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Stability and Error Analysis of Operator Splitting Methods for American Options Under the Black–Scholes Model

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Abstract
The operator splitting method has shown to be an effective approach for solving the linear complementarity problem for pricing American options. It has been successfully applied to various Black–Scholes models, and it is implementation friendly because the differential equation and the complementarity conditions are decoupled and easily solved on its own part. However, despite its popularity, no stability and error analysis is available for these operator splitting methods. The challenge mainly arises from the special splitting associated with the slack function and the complementarity constraints. In this paper, we establish stability results for the operator splitting schemes based on the backward Euler and BDF2 methods, as well as an error estimate for the scheme based on the backward Euler method. We also provide numerical experiments to demonstrate the convergence behaviors of the two operator splitting methods.

Keywords Operator splitting · Black–Scholes · American option · Linear complementarity problem · Stability

1 Introduction

The Black–Sholes model is one of the most important models for pricing option contracts. The first mathematical formulation of the model was proposed in [4]. The Black–Sholes partial differential equation for European options is an initial boundary value problem with advection and diffusion terms. And the system that governs the price of an American option consists of a differential inequality and its complementarity conditions, because the American option

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allows its holder to exercise it earlier than the maturity. It can be categorized as a linear complementarity problem, which is harder to solve than a standard initial boundary value problem. Many interesting questions have been raised surrounding this ubiquitous mathematical problem, including uniqueness and regularity, analytic form of the exact solution, numerical approximation, and its link to quadratic programming, variational calculus, and optimization. In this paper, we focus on numerical approximations of the classic Black–Scholes model for American options, and carry on rigorous analysis for the associated numerical methods.

Numerical approximations of American options are distinctly different from those of European options, due to the constraints and the moving boundary. In this paper, we primarily consider the approximation in time. The projected Gaussian elimination in [5] is the first attempt to this problem in a complementarity formulation. Later we have the projected SOR method in [11], the penalty method in [8], the front fixing method with a spectral accuracy in [29], and the front tracking method in [30]. Recently, the two phase method in [10], the projection and contraction method in [28], the active set method in [27], and the reduced basis method in [2] have been introduced.

In this paper, we consider an important class of methods for solving the LCP, the operator splitting method introduced in [18]. The idea was originated from the splitting and projection schemes in computational fluid dynamics. In essence, it decouples the complementarity conditions and the differential equation, so that they can be solved separately. Therefore, this type of methods is implementation friendly, since it does not require any iterative procedures. It has been successfully applied to many variants of the Black–Scholes model, including the Heston model in [16,19,20,25], the Merton and the Kon's jump diffusion model in [22], the infinite jump-diffusion model in [23], the regime-switching model in [24], and the fractional Black–Scholes model in [6]. However, no stability and convergence results are established in any of the above mentioned operator-splitting methods. We should mention that authors of [20] showed that the difference between the coupled Crank–Nicolson method and the operator splitting Crank–Nicolson method is of second-order in time for the option price, but no stability result is established. We refer to [17] for a review of the topic.

The goal of this paper is to establish stability results for the first- and second-order splitting schemes, and prove an error estimate for the first-order splitting scheme. The main challenge is how to prove the stability of the operator splitting schemes. The key is to recognize the similarity between these operator splitting schemes and the pressure-correction method for incompressible Navier–Stokes equations (cf. [12,13,31]), and adopt some of the essential procedures in the stability proofs of the pressure-correction scheme in [13,26]. We also carry out ample numerical experiments to demonstrate the convergence behaviors of the backward Euler and the BDF2 methods under different parameter regimes.

The paper is organized as follows. In the next section, we describe basic notations and assumptions used throughout the paper. In Sect. 3, we describe the formulation of the linear complementarity problem and related properties. In Sect. 4, we present the operator splitting methods, and prove their stability. In Sect. 5, we carry out an error analysis for the first-order method. In Sect. 6, we present some numerical results and related discussions.

2 Notations and Preliminaries

In this section, we introduce notations as well as some background materials that will be used. Throughout the paper, c denotes a generic constant that is independent of the time step size, Δt, but may depend on the data and the regularity of the exact solution.

We denote the L^2 norm by $\|\cdot\|$. We also introduce the discrete norms. Let $\{\phi^0, \phi^1, \ldots, \phi^n\}$ be a sequence of functions in a Hilbert space W, and $\Delta t = T/n$. We introduce the following discrete norms:

$$\|\phi\|^2_{l^2(W)} = \Delta t \sum_{i=0}^{n} \|\phi^i\|^2_W, \quad \|\phi\|^2_{l^\infty(W)} = \max_{0\leqslant i\leqslant n} \|\phi^i\|^2_W. \tag{2.1}$$

The following notations are used for discrete differences:

$$\delta\phi^{i+1} = \phi^{i+1} - \phi^i, \tag{2.2}$$
$$\delta^2\phi^{i+1} = \phi^{i+1} - 2\phi^i + \phi^{i-1}, \tag{2.3}$$
$$\mathcal{D}^2\phi^{i+1} = 3\phi^{i+1} - 4\phi^i + \phi^{i-1}. \tag{2.4}$$

We will frequently apply these algebraic identities:

$$2(a-b)a = a^2 - b^2 + (a-b)^2, \tag{2.5}$$
$$2(3a-4b+c)a = a^2 + (2a-b)^2 - b^2 - (2b-c)^2 + (a-2b+c)^2, \tag{2.6}$$
$$2(3a-4b+c)(a-b) = (a-b)^2 - (b-c)^2 + (a-2b+c)^2 + 4(a-b)^2. \tag{2.7}$$

We will frequently apply the following version of discrete Gronwall's Lemma (cf. [14,26]):

Lemma 2.1 (Discrete Gronwall's Lemma) *Let y^n, h^n, g^n, and f^n be nonnegative series such that*

$$y^m + \Delta t \sum_{n=0}^{m} h^n \leqslant B + \Delta t \sum_{n=0}^{m} (g^n y^n + f^n), \quad \Delta t \sum_{n=0}^{M} g^n \leqslant c, \quad 0 \leqslant m \leqslant M = \left[\frac{T}{\Delta t}\right]. \tag{2.8}$$

In addition, assume that $g^n \Delta t < 1$ for every n. Define $\nu = \max_{0\leqslant n\leqslant M}(1 - g^n\Delta t)^{-1}$. Then

$$y^m + \Delta t \sum_{n=0}^{m} h^n \leqslant e^{\nu K}\left(B + \Delta t \sum_{n=0}^{m} f^n\right), \quad 0 \leqslant m \leqslant M. \tag{2.9}$$

3 The Linear Complementarity Problem

Let S be the stock price, τ is the time and $t = T - \tau$ be the time to maturity T. The volatility σ and the interest rate r are assumed as constants. For any value of S and t, the price of an American option $V(S,t)$ satisfies the following linear complementarity problem (LCP):

$$V_t \geqslant \mathcal{L}V, \tag{3.1a}$$
$$V \geqslant g, \tag{3.1b}$$
$$(V_t - \mathcal{L}V)(V - g) = 0, \tag{3.1c}$$

where the spatial differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}V(S,t) = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV. \tag{3.2}$$

In the above $g = g(S)$ is a final pay-off function. For example, the put option is equipped with

$$g(S) = \max\{K - S, 0\}. \tag{3.3}$$

We briefly explain the meaning of (3.1). First, the return on a delta-hedged portfolio cannot be greater than that from a risk-free bank interest, because the American option may be exercised any time before expiration [32]. Thus we have (3.1a). Secondly, the option holder has the right to exercise it any time according the payoff, implying the option value is at least its payoff or (3.1b). Lastly, the option value is simply equal to its payoff when the option holder chooses to exercise it. Otherwise, the option value will be equal to the risk-neutral evaluation of a European option at that time period. One of the above two cases has to happen. It is why we have a quadratic expression in (3.1c). The above arguments hold for any (S, t), so system (3.1) always holds.

By introducing a slack function $\psi(S, t)$, we reformulate (3.1) into

$$V_t = \mathcal{L}V + \psi, \tag{3.4a}$$
$$\psi \geqslant 0, \tag{3.4b}$$
$$V \geqslant g, \tag{3.4c}$$
$$\psi(V - g) = 0. \tag{3.4d}$$

We assume the following initial and boundary conditions:

$$V(S, 0) = g(S), \tag{3.5a}$$
$$V(0, t) = K, \quad \lim_{S\to\infty} V(S, t) = 0. \tag{3.5b}$$

Regarding the operator $\mathcal{L}$, two integrations by parts lead to

$$\begin{aligned}(\mathcal{L}V, V) &= \frac{\sigma^2}{2}(S^2 V_{SS}, V) + r(SV_S, V) - r(V, V) \\ &= -\frac{\sigma^2}{2}(SV_S, SV_S) + (r - \sigma^2)(SV_S, V) - r(V, V) \\ &= -\frac{\sigma^2}{2}\|SV_S\|^2 + \frac{\sigma^2 - 3r}{2}\|V\|^2,\end{aligned} \tag{3.6}$$

where we have assumed $\lim_{S\to\infty} S^2 V_S V(S, t) = 0$ and $\lim_{S\to\infty} SV^2(S, t) = 0$ (cf. [1]). Therefore, $-\mathcal{L}$ is a strongly elliptic operator if $\sigma^2 - 3r \leq 0$. Formally, one can show that the system (3.4) and (3.5) possesses a unique solution through a discounted expectation evaluated at the optimal stopping time [21]. And its precise meaning can be given through the variational inequality [3,9,21]. The LCP approach was generalized to many kinds of options in [32] and was later studied in [15].

According to [21], the solution to the LCP (3.4) and (3.5) is continuously differentiable in the stock price. Regarding the time variable, we assume the following regularity results

$$\frac{V(S, t + k) - V(S, t)}{k} - V_t(S, t) = ck, \quad \psi(S, t + k) - \psi(S, t) = ck, \quad \forall t. \tag{3.7}$$

In this study, both σ and r are assumed as constants.

4 Numerical Methods and Stability Analysis

In this section, we describe a first-order and a second-order operator splitting methods, and prove that they are unconditionally stable.

The system (3.4) and (3.5) is first discretized in time on a uniform grid:

$$\Delta t = k = T/M, \quad t_n = n\Delta t, \quad 0 \leqslant n \leqslant M. \tag{4.1}$$

We start from $V_0(S) = g(S)$ and seek a sequence of functions,

$$V_1(S),\ V_2(S), \ldots,\ V_n(S), \ldots,\ V_M(S), \tag{4.2}$$

each of which is an approximation to $V(S,t)$ at the corresponding grid point t_n. $V_n(S)$ is often abbreviated as V_n. Similar notations are used for $\psi(S,t)$ and other involved variables. The initial conditions are $V_0 = g$ and $\psi_0 = 0$.

Let $S_{\max}$ denote the domain truncation for the semi-infinite domain, which is typically set as a multiple of K in numerical experiments.

4.1 The Operator Splitting Methods

We describe first the operator splitting method based on the backward Euler scheme [18]. Assume that V_n and ψ_n have been obtained from the time step of t_n. Two substeps will be performed at the step t_{n+1}. In the first substep, we solve for an intermediate solution $\tilde{V}_{n+1}$ from the boundary value problem (BVP):

$$\begin{cases} \dfrac{\tilde{V}_{n+1} - V_n}{\Delta t} - \mathcal{L}\tilde{V}_{n+1} = \psi_n, \\ \tilde{V}_{n+1}(0) = K, \quad \tilde{V}_{n+1}(S_{\max}) = 0. \end{cases} \tag{4.3}$$

In the second substep, we project $\tilde{V}_{n+1}$ to the constraint space to obtain V_{n+1}. Specifically, we perform the following correction:

$$\frac{V_{n+1} - \tilde{V}_{n+1}}{\Delta t} = \psi_{n+1} - \psi_n, \tag{4.4a}$$

$$\psi_{n+1} \geqslant 0, \quad V_{n+1} \geqslant g, \quad \psi_{n+1}(V_{n+1} - g) = 0. \tag{4.4b}$$

The system (4.4) is easy to solve in the (ψ_{n+1}, V_{n+1}) plane. Indeed, we have

$$(\psi_{n+1}, V_{n+1}) = \begin{cases} (0, \tilde{V}_{n+1} - \Delta t\psi_n), & \text{if } -\Delta t\psi_n + \tilde{V}_{n+1} \geqslant g, \\ \left(\dfrac{g - \tilde{V}_{n+1}}{\Delta t} + \psi_n, g\right), & \text{otherwise.} \end{cases} \tag{4.5}$$

To validate the discretization, we sum up (4.3) and (4.4) to get

$$\begin{cases} \dfrac{V_{n+1} - V_n}{\Delta t} - \mathcal{L}\tilde{V}_{n+1} = \psi_{n+1}, \\ \psi_j \geqslant 0, \quad V_{n+1} \geqslant g, \quad \psi_{n+1}(V_{n+1} - g) = 0. \end{cases} \tag{4.6}$$

According to (4.4a), we have

$$\tilde{V}_{n+1} = V_{n+1} - \Delta t(\psi_{n+1} - \psi_n), \tag{4.7}$$

which implies that $\tilde{V}_{n+1}$ is also an approximation to $V(S, t_{n+1})$.

Note that the scheme (4.3), (4.4) is reminiscent of the pressure-correction method for incompressible Navier–Stokes equations [12,13,31]. Indeed, the slack function ψ plays a similar role as the pressure, and is introduced to enforce (4.4b) which is similar to the divergence-free condition in the incompressible Navier–Stokes equations.

We now describe a second-order method based on the BDF2 discretization [18]. Assume that $\{V_n, V_{n-1}\}$ and $\{\psi_n, \psi_{n-1}\}$ have been obtained from time steps of t_n and t_{n-1}. Two substeps will be performed at the step t_{n+1}. In the first substep, we solve for an intermediate solution $\tilde{V}_{n+1}$ from the BVP:

$$\begin{cases} \dfrac{3\tilde{V}_{n+1} - 4V_n + V_{n-1}}{2\Delta t} - \mathcal{L}\tilde{V}_{n+1} = \psi_n, \\ \tilde{V}_{n+1}(0) = K, \quad \tilde{V}_{n+1}(S_{\max}) = 0. \end{cases} \tag{4.8}$$

In the second substep, we project $\tilde{V}_{n+1}$ to the constraint space to obtain V_{n+1} with the following correction:

$$\frac{3(V_{n+1} - \tilde{V}_{n+1})}{2\Delta t} = \psi_{n+1} - \psi_n, \tag{4.9a}$$

$$\psi_{n+1} \geqslant 0, \quad V_{n+1} \geqslant g, \quad \psi_{n+1}(V_{n+1} - g) = 0. \tag{4.9b}$$

Similarly as before, the system (4.9) is easy to solve in the (ψ_{n+1}, V_{n+1}) plane, since we have

$$(\psi_{n+1}, V_{n+1}) = \begin{cases} (0, \tilde{V}_{n+1} - \dfrac{2\Delta t}{3}\psi_n), & \text{if } -\dfrac{2\Delta t}{3}\psi_n + \tilde{V}_{n+1} \geqslant g, \\ (\dfrac{3(g - \tilde{V}_{n+1})}{2\Delta t} + \psi_n, g), & \text{otherwise.} \end{cases} \tag{4.10}$$

To validate the discretization, we sum up (4.8) and (4.9) to get

$$\begin{cases} \dfrac{3V_{n+1} - 4V_n + V_{n-1}}{2\Delta t} - \mathcal{L}\tilde{V}_{n+1} = \psi_{n+1}, \\ \psi_j \geqslant 0, \quad V_{n+1} \geqslant g, \quad \psi_{n+1}(V_{n+1} - g) = 0. \end{cases} \tag{4.11}$$

Similar to the first-order case, it is clear now that the scheme (4.8), (4.9) is reminiscent of the second-order pressure-correction method for incompressible Navier–Stokes equations [12,13,31].

4.2 Stability Results

We introduce new sequences $\{Z_n\}_{n=0}^M$ and $\{\tilde{Z}_n\}_{n=0}^M$ defined as

$$Z_n = V_n - g, \quad \tilde{Z}_n = \tilde{V}_n - g, \tag{4.12}$$

and rewrite the method (4.3) and (4.4) as

$$\begin{cases} \dfrac{\tilde{Z}_{n+1} - Z_n}{\Delta t} = \mathcal{L}\tilde{Z}_{n+1} + \mathcal{L}g + \psi_n, \\ \tilde{Z}_{n+1}(0) = 0, \quad \tilde{Z}_{n+1}(S_{\max}) = 0. \end{cases} \tag{4.13}$$

and

$$\frac{Z_{n+1} - \tilde{Z}_{n+1}}{\Delta t} = \psi_{n+1} - \psi_n, \tag{4.14a}$$

$$\psi_{n+1} \geqslant 0, \quad Z_{n+1} \geqslant 0, \quad \psi_{n+1}Z_{n+1} = 0. \tag{4.14b}$$

We have the following stability result for the above scheme.

Theorem 4.1 *We assume either $\sigma^2 - 3r \leqslant 0$ or $\Delta t \leqslant \frac{1}{4(\sigma^2-3r)+2}$ if $\sigma^2 - 3r > 0$. Then, the scheme (4.13) and (4.14) (or (4.3) and (4.4)) is stable in the sense that for all $m \geqslant 0$ we have*

$$\|Z_m\|^2 + \frac{1}{2}\sum_{n=1}^{m}\|\tilde{Z}_n - Z_{n-1}\|^2 + \Delta t^2\|\psi_m\|^2 + \sigma^2\Delta t\sum_{n=0}^{m}\|S(\tilde{Z}_n)_S\|^2$$
$$\leqslant c(B + \Delta t\sum_{n=0}^{m}\|\mathcal{L}g\|^2), \quad \forall 1 \le m \le T/\Delta t,$$

where B is related to the initial data:

$$B = \|Z_0\|^2 + \Delta t^2\|\psi_0\|^2 + \sigma^2\Delta t\|S(\tilde{Z}_0)_S\|^2 = \|g\|^2 + \Delta t^2\|\psi_0\|^2 + \sigma^2\Delta t\|Sg_S\|^2. \tag{4.15}$$

Proof First, take the inner product of (4.13) with $2\Delta t\tilde{Z}_{n+1}$ to have

$$\|\tilde{Z}_{n+1}\|^2 - \|Z_n\|^2 + \|\tilde{Z}_{n+1} - Z_n\|^2 = 2\Delta t(\mathcal{L}\tilde{Z}_{n+1}, \tilde{Z}_{n+1}) + 2\Delta t(\mathcal{L}g, \tilde{Z}_{n+1}) + 2\Delta t(\psi_n, \tilde{Z}_{n+1}). \tag{4.16}$$

For the term $2\Delta t(\mathcal{L}\tilde{Z}_{n+1}, \tilde{Z}_{n+1})$ we have

$$2\Delta t(\mathcal{L}\tilde{Z}_{n+1}, \tilde{Z}_{n+1}) = -\Delta t\sigma^2\|S(\tilde{Z}_{n+1})_S\|^2 + \Delta t(\sigma^2 - 3r)\|\tilde{Z}_{n+1}\|^2. \tag{4.17}$$

If $\beta = \sigma^2 - 3r > 0$, we can bound the last term by

$$\Delta t(\sigma^2 - 3r)\|\tilde{Z}_{n+1}\|^2 \leqslant 2\Delta t\beta\|(\tilde{Z}_{n+1} - Z_n\|^2 + \|Z_n\|^2). \tag{4.18}$$

For the term $2\Delta t(\mathcal{L}g, \tilde{Z}_{n+1})$, we have

$$\begin{aligned} 2\Delta t(\mathcal{L}g, \tilde{Z}_{n+1}) &= 2\Delta t(\mathcal{L}g, \tilde{Z}_{n+1} - Z_n) + 2\Delta t(\mathcal{L}g, Z_n) \\ &\leqslant 2\Delta t^2\|\mathcal{L}g\|^2 + \frac{1}{2}\|\tilde{Z}_{n+1} - Z_n\|^2 + \Delta t(\|\mathcal{L}g\|^2 + \|Z_n\|^2). \end{aligned} \tag{4.19}$$

Next, we rewrite (4.14a) as

$$Z_{n+1} - \Delta t\psi_{n+1} = \tilde{Z}_{n+1} - \Delta t\psi_n. \tag{4.20}$$

Taking the inner product on each side of (4.20) with itself, we obtain

$$\|Z_{n+1}\|^2 - 2\Delta t(Z_{n+1}, \psi_{n+1}) + \Delta t^2\|\psi_{n+1}\|^2 = \|\tilde{Z}_{n+1}\|^2 - 2\Delta t(\tilde{Z}_{n+1}, \psi_n) + \Delta t^2\|\psi_n\|^2. \tag{4.21}$$

Notice that $Z_{n+1}\psi_{n+1} = 0$ and the term $2\Delta t(\tilde{Z}_{n+1}, \psi_n)$ will cancel between (4.16) and (4.21). We add up the above results to obtain: (i) if $\sigma^2 - 3r \leqslant 0$, we have

$$\|Z_{n+1}\|^2 - \|Z_n\|^2 + \frac{1}{2}\|\tilde{Z}_{n+1} - Z_n\|^2 + \Delta t^2\|\psi_{n+1}\|^2 - \Delta t^2\|\psi_n\|^2 + \Delta t\sigma^2\|S(\tilde{Z}_{n+1})_S\|^2 \leqslant \Delta t\|Z_n\|^2 + c\Delta t\|\mathcal{L}g\|^2; \tag{4.22}$$

and (ii) if $\sigma^2 - 3r > 0$, we have

$$\|Z_{n+1}\|^2 - \|Z_n\|^2 + \|\tilde{Z}_{n+1} - Z_n\|^2 + \Delta t^2\|\psi_{n+1}\|^2 - \Delta t^2\|\psi_n\|^2 + \Delta t\sigma^2\|S(\tilde{Z}_{n+1})_S\|^2 \leqslant \Delta t(2\beta + 1)\|\tilde{Z}_{n+1} - Z_n\|^2 + \Delta t(2\beta + 1)\|Z_n\|^2 + c\Delta t\|\mathcal{L}g\|^2, \tag{4.23}$$

which, under the assumption that $\Delta t(2\beta+1) \leqslant \frac{1}{2}$, implies

$$\|Z_{n+1}\|^2 - \|Z_n\|^2 + \frac{1}{2}\|\tilde{Z}_{n+1} - Z_n\|^2 + \Delta t^2\|\psi_{n+1}\|^2 - \Delta t^2\|\psi_n\|^2 + \Delta t\sigma^2\|S(\tilde{Z}_{n+1})_S\|^2 \leqslant \Delta t(2\beta+1)\|Z_n\|^2 + c\Delta t\|\mathcal{L}g\|^2. \tag{4.24}$$

Summing up the inequalities (4.22) and (4.24) from $n = 0$ to $n = m-1$, respectively, and applying the discrete Gronwall's Lemma (Lemma 2.1), we obtain the desired result. □

Next, we consider the second-order operator splitting method based on BDF2. We introduce the same variables defined in (4.12) and rewrite the BDF2 method (4.8) and (4.9) into the following procedure. In the first sub-step, we solve

$$\begin{cases} \dfrac{3\tilde{Z}_{n+1} - 4Z_n + Z_{n-1}}{2\Delta t} = \mathcal{L}\tilde{Z}_{n+1} + \mathcal{L}g + \psi_n, \\ \tilde{Z}_{n+1}(0) = 0, \quad \tilde{Z}_{n+1}(S_{\max}) = 0. \end{cases} \tag{4.25}$$

In the second substep, we project $\tilde{Z}_{n+1}$ to the constraint space to obtain Z_{n+1} with the following correction:

$$\frac{3(Z_{n+1} - \tilde{Z}_{n+1})}{2\Delta t} = \psi_{n+1} - \psi_n, \tag{4.26a}$$

$$\psi_{n+1} \geqslant 0, \quad Z_{n+1} \geqslant 0, \quad \psi_{n+1}Z_{n+1} = 0. \tag{4.26b}$$

Theorem 4.2 *We assume either $\sigma^2 - 3r \leqslant 0$ or $\Delta t \leqslant \frac{1}{4(\sigma^2-3r)+2}$ if $\sigma^2 - 3r > 0$. Then, the second-order scheme (4.25) and (4.26) (or (4.8) and (4.9)) is stable in the sense that for all $m \geqslant 0$ we have*

$$\|Z_m\|^2 + \|2Z_m - Z_{m-1}\|^2 + \frac{4\Delta t^2}{3}\|\psi_m\|^2 + \sum_{n=0}^{m-1}\|\tilde{Z}_{n+1} - Z_{n+1}\|^2 + 2\Delta t\sigma^2\sum_{n=0}^{m}\|S(\tilde{Z}_n)_S\|^2 \leqslant c\Big(B + \Delta t\sum_{n=0}^{m}\|\mathcal{L}g\|^2\Big), \quad \forall 1 \le m \le T/\Delta t,$$

where B is related to the initial data:

$$B = \|Z_0\|^2 + \|2Z_1 - Z_0\|^2 + \frac{4\Delta t^2}{3}\|\psi_0\|^2 + 2\Delta t\sigma^2\|Sg_S\|^2. \tag{4.27}$$

Proof First, take the inner product of (4.25) with $4\Delta t\tilde{Z}_{n+1}$, to obtain

$$2(3\tilde{Z}_{n+1} - 4Z_n + Z_{n-1}, \tilde{Z}_{n+1}) = 4\Delta t(\mathcal{L}\tilde{Z}_{n+1}, \tilde{Z}_{n+1}) + 4\Delta t(\mathcal{L}g, \tilde{Z}_{n+1}) + 4\Delta t(\psi_n, \tilde{Z}_{n+1}). \tag{4.28}$$

We rewrite the left hand side of (4.28) into

$$\begin{aligned} 2(3\tilde{Z}_{n+1} - 4Z_n + Z_{n-1}, \tilde{Z}_{n+1}) &= 2(3Z_{n+1} - 4Z_n + Z_{n-1}, \tilde{Z}_{n+1}) + 6(\tilde{Z}_{n+1} - Z_{n+1}, \tilde{Z}_{n+1}) \\ &= 2(\mathcal{D}^2 Z_{n+1}, Z_{n+1}) + 2(3Z_{n+1} - 4Z_n + Z_{n-1}, \tilde{Z}_{n+1} - Z_{n+1}) + 6(\tilde{Z}_{n+1} - Z_{n+1}, \tilde{Z}_{n+1}). \end{aligned}$$

For the term $2(\mathcal{D}^2 Z_{n+1}, Z_{n+1})$ we apply (2.6) and have

$$2(\mathcal{D}^2 Z_{n+1}, Z_{n+1}) = \|Z_{n+1}\|^2 - \|Z_n\|^2 + \|2Z_{n+1} - Z_n\|^2 - \|2Z_n - Z_{n-1}\|^2 + \|\delta^2 Z_{n+1}\|^2. \tag{4.29}$$

Next, we have

$$\begin{aligned} &2(3Z_{n+1} - 4Z_n + Z_{n-1}, \tilde{Z}_{n+1} - Z_{n+1}) \\ &= 2(\delta^2 Z_{n+1}, \tilde{Z}_{n+1} - Z_{n+1}) + 4(Z_{n+1} - Z_n, \tilde{Z}_{n+1} - Z_{n+1}). \end{aligned} \tag{4.30}$$

For the term $2(\delta^2 Z_{n+1}, \tilde{Z}_{n+1} - Z_{n+1})$, we use Cauchy-Schwarz to get

$$\left|2(\delta^2 Z_{n+1}, \tilde{Z}_{n+1} - Z_{n+1})\right| \leqslant \|\delta^2 Z_{n+1}\|^2 + \|\tilde{Z}_{n+1} - Z_{n+1}\|^2. \tag{4.31}$$

For the cross term $4(Z_{n+1} - Z_n, \tilde{Z}_{n+1} - Z_{n+1}))$, we use (4.26) to obtain

$$\begin{aligned} 4(Z_{n+1} - Z_n, \tilde{Z}_{n+1} - Z_{n+1})) &= -\frac{8\Delta t}{3}(Z_{n+1} - Z_n, \psi_{n+1} - \psi_n) \\ &= -\frac{8\Delta t}{3}\Big((Z_{n+1}, \psi_{n+1}) + (Z_n, \psi_n) - (Z_{n+1}, \psi_n) - (Z_n, \psi_{n+1})\Big) \\ &= \frac{8\Delta t}{3}\Big((Z_{n+1}, \psi_n) + (Z_n, \psi_{n+1})\Big) \geqslant 0, \end{aligned} \tag{4.32}$$

since $(Z_n, \psi_n) = 0$ and $\psi_n,\ Z_n \geqslant 0$ for all n.

For the term $6(\tilde{Z}_{n+1} - Z_{n+1}, \tilde{Z}_{n+1})$, we have

$$6(\tilde{Z}_{n+1} - Z_{n+1}, \tilde{Z}_{n+1}) = 3(\|\tilde{Z}_{n+1}\|^2 - \|Z_{n+1}\|^2 + \|\tilde{Z}_{n+1} - Z_{n+1}\|^2). \tag{4.33}$$

Next, we rewrite the second sub-step as

$$3Z_{n+1} - 2\Delta t \psi_{n+1} = 3\tilde{Z}_{n+1} - 2\Delta t \psi_n. \tag{4.34}$$

Taking the inner product of each side of the above equation with itself, we obtain, after dividing both by 3 and since $(Z_{n+1}, \psi_{n+1}) = 0$, that

$$3\|Z_{n+1}\|^2 + \frac{4}{3}\Delta t^2\|\psi_{n+1}\|^2 = 3\|\tilde{Z}_{n+1}\|^2 - 4\Delta t(\tilde{Z}_{n+1}, \psi_n) + \frac{4}{3}\Delta t^2\|\psi_n\|^2. \tag{4.35}$$

Combining the above relations, we obtain

$$\begin{aligned} &\|Z_{n+1}\|^2 - \|Z_n\|^2 + \|2Z_{n+1} - Z_n\|^2 - \|2Z_n - Z_{n-1}\|^2 + \frac{4\Delta t^2}{3}(\|\psi_{n+1}\|^2 - \|\psi_n\|^2) \\ &\quad + 2\|\tilde{Z}_{n+1} - Z_{n+1}\|^2 \leqslant 4\Delta t(\mathcal{L}\tilde{Z}_{n+1}, \tilde{Z}_{n+1}) + 4\Delta t(\mathcal{L}g, \tilde{Z}_{n+1}). \end{aligned} \tag{4.36}$$

For the term $4\Delta t(\mathcal{L}\tilde{Z}_{n+1}, \tilde{Z}_{n+1})$, we proceed as in the first-order case. Namely, if $\beta = \sigma^2 - 3r > 0$, we bound it by

$$4\Delta t(\mathcal{L}\tilde{Z}_{n+1}, \tilde{Z}_{n+1}) \leqslant -2\Delta t\sigma^2\|S(\tilde{Z}_{n+1})_S\|^2 + 4\Delta t\beta\|\tilde{Z}_{n+1} - Z_{n+1}\|^2 + 4\Delta t\beta\|Z_{n+1}\|^2. \tag{4.37}$$

Similarly, for the term $4\Delta t(\mathcal{L}g, \tilde{Z}_{n+1})$ we have

$$\begin{aligned} 4\Delta t(\mathcal{L}g, \tilde{Z}_{n+1}) &= 4\Delta t(\mathcal{L}g, \tilde{Z}_{n+1} - Z_{n+1}) + 4\Delta t(\mathcal{L}g, Z_{n+1}) \\ &\leqslant 4\Delta t^2\|\mathcal{L}g\|^2 + \|\tilde{Z}_{n+1} - Z_{n+1}\|^2 + 2\Delta t(\|\mathcal{L}g\|^2 + \|Z_{n+1}\|^2). \end{aligned} \tag{4.38}$$

Then under the assumption that $\sigma^2 - 3r \leqslant 0$ or $\Delta t(2\beta+1) \leqslant \frac{1}{2}$ if $\sigma^2 - 3r > 0$, we combine all the above results into (4.36) to get

$$\begin{aligned}&\|Z_{n+1}\|^2 - \|Z_n\|^2 + \|2Z_{n+1} - Z_n\|^2 - \|2Z_n - Z_{n-1}\|^2 \\ &\quad + \|\delta^2 Z_{n+1}\|^2 + \frac{4\Delta t^2}{3}(\|\psi_{n+1}\|^2 - \|\psi_n\|^2) \\ &\quad + \|\tilde{Z}_{n+1} - Z_{n+1}\|^2 + 2\Delta t\sigma^2 \|S(\tilde{Z}_{n+1})_S\|^2 \leqslant c\Delta t\|\mathcal{L}g\|^2 + c\Delta t\|Z_{n+1}\|^2.\end{aligned}$$

Summing up the above inequality from $n = 0$ to $n = m-1$ and applying the discrete Gronwall's Lemma (Lemma 2.1), we obtain the desired result. □

Remark 4.3 Note that the conditions on Δt in the above theorems are just sufficient conditions for the stability. No special effort is made to optimize these conditions, and they can be slightly relaxed with a refined analysis.

5 Error Estimates

With the stability results established in the last section, one can then derive corresponding error estimates by assuming adequate regularity on the exact solution. For the sake of brevity, we establish in this section error estimates for the first-order method only.

First, we introduce error functions and error equations. Denote the exact solution $Z(S,t)$ at t_n by $Z(\cdot, t_n)$, and similar for other related variables. We define

$$\begin{aligned}&e_n = Z(\cdot, t_n) - Z_n, \quad \tilde{e}_n = Z(\cdot, t_n) - \tilde{Z}_n, \\ &h_n = \psi(\cdot, t_n) - \psi_n, \quad f_n = h_n + \psi(\cdot, t_{n+1}) - \psi(\cdot, t_n),\end{aligned} \tag{5.1}$$

and assume that ψ is sufficiently smooth such that

$$\|f_n\| \leq \|h_n\| + c\Delta t, \tag{5.2}$$

where c is independent of n. From the continuous system (3.4), we have

$$\frac{Z(\cdot, t_{n+1}) - Z(\cdot, t_n)}{\Delta t} = \mathcal{L}Z(\cdot, t_{n+1}) + \mathcal{L}g + \psi(\cdot, t_{n+1}) + R_{n+1}. \tag{5.3}$$

To obtain the error equation, we subtract (5.3) from (4.13), to have

$$\frac{\tilde{e}_{n+1} - e_n}{\Delta t} = \mathcal{L}\tilde{e}_{n+1} + f_n + R_{n+1}. \tag{5.4}$$

We can obtain another error equation from the second sub-step (4.4), as follows,

$$\frac{e_{n+1} - \tilde{e}_{n+1}}{\Delta t} = h_{n+1} - h_n - (\psi(\cdot, t_{n+1}) - \psi(\cdot, t_n)), \tag{5.5}$$

or,

$$e_{n+1} - \Delta t\, h_{n+1} = \tilde{e}_{n+1} - \Delta t f_n, \tag{5.6}$$

Theorem 5.1 *We assume either $\sigma^2 - 3r \leqslant 0$ or $\Delta t \leqslant \frac{1}{4(\sigma^2-3r)+2}$ if $\sigma^2 - 3r > 0$, and that the solution (Z, ψ) is sufficiently smooth. Then, the following error estimates hold for the method* (4.13), (4.14) (*or* (4.3), (4.4)):

$$\|e_m\|^2 + \Delta t \sum_{n=0}^{m-1} \|S(\tilde{e}_{n+1})_S\|^2 \leqslant c\Delta t^2, \quad \forall 1 \leq m \leq T/\Delta t.$$

Proof First, take the inner product of (5.4) with $2\Delta t \tilde{e}_{n+1}$ to have

$$\begin{aligned}\|\tilde{e}_{n+1}\|^2 - \|e_n\|^2 + \|\tilde{e}_{n+1} - e_n\|^2 &= 2\Delta t(\mathcal{L}\tilde{e}_{n+1}, \tilde{e}_{n+1}) \\ &\quad + 2\Delta t(f_n, \tilde{e}_{n+1}) + 2\Delta t(R_{n+1}, \tilde{e}_{n+1}).\end{aligned} \tag{5.7}$$

Next, take the inner product of (5.6) on each side with itself, we obtain

$$\|e_{n+1}\|^2 - 2\Delta t(e_{n+1}, h_{n+1}) + \Delta t^2\|h_{n+1}\|^2 = \|\tilde{e}_{n+1}\|^2 - 2\Delta t(\tilde{e}_{n+1}, f_n) + \Delta t^2\|f_n\|^2. \tag{5.8}$$

Combine (5.7) and (5.8) to get

$$\|e_{n+1}\|^2 - \|e_n\|^2 + \|\tilde{e}_{n+1} - e_n\|^2 + \Delta t^2(\|h_{n+1}\|^2 - \|f_n\|^2) \tag{5.9}$$
$$= 2\Delta t(\mathcal{L}\tilde{e}_{n+1}, \tilde{e}_{n+1}) + 2\Delta t(e_{n+1}, h_{n+1}) + 2\Delta t(R_{n+1}, \tilde{e}_{n+1}). \tag{5.10}$$

For the term $2\Delta t(\mathcal{L}\tilde{e}_{n+1}, \tilde{e}_{n+1})$, similar to (4.17), we have

$$2\Delta t(\mathcal{L}\tilde{e}_{n+1}, \tilde{e}_{n+1}) = -\Delta t\sigma^2\|S(\tilde{e}_{n+1})_S\|^2 + \Delta t(\sigma^2 - 3r)\|\tilde{e}_{n+1}\|^2. \tag{5.11}$$

If $\beta = \sigma^2 - 3r > 0$, we bound the last term by

$$\Delta t(\sigma^2 - 3r)\|\tilde{e}_{n+1}\|^2 \leqslant 2\Delta t\beta\|(\tilde{e}_{n+1} - e_n\|^2 + \|e_n\|^2). \tag{5.12}$$

For the term $2\Delta t(e_{n+1}, h_{n+1})$ we notice that

$$\begin{aligned}(e_{n+1}, h_{n+1}) &= (Z(\cdot, t_{n+1}) - Z_{n+1}, \psi(\cdot, t_{n+1}) - \psi_{n+1}) \\ &= (Z(\cdot, t_{n+1}), \psi(\cdot, t_{n+1})) - (Z(\cdot, t_{n+1}), \psi_{n+1}) - (Z_{n+1}, \psi(\cdot, t_{n+1})) + (Z_{n+1}, \psi_{n+1}).\end{aligned}$$

Since $Z(\cdot, t_{n+1})\psi(\cdot, t_{n+1}) = 0$ and $Z_{n+1}\psi_{n+1} = 0$, we have

$$(e_{n+1}, h_{n+1}) = -(Z(\cdot, t_{n+1}), \psi_{n+1}) - (Z_{n+1}, \psi(\cdot, t_{n+1})). \tag{5.13}$$

Furthermore, since $Z(\cdot, t_{n+1})$, $\psi(\cdot, t_{n+1})$, Z_{n+1}, and ψ_{n+1} are all nonnegative, we derive from the above that

$$2\Delta t(e_{n+1}, h_{n+1}) \leqslant 0. \tag{5.14}$$

For the term $2\Delta t(R_{n+1}, \tilde{e}_{n+1})$, we have

$$\begin{aligned}2\Delta t(R_{n+1}, \tilde{e}_{n+1}) &= 2\Delta t(R_{n+1}, \tilde{e}_{n+1} - e_n) + 2\Delta t(R_{n+1}, e_n) \\ &\leqslant \Delta t(\|R_{n+1}\|^2 + \|\tilde{e}_{n+1} - e_n\|^2) + \Delta t(\|R_{n+1}\|^2 + \|e_n\|^2).\end{aligned} \tag{5.15}$$

We derive from (5.2) that

$$\|f_n\|^2 \leqslant \|h_n\|^2 + c\Delta t\|h_n\| + c\Delta t^2 \leq \|h_n\|^2 + c\Delta t\|h_n\|^2 + c\Delta t.$$

If $\beta = \sigma^2 - 3r \leqslant 0$ or $\Delta t(2\beta + 1) \leqslant \frac{1}{2}$ if $\beta = \sigma^2 - 3r > 0$, we add up the above results to obtain

$$\begin{aligned}&\|e_{n+1}\|^2 - \|e_n\|^2 + \frac{1}{2}\|\tilde{e}_{n+1} - e_n\|^2 + \Delta t^2\|h_{n+1}\|^2 - \Delta t^2\|h_n\|^2 + \Delta t\sigma^2\|S(\tilde{e}_{n+1})_S\|^2 \\ &\leqslant \Delta t(2\beta + 1)\|e_n\|^2 + c\Delta t^3\|h_n\|^2 + c\Delta t^3 + 2\Delta t\|R_{n+1}\|^2.\end{aligned}$$

Summing up the above inequality from $n = 0$ to $n = m - 1$ and applying the discrete Gronwall's Lemma (Lemma 2.1), we obtain the desired result. □

6 Numerical Results and Discussions

In this section, we present some numerical results and discuss their performance of the two schemes.

6.1 Spatial Discretization

At each time step, we need to solve a boundary value problem (4.13) or (4.25). We employ two methods in space: a central finite difference method on the truncated domain $(0, S_{\max})$ with uniform grid and a spectral element method on the semi-infinite domain with Gaussian points [7].

The mesh for the finite difference method is given as

$$0 = S_0 < S_1 < \cdots < S_N = S_{\max}, \quad S_{i+1} - S_i = h = \Delta S = S_{\max}/N. \tag{6.1}$$

Then at each time step, we need to solve the following equation:

$$\begin{aligned} &c_1 S^2 u''(S) + c_2 S u'(S) + c_3 u = f, \quad S \in (0, S_N), \\ &a_1 u(0) + b_1 u'(0) = g_1, \quad a_2 u(S_N) + b_2 u'(S_N) = g_2. \end{aligned} \tag{6.2}$$

We discretize the above equation with the second-order centered finite difference method:

$$c_1 S_i^2 \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + c_2 S_i \frac{u_{i+1} - u_{i-1}}{2h} + c_3 u_i = f_i, \quad 1 \leqslant i \leqslant N-1, \tag{6.3}$$

where u_i denotes the numerical approximation to $u(S_i)$. And the boundary conditions are discretized with BDF2 formulas:

$$\begin{aligned} a_1 u_0 - b_1 \frac{u_2 - 4u_1 + 3u_0}{2h} &= g_1, \\ a_2 u_N + b_2 \frac{3u_N - 4u_{N-1} + u_{N-2}}{2h} &= g_2. \end{aligned} \tag{6.4}$$

As indicated in the previous section, we use the Dirichlet boundary condition where $u_0 = K$ and $u_N = 0$. Hence, the linear system is tridiagonal.

For the spectral element method, we use Legendre polynomials in finite subdomains and Laguerre functions on the semi-infinite subdomain. Linear hat functions were used to connect basis on neighboring subdomains. Detailed descriptions of the spectral element method refer to Sect. 4.1 in [7].

6.2 Stability and Order of Convergence

We perform numerical experiments with the following set of parameters:

$$T = 1, \quad K = 50, \quad r = 0.01, \quad S_{\max} = 2K. \tag{6.5}$$

We measure the point-wise error at the strike K (as $|\cdot|_K$), the l_∞ error (as $\|\cdot\|_\infty$), and the l_2 error (as $\|\cdot\|_2$). In all our numerical tests, the two operator splitting schemes are always stable with with a wide range of σ and r.

For the finite difference method, we use a sufficiently fine mesh $h = 1/2^{10}$ in space so that the errors are dominated by time discretization. In the following, we take $\sigma = 0.01$ or 0.2. Note that for $\sigma = 0.01$ we have $\sigma^2 - 3r < 0$, while for $\sigma = 0.2$ we have $\sigma^2 - 3r > 0$.

Table 1 (With the finite difference method) The order of convergence Ω for the first-order method (4.3) and (4.4), with parameters in (6.5) and $\sigma = 0.01$

$M = 2^i$	$\|e_{(2k)}\|_2$	Ω	$\|e_{(2k)}\|_\infty$	Ω	$\lvert e_{(2k)}\rvert_K$	Ω
$i = 4$	2.92E−04	–	4.86E−04	–	3.15E−04	–
$i = 5$	1.76E−04	0.73	2.89E−04	0.75	2.46E−04	0.35
$i = 6$	9.49E−05	0.89	1.55E−04	0.91	1.39E−04	0.83
$i = 7$	4.95E−05	0.94	8.00E−05	0.95	7.34E−05	0.92
$i = 8$	2.47E−05	1.00	3.98E−05	1.01	3.68E−05	0.99
$i = 9$	1.17E−05	1.08	1.88E−05	1.08	1.75E−05	1.07
$i = 10$	5.06E−06	1.21	8.13E−06	1.21	7.57E−06	1.21

Table 2 (With the finite difference method) The order of convergence Ω for the first-order method (4.3) and (4.4), with parameters in (6.5) and $\sigma = 0.2$

$M = 2^i$	$\|e_{(2k)}\|_2$	Ω	$\|e_{(2k)}\|_\infty$	Ω	$\lvert e_{(2k)}\rvert_K$	Ω
$i = 4$	1.30E−01	–	3.73E−02	–	3.73E−02	–
$i = 5$	6.70E−02	0.95	1.92E−02	0.96	1.92E−02	0.96
$i = 6$	3.42E−02	0.97	9.78E−03	0.97	9.78E−03	0.97
$i = 7$	1.73E−02	0.99	4.93E−03	0.99	4.93E−03	0.99
$i = 8$	8.54E−03	1.02	2.44E−03	1.02	2.44E−03	1.02
$i = 9$	4.06E−03	1.07	1.16E−03	1.07	1.16E−03	1.07
$i = 10$	1.77E−03	1.20	5.04E−04	1.20	5.03E−04	1.20

We shall use $M = 2^k$ uniform points in time. The order of convergence Ω is computed by

$$\Omega = \log_2 \frac{\|e_{(2k)}\|}{\|e_{(k)}\|}. \tag{6.6}$$

In the above, $e_{(k)}$ is defined as

$$e_{(k)} = V_{(k)} - V_\star, \tag{6.7}$$

where $V_{(k)}$ denotes the numerical solution with time step size k and $V_\star$ denotes a reference solution computed with $M_\star = 2^{12}$ points.

Results from Table 1, 2, 3 and 4 are obtained with the finite difference method. In Table 1, we present the order of convergence of the first-order method (4.3) and (4.4) with $\sigma = 0.01$. Table 2 lists the results with $\sigma = 0.2$. In both cases, the convergence rate reaches its asymptotic rate of first-order when the time step size is sufficiently fine ($\Delta t \le 2^{-7}$).

In Tables 3 and 4 , we list the order of convergence for the BDF2 method with $\sigma = 0.01$ and $\sigma = 0.2$, respectively. We observe that, when $\sigma = 0.01$, the order of convergence quickly reaches its asymptotic rate of 2. However, when $\sigma = 0.2$, the order of convergence would only reach its asymptotic rate with much smaller time step sizes.

Results from Table 5, 6, 7 and 8 are obtained with the spectral element method. We use 3 elements as follows:

$$I_1 = [0, K], \quad I_2 = [K, 2K], \quad I_3 = [2K, \infty), \tag{6.8}$$

Table 3 (With the finite difference method) The order of convergence Ω for the BDF2 method (4.8) and (4.9), with parameters in (6.5) and $\sigma = 0.01$

$M = 2^i$	$\|e_{(2k)}\|_2$	Ω	$\|e_{(2k)}\|_\infty$	Ω	$\|e_{(2k)}\|_K$	Ω
$i = 4$	9.21E−05	–	2.33E−04	–	1.41E−04	–
$i = 5$	2.06E−05	2.16	5.17E−05	2.17	3.21E−05	2.13
$i = 6$	4.93E−06	2.06	1.24E−05	2.06	7.56E−06	2.08
$i = 7$	1.25E−06	1.99	3.11E−06	1.99	1.95E−06	1.96
$i = 8$	2.97E−07	2.07	7.40E−07	2.07	4.33E−07	2.17
$i = 9$	7.14E−08	2.06	1.75E−07	2.08	9.58E−08	2.18
$i = 10$	1.69E−08	2.08	4.08E−08	2.10	2.15E−08	2.16

Table 4 (With the finite difference method) The order of convergence Ω for the BDF2 method (4.8) and (4.9), with parameters in (6.5) and $\sigma = 0.2$

$M = 2^i$	$\|e_{(2k)}\|_2$	Ω	$\|e_{(2k)}\|_\infty$	Ω	$\|e_{(2k)}\|_K$	Ω
$i = 4$	1.97E−02	–	4.93E−03	–	4.76E−03	–
$i = 5$	7.03E−03	1.48	1.77E−03	1.48	1.77E−03	1.43
$i = 6$	2.63E−03	1.42	6.56E−04	1.43	6.49E−04	1.45
$i = 7$	1.00E−03	1.39	2.45E−04	1.42	2.34E−04	1.47
$i = 8$	3.87E−04	1.37	1.00E−04	1.29	8.34E−05	1.49
$i = 9$	1.43E−04	1.44	3.50E−05	1.51	2.84E−05	1.55
$i = 10$	4.89E−05	1.54	1.30E−05	1.43	9.13E−06	1.64
$i = 11$	1.25E−05	1.97	2.90E−06	2.16	2.28E−06	2.00

Table 5 (With the spectral elmement method) The order of convergence Ω for the first-order method (4.3) and (4.4), with parameters in (6.5) and $\sigma = 0.04$

$M = 2^i$	$\|e_{(2k)}\|_2$	Ω	$\|e_{(2k)}\|_\infty$	Ω	$\|e_{(2k)}\|_K$	Ω
$i = 2$	1.40E−1	1.10	9.36E−2	1.21	9.30E−2	1.22
$i = 3$	6.76E−2	1.05	4.44E−2	1.08	4.41E−2	1.08
$i = 4$	3.36E−2	1.01	2.18E−2	1.03	2.17E−2	1.02
$i = 5$	1.68E−2	1.00	1.09E−2	1.01	1.08E−2	1.00
$i = 6$	8.48E−3	0.99	5.43E−3	1.00	5.41E−3	1.00
$i = 7$	4.26E−3	0.99	2.72E−3	1.00	2.70E−3	1.00
$i = 8$	2.10E−3	1.02	1.34E−3	1.02	1.33E−3	1.02
$i = 9$	9.98E−4	1.08	6.33E−4	1.08	6.31E−4	1.08
$i = 10$	4.33E−4	1.20	2.75E−4	1.20	2.74E−4	1.20
$i = 11$	1.46E−4	1.57	9.22E−5	1.57	9.20E−5	1.57

with $N_1 = 128$, $N_2 = 512$, and $N_3 = 128$ ($N_i + 1$ is the number of quadrature points in the subdomain I_i.) For I_3, we set the largest Laguerre point on the right end to be $3K$. In terms of convergence in time, we can observe behaviors similar to those from the finite difference method.

Table 6 (With the spectral elmement method) The order of convergence Ω for the first-order method (4.3) and (4.4), with parameters in (6.5) and $\sigma = 0.2$

$M = 2^i$	$\|e_{(2k)}\|_2$	Ω	$\|e_{(2k)}\|_\infty$	Ω	$\|e_{(2k)}\|_K$	Ω
$i = 2$	2.26E+0	1.03	6.29E−1	1.14	6.26E−1	1.14
$i = 3$	1.13E+0	1.00	3.03E−1	1.05	3.02E−1	1.05
$i = 4$	5.65E−1	1.00	1.50E−1	1.02	1.49E−1	1.02
$i = 5$	2.83E−1	1.00	7.45E−2	1.01	7.43E−2	1.01
$i = 6$	1.41E−1	1.00	3.71E−2	1.01	3.70E−2	1.01
$i = 7$	7.01E−2	1.01	1.83E−2	1.02	1.83E−2	1.02
$i = 8$	3.42E−2	1.04	8.91E−3	1.04	8.89E−3	1.04
$i = 9$	1.60E−2	1.09	4.18E−3	1.09	4.17E−3	1.09
$i = 10$	6.92E−3	1.21	1.80E−3	1.22	1.80E−3	1.21
$i = 11$	2.32E−3	1.58	6.03E−4	1.58	6.01E−4	1.58

Table 7 (With the spectral elmement method) The order of convergence Ω for the BDF2 method (4.8) and (4.9), with parameters in (6.5) and $\sigma = 0.04$

$M = 2^i$	$\|e_{(2k)}\|_2$	Ω	$\|e_{(2k)}\|_\infty$	Ω	$\|e_{(2k)}\|_K$	Ω
$i = 2$	2.08E−2	1.96	1.66E−2	2.01	1.10E−2	2.04
$i = 3$	4.49E−3	2.21	2.86E−3	2.54	2.38E−3	2.21
$i = 4$	2.33E−3	0.95	3.09E−3	-0.11	4.96E−4	2.26
$i = 5$	8.21E−4	1.50	8.31E−4	1.89	8.34E−5	2.57
$i = 6$	2.25E−4	1.87	2.11E−4	1.98	1.00E−5	3.06
$i = 7$	6.96E−5	1.69	5.48E−5	1.94	6.79E−6	0.56
$i = 8$	2.20E−5	1.66	1.29E−5	2.09	6.05E−6	0.17
$i = 9$	9.02E−6	1.29	5.56E−6	1.21	4.24E−6	0.51
$i = 10$	4.72E−6	0.94	2.89E−6	0.94	2.63E−6	0.69
$i = 11$	1.46E−6	1.69	8.89E−7	1.70	8.37E−7	1.65

Table 8 (With the spectral elmement method) The order of convergence Ω for the BDF2 method (4.8) and (4.9), with parameters in (6.5) and $\sigma = 0.2$

$M = 2^i$	$\|e_{(2k)}\|_2$	Ω	$\|e_{(2k)}\|_\infty$	Ω	$\|e_{(2k)}\|_K$	Ω
$i = 2$	1.96E−1	1.64	5.03E−2	1.65	3.62E−2	2.01
$i = 3$	5.98E−2	1.71	1.46E−2	1.79	1.27E−2	1.51
$i = 4$	1.98E−2	1.59	4.89E−3	1.57	4.75E−3	1.42
$i = 5$	7.02E−3	1.50	1.77E−3	1.46	1.77E−3	1.42
$i = 6$	2.64E−3	1.41	8.39E−4	1.08	6.38E−4	1.47
$i = 7$	9.96E−4	1.41	2.73E−4	1.62	2.35E−4	1.44
$i = 8$	3.72E−4	1.42	9.31E−5	1.55	8.76E−5	1.42
$i = 9$	1.47E−4	1.34	3.70E−5	1.33	3.50E−5	1.32
$i = 10$	5.67E−5	1.37	1.41E−5	1.39	1.36E−5	1.37
$i = 11$	1.71E−5	1.73	4.21E−6	1.75	4.09E−6	1.73

6.3 Summary

We considered two operator splitting methods for American options under the classic Black–Scholes model. The two operator splitting methods that we consider are reminiscent to the first and second-order pressure-correction methods. By adopting some essential procedures in the stability proof of pressure-correction methods, we were able to establish the first rigorous stability results for the first and second-order splitting methods. We have also derived error estimates for the first-order splitting method, and presented numerical results to demonstrate the convergence behaviors of the two operating splitting methods.

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