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LAGUERRE AND COMPOSITE LEGENDRE-LAGUERRE DUAL-PETROV-GALERKIN METHODS FOR THIRD-ORDER EQUATIONS

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Abstract. Dual-Petrov-Galerkin approximations to linear third-order equations and the Korteweg-de Vries equation on semi-infinite intervals are considered. It is shown that by choosing appropriate trial and test basis functions the Dual-Petrov-Galerkin method using Laguerre functions leads to strongly coercive linear systems which are easily invertible and enjoy optimal convergence rates. A novel multi-domain composite Legendre-Laguerre dual-Petrov-Galerkin method is also proposed and implemented. Numerical results illustrating the superior accuracy and effectiveness of the proposed dual-Petrov-Galerkin methods are presented.

1. **Introduction.** For numerical approximations of partial differential equations which are set on semi-infinite intervals, an effective tool is to use the Laguerre polynomials/functions which are mutually orthogonal with respect to appropriate inner product in $(0,\infty)$. There have been a number of investigations on using Laguerre polynomials/functions for elliptic type equations (cf. [15, 7, 10, 16, 9]), but not many results are available on using Laguerre polynomials/functions for equations of other type. However, some physically interesting equations, e.g., the Korteweg-de Vries (KDV) equation, are naturally set on a semi-infinite interval. Hence, it would be desirable to have an accurate and efficient numerical method for third-order equations on a semi-infinite interval. This is a challenging task since it involves two distinct difficulties associated with unbounded domain and third-order operator.

Recently, the first author introduced in [17] a dual-Petrov-Galerkin method for third and higher odd-order equations in a finite interval. The key idea is to use trial functions satisfying the underlying boundary conditions of the differential equations

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and test functions satisfying a set of "dual" boundary conditions. The resulted variational formulation for third and higher odd-order dispersive equations becomes strongly coercive. Consequently, it leads to optimal spectral convergence rates and a very efficient and accurate algorithm. We note that the well-posedness and decay properties of this dual-Petrov-Galerkin formulation for the KDV equation has been studied recently in [4].

The purpose of this paper is two-fold: (i) to develop and analyze a dual-Petrov-Galerkin method on a semi-infinite interval using Laguerre functions; and (ii) to develop a well-posed multi-domain composite Legendre-Laguerre method which is better suited in practical use than the single domain Laguerre method. We note that although multi-domain techniques are well developed for second-order equations, it is a non-trivial task to design a well-posed multi-domain spectral algorithm for third-order equations.

We now introduce some notations. Let $\Lambda = (a, b)$ with $-\infty < a < b \leq +\infty$, and $\omega(x)$ be a given weight function in Λ, which is not necessary in $L^1(\Lambda)$. We shall use the weighted Sobolev spaces $H^r_\omega(\Lambda)$ $(r = 0, 1, 2, \cdots)$, whose inner products, norms and semi-norms are denoted by $(\cdot,\cdot)_{r,\omega}$, $\|\cdot\|_{r,\omega}$ and $|\cdot|_{r,\omega}$. For real $r > 0$, we define the space $H^r_\omega(\Lambda)$ by space interpolation. In particular, the norm and inner product of $L^2_\omega(\Lambda) = H^0_\omega(\Lambda)$ are denoted by $\|\cdot\|_\omega$ and $(\cdot,\cdot)_\omega$, respectively. The subscript ω will be omitted from the notations in case of $\omega \equiv 1$. For simplicity, we denote $\partial^k_x v = \frac{d^k v}{dx^k}, k \geq 1$.

We denote by c a generic positive constant independent of any function and N. The expression $A \lesssim B$ means that there exists a generic positive constant c such that $A \leq cB$.

The remainder of the paper is organized as follows. In the next section, we introduce the Laguerre Dual-Petrov-Galerkin method, provide details for its efficient implementation and present illustrative numerical experiments. In Section 3, we prove the error estimates for both a third-order linear equation and the KDV equation. In Section 4, we develop a multi-domain composite Legendre-Laguerre dual-Petrov-Galerkin method and present some numerical results.

2. **Laguerre Dual-Petrov-Galerkin Method.**

In this section, we propose a Laguerre dual-Petrov-Galerkin (LDPG) method for third-order equations, and provide a theoretical and numerical study on the third-order derivative operator.

Let us first recall some basic properties of the Laguerre polynomial which is denoted by $\mathcal{L}_n(x)$ (cf. [21]):

$$\int_0^\infty \mathcal{L}_m(x)\mathcal{L}_n(x)e^{-x}dx = \delta_{m,n}; \tag{2.1}$$

$$\mathcal{L}_n(0) = 1, \;\; \mathcal{L}'_n(0) = -n, \;\; \mathcal{L}_n(x) = \partial_x\mathcal{L}_n(x) - \partial_x\mathcal{L}_{n+1}(x). \tag{2.2}$$

It is well-known that the Laguerre polynomials are not suitable for practical use because their wild behaviors as $x \to +\infty$. On the other hand, the Laguerre functions, defined as $\hat{\mathcal{L}}_n(x) = \mathcal{L}_n(x)e^{-x/2}$, have desirable properties which are preferable in practice. Let $\mathcal{R}_+ := (0, +\infty)$, one derives from (2.1) that $\{\hat{\mathcal{L}}_n(x)\}$ form a sequence of orthogonal basis in $L^2(\mathcal{R}_+)$, i.e.,

$$\int_0^\infty \hat{\mathcal{L}}_n(x)\hat{\mathcal{L}}_m(x)dx = \delta_{n,m}. \tag{2.3}$$

We emphasize that in contrast to the Laguerre polynomials, the Laguerre functions are well-behaved, as indicated by the following relations (cf. page 40 in [6])

$$\hat{\mathcal{L}}_n(0) = 1, \quad |\hat{\mathcal{L}}_n(x)| \le 1, \quad n \ge 0,\ x \in \mathcal{R}_+, \tag{2.4}$$

and (cf. Thm. 8.22.1 in [21])

$$\hat{\mathcal{L}}_n(x) = \pi^{-1/2}(nx)^{-1/4}\cos(2(nx)^{1/2} - \pi/4) + O(e^{-x/2}n^{-3/4}), \qquad \forall x \in \mathcal{R}_+. \tag{2.5}$$

2.1. **A linear third-order equation.** Let us consider first the following model third-order equation

$$\begin{aligned} &u_{xxx} + \beta u = f, \qquad \beta > 0, \quad x \in \mathcal{R}_+,\\ &u(0) = 0, \ \lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} \partial_x u(x) = 0. \end{aligned} \tag{2.6}$$

Since the third-order operator is not symmetric, it is natural to use a Petrov-Galerkin method, in which the trial and test function spaces are different. It is shown in [17] that for third and higher odd-order equation, it is advantageous to choose the trail and test function spaces satisfying "dual" boundary conditions.

Let us denote $\widehat{P}_N := \text{span}\{\hat{\mathcal{L}}_n : n = 0, 1, \cdots, N\}$. Then, thanks to (2.5), the asymptotic "boundary" conditions at infinity are automatically satisfied by functions in $\widehat{\mathcal{P}}_N$. Hence, it is natural to define the "dual" approximation spaces as follows:

$$X_N := \{u \in \widehat{\mathcal{P}}_N : u(0) = 0\}, \quad X_N^* := \{u \in \widehat{\mathcal{P}}_{N+1} : u(0) = u_x(0) = 0\}. \tag{2.7}$$

The Laguerre dual-Petrov-Galerkin approximation to (2.6) is to: Find $u_N \in X_N$ such that

$$(\partial_x u_N, \partial_x^2 v_N) + \beta(u_N, v_N) = (f, v_N), \qquad \forall v_N \in X_N^*. \tag{2.8}$$

It is clear that $xu_N \in X_N^*$ for any $u_N \in X_N$. We denote hereafter

$$\omega_\alpha(x) = x^\alpha e^{-x}, \quad \hat{\omega}_\alpha(x) = x^\alpha. \tag{2.9}$$

For simplicity, we also denote $\omega(x) = \omega_0(x)$ and $\hat{\omega}(x) = \hat{\omega}_0(x)$. We have the following result on the stability and well-posedness of the LDGP scheme (2.8).

Lemma 2.1.

$$\frac{\beta}{2}\|u_N\|^2_{\hat{\omega}_1} + \frac{3}{2}|u_N|_1^2 \le \frac{1}{2\beta}\|f\|^2_{\hat{\omega}_1}. \tag{2.10}$$

Proof. Thanks to the homogeneous boundary conditions built in X_N, integration by parts yields

$$\begin{aligned} (\partial_x u_N, \partial_x^2(xu_N)) &= (\partial_x u_N, x\partial_x^2 u_N + 2\partial_x u_N)\\ &= \frac{1}{2}(\partial_x(\partial_x u_N)^2, x) + 2|u_N|_1^2 = \frac{3}{2}|u_N|_1^2. \end{aligned} \tag{2.11}$$

On the other hand,

$$|(f, xu_N)| \le \frac{\beta}{2}\|u_N\|^2_{\hat{\omega}_1} + \frac{1}{2\beta}\|f\|^2_{\hat{\omega}_1}.$$

We obtain the desired result by taking $v_N = xu_N$ in (2.8). □

As suggested in [18, 19, 17], one should choose appropriate basis functions to minimize the band-width and the condition number of the underlying matrix. Using (2.2)-(2.5), one verifies readily that

$$\begin{aligned} &\hat{\phi}_k(x) = \hat{\mathcal{L}}_k(x) - \hat{\mathcal{L}}_{k+1}(x) \in X_{k+1},\\ &\hat{\psi}_k(x) = \hat{\mathcal{L}}_k(x) - 2\hat{\mathcal{L}}_{k+1}(x) + \hat{\mathcal{L}}_{k+2}(x) \in X_{k+1}^*. \end{aligned} \tag{2.12}$$

Therefore,

$$X_N = \text{span}\{\hat{\phi}_0, \hat{\phi}_1, \cdots, \hat{\phi}_{N-1}\}, \quad X_N^* = \text{span}\{\hat{\psi}_0, \hat{\psi}_1, \cdots, \hat{\psi}_{N-1}\}. \tag{2.13}$$

We now consider the linear system of (2.8) associated with the above basis functions. Thanks to (2.2), we have

$$\hat{\phi}_k'(x) = \frac{1}{2}(\hat{\mathcal{L}}_k(x) + \hat{\mathcal{L}}_{k+1}(x)), \quad \hat{\psi}_k''(x) = \frac{1}{4}(\hat{\mathcal{L}}_k(x) + 2\hat{\mathcal{L}}_{k+1}(x) + \hat{\mathcal{L}}_{k+2}(x)). \tag{2.14}$$

Hence, by setting

$$s_{ij} = (\hat{\phi}_j', \hat{\psi}_i''), \qquad m_{ij} = (\hat{\phi}_j, \hat{\psi}_i),$$

one can use (2.14) and the orthogonality of the Laguerre functions to verify that

$$s_{ij} = \begin{cases} \frac{1}{8}, & j = i+2, \\ \frac{3}{8}, & j = i+1, \\ \frac{3}{8}, & j = i, \\ \frac{1}{8}, & j = i-1, \\ 0, & \text{otherwise}, \end{cases} \qquad m_{ij} = \begin{cases} 1, & j = i+2, \\ -3, & j = i+1, \\ 3, & j = i, \\ -1, & j = i-1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.15}$$

Let use denote

$$\begin{aligned} &u_N(x) = \sum_{k=0}^{N-1} \hat{u}_k \hat{\phi}_k(x), \quad \bar{u} = (\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_{N-1})^t; \\ &\hat{f}_k = (f, \hat{\psi}_k), \qquad\qquad \bar{f} = (\hat{f}_0, \hat{f}_1, \cdots, \hat{f}_{N-1})^t, \\ &S = (s_{ij})_{i,j=0,1,\cdots,N-1}, \quad M = (m_{ij})_{i,j=0,1,\cdots,N-1}. \end{aligned} \tag{2.16}$$

Then, the linear system (2.8) becomes

$$(S + \beta M)\bar{u} = \bar{f}, \tag{2.17}$$

which can be efficiently solved.

We state below a convergence result which will be proved in Section 3.

Theorem 2.1. *Let u and u_N be the solutions of* (2.6) *and* (2.8)*, respectively. If $u \in L^2_{\hat{\omega}_{-1}}(\mathcal{R}_+) \cap H^m_{\hat{\omega}_{m-1}}(\mathcal{R}_+) \cap H^m_{\hat{\omega}_m}(\mathcal{R}_+)$ with $m \geq 2$, then*

$$\frac{\beta}{2}\|u - u_N\|_{\hat{\omega}_1} + |u - u_N|_1 \lesssim N^{1-m/2}(\|u\|_{m,\hat{\omega}_m} + N^{-1/2}\|u\|_{m,\hat{\omega}_{m-1}}). \tag{2.18}$$

2.2. **Application to the KDV equation.** There exist a large body of literature on the theoretical and numerical results of the KDV type equations. Although most of the studies were concerned with the Cauchy problems of the KDV equations, the initial-boundary problems also received considerable attention (see, for instance, [20, 13, 1, 3, 12, 11, 2] and the references therein).

As an example of the LDPG method for nonlinear problems, we consider the KDV equation on the half line

$$\begin{cases} \partial_t u + \alpha u u_x + \beta u_{xxx} = f, & x \in (0, \infty), t \in (0, T], \\ u(0,t) = 0, \ \lim\limits_{x \to +\infty} u(x,t) = \lim\limits_{x \to +\infty} u_x(x,t) = 0, & t \in [0, T], \\ u(x, 0) = u_0(x), & x \in [0, \infty). \end{cases} \tag{2.19}$$

The two positive constants α and β are introduced to accommodate the scaling of spatial interval. For the sake of simplicity, we consider here a homogeneous boundary condition. Non-homogeneous boundary conditions can be easily handled by subtracting a simple function from the solution (cf. [17]).

The semi-discrete LDPG approximation to (2.19) is: Find $u_N(x,t) \in X_N$ such that

$$(\partial_t u_N(\cdot,t), v_N) - \frac{\alpha}{2}((u_N(\cdot,t))^2, \partial_x v_N) \\ + \beta(\partial_x u_N(\cdot,t), \partial_x^2 v_N) = (f, v_N), \qquad \forall v_N \in X_N^*, \quad t \in (0,T], \tag{2.20}$$

with initial condition $u_N(0) = \hat{\pi}_N^0 u_0$, where $\hat{\pi}_N^0 : L^2_{\hat{\omega}_{-1}}(\mathcal{R}_+) \to X_N$ is an orthogonal projection defined by

$$(\hat{\pi}_N^0 u - u, v_N)_{\hat{\omega}_{-1}} = 0, \quad \forall v_N \in X_N.$$

The approximation properties of this projector will be studied in Section 4.

Next, let τ be the step size in time, and $t_k = k\tau$ $(k = 0, 1, \cdots, n_T = [T/\tau])$. For simplicity, we denote $u^k := u(x, t_k)$, and

$$D_\tau u^k := \frac{1}{2\tau}(u^{k+1} - u^{k-1}), \qquad \hat{u}^k := \frac{1}{2}(u^{k+1} + u^{k-1}). \tag{2.21}$$

We consider the following Crank-Nicolson leap-frog LDPG scheme: Find $u_N^k \in X_N$ such that

$$(D_\tau u_N^k, v_N) - \frac{\alpha}{2}((u_N^k)^2, \partial_x v_N) + \beta(\partial_x \hat{u}_N^k, \partial_x^2 v_N) \\ = (f^k, v_N), \;\; \forall v_N \in X_N^*, \; 1 \le k \le n_T, \tag{2.22}$$

with $u_N^0 = \hat{\pi}_N^0 u_0$, and

$$u_N^1 = \hat{\pi}_N^0(u_0 + \tau \partial_t u|_{t=0}) = \hat{\pi}_N^0[u_0 + \tau(f|_{t=0} - \beta\partial_x^3 u_0 - \alpha u_0 \partial_x u_0)].$$

Note that in the scheme (2.22), we only need to solve a linear equation of the form (2.8) at each time step.

The following convergence results will be proved in Section 3.

Theorem 2.2. *Let u and u_N be the solutions of* (2.19) *and* (2.20)*, respectively. We assume that*

$$\begin{aligned} u \in \quad & L^2(0,T; L^2_{\hat{\omega}_{-1}}(\mathcal{R}_+) \cap H^m_{\hat{\omega}_{m-1}}(\mathcal{R}_+) \cap H^m_{\hat{\omega}_m}(\mathcal{R}_+)) \cap \\ & L^\infty(0,T; L^2_{\hat{\omega}_1}(\mathcal{R}_+) \cap H^1(\mathcal{R}_+) \cap H^3_{\hat{\omega}_2}(\mathcal{R}_+) \cap W^{1,\infty}(\mathcal{R}_+)), \\ \partial_t u \quad & \in L^2(0,T; H^{m-1}_{\hat{\omega}_{m-1}}(\mathcal{R}_+)), \qquad m \ge 3. \end{aligned} \tag{2.23}$$

Then,

$$\|u - u_N\|_{L^\infty(0,T;L^2_{\hat{\omega}_1}(\mathcal{R}_+))} + \|\partial_x(u - u_N)\|_{L^2(0,T;L^2(\mathcal{R}_+))} \le c^* N^{1-m/2}, \tag{2.24}$$

where c^* is a positive constant depending only on α, β, T, and the norms of u and $\partial_t u$ in the spaces mentioned in (2.23)*.*

2.3. **Numerical results.** To examine numerically the convergence behavior, we first consider the following two exact solutions of (2.6) with $\beta = 1$:

Example 2.1. $u(x) = \sin(kx)e^{-x}$ (exponential decay at infinity).

Example 2.2. $u(x) = x/(1+x)^h$ (algebraic decay without essential singularity at infinity).

In Figure 2.1, we compare the exact solutions in Examples 2.1 and 2.2 with numerical solutions obtained by the LDPG scheme (2.8). It shows that this scheme provides accurate numerical results. To illustrate the rate of the convergence, we

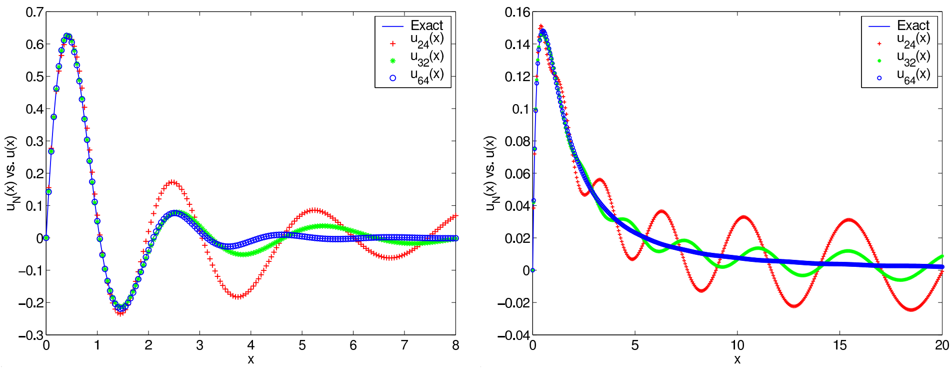

FIGURE 2.1. LDPG approximation: Example 2.1 with $k = 3$ (left) and Example 2.2 with $h = 3$ (right).

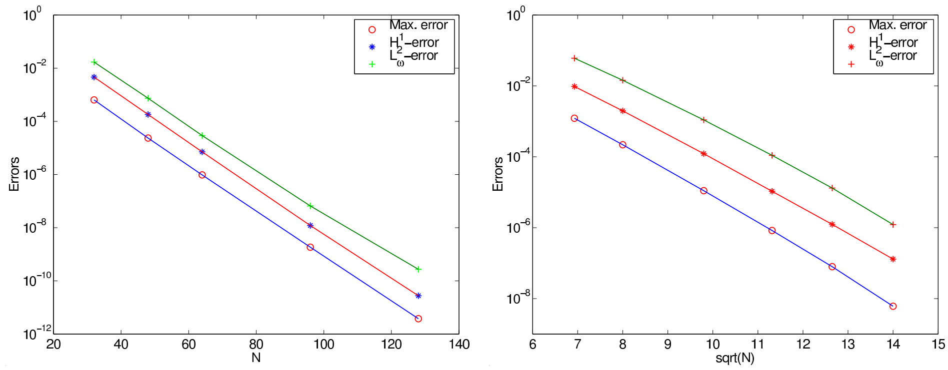

FIGURE 2.2. Convergence rate: Example 2.1 with $k = 2$ (left) and Example 2.2 with $h = 3$ (right).

plot in Figure 2.2 the maximum errors at the Laguerre-Gauss-Radau nodes, and the discrete $L^2_{\hat{\omega}_1}(\mathcal{R}_+)$ and $H^1(\mathcal{R}_+)$ errors.

From Figure 2.2 (left), we observe a geometric convergence rate (like e^{-cN}) for Example 2.1. This is consistent with Theorem 2.1 which asserts that the approximate solution will converge faster than any algebraic power of N. On the other hand, we find from Figure 2.2 (right) that the maximum and the discrete $L^2_{\hat{\omega}_1}(\mathcal{R}_+)$ and $H^1(\mathcal{R}_+)$ errors for Example 2.2 behave like $e^{-c\sqrt{N}}$, while Theorem 2.1 only predicts a convergence of no more than $h - \frac{1}{2}$. This error behavior is also observed for Laguerre approximation of second-order equations (cf. [16]).

Example 2.3. We consider the KDV equation (2.19) with $\alpha = \beta = 1$, $f \equiv 0$ and the exact soliton solution:

$$u(x,t) = 12\kappa^2 \mathrm{sech}^2(\kappa x - 4\kappa^3 t - x_0). \tag{2.25}$$

Here, we take $\kappa = 0.3, x_0 = 4$, and use the scheme (2.22) with $\tau = 10^{-3}$. The maximum absolute errors at $t = 1$ and $t = 10$ with various N are plotted in Figure 2.3 (left). It also indicates a sub-geometric convergence rate (like $e^{-c\sqrt{N}}$) which is consistent with Theorem 2.2.

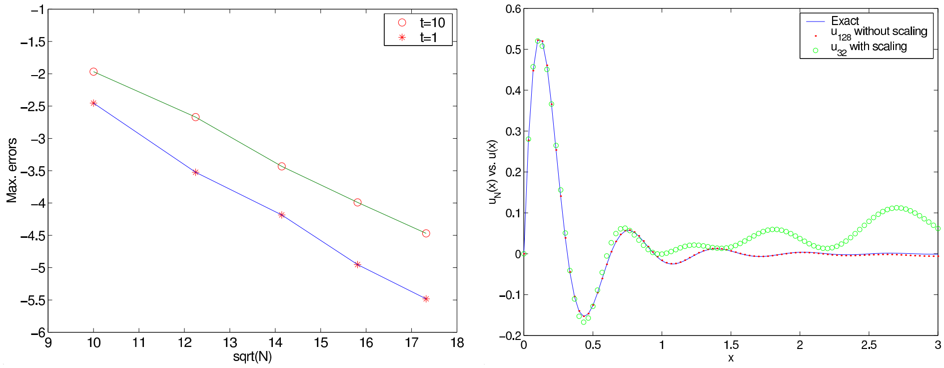

FIGURE 2.3. Left: Convergence rate of LDPG to KDV; Right: Approximation by LDPG with scaling.

Although the Laguerre dual-Petrov-Galerkin method presented above have a theoretical spectral convergence rate, the poor resolution property of Laguerre polynomials/functions, which was pointed out in [8], is one of the main reasons why Laguerre polynomials/functions are not used very frequently in practice. However, it is shown in [16] the resolution of Laguerre functions for second-order equations can be significantly improved by using a scaling factor. We illustrate with an example below that the same is true for the LDPG method.

Example 2.4. $u(x) = \sin kx/(1+x)^h$ (algebraic decay with essentially singularity at infinity).

We choose a scaling factor M such that $|u(x_N/M)| < \varepsilon$, where x_N is the maximum Laguerre-Gauss-Radau node, and ε is a given accuracy threshold (cf. [16]). The approximations of Example 2.4 with $k = 10$ and $h = 5$ using the LPDG scheme (2.8) with scaling factor 15 and without scaling are plotted in Figure 2.3 (right). Notice that if no scaling is used, the approximation with $N = 128$ still exhibits a noticeable error, while the approximation with a scaling using only 32 nodes is virtually indistinguishable with the exact solution.

3. **Approximation Results and Error Estimates.**

In this section, we provide proofs for the two main theorems stated in the previous section.

We need to first establish some approximation properties of several orthogonal projection operators associated with the dual-Petrov-Galerkin method. We note that although there exist many results on approximations by Laguerre polynomials/functions (cf. [15, 7, 10, 16, 9]), but most of them are not applicable here. Since the trial and test spaces in our dual-Petrov-Galerkin formulations are linked by a weight function such as x or x^{-1}, it is convenient to consider generalized Laguerre polynomials, $\mathcal{L}_n^{(\alpha)}(x)$ $(\alpha > -1, n \geq 0)$, which are defined by

$$\begin{cases} \mathcal{L}_n^{(\alpha)}(x) = \dfrac{2n+\alpha-1-x}{n}\mathcal{L}_{n-1}^{(\alpha)}(x) - \dfrac{n+\alpha-1}{n}\mathcal{L}_{n-2}^{(\alpha)}(x), \quad n \geq 2, \\ \mathcal{L}_0^{(\alpha)}(x) = 1, \quad \mathcal{L}_1^{(\alpha)}(x) = -x+\alpha+1. \end{cases} \tag{3.1}$$

We note that the Laguerre polynomial $\mathcal{L}_n(x)$ is $\mathcal{L}_n^{(0)}(x)$.

We recall some basic properties of the generalized Laguerre polynomials below (cf. [21]):

$$\int_0^\infty \mathcal{L}_m^{(\alpha)}(x)\mathcal{L}_n^{(\alpha)}(x)\omega_\alpha(x)dx = \gamma_n^{(\alpha)}\delta_{n,m}, \quad \text{with} \quad \gamma_n^{(\alpha)} = \frac{(n+\alpha)!}{n!}; \tag{3.2}$$

$$\mathcal{L}_n^{(\alpha+1)}(x) = \sum_{k=0}^{n} \mathcal{L}_k^{(\alpha)}(x); \tag{3.3}$$

$$\partial_x \mathcal{L}_n^{(\alpha)}(x) = -\mathcal{L}_{n-1}^{(\alpha)}(x), \qquad n \geq 1; \tag{3.4}$$

$$\mathcal{L}_n^{(\alpha)}(x) = \partial_x \mathcal{L}_n^{(\alpha)}(x) - \partial_x \mathcal{L}_{n+1}^{(\alpha)}(x), \qquad n \geq 0; \tag{3.5}$$

$$x\partial_x \mathcal{L}_n^{(\alpha)}(x) = n\mathcal{L}_n^{(\alpha)}(x) - (n+\alpha)\mathcal{L}_{n-1}^{(\alpha)}(x), \qquad n \geq 1; \tag{3.6}$$

$$x\mathcal{L}_n^{(\alpha+1)}(x) = (n+\alpha+1)\mathcal{L}_n^{(\alpha)}(x) - (n+1)\mathcal{L}_{n+1}^{(\alpha)}(x), \qquad n \geq 0. \tag{3.7}$$

3.1. **Approximation results.** Although we are interested in the approximation properties of Laguerre functions, it is convenient to study first the approximation properties of Laguerre polynomials. Let

$$\phi_n(x) = \mathcal{L}_n(x) - \mathcal{L}_{n+1}(x), \qquad x \in \mathcal{R}_+. \tag{3.8}$$

Using (3.4)-(3.6) yields

$$\phi_n(x) = \frac{1}{n+1} x\mathcal{L}_n^{(1)}(x), \quad \forall x \in \mathcal{R}_+; \tag{3.9}$$

$$\partial_x^k \phi_n(x) = \partial_x^{k-1}\mathcal{L}_n(x) = (-1)^{k-1}\mathcal{L}_{n-k+1}^{(k-1)}(x), \quad 1 \leq k \leq n+1. \tag{3.10}$$

Hence, as a consequence of (3.2) and (3.9), $\{\phi_n\}$ forms a $L^2_{\omega_{-1}}(\mathcal{R}_+)-$orthogonal system, and by (3.10),

$$\int_0^\infty \partial_x^k \phi_m(x)\partial_x^k \phi_n(x)\omega_{k-1}(x)dx = \eta_{n,k}\delta_{m,n}, \quad 0 \leq k \leq n+1, \tag{3.11}$$

where

$$\eta_{n,k} = \gamma_{n-k+1}^{(k-1)} = \frac{n!}{(n-k+1)!}. \tag{3.12}$$

Let $\mathcal{P}_N^0 := \{v : v \in \mathcal{P}_N,\ v(0) = 0\}$. We consider the orthogonal projection $\pi_N^0 : L^2_{\omega_{-1}}(\mathcal{R}_+) \to \mathcal{P}_N^0$ defined by

$$(\pi_N^0 v - v, v_N)_{\omega_{-1}} = 0, \qquad \forall v_N \in \mathcal{P}_N^0. \tag{3.13}$$

Lemma 3.1. *If* $v \in L^2_{\omega_{-1}}(\mathcal{R}_+)$ *and* $\partial_x^m v \in L^2_{\omega_{m-\mu}}(\mathcal{R}_+)$, *then*

$$\|\partial_x^l(\pi_N^0 v - v)\|_{\omega_{l-\mu}} \lesssim N^{l/2-r/2}\|\partial_x^m v\|_{\omega_{m-\mu}}, \quad 0 \leq l \leq m,\ \mu = 0, 1. \tag{3.14}$$

Proof. We first consider the case $\mu = 1$. For any $v \in L^2_{\omega_{-1}}(\mathcal{R}_+)$, we write

$$v(x) = \sum_{n=0}^{\infty} \hat{v}_n \phi_n(x), \quad \text{with} \quad \hat{v}_n = \frac{1}{\eta_{n,0}}(v, \phi_n)_{\omega_{-1}}. \tag{3.15}$$

So formally by (3.11),

$$\|\partial_x^k v\|^2_{\omega_{k-1}} = \sum_{n=k-1}^{\infty} \eta_{n,k}\hat{v}_n^2. \tag{3.16}$$

On the other hand,

$$\pi_N^0 v(x) - v(x) = -\sum_{n=N}^{\infty} \hat{v}_n \phi_n(x).$$

Therefore, we derive from (3.16) that

$$\|\partial_x^l(\pi_N^0 v - v)\|_{\omega_{l-1}}^2 = \sum_{n=N}^{\infty} \eta_{n,l}\hat{v}_n^2 = D_N^{l,m} \sum_{n=N}^{\infty} \eta_{n,m}\hat{v}_n^2 \le D_N^{l,m}\|\partial_x^m v\|_{\omega_{m-1}}^2 \tag{3.17}$$

where (by (3.12))

$$D_N^{l,m} = \max_{n\ge N}\Big(\frac{\eta_{n,l}}{\eta_{n,m}}\Big) = \frac{(N-m+1)!}{(N-l+1)!} \lesssim N^{l-m}.$$

This implies (3.14) with $\mu = 1$.

Next, by the definition of ϕ_n and (3.15),

$$v(x) = \sum_{n=0}^{\infty} \hat{v}_n(\mathcal{L}_n(x) - \mathcal{L}_{n+1}(x)) = \sum_{n=0}^{\infty} \tilde{v}_n\mathcal{L}_n(x) \tag{3.18}$$

where

$$\tilde{v}_n = \hat{v}_n - \hat{v}_{n-1}, \quad \hat{v}_{-1} = 0.$$

By (3.2) and (3.4), we have

$$\|\partial_x^k v\|_{\omega_k}^2 = \sum_{n=k}^{\infty} \tilde{v}_n^2\|\mathcal{L}_{n-k}^{(k)}\|_{\omega_k}^2 = \sum_{n=k}^{\infty} \tilde{v}_n^2\gamma_{n-k}^{(k)}. \tag{3.19}$$

Therefore,

$$\|\partial_x^l(\pi_N^0 v - v)\|_{\omega_l}^2 = \sum_{n=l}^{\infty} \tilde{v}_n^2\gamma_{n-l}^{(l)} \le C_N^{l,m} \sum_{n=N}^{\infty} \tilde{v}_n^2\gamma_{n-m}^{(m)} \le C_N^{l,m}\|\partial_x^m v\|_{\omega_m}^2$$

where (by (3.12))

$$C_N^{l,m} = \max_{n\ge N}\Big(\frac{\gamma_{n-l}^{(l)}}{\gamma_{n-m}^{(m)}}\Big) = \frac{(N-m)!}{(N-l)!} \lesssim N^{l-m}.$$

This yields (3.14) with $\mu = 0$. □

For the error analysis, we also need the following result:

Lemma 3.2. *If $v \in L^2_{\omega_{-1}}(\mathcal{R}_+)$ and $\partial_x^m v \in L^2_{\omega_{m+1}}(\mathcal{R}_+)$, then for $m \ge 0$,*

$$\|\pi_N^0 v - v\|_{\omega_1} \lesssim N^{-m/2}\|\partial_x^m v\|_{\omega_{m+1}}. \tag{3.20}$$

Proof. By (3.1) and (3.9),

$$\phi_n(x) = -\mathcal{L}_{n-1}^{(1)}(x) + 2\mathcal{L}_n^{(1)}(x) - \mathcal{L}_{n+1}^{(1)}(x), \quad \mathcal{L}_{-1}^{(1)}(x) := 0, \quad n \ge 0.$$

So for any $v \in L^2_{\omega_{-1}}(\mathcal{R}_+)$, we derive from (3.15) that

$$v(x) = \sum_{n=0}^{\infty} \bar{v}_n\mathcal{L}_n^{(1)}(x), \quad \text{with} \quad \bar{v}_n = -\hat{v}_{n-1} + 2\hat{v}_n - \hat{v}_{n+1}, \;\; \hat{v}_{-1} := 0.$$

Thus, by (3.2) and (3.4),

$$\|\partial_x^m v\|_{\omega_{m+1}}^2 = \sum_{n=m}^{\infty} \bar{v}_n^2\|\mathcal{L}_{n-m}^{(m+1)}\|_{\omega_{m+1}}^2 = \sum_{n=m}^{\infty} \bar{v}_n^2\gamma_{n-m}^{(m+1)}. \tag{3.21}$$

Consequently,

$$\|\pi_N^0 v - v\|_{\omega_1}^2 = \sum_{n=0}^{\infty} \bar{v}_n^2\gamma_n^{(1)} \le \bar{C}_{N,m} \sum_{n=N}^{\infty} \bar{v}_n^2\gamma_{n-m}^{(m+1)} \le \bar{C}_{N,m}\|\partial_x^m v\|_{\omega_{m+1}}^2$$

where (by (3.12))

$$\bar{C}_{N,m} = \max_{n\geq N}\Big(\frac{\gamma_n^{(1)}}{\gamma_{n-m}^{(m+1)}}\Big) = \frac{(N-m)!}{N!} \lesssim N^{-m}.$$

This completes the proof. □

We will also need the following imbedding results:

Lemma 3.3. *If $v \in L^2_{\omega_\alpha}(\mathcal{R}_+)$ and $\partial_x v \in L^2_{\omega_{\alpha+1}}(\mathcal{R}_+)$ with $\alpha > -1$, then*

$$\|v\|_{\omega_{\alpha+1}} \leq \sqrt{2(\alpha+1)}\|v\|_{\omega_\alpha} + 2\|\partial_x v\|_{\omega_{\alpha+1}}. \tag{3.22}$$

Moreover, for any $v \in H^1_{\omega_0}(\mathcal{R}_+)$ with $v(0) = 0$,

$$\|v\|_{\omega_0} \leq 2\|\partial_x v\|_{\omega_0}. \tag{3.23}$$

Proof. We derive from

$$x^{\alpha+1}e^{-x}v^2(x) = \int_0^x \partial_y(y^{\alpha+1}e^{-y}v^2(y))dy,$$

that

$$\begin{aligned}\omega_{\alpha+1}(x)v^2(x) &+ \int_0^x \omega_{\alpha+1}(y)v^2(y)dy \\ &= 2\int_0^x \omega_{\alpha+1}(y)v(y)\partial_y v(y)dy + (\alpha+1)\int_0^x \omega_\alpha(y)v^2(y)dy \\ &\leq \frac{1}{2}\int_0^x \omega_{\alpha+1}(y)v^2(y)dy + 2\|\partial_x v\|^2_{\omega_{\alpha+1}} + (\alpha+1)\|v\|^2_{\omega_\alpha}.\end{aligned}$$

Letting $x \to +\infty$, we obtain (3.22). We recall that (3.23) was proved in [10]. □

We now consider the approximation properties of Laguerre functions under the projection operator $\hat{\pi}^0_N : L^2_{\hat{\omega}_{-1}}(\mathcal{R}_+) \to X_N$ defined by

$$\hat{\pi}^0_N u = e^{-x/2}\pi^0_N(ue^{x/2}). \tag{3.24}$$

Clearly, by (3.13), we have that

$$(\hat{\pi}^0_N u - u, v_N)_{\hat{\omega}_{-1}} = (\pi^0_N(ue^{x/2}) - (ue^{x/2}), v_N e^{x/2})_{\omega_{-1}} = 0, \quad \forall v_N \in X_N. \tag{3.25}$$

It is straightforward to verify that

$$\|\partial_x^m(ue^{\frac{x}{2}})\|_{\omega_k} \lesssim \|u\|_{m,\hat{\omega}_k},\ \forall u \in H^m_{\hat{\omega}_k}(\mathcal{R}_+). \tag{3.26}$$

We have the following results related to the projection operator $\hat{\pi}^0_N$.

Theorem 3.1.

$$\begin{aligned}&\|\hat{\pi}^0_N u - u\|_{\hat{\omega}_\mu} \lesssim N^{-m/2}\|u\|_{m,\hat{\omega}_{m+\mu}}, \\ &\qquad \forall u \in L^2_{\hat{\omega}_{-1}}(\mathcal{R}_+) \cap H^m_{\hat{\omega}_{m+\mu}}(\mathcal{R}_+), \quad m \geq 0, \quad \mu = -1, 0, 1,\end{aligned} \tag{3.27}$$

$$\begin{aligned}&\|\partial_x(\hat{\pi}^0_N u - u)\|_{\hat{\omega}_\mu} \lesssim N^{1/2-m/2}\|u\|_{m,\hat{\omega}_{m+\mu-1}}, \\ &\qquad \forall u \in L^2_{\hat{\omega}_{-1}}(\mathcal{R}_+) \cap H^m_{\hat{\omega}_{m+\mu-1}}(\mathcal{R}_+), \quad m \geq 1, \quad \mu = 0, 1,\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\|\partial_x^2(\hat{\pi}^0_N u - u)\|_{\hat{\omega}_2} \lesssim N^{1-m/2}\|u\|_{m,\hat{\omega}_m}, \\ &\qquad \forall u \in L^2_{\hat{\omega}_{-1}}(\mathcal{R}_+) \cap H^m_{\hat{\omega}_m}(\mathcal{R}_+), \quad m \geq 2.\end{aligned} \tag{3.29}$$

Proof. Let $v = ue^{x/2}$. By Lemma 3.1 with $l = 0, \mu = 0, 1$, and Lemma 3.2,

$$\|\hat{\pi}_N^0 u - u\|_{\hat{\omega}_\mu} = \|\pi_N^0 v - v\|_{\omega_\mu} \lesssim N^{-m/2}\|\partial_x^m v\|_{\omega_{m+\mu}}.$$

Thus, (3.27) follows from above and (3.26).

We now prove (3.28). Due to $(\pi_N^0 v - v)(0) = 0$, we derive from (3.23) that

$$\|\pi_N^0 v - v\|_{\omega_0} \le 2\|\partial_x(\pi_N^0 v - v)\|_{\omega_0}. \tag{3.30}$$

Since

$$\partial_x(\hat{\pi}_N^0 u - u) = e^{-x/2}(\partial_x(\pi_N^0 v - v) - \frac{1}{2}(\pi_N^0 v - v)), \tag{3.31}$$

we obtain from Lemma 3.1 with $\mu = 0, l = 1$, and (3.30) that

$$\begin{aligned}\|\partial_x(\hat{\pi}_N^0 u - u)\| &\lesssim \|\partial_x(\pi_N^0 v - v)\|_{\omega_0} + \|\pi_N^0 v - v\|_{\omega_0} \lesssim \|\partial_x(\pi_N^0 v - v)\|_{\omega_0}\\ &\lesssim N^{1/2-m/2}\|\partial_x^m(ue^{x/2})\|_{\omega_{m-1}} \lesssim N^{1/2-m/2}\|u\|_{m,\hat{\omega}_{m-1}}.\end{aligned}$$

This yields (3.28) with $\mu = 0$. Next, by (3.22) with $\alpha = 0$,

$$\|\pi_N^0 v - v\|_{\omega_1} \lesssim \|\pi_N^0 v - v\|_{\omega_0} + \|\partial_x(\pi_N^0 v - v)\|_{\omega_1}. \tag{3.32}$$

This fact with (3.31) and Lemma 3.2 with $\mu = 1$, $l = 0, 1$ leads to

$$\begin{aligned}\|\partial_x(\hat{\pi}_N^0 u - u)\|_{\omega_1} &\lesssim \|\partial_x(\pi_N^0 v - v)\|_{\omega_1} + \|\pi_N^0 v - v\|_{\omega_1}\\ &\lesssim \|\partial_x(\pi_N^0 v - v)\|_{\omega_1} + \|\pi_N^0 v - v\|_{\omega_0}\\ &\lesssim N^{1/2-m/2}\|\partial_x^m(ue^{x/2})\|_{\omega_m} \lesssim N^{1/2-m/2}\|u\|_{m,\hat{\omega}_m}.\end{aligned}$$

This implies (3.28) with $\mu = 1$.

Next, we prove (3.29). By (3.22) with $\alpha = 1$,

$$\|\partial_x(\pi_N^0 v - v)\|_{\omega_2} \lesssim \|\partial_x^2(\pi_N^0 v - v)\|_{\omega_2} + \|\partial_x(\pi_N^0 v - v)\|_{\omega_1}. \tag{3.33}$$

Also by (3.22) with $\alpha = 1$, (3.32) and (3.33),

$$\begin{aligned}\|\pi_N^0 v - v\|_{\omega_2} &\lesssim \|\partial_x(\pi_N^0 v - v)\|_{\omega_2} + \|\pi_N^0 v - v\|_{\omega_1}\\ &\lesssim \|\partial_x(\pi_N^0 v - v)\|_{\omega_2} + \|\partial_x(\pi_N^0 v - v)\|_{\omega_1} + \|\pi_N^0 v - v\|_{\omega_0}\\ &\lesssim \|\partial_x^2(\pi_N^0 v - v)\|_{\omega_2} + 2\|\partial_x(\pi_N^0 v - v)\|_{\omega_1} + \|\pi_N^0 v - v\|_{\omega_0}.\end{aligned} \tag{3.34}$$

Since

$$\partial_x^2(\hat{\pi}_N^0 u - u) = e^{-x/2}(\partial_x^2(\pi_N^0 v - v) - \partial_x(\pi_N^0 v - v) + \frac{1}{4}(\pi_N^0 v - v)),$$

using (3.33), (3.34) and Lemma 3.2 with $\mu = 1$, $l = 0, 1, 2$, we derive that

$$\begin{aligned}&\|\partial_x^2(\hat{\pi}_N^0 u - u)\|_{\hat{\omega}_2}\\ &\lesssim \|\partial_x^2(\pi_N^0 v - v)\|_{\omega_2} + \|\partial_x(\pi_N^0 v - v)\|_{\omega_2} + \|\pi_N^0 v - v\|_{\omega_2}\\ &\lesssim \|\partial_x^2(\pi_N^0 v - v)\|_{\omega_2} + \|\partial_x(\pi_N^0 v - v)\|_{\omega_1} + \|\pi_N^0 v - v\|_{\omega_0}\\ &\lesssim N^{1-m/2}\|\partial_x^m(ue^{x/2})\|_{\omega_m} \lesssim N^{1-m/2}\|u\|_{m,\hat{\omega}_m}.\end{aligned} \tag{3.35}$$

The proof is complete. □

3.2. **Proof of Theorem** 2.1. Let u and u_N be the solutions of (2.6) and (2.8), respectively. Let $\hat{\pi}_N^0$ be the projector as in (3.24), and set $\hat{e}_N = \hat{\pi}_N^0 u - u_N$. Then, by (2.6) and (2.8),

$$\begin{aligned}(\partial_x \hat{e}_N, \partial_x^2 v_N) + \beta(\hat{e}_N, v_N) = -(\partial_x(\hat{\pi}_N^0 u - u), \partial_x^2 v_N) \\ + \beta(\hat{\pi}_N^0 u - u, v_N), \quad \forall v_N \in X_N^*.\end{aligned} \tag{3.36}$$

Taking $v_N = x\hat{e}_N \in X_N^*$ in (3.36), we derive from (2.11) that

$$\frac{3}{2}|\hat{e}_N|_1^2 + \beta\|\hat{e}_N\|_{\hat{\omega}_1}^2 \le |(\partial_x(\hat{\pi}_N^0 u - u), \partial_x^2(x\hat{e}_N))| + \beta|(\hat{\pi}_N^0 u - u, x\hat{e}_N)|. \tag{3.37}$$

By (3.28) with $\mu = 0$ and (3.29),

$$\begin{aligned}|(\partial_x^2(\hat{\pi}_N^0 u - u), \partial_x(x\hat{e}_N))| &\le |(\partial_x^2(\hat{\pi}_N^0 u - u), x\partial_x\hat{e}_N)| + |(\partial_x^2(\hat{\pi}_N^0 u - u), \hat{e}_N)| \\ &\le |(x\partial_x^2(\hat{\pi}_N^0 u - u), \partial_x\hat{e}_N)| + |(\partial_x(\hat{\pi}_N^0 u - u), \partial_x\hat{e}_N)| \\ &\le \frac{1}{2}|\hat{e}_N|_1^2 + \|\partial_x^2(\hat{\pi}_N^0 u - u)\|_{\hat{\omega}_2}^2 + \|\partial_x(\hat{\pi}_N^0 u - u)\|^2 \\ &\le \frac{1}{2}|\hat{e}_N|_1^2 + cN^{2-m}(\|u\|_{m,\hat{\omega}_m}^2 + N^{-1}\|u\|_{m,\hat{\omega}_{m-1}}^2).\end{aligned} \tag{3.38}$$

Moreover, by Lemma 3.1, (3.22) with $\alpha = 0$ and Lemma 3.2 with $\mu = 0, l = 0, 1$, we obtain that

$$\begin{aligned}\|\hat{\pi}_N^0 u - u\|_{\hat{\omega}_1} &= \|\pi_N^0(ue^{x/2}) - ue^{x/2}\|_{\omega_1} \\ &\lesssim \|\partial_x(\pi_N^0(ue^{x/2}) - ue^{x/2})\|_{\omega_1} + \|\pi_N^0(ue^{x/2}) - ue^{x/2}\|_{\omega_0} \\ &\lesssim N^{1/2-m/2}\|\partial_x^m(ue^{x/2})\|_{\omega_m} \lesssim N^{1/2-m/2}\|u\|_{m,\hat{\omega}_m}.\end{aligned} \tag{3.39}$$

Thus,

$$\begin{aligned}|(\hat{\pi}_N^0 u - u, x\hat{e}_N)| &\le \frac{\beta}{2}\|\hat{e}_N\|_{\hat{\omega}_1}^2 + \frac{1}{2\beta}\|\hat{\pi}_N^0 u - u\|_{\hat{\omega}_1}^2 \\ &\le \frac{\beta}{2}\|\hat{e}_N\|_{\hat{\omega}_1}^2 + cN^{1-m}\|u\|_{m,\hat{\omega}_m}^2.\end{aligned} \tag{3.40}$$

A combination of (3.37), (3.39) and (3.40) leads to that

$$\frac{\beta}{2}\|\hat{e}_N\|_{\hat{\omega}_1}^2 + |\hat{e}_N|_1^2 \lesssim N^{2-m}(\|u\|_{m,\hat{\omega}_m}^2 + N^{-1}\|u\|_{m,\hat{\omega}_{m-1}}^2). \tag{3.41}$$

Finally, we obtain from (3.28) with $\mu = 0$, (3.39) and (3.41) that

$$\begin{aligned}&\frac{\beta}{2}\|u - u_N\|_{\hat{\omega}_1} + |u - u_N|_1 \\ &\le \frac{\beta}{2}\|\hat{e}_N\|_{\hat{\omega}_1} + |\hat{e}_N|_1 + \frac{\beta}{2}\|\hat{\pi}_N^0 u - u\|_{\hat{\omega}_1} + |\hat{\pi}_N^0 u - u|_1 \\ &\lesssim N^{1-m/2}(\|u\|_{m,\hat{\omega}_m} + N^{-1/2}\|u\|_{m,\hat{\omega}_{m-1}}).\end{aligned} \tag{3.42}$$

Thus, the proof of Theorem 2.1 is complete.

3.3. **Proof of Theorem 2.2.** Some additional lemmas are needed for the numerical analysis of nonlinear problems such as KDV equation.

Lemma 3.4.

$$\|x^{\frac{1}{2}}\hat{\pi}_N^0 u\|_{L^\infty} \lesssim \|u\|_{\hat{\omega}_1} + \|u\|_1, \quad \forall u \in L^2_{\hat{\omega}_1}(\mathcal{R}_+) \cap H^1(\mathcal{R}_+), \tag{3.43}$$

$$\|\partial_x \hat{\pi}_N^0 u\|_{L^\infty} \lesssim \|u\|_{3,\hat{\omega}_2}, \qquad \forall u \in H^3_{\hat{\omega}_2}(\mathcal{R}_+). \tag{3.44}$$

Proof. We have

$$\begin{aligned} xv^2(x) = \int_0^x \partial_y(yv^2(y))dy \quad &= \int_0^x v^2(y)dy + 2\int_0^x yv(y)\partial_y v(y)dy \\ &\le \|v\|^2 + 2\|v\|_{\hat{\omega}_1}|v|_{1,\hat{\omega}_1}. \end{aligned}$$

Thus by (3.28) with $\mu = 0, 1$, and (3.29),

$$\begin{aligned} \|x^{\frac{1}{2}}\hat{\pi}_N^0 u\|^2_{L^\infty} \quad &\le \|\hat{\pi}_N^0 u\|^2 + 2\|\hat{\pi}_N^0 u\|_{\hat{\omega}_1}|\hat{\pi}_N^0 u|_{1,\hat{\omega}_1} \\ &\lesssim \|u\|^2 + \|u\|_{\hat{\omega}_1}\|u\|_1 \lesssim \|u\|^2_{\hat{\omega}_1} + \|u\|_1^2. \end{aligned}$$

This leads to (3.43).

Next, let $v = ue^{x/2}$. Then, by (3.24), (2.14) and (3.15),

$$\partial_x \hat{\pi}_N^0 u = \partial_x(e^{-x/2}\pi_N^0 v) = \sum_{n=0}^{N-1} \hat{v}_n \partial_x \hat{\phi}_n(x) = \frac{1}{2}\sum_{n=0}^{N-1} \hat{v}_n(\hat{\mathcal{L}}_n(x) + \hat{\mathcal{L}}_{n+1}(x)).$$

Therefore, by (2.4), (3.12), (3.16) and Lemma 3.1 with $\mu = 1, l = 3$,

$$\begin{aligned} \|\partial_x \hat{\pi}_N^0 u\|_{L^\infty} \quad &\le \sum_{n=0}^{N-1} |\hat{v}_n| \le \Big(\sum_{n=0}^{N-1} \eta_{n,3}^{-1}\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{N-1} \hat{v}_n^2 \eta_{n,3}\Big)^{\frac{1}{2}} \\ &\lesssim \|\partial_x^3(\pi_N^0 v)\|_{\omega_2} \lesssim \|\partial_x^3 v\|_{\omega_2} \lesssim \|u\|_{3,\hat{\omega}_2}. \end{aligned}$$

The proof is complete. □

Lemma 3.5.

$$\|u\| \le \|u\|_{\hat{\omega}_1} + 2|u|_1, \quad \forall u \in L^2_{\hat{\omega}_1}(\mathcal{R}_+) \cap H^1(\mathcal{R}_+), \ \ \textit{with} \ \ u(0) = 0, \tag{3.45}$$

$$\|x^{-1}u\|_{L^\infty} \lesssim N^{\frac{1}{2}}(\|u\|_{\hat{\omega}_1} + |u|_1), \qquad \forall u \in X_N. \tag{3.46}$$

Proof. We first prove (3.45). Thanks to $u(0) = 0$, we obtain from the Hardy inequality (cf. [14]) that

$$\int_0^1 u^2(x)x^{d-2}dx \le \frac{4}{1-d}\int_0^1 (\partial_x u(x))^2 x^d dx, \quad d < 1.$$

Thus,

$$\int_0^1 u^2(x)dx \le \int_0^1 u^2(x)x^{-2}dx \le 4\int_0^1 (\partial_x u(x))^2 dx \le 4|u|_1^2.$$

On the other hand, it is clear that

$$\int_1^\infty u^2(x)dx \le \int_1^\infty xu^2(x)dx \le \|u\|^2_{\hat{\omega}_1}.$$

A combination of the above two inequalities leads to (3.45).

Next, let $\phi_n(x)$ be the same as in (3.8), and set $v = ue^{x/2}$. Then by (3.9), (3.3) and (2.4),

$$\begin{aligned}|x^{-1}u(x)| &= |x^{-1}v(x)e^{-x/2}| \le \sum_{n=0}^{N-1} \frac{|\hat{v}_n|}{n+1} |\mathcal{L}_n^{(1)}(x)| e^{-x/2} \\ &\le \sum_{n=0}^{N-1} \frac{|\hat{v}_n|}{n+1} \sum_{k=0}^{n} |\hat{\mathcal{L}}_k(x)| \le \sum_{n=0}^{N-1} |\hat{v}_n|.\end{aligned}$$

Furthermore, by (3.12), (3.16) and (3.45),

$$\begin{aligned}|x^{-1}u(x)| &\le \Big(\sum_{n=0}^{N-1} \eta_{n,1}^{-1}\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{N-1} \hat{v}_n^2 \eta_{n,1}\Big)^{\frac{1}{2}} \le N^{\frac{1}{2}} \|\partial_x v\|_{\omega_0} = N^{\frac{1}{2}} \|\partial_x(ue^{x/2})\|_{\omega_0} \\ &\lesssim N^{\frac{1}{2}}(|u|_1 + \|u\|) \lesssim N^{\frac{1}{2}}(|u|_1 + \|u\|_{\hat{\omega}_1}).\end{aligned}$$

This completes the proof. □

Now, let u and u_N be the solutions of (2.19) and (2.20), respectively. Set $U_N = \hat{\pi}_N^0 u$ and $e_N = u_N - U_N$. By (2.19),

$$\begin{aligned}(\partial_t U_N, v_N) &- \frac{\alpha}{2}(U_N^2, \partial_x v_N) + \beta(\partial_x U_N, \partial_x^2 v_N) \\ &+ \sum_{j=1}^{3} G_j(u, U_N; v_N) = (f, v_N), \quad \forall v_N \in X_N^*,\ t \in (0, T],\end{aligned} \tag{3.47}$$

where

$$\begin{aligned}G_1(u, U_N; v_N) &= (\partial_t u - \partial_t U_N, v_N), \\ G_2(u, U_N; v_N) &= -\frac{\alpha}{2}(u^2 - U_N^2, \partial_x v_N), \\ G_3(u, U_N; v_N) &= \beta(\partial_x u - \partial_x U_N, \partial_x^2 v_N).\end{aligned}$$

Subtracting (3.47) from (2.20) yields

$$\begin{aligned}(\partial_t e_N, v_N) &- \frac{\alpha}{2}(e_N^2, \partial_x v_N) + \beta(\partial_x e_N, \partial_x^2 v_N) \\ &= \sum_{j=1}^{4} G_j(u, U_N; v_N), \quad \forall v_N \in X_N^*,\end{aligned} \tag{3.48}$$

where

$$G_4(u, U_N; v_N) = \alpha(U_N e_N, \partial_x v_N).$$

Taking $v_N = xe_N \in X_N^*$ in (3.48), we derive from Lemma 2.1 that

$$\frac{1}{2}\frac{d}{dt}\|e_N\|_{\hat{\omega}_1}^2 + \frac{3\beta}{2}|e_N|_1^2 \le \frac{\alpha}{2}|(e_N^2, \partial_x(xe_N))| + \sum_{j=1}^{4} |G_j(u, U_N; xe_N)|. \tag{3.49}$$

Now, we estimate the terms at the right side of (3.49). An integration by parts yields that

$$(e_N^2, \partial_x(xe_N)) = \int_0^\infty e_N^3(x,t)dx + \frac{1}{3}\int_0^\infty x\partial_x e_N^3(x,t)dx = \frac{2}{3}\int_0^\infty e_N^3(x,t)dx.$$

By Lemma 3.5,

$$
\begin{aligned}
|(e_N^2, \partial_x(xe_N))| &\lesssim \|x^{-1}e_N\|_{L^\infty}\|e_N\|_{\hat{\omega}_1}^2 \\
&\lesssim N^{\frac{1}{2}}(|e_N|_1 + \|e_N\|_{\hat{\omega}_1})\|e_N\|_{\hat{\omega}_1}^2 \\
&\leq \frac{\beta}{4}|e_N|_1^2 + \frac{c}{\beta}N\|e_N\|_{\hat{\omega}_1}^4 + cN^{\frac{1}{2}}\|e_N\|_{\hat{\omega}_1}^3.
\end{aligned}
\tag{3.50}
$$

Next, by (3.28) with $\mu = 1$, we obtain

$$
\begin{aligned}
|G_1(u, U_N; xe_N)| &\leq \|e_N\|_{\hat{\omega}_1}^2 + \|\partial_t(u - U_N)\|_{\hat{\omega}_1}^2 \\
&\lesssim \|e_N\|_{\hat{\omega}_1}^2 + N^{1-m}\|\partial_t u\|_{m-1,\hat{\omega}_m}^2.
\end{aligned}
\tag{3.51}
$$

Then, by (3.28) with $\mu = 1$, (3.29) with $\mu = 0$ and (3.43),

$$
\begin{aligned}
|G_2(u, U_N; xe_N)| &= \alpha|(u\partial_x u - U_N\partial_x U_N, xe_N)| \\
&\leq \alpha(\|e_N\|_{\hat{\omega}_1}^2 + \|(u - U_N)\partial_x u\|_{\hat{\omega}_1}^2 + \|U_N(\partial_x u - \partial_x U_N)\|_{\hat{\omega}_1}^2) \\
&\lesssim \|e_N\|_{\hat{\omega}_1}^2 + \|\partial_x u\|_{L^\infty}^2\|u - U_N\|_{\hat{\omega}_1}^2 \\
&\quad + \|x^{\frac{1}{2}}U_N\|_{L^\infty}^2\|\partial_x(u - U_N)\|^2 \\
&\lesssim \|e_N\|_{\hat{\omega}_1}^2 + N^{1-m}\|\partial_x u\|_{L^\infty}^2\|u\|_{m-1,\hat{\omega}_m}^2 \\
&\quad + N^{1-m}(\|u\|_{\hat{\omega}_1}^2 + \|u\|_1^2)\|u\|_{m,\hat{\omega}_m}^2 \\
&\lesssim \|e_N\|_{\hat{\omega}_1}^2 + N^{1-m}(\|\partial_x u\|_{L^\infty}^2 + \|u\|_{\hat{\omega}_1}^2 + \|u\|_1^2)\|u\|_{m,\hat{\omega}_m}^2.
\end{aligned}
$$

By (3.38), we find

$$
|G_3(u, U_N; xe_N)| \leq \frac{\beta}{4}|e_N|_1^2 + \frac{c}{\beta}N^{2-m}(\|u\|_{m,\hat{\omega}_m}^2 + N^{-1}\|u\|_{m,\hat{\omega}_{m-1}}^2).
$$

By Lemma 3.4, we obtain

$$
\begin{aligned}
|G_4(u, U_N; xe_N)| &= \alpha|(\partial_x(U_N e_N), xe_N)| = \alpha|(\partial_x U_N e_N + U_N\partial_x e_N, xe_N)| \\
&\leq \alpha(\|\partial_x U_N\|_{L^\infty}\|e_N\|_{\hat{\omega}_1}^2 + \frac{\beta}{4\alpha}|e_N|_1^2 + \frac{\alpha}{\beta}\|xU_N e_N\|^2) \\
&\leq \frac{\beta}{4}|e_N|_1^2 + c(\|\partial_x U_N\|_{L^\infty} + \|x^{\frac{1}{2}}U_N\|_{L^\infty}^2)\|e_N\|_{\hat{\omega}_1}^2 \\
&\leq \frac{\beta}{4}|e_N|_1^2 + c(\|u\|_{\hat{\omega}_1}^2 + \|u\|_1^2 + \|u\|_{3,\hat{\omega}_2}^2)\|e_N\|_{\hat{\omega}_1}^2.
\end{aligned}
$$

Hence, a combination of the above estimates yields

$$
\begin{aligned}
\frac{1}{2}\|e_N\|_{\hat{\omega}_1}^2 + \beta\int_0^t |e_N(s)|_1^2 ds &\leq b_1^* N^{2-m} \\
+ \int_0^t \|e_N(s)\|_{\hat{\omega}_1}^2 &(b_2^* + c_1 N^{\frac{1}{2}}\|e_N(s)\|_{\hat{\omega}_1}^{\frac{3}{2}} + c_2 N\|e_N(s)\|_{\hat{\omega}_1}^2)ds
\end{aligned}
\tag{3.52}
$$

where c_1, c_2 are two generic positive constants, and b_1^*, b_2^* are two constants depending only on the norms of u in the mentioned spaces.

We are now in position to apply a Gronwall type lemma. Indeed, for $m \geq 3$, the conditions of Lemma 3.6 below hold for all $t \in [0, T]$. As a consequence of (3.52) and Lemma 3.6, we obtain that

$$\frac{1}{2}\|e_N\|^2_{L^\infty(0,t;L^2_{\hat{\omega}_1}(\mathcal{R}_+))} + \beta\|\partial_x e_N\|^2_{L^2(0,t;L^2(\mathcal{R}_+))} \leq b_1^* N^{2-m} e^{2b_2^* t}, \quad \forall t \in [0, T]. \tag{3.53}$$

Finally, the desired results follow from a similar procedure as in the derivation of (3.42).

Lemma 3.6. (cf. [5]) *Assume that*

- *z(t) is a nonnegative function of t,*
- *the constants* $b_j, d_j, r_j > 0, j = 1, 2,$
- *for certain* $t_1 > 0$, $b_1 e^{2b_2 t_1} \leq \min_{j=1,2} \left(\frac{b_2}{d_j}\right)^{\frac{1}{r_j}}$,
- $\forall t \in [0, t_1]$,

$$z(t) \leq b_1 + \int_0^t z(s)(b_1 + d_1 z^{r_1}(s) + d_2 z^{r_2}(s))ds,$$

Then for all $t \in [0, t_1]$, *we have* $z(t) \leq b_0 e^{2b_2 t}$.

Remark 3.1. By a procedure similar to that used in the proof of Theorem 2.2 and in [17], we can also obtain an error estimate for the fully-discrete LDPG scheme (2.22). More precisely, it can be proved that if u possesses the similar regularity as in Theorem 2.2, and $\tau N \leq c_0$, then for $1 \leq k \leq n_T$,

$$\|u(t_k) - u_N^k\|_{\hat{\omega}_1} + \Big(\tau \sum_{0\leq l\leq k-1} \|\partial_x(u(t_l) - u_N^l)\|^2\Big)^{\frac{1}{2}} \lesssim d^*(\tau^2 + N^{1-m/2}). \tag{3.54}$$

We leave the details of the proof to the interested readers.

4. **Composite Legendre-Laguerre Dual-Petrov-Galerkin Method.** It should be noted that even with a proper scaling, a single domain Laguerre method is not suitable to resolve solutions with sharp interfaces or multiple internal layers. Hence, it is necessary to develop a multi-domain spectral method for such problems.

A natural choice for a multi-domain spectral method in a semi-infinite interval is to use Legendre polynomials for all but one subdomain in which the Laguerre functions should be used. Such an approach is relatively straightforward for second-order equations and has been studied in [9]. However, it is not obvious how to properly design a multi-domain spectral algorithm for third-order equations. In this section, we propose a well-posed composite Legendre-Laguerre multi-domain approach.

We recall first some basic properties of Legendre polynomials $\{L_k(x)\}$:

$$\int_{-1}^1 L_k(x)L_j(x)dx = \frac{2}{2k+1}\delta_{k,j}; \tag{4.1}$$

$$(2k+1)L_k(x) = L'_{k+1}(x) - L'_{k-1}(x), \qquad k \geq 1; \tag{4.2}$$

$$L_k(\pm 1) = (\pm 1)^k, \quad L'_k(\pm 1) = \frac{1}{2}(\pm 1)^{k-1}k(k+1). \tag{4.3}$$

We consider first the model equation (2.6) in the interval $I := (-1, \infty)$. In order to design a well-posed multi-domain formulation, we will start with a a proper variational formulation for (2.6). Let us denote

$$\begin{aligned} V &:= \{v \in H^1(I) : v(-1) = 0,\ \lim_{x\to+\infty} v(x) = \lim_{x\to+\infty} \partial_x v(x) = 0\} \\ W &:= \{v \in H^2(I) : v(-1) = v_x(-1) = 0,\ \lim_{x\to+\infty} v(x) = 0\} \end{aligned} \tag{4.4}$$

and

$$a(u, v) := (\partial_x u, \partial_x^2 v) + \beta(u, v),\ u \in V,\ v \in W. \tag{4.5}$$

Then, a dual-Petrov-Galerkin formulation for (2.6) is: Find $u \in V$ such that

$$a(u, v) = (f, v),\ \forall v \in W. \tag{4.6}$$

Thus, it is clear that when constructing approximation spaces (V_N, W_N) for (V, W), it is natural to require that $V_N \in C(I)$ and $W_N \in C^1(I)$.

For the sake of clarity, we will concentrate on the case of two subdomains. The approach can be extended to more than two subdomains in a straightforward manner.

Let $I_1 := (-1, 1)$, $I_2 := (1, \infty)$, $u^{I_1} := u|_{I_1}$ and $u^{I_2} := u|_{I_2}$. Further, let $N := (N_1, N_2)$, $\mathcal{P}_{N_1}(I_1)$ be the space of all polynomials of degree $\le N_1$ on I_1, and $\widehat{\mathcal{P}}_{N_2}(I_2) := \text{span}\{\hat{\mathcal{L}}_n(x+1) : n = 0, 1, \cdots, N\}$. Then, a set of proper trial and test function spaces are

$$\begin{aligned} V_N &:= \{u : u^{I_1} \in \mathcal{P}_{N_1}(I_1); u^{I_2} \in \widehat{\mathcal{P}}_{N_2}(I_2); u \in C^0(I),\ u(-1) = 0\}, \\ W_N &:= \{u : u^{I_1} \in \mathcal{P}_{N_1+1}(I_1);\ u^{I_2} \in \widehat{\mathcal{P}}_{N_2+1}(I_2);\ u \in C^1(I),\ u(-1) = u_x(-1) = 0\}. \end{aligned} \tag{4.7}$$

We note that $\dim(V_N) = \dim(W_N) = N_1 + N_2$. Hence, the composite Legendre-Laguerre dual-Petrov-Galerkin (LLDPG) approximation to (2.6) is : Find $u_N \in V_N$ such that

$$a(u_N, v_N) = (f, v_N), \qquad \forall v_N \in W_N. \tag{4.8}$$

We will show below that the choice of V_N and W_N guarantees the well-posedness of the above variational formulation.

4.1. **Basis functions and implementations.** In this subsection, we are concerned with the implementation details for (4.8).

We introduce two pairs of trial and test functions spaces for the two subdomains as follows:

$$\begin{aligned} \mathring{V}_{N_1}^{I_1} &:= \{u^{I_1} : u^{I_1} \in \mathcal{P}_{N_1}(I_1);\ u^{I_1}(\pm 1) = u_x^{I_1}(1) = 0\}, \\ \mathring{W}_{N_1}^{I_1} &:= \{u^{I_1} : u^{I_1} \in \mathcal{P}_{N_1+1}(I_1);\ u^{I_1}(\pm 1) = u_x^{I_1}(\pm 1) = 0\}; \end{aligned} \tag{4.9}$$

$$\begin{aligned} \mathring{V}_{N_2}^{I_2} &:= \{u^{I_2} : u^{I_2} \in \widehat{\mathcal{P}}_{N_2}(I_2);\ u^{I_2}(1) = 0\}, \\ \mathring{W}_{N_2}^{I_2} &:= \{u^{I_2} : u^{I_2} \in \widehat{\mathcal{P}}_{N_2+1}(I_2);\ u^{I_2}(1) = u_x^{I_2}(1) = 0\}. \end{aligned} \tag{4.10}$$

Using (4.2) and (4.3), one verifies readily that

$$\begin{aligned} \phi_k^{I_1}(x) &= L_k(x) - \frac{2k+3}{2k+5} L_{k+1}(x) - L_{k+2}(x) + \frac{2k+3}{2k+5} L_{k+3}(x) \in \mathring{V}_{k+3}^{I_1}, \\ \psi_k^{I_1}(x) &= L_k(x) - \Big(1 + \frac{2k+3}{2k+7}\Big) L_{k+2}(x) + \frac{2k+3}{2k+7} L_{k+4}(x) \in \mathring{W}_{k+3}^{I_1}. \end{aligned} \tag{4.11}$$

Note that $\{\phi_k^{I_1}\}$ are the basis functions used in [17] for third-order equations, while $\{\psi_k^{I_1}\}$ are used in [18] for forth-order equations. As in (2.12), we choose

$$\begin{aligned}\phi_k^{I_2}(x) &= \hat{\mathcal{L}}_k(x+1) - \hat{\mathcal{L}}_{k+1}(x+1) \in \mathring{V}_{k+1}^{I_2},\\ \psi_k^{I_2}(x) &= \hat{\mathcal{L}}_k(x+1) - 2\hat{\mathcal{L}}_{k+1}(x+1) + \hat{\mathcal{L}}_{k+2}(x+1) \in \mathring{W}_{k+1}^{I_2}.\end{aligned} \tag{4.12}$$

Therefore,

$$\begin{aligned}&\mathring{V}_{N_j}^{I_j} = \mathrm{span}\{\phi_0^{I_j}, \phi_1^{I_j}, \cdots, \phi_{N_j'}^{I_j}\}, \qquad \mathring{W}_{N_j}^{I_j} = \mathrm{span}\{\psi_0^{I_j}, \psi_1^{I_j}, \cdots, \psi_{N_j'}^{I_j}\},\\ &\qquad j = 1,2,\ N_1' = N_1 - 3,\ N_2' = N_2 - 1.\end{aligned} \tag{4.13}$$

Note that these basis functions are local, i.e., their support is restricted in one subdomain.

Let us denote

$$\mathring{V}_N := \{u : u^{I_1} \in \mathring{V}_{N_1}^{I_1},\ u^{I_2} \in \mathring{V}_{N_2}^{I_2}\}, \quad \mathring{W}_N := \{u : u^{I_1} \in \mathring{W}_{N_1}^{I_1},\ u^{I_2} \in \mathring{W}_{N_2}^{I_2}\}. \tag{4.14}$$

Next, we construct global basis function $\{\Phi_1, \Phi_2\} \in V_N$ and $\{\Psi_1, \Psi_2\} \in W_N$ such that the spaces V_N and W_N can be decomposed into

$$V_N = \mathring{V}_N \cup \mathrm{span}\{\Phi_1, \Phi_2\}, \quad W_N = \mathring{W}_N \cup \mathrm{span}\{\Psi_1, \Psi_2\}. \tag{4.15}$$

More precisely, we seek $\Phi_j(x)$ and $\Psi_j(x), j = 1,2, x \in I$ such that

$$\begin{aligned}&\Phi_1 \in V_N,\ \ \Phi_1(-1) = 0,\ \ \Phi_1(1) = 1,\ \ \Phi_1'(1) = 0,\\ &\Phi_2 \in V_N,\ \ \Phi_2(-1) = 0,\ \ \Phi_2(1) = 0,\ \ \Phi_2'(1) = 1,\ \ \Phi_2^{I_2}(x) \equiv 0,\ x \in I_2;\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\Psi_1 \in W_N,\ \ \Psi_1(-1) = \Psi_1'(-1) = 0,\ \ \Psi_1(1) = 1,\ \ \Psi_1'(1) = 0,\\ &\Psi_2 \in W_N,\ \ \Psi_2(-1) = \Psi_2'(-1) = 0,\ \ \Psi_2(1) = 0,\ \ \Psi_2'(1) = 1.\end{aligned} \tag{4.17}$$

A simple set of functions satisfying these conditions are given below:

$$\Phi_1(x) = \begin{cases} \frac{2}{3}L_0(x) + \frac{1}{2}L_1(x) - \frac{1}{6}L_2(x), & x \in I_1,\\ \hat{\mathcal{L}}_0(x-1), & x \in I_2,\end{cases} \tag{4.18}$$

$$\Phi_2(x) = \begin{cases} -\frac{1}{3}L_0(x) + \frac{1}{3}L_2(x), & x \in I_1,\\ 0, & x \in I_2,\end{cases} \tag{4.19}$$

$$\Psi_1(x) = \begin{cases} \frac{1}{2}L_0(x) + \frac{3}{5}L_1(x) - \frac{1}{10}L_3(x), & x \in I_1,\\ \frac{3}{2}\hat{\mathcal{L}}_0(x-1) - \frac{1}{2}\hat{\mathcal{L}}_1(x-1), & x \in I_2,\end{cases} \tag{4.20}$$

$$\Psi_2(x) = \begin{cases} \frac{1}{6}(L_2(x) - L_0(x)) + \frac{1}{10}(L_3(x) - L_1(x)), & x \in I_1,\\ \hat{\mathcal{L}}_0(x-1) - \hat{\mathcal{L}}_1(x-1), x \in I_2.\end{cases} \tag{4.21}$$

With this set of basis functions, the linear system associated with (4.8) can be solved using the procedure below.

- Pre-computation: We construct the orthogonal compliment of V_N with respect to the bilinear form $a(\cdot,\cdot)$. To this end, let $\mathring{\Phi}_1, \mathring{\Phi}_2 \in \mathring{V}_N$ be the solutions of the following problems:

$$a(\mathring{\Phi}_j, \mathring{v}_N) = -a(\Phi_j, \mathring{v}_N), \quad \forall \mathring{v}_N \in \mathring{W}_N,\ \ j = 1,2. \tag{4.22}$$

  Setting $\Theta_j = \mathring{\Phi}_j + \Phi_j, j = 1,2$, and $V_N^H := \mathrm{span}\{\Theta_1, \Theta_2\}$. By construction, we have $\mathring{W}_N \perp V_N^H$ in the sense that

$$a(v_N, w_N) = 0, \quad \forall v_N \in V_N^H,\ \forall w_N \in \mathring{W}_N.$$

- First step: Find $\mathring{u}_N \in \mathring{V}_N$ such that

$$a(\mathring{u}_N, \mathring{v}_N) = (f, \mathring{v}_N), \quad \forall \mathring{v}_N \in \mathring{W}_N. \tag{4.23}$$

- Second step: We determine the unknowns $(u_N(1), u_N'(1))$ at the interface by

$$a(\Theta_1, \Psi_j)u_N(1) + a(\Theta_2, \Psi_j)u_N'(1) = (f, \Psi_j) - a(\mathring{u}_N, \Psi_j), \quad j = 1, 2. \tag{4.24}$$

We observe from (4.23) and (4.24) that

$$a(\mathring{u}_N + u_N(1)\Theta_1 + u_N'(1)\Theta_2, v_N) = (f, v_N), \quad \forall v_N \in W_N, \tag{4.25}$$

which implies that the solution of (4.8) is

$$u_N = \mathring{u}_N + u_N(1)\Theta_1 + u_N'(1)\Theta_2. \tag{4.26}$$

Note that the equation (4.23) (and (4.22)) can be solved separately on each subdomain. We have already showed in Section 2 that the problem (4.23) on I_2 is well-posed and can be efficiently solved.

Note that the subproblem on I_1 is different from the dual-Petrov-Galerkin formulation studied in [17]. However, we still have the following result:

Lemma 4.1. *The dual-Petrov-Galerkin formulation: find* $u_N^{I_1} \in \mathring{V}_{N_1}^{I_1}$ *such that*

$$(\partial_x u_{N_1}^{I_1}, \partial_x^2 v_{N_1}^{I_1}) + \beta(u_{N_1}^{I_1}, v_{N_1}^{I_1}) = (f^{I_1}, v_{N_1}^{I_1}), \quad \forall v_{N_1}^{I_1} \in \mathring{W}_{N_1}^{I_1}, \tag{4.27}$$

admits a unique solution. Furthermore,

$$\frac{\beta}{2}\|(1+x)^{1/2}u_{N_1}^{I_1}\|^2 + \frac{3}{2}|u_{N_1}^{I_1}|_1^2 \le \frac{1}{2\beta}\|(1+x)^{1/2}f^{I_1}\|^2. \tag{4.28}$$

Proof. Given $u_{N_1}^{I_1} \in \mathring{V}_{N_1}^{I_1}$, we have $(1+x)u_{N_1}^{I_1} \in \mathring{W}_{N_1}^{I_1}$. Taking $v_{N_1}^{I_1} = (1+x)u_{N_1}^{I_1}$ in (4.27), one can verify that

$$\begin{aligned}(\partial_x u_{N_1}^{I_1}, \partial_x^2((1+x)u_{N_1}^{I_1})) &= (\partial_x u_{N_1}^{I_1}, (1+x)\partial_x^2 u_{N_1}^{I_1} + 2\partial_x u_{N_1}^{I_1})\\ &= \frac{1}{2}(\partial_x(\partial_x u_{N_1}^{I_1})^2, (1+x)) + 2|u_{N_1}^{I_1}|_1^2 = \frac{3}{2}|u_{N_1}^{I_1}|_1^2,\end{aligned}$$

and

$$(f^{I_1}, (1+x)u_{N_1}^{I_1}) \le \frac{\beta}{2}\|(1+x)^{1/2}u_{N_1}^{I_1}\|^2 + \frac{1}{2\beta}\|(1+x)^{1/2}f^{I_1}\|^2.$$

The desired result follows from the Lax-Milgram Lemma. □

Setting

$$d_{ij}^{I_1} = (\partial_x \phi_j^{I_1}, \partial_x^2 \psi_i^{I_1}) + \beta(\phi_j^{I_1}, \psi_i^{I_1}),$$

one can readily obtain that

$$d_{ij}^{I_1} = \begin{cases} \beta c_i \gamma_{i+4}, & j = i+4,\\ -\beta a_{i+3} c_i \gamma_{i+4}, & j = i+3,\\ \beta(b_i \gamma_{i+2} - c_i \gamma_{i+4}), & j = i+2,\\ -2(2i+3)(2i+5) + \beta(-a_{i+1} b_i \gamma_{i+2} - a_{i+1} c_i \gamma_{i+4}), & j = i+1,\\ 2(2i+3)^2 + \beta(\gamma_i - b_i \gamma_{i+2}), & j = i,\\ \beta(-a_{i-1}\gamma_i + a_{i-1} b_i \gamma_{i+2}), & j = i-1,\\ -\beta\gamma_i, & j = i-2,\\ \beta a_{i-3}\gamma_i, & j = i-3,\\ 0, & \text{otherwise}. \end{cases} \tag{4.29}$$

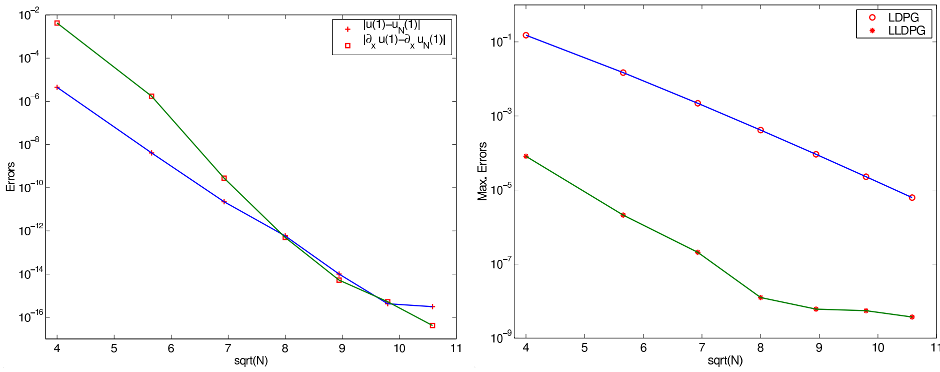

FIGURE 4.1. Left: Errors at interface; Right: LDPG vs. LLDPG

where

$$a_k = \frac{2k+3}{2k+5}, \quad b_k = -1 - \frac{2k+3}{2k+7}, \quad c_k = \frac{2k+3}{2k+7}, \quad \gamma_k = \frac{2}{2k+1}.$$

Thus, the problem (4.23) on I_1 can also be efficiently solved.

Remark 4.1. By combining the techniques in Section 2 & [17] and [9], one can derive error estimates for the composite Legendre-Laguerre scheme (2.20) and (2.22). However, the details are beyond the scope of this paper.

4.2. **Numerical results.** In order to examine the convergence rate of the LLDPG method, we first compare it with the LDPG method.

Example 4.1. We consider linear equation (2.6) with the exact solution given in Example 2.2 ($h = 3.5$). We take $N_1 = N_2 = N/2$, where N is the mode used in the LDPG scheme (2.8). In Figure 4.1 (left), we plot the errors at the interface, which shows a very accurate approximation to the values $u(1)$ and $u'(1)$.

The maximum absolute errors at the nodes for the LDPG scheme (2.8) and the LLDPG (4.8) are illustrated in Figure 4.1 (right). Note that much better numerical results can be obtained with the LLDPG method.

We now consider the application of the LLDPG method to the KDV equation. As in Section 2, we use the Crank-Nicolson leap-frog scheme for the time discretization. Notice that at each time step, we only need to solve an equation of the form (4.8).

Example 4.2. We consider the initial value KDV problem:

$$u_t + uu_x + u_{xxx} = 0, \quad u(x,0) = u_0(x), \tag{4.30}$$

with the exact soliton solution given in (2.25). Since $|u(x,t)|$ tends to 0 exponentially as $|x| \to +\infty$, we can approximate the initial value problem (4.30) by an initial boundary value problem in $(-S, +\infty)$, where $S > 0$ such that $|u(-S,t)|$ is negligibly small. We take $\kappa = 0.3, x_0 = -5, S = 30, \tau = 10^{-3}$ and apply the LLDPG method with two subdomains $(-30, 30)$ and $(30, \infty)$ and with $N_1 = 2N/3, N_2 = N/3$. On the left of Figure 4.2, we plot time evaluation of the approximate solution ($N = 160$), and on the right, we plot the maximum errors at $t = 1, 10, 20$. (Here, the wave did not reach the interface $x = 30$). A geometric convergence rate is observed in this case.

Example 4.3. We still consider problem (4.30) with exact solution (2.25), and take $\kappa = 0.3, x_0 = 1$ We use the LLDPG scheme with two subdomains $(-10, 10)$

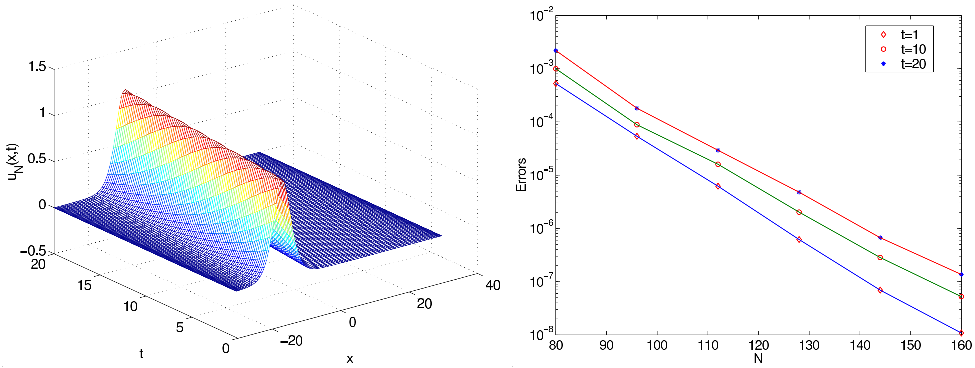

FIGURE 4.2. Left: Numerical solution by LLDPG; Right: Maximum errors vs. N

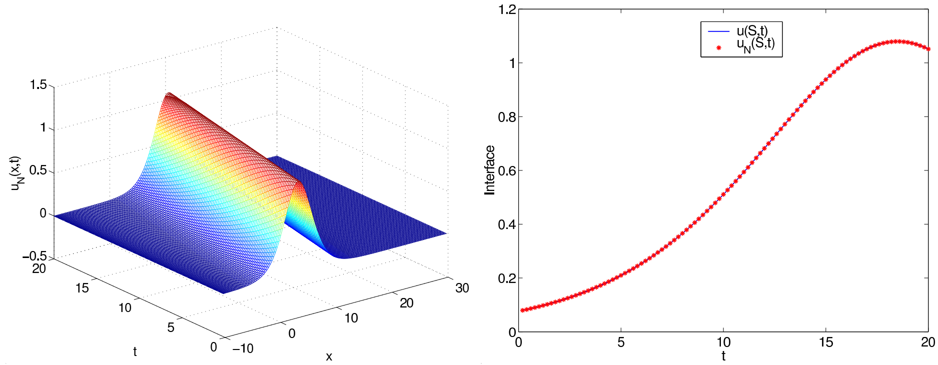

FIGURE 4.3. Left: Numerical solution by LLDPG; Right: $u_N(S,t)$ vs. $u(S,t)$

and $(10,\infty)$, and we take $\tau = 10^{-3}$, $N_1 = N_2 = 100$. We plot in Figure 4.3 (left), the time evaluation of the solution, and on the right, we plot the interface $u_N(S,t)$ vs. $u(S,t)$ at different time t. It shows that we can also get accurate numerical results when the wave passes through the interface.

5. **Concluding Remarks.** We proposed in this paper a dual-Petrov-Galerkin method for linear third-order equations and the Korteweg-de Vries equation in semi-infinite intervals.

We first presented a single domain dual-Petrov-Galerkin method using Laguerre functions and carried out a complete error analysis for a linear third-order equation and the KDV equation. It is shown that the dual-Petrov-Galerkin method leads to an efficient numerical algorithm and optimal error estimates.

We then presented a multi-domain dual-Petrov-Galerkin method using Legendre polynomials in the finite interval(s) and Laguerre functions in the infinite interval. By carefully choosing trial and test function spaces, we developed a well-posed and efficient multi-domain algorithm for third-order equations.

We also presented ample numerical results for both single domain and multi-domain approaches which illustrated the superior accuracy and effectiveness of the proposed dual-Petrov-Galerkin methods for third-order equations in semi-infinite intervals.

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