymbol{H}^1$-norm of $\tilde{\boldsymbol{e}}^{k+1}$:

$$\frac{\nu}{2}\|q^{k+1} - q^k\|^2 + \nu\|\nabla \times \tilde{\boldsymbol{e}}^{k+1}\|^2 = \frac{\nu}{2}\|\nabla\tilde{\boldsymbol{e}}^{k+1}\|^2 + \frac{\nu}{2}\|\nabla \times \tilde{\boldsymbol{e}}^{k+1}\|^2.$$

For the right-hand side of the error equation, we estimate it as:

$$(\boldsymbol{R}^{k+1}, \boldsymbol{e}^{k+1}) = (\boldsymbol{R}^{k+1}, \boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k) + (\boldsymbol{R}^{k+1}, \tilde{\boldsymbol{e}}^k) \leqslant c\|\boldsymbol{R}^{k+1}\|^2 + \frac{1}{2}\|\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k\|^2 + \|\tilde{\boldsymbol{e}}^k\|^2.$$

We collect all terms with $\|\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k\|^2$ and obtain its coefficient as $\big(1-(c+1)\Delta t\big)$. We choose a sufficiently small Δt so that the coefficient is larger than $1/2$. We combine all the above estimates to obtain:

$$\begin{aligned}\|\tilde{\boldsymbol{e}}^{k+1}\|^2 - \|\tilde{\boldsymbol{e}}^k\|^2 &+ \tfrac{1}{2}\|\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k\|^2 + \Delta t^2(\|\boldsymbol{s}^{k+1}\|^2 - \|\boldsymbol{s}^k\|^2) + \tfrac{\nu\Delta t}{2}\|\nabla\tilde{\boldsymbol{e}}^{k+1}\|^2 \\ &+ \nu\Delta t\|\nabla \times \tilde{\boldsymbol{e}}^{k+1}\|^2 \\ \leqslant c\Delta t^3 &+ c\Delta t\|\boldsymbol{R}^{k+1}\|^2 + c\Delta t\|\tilde{\boldsymbol{e}}^k\|^2 + c\Delta t^3\|\boldsymbol{s}^k\|^2.\end{aligned}$$

We sum up the inequality from $k = 0$ to $k = m - 1$ and apply the discrete Gronwall's Lemma (Lemma 2.1), to obtain:

$$\|\tilde{\boldsymbol{e}}^{m+1}\|^2 + \Delta t^2\|\boldsymbol{s}^{m+1}\|^2 + \frac{1}{2}\sum_{k=0}^{m}(\nu\Delta t\|\nabla\tilde{\boldsymbol{e}}^{k+1}\|^2 + \|\boldsymbol{e}^{k+1} - \tilde{\boldsymbol{e}}^k\|^2) \leq c\Delta t^2. \quad (5.7)$$

From Eq. 5.7, we also notice that $\|\boldsymbol{s}^{m+1}\| \leq c$. Therefore, we can conclude $\|\boldsymbol{e}^{m+1}\|^2 \leq c\Delta t^2$, thanks to Eq. 5.4, Eq. 5.7, and:

$$\begin{aligned}\|\boldsymbol{e}^{k+1}\| &= \|\tilde{\boldsymbol{e}}^{k+1} + \Delta t(\boldsymbol{s}^{k+1} - \boldsymbol{z}^k)\| \\ &= \|\tilde{\boldsymbol{e}}^{k+1} + \Delta t[(\boldsymbol{s}^{k+1} - \boldsymbol{s}^k) + (\boldsymbol{z}^k - \boldsymbol{s}^k)]\| \\ &\leq \|\tilde{\boldsymbol{e}}^{k+1}\| + \Delta t(\|\boldsymbol{s}^{k+1}\| + \|\boldsymbol{s}^k\| + \|\boldsymbol{z}^k - \boldsymbol{s}^k\|) \\ &\leq c\Delta t.\end{aligned}$$

□

6 Numerical experiments

We present below some numerical experiments to validate the accuracy of the rotational velocity correction methods. To test the order of convergence, we use the following exact solution in $\Omega = (-1, 1)^2$:

$$\begin{cases} u_1(x, y, t) = \sin^2 \pi x \sin 2\pi y \sin t, \\ u_2(x, y, t) = -\sin 2\pi x \sin^2 \pi y \sin t, \\ p(x, y, t) = \cos x \cos y \sin t. \end{cases} \quad (6.1)$$

The external field $\boldsymbol{f}$ is calculated according to Eq. 6.1 and then given as inputs to the program. We employ the Legendre-spectral method [21] in space with a mesh size $(n_x, n_y) = (32, 32)$. We take $\nu = 1$ and computed the solution using the first- and second-order schemes up to time $T = 1$, and measured the errors in the discrete $L^2(0, T; \boldsymbol{L}^2)$- and $L^2(0, T; \boldsymbol{H}^1)$-norms for the velocity and in the discrete $L^2(0, T; L^2)$ norm for the pressure. The results are plotted in Fig. 1 (resp. Fig. 1)

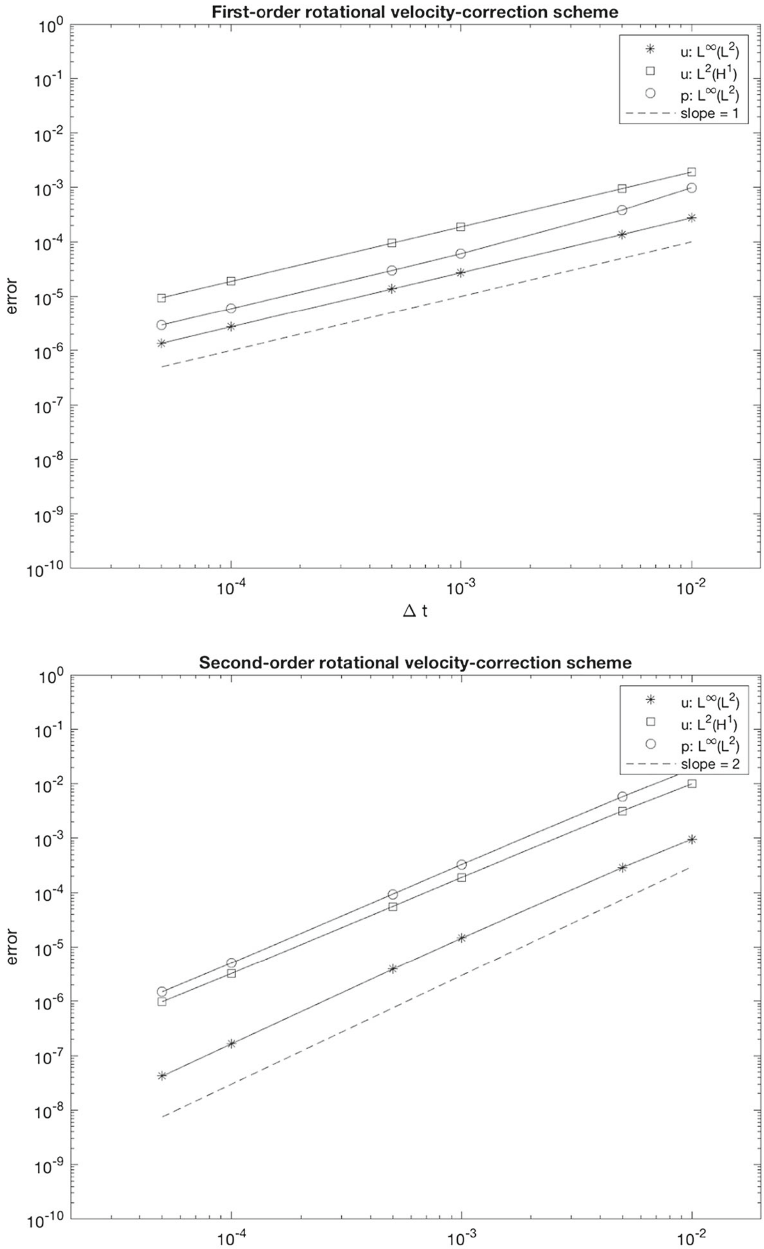

Fig. 1 Convergence rate of the rotational velocity correction schemes: left, first-order scheme; right, second-order scheme

from the first-order scheme (3.1)–(3.2) (resp. the second-order scheme (3.3)–(3.4)). We observe the convergence rate of first-order in the left figure, and of second-order in the right figure. These convergence rates for the velocity are consistent with our theoretical estimates in the previous section. Although we did not prove a convergence rate for the pressure, we observe that its convergence rates in the discrete $L^2(0, T; L^2)$ norm behave essentially the same as the velocity error in the discrete $L^2(0, T; \boldsymbol{H}^1)$-norm.

7 Concluding remarks

We studied in this paper the stability and convergence of the rotational velocity correction method for the Navier–Stokes equations. Our analysis is based on a new Gauge–Uzawa formulation that yields an elegant treatment to the nonlinear term and rotational term. The stability results were established for the first-order method in the nonlinear case and the second-order method in the linear case. However, the stability of the second-order scheme in the nonlinear case is still illusive due to the additional difficulty associated with the BDF2 treatment.

For the convergence, we proved the $O(\Delta t)$ accuracy for the velocity in both $l^\infty(\boldsymbol{L}^2)$-norm and $l^2(\boldsymbol{H}^1)$-norm for the first-order method. We also provided numerical experiments which confirm that that the first-order (resp. second-order) rotational velocity correction scheme leads to the first-order (resp. second-order) convergence rate for the velocity in both $l^\infty(\boldsymbol{L}^2)$-norm and $l^2(\boldsymbol{H}^1)$-norm and for the pressure in $l^\infty(\boldsymbol{L}^2)$-norm.

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