DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **8**, Number **3**, October **2007**

Website: http://AIMsciences.org
pp. **663–676**

ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS OF CONSISTENT SPLITTING SCHEMES FOR INCOMPRESSIBLE FLOWS

Jie Shen

Department of Mathematics, Purdue University
West Lafayette, IN 47907, USA

Xiaofeng Yang

Department of Mathematics, Purdue University
West Lafayette, IN 47907, USA

Abstract. We study a finite element approximation for the consistent splitting scheme proposed in [11] for the time dependent Navier-Stokes equations. At each time step, we only need to solve a Poisson type equation for each component of the velocity and the pressure. We cast the finite element approximation in an abstract form using appropriately defined discrete differential operators, and derive optimal error estimates for both velocity and pressure under the inf-sup assumption.

1. **Introduction.** A main difficulty for the numerical simulation of incompressible flows is caused by the incompressibility constraint which couples the velocity and the pressure. Since Chorin [4] and Temam [28] introduced in the later sixties the original projection method which provided a strategy to decouple the computation of the pressure from that of the velocity, an enormous body of work has been devoted to the analysis and the implementation of various versions of the projection type method. For the up-to-date review on this subject, we refer to [9], where, the authors classified various projection type schemes into three classes, namely the pressure-correction (cf., for instance, [5, 8, 13, 22, 25, 26, 27, 20]), the velocity correction (cf. [12, 19, 21, 24]) and the consistent splitting [11, 18] (which is equivalent, only in semi-discretized form, to the gauge method [6, 23]).

The consistent splitting schemes we consider in this paper were proposed by Guermond and Shen in [11]. The consistent splitting scheme is based on replacing the divergence free condition in the time-dependent Stokes equations (6) by a formally equivalent pressure Poisson equation

$$(\nabla p, \nabla q) = (f - \nabla \times \nabla \times u, \nabla q),\ \forall q \in H^1(\Omega). \tag{1}$$

It updates the velocity through the momentum equation with an explicit treatment for the pressure and then updates the pressure through (1). Hence, it does not involve a projection step, and consequently, the velocity approximation is never

2000 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Navier–Stokes equations, finite elements, incompressibility, consistent splitting, error estimates.

This work is supported in part by NSF grants DMS-0311915, DMS-0610646.

divergence free but the divergence is slow to converge to zero as the discretization parameters tend to zero.

It is shown in [11], by ample numerical results, that the consistent splitting schemes are free of splitting errors and lead to optimal (in time) results for the velocity as well as for the pressure. In this regard, the consistent splitting schemes are more attractive than the corresponding pressure-correction [20, 13] or velocity-correction schemes [19, 12] which involve a projection step and the accuracy of the pressure approximation is affected by a splitting error (see, for instance, [9] for a recent review on this subject).

However, there are only very limited analytical results on the stability and error analysis for the consistent splitting schemes. In [11], the authors provided an *a priori* estimate, which could be regarded as a proof of stability, for the semi-discrete first-order consistent splitting scheme. In [18], a normal mode analysis for a second-order consistent splitting scheme was carried out for the case of a periodic channel. In particular, how to prove the stability of the second-order consistent splitting scheme (which is found to be unconditionally stable in practice) in the general setting remains to be open. Hence, we shall concentrate in this paper on the first-order consistent splitting scheme.

For the fully discrete case, the issues are further complicated by the fact that there are two entirely different ways to discretize the consistent splitting scheme (cf. [11]). Given a suitable set of approximation spaces $X_h \times M_h$ for the velocity and the pressure, the first version of the first-order consistent splitting scheme for the time dependent Stokes equations (6) is: find $u_h^k \in X_h, p_h^k \in M_h$ such that

$$\left(\frac{u_h^{k+1}-u_h^k}{\delta t}, v_h\right) + (\nabla u_h^{k+1}, \nabla v_h) - (p_h^k, \nabla \cdot v_h) = (f(t^{k+1}), v_h), \forall v_h \in X_h, \quad (2)$$

$$(\nabla(p_h^{k+1} - p_h^k + \Pi_h \nabla \cdot u_h^{k+1}), \nabla q_h) = (\frac{u_h^{k+1}-u_h^k}{\delta t}, \nabla q_h), \forall q_h \in M_h. \quad (3)$$

(where Π_h is a projection operator defined in (22)) while the second version is: find $u_h^k \in X_h, p_h^k \in M_h$ such that

$$\left(\frac{u_h^{k+1}-u_h^k}{\delta t}, v_h\right) + (\nabla u_h^{k+1}, \nabla v_h) - (p_h^k, \nabla \cdot v_h) = (f(t^{k+1}), v_h), \forall v_h \in X_h, \quad (4)$$

$$(\nabla p_h^{k+1}, \nabla q_h) = (f(t^{k+1}), \nabla q_h) - \int_{\partial\Omega} \nabla \times u_h^{k+1} \cdot \nabla \times q_h, \forall q_h \in M_h. \quad (5)$$

It is argued in [11] (see also [9]) by some heuristic arguments that the inf-sup condition for $X_h \times M_h$ is needed for the first version to achieve optimal error estimates for both velocity and pressure while the inf-sup condition is not needed for the second version. The main purpose of this paper is to carry out a rigorous stability and error analysis for the finite-element approximation of the time-dependent Stokes equation using the first version of the consistent splitting method. We shall show that it indeed leads to optimal error estimates for both velocity and pressure provided that the inf-sup condition is satisfied. The stability and error analysis for the second version, with or without the inf-sup condition, remains to be a major open problem.

The rest of paper is organized as follows. In section 2, we introduce notations and present some preliminary results for the finite element approximations. In Section 3, we prove a stability result as well as optimal error estimates for the first-order consistent splitting scheme. In Section 4, we present some numerical results confirming our analysis and conclude with a few remarks.

2. **Preliminaries.** Since it is now well-known that a consistent treatment of nonlinear terms in the Navier-Stokes equations will not affect the formal accuracy of a splitting scheme (cf. [9, 15]), we shall restrict ourselves to the time-dependent Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla^2 u + \nabla p = f \qquad \text{in } \Omega \times [0,T], \\ \text{div } u = 0 \qquad \text{in } \Omega \times [0,T]. \end{cases} \tag{6}$$

with initial and boundary conditions

$$u|_{t=0} = v_0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{7}$$

where f is the body force, Ω is an open bounded domain in R^d ($d = 2$ or 3) with a sufficiently smooth boundary.

We now introduce some notations. Let $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ denote the usual Sobolev spaces equipped with the norm $\|\cdot\|_{s,p}$ for $0 \le s \le \infty, 0 \le p \le \infty$. In particular, we denote the Hilbert spaces $W^{s,2}(\Omega)$ by $H^s(\Omega)$ ($s = 0, \pm 1, \cdots$) with norm $\|\cdot\|_s$ and semi norm $|\cdot|_s$. The norm and inner product of $L^2(\Omega) = H^0(\Omega)$ are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$ respectively.

We use d_t and ∂_t to denote the derivative and partial derivative with respect to time, respectively. Let $\delta t > 0$ be the time step and set $t^k = k\delta t$ for $0 \le k \le K = [T/\delta t]$. For any function which is continuous in time, $\phi(t)$, we denote $\phi^k = \phi(t^k)$ and define the difference operator δ by $\delta\phi^k = \phi^k - \phi^{k-1}$. Let W be a Banach space, we set $L^p(W) = L^p(0,T;W)$. We denote by $\ell^p(W)$ the discrete L^p space for the vector $\{w = (w^0, w^1, \cdots, w^K), w^k \in W, 0 \le k \le K\}$ with norm:

$$\|\phi\|_{\ell^p(W)} := \left(\delta t \sum_{k=0}^{K} \|\phi^k\|_W^p\right)^{\frac{1}{p}}, \qquad \|\phi\|_{\ell^\infty(W)} := \max_{0\le k\le K}\left(\|\phi^k\|_W\right). \tag{8}$$

We denote by c a generic constant that is independent of h and δt but possibly depends on the data and the solution. We shall use the expression $A \lesssim B$ to say that there is a generic constant such that $A \le cB$.

We define the following Hilbert spaces

$$X = H_0^1(\Omega)^d = \{v \in H^1(\Omega)^d, v|_{\partial\Omega} = 0\}, \tag{9}$$

$$L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0\}, \tag{10}$$

$$Q = H^1(\Omega), \; M = L_0^2(\Omega) \cap H^1(\Omega). \tag{11}$$

Let $\Sigma_h = \{J\}$ be a quasi-uniform triangulation of the domain Ω. We denote by (X_h, Q_h) a suitable internal approximation to (X, Q) satisfying the following approximation properties: there exists an integer $l > 0$ such that

$$\begin{cases} \inf_{v_h\in X_h}\{\|v - v_h\|_0 + h\|v - v_h\|_1\} \lesssim h^{r+1}\|v\|_{r+1}, \\ \qquad\qquad \forall v \in H^{r+1}(\Omega)^d \cap H_0^1(\Omega)^d, \; r \in [0,l]. \\ \inf_{q_h\in Q_h}\{\|q - q_h\|_0 + h\|q - q_h\|_1\} \lesssim h^r\|q\|_r, \forall q \in H^r(\Omega), \; r \in [1,l]. \end{cases} \tag{12}$$

Setting $M_h = Q_h \cap L_0^2(\Omega)$, we also assume the pair $X_h \times M_h$ satisfies the Babŭska-Brezzi inf-sup condition (cf. [1, 3]):

$$\exists c > 0, \quad \inf_{q_h\in M_h}\sup_{v_h\in X_h}\frac{(\nabla\cdot v_h, q_h)}{\|\nabla v_h\|_0} \ge c\|q_h\|_0. \tag{13}$$

An example of such a pair is the Taylor-Hood element with $l \geq 2$:

$$X_h = \{v_h \in C^0(\Omega)^d \cap X, \quad v_h|_J \in P_l(J)^d, \forall J \in \Sigma_h\}, \tag{14}$$

$$M_h = \{q_h \in C^0(\Omega) \cap M, \quad q_h|_J \in P_{l-1}(J), \forall J \in \Sigma_h\}. \tag{15}$$

In order to reformulate the finite element approximation of the consistent splitting scheme into a suitable operator form, we introduce several discrete differential operators as in [14].

- Discrete Laplacian operator: $A_h : X_h \to X_h'$ such that

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h), \ \forall (u_h, v_h) \in X_h \times X_h, \tag{16}$$

  where X_h' is the dual space of X_h; X_h' is identical to X_h in term of vector space but is equipped with the dual norm.
- Discrete divergence operator: $B_h : X_h \to M_h$ and discrete gradient operator $B_h^t : M_h \to X_h'$ such that

$$(B_h v_h, p_h) = -(\nabla \cdot v_h, p_h) = (v_h, B_h^t p_h), \ \forall (v_h, p_h) \in X_h \times M_h. \tag{17}$$

We also define $\pi_h : H^{-1}(\Omega)^d \to X_h'$ such that

$$(\pi_h f, v_h) = (f, v_h), v_h \in X_h. \tag{18}$$

In the functional framework defined above, the mixed finite element approximation to the time-dependent Stokes problem (6) can be formulated as follows. For $f_h(t) \in X_h'$ and $v_{0,h} \in X_h$, find $(u_h(t), p_h(t)) \in X_h \times M_h$ such that

$$\begin{cases} \dfrac{du_h}{dt} + A_h u_h + B_h^t p_h = f_h, \quad 0 < t \leq T, \\ B_h u_h = 0, \\ u_h|_{t=0} = v_{0,h}. \end{cases} \tag{19}$$

Where $f_h = \pi_h f$ and $v_{0,h}$ is a suitable approximation of v_0 in X_h. It is well-known that the problem has the unique solution $(u_h(t), p_h(t))$, and the solution is stable with respect to the data. Furthermore, because X_h and M_h are convergent and stable approximations of $H_0^1(\Omega)^d$ and $H^1(\Omega)/\mathbb{R}$, the solution of (19) converges in an appropriate sense to that of the continuous problem (6). For more details on finite element approximations to the Stokes/Navier-Stokes equations, we refer to, for instance, [7, 16, 17].

In order to describe the consistent splitting scheme, we introduce the vector space $Y_h = X_h + \nabla M_h$ and equip it with the norm of $L^2(\Omega)^d$. It is clear that $X_h \subset Y_h$. We denote $i_h : X_h \to Y_h$ the natural injection of X_h into Y_h, and i_h^t the L^2 projection of Y_h in X_h. Obviously,

$$\forall v_h \in Y_h, \quad \|i_h^t v_h\|_0 \leq \|v_h\|_0. \tag{20}$$

We define another discrete divergence operator $C_h : Y_h \to M_h$ and its transpose $C_h^t : M_h \to Y_h$ by

$$(C_h v_h, q_h) = (v_h, \nabla q_h) = (v_h, C_h^t q_h), \ \forall (v_h, q_h) \in Y_h \times M_h. \tag{21}$$

The following relationships between B_h and C_h are proved in [14]:

Lemma 2.1. *C_h is an extension of B_h, i.e., $C_h i_h = B_h$, $i_h^t C_h^t = B_h^t$. C_h^t is the restriction of ∇ to M_h, i.e. $C_h^t q_h = \nabla q_h$, $\forall q_h \in M_h$.*

Finally, we define Π_h as the orthogonal projector from $L^2(\Omega)$ onto Q_h by

$$(\Pi_h \phi - \phi, \psi_h) = 0, \qquad \forall \psi_h \in Q_h. \tag{22}$$

3. **Stability and error estimates.**

3.1. **Stability analysis.** The first step is to rewrite the FEM approximation to the first-order consistent splitting scheme (2-3), which is in differential form, into the following operator form using the discrete operators introduced in the last section: find $u_h^k \in X_h, p_h^k \in M_h$ such that

$$\frac{u_h^{k+1} - u_h^k}{\delta t} + A_h u_h^{k+1} + B_h^t p_h^k = f_h^{k+1}, \tag{23}$$

$$(C_h^t(p_h^{k+1} - p_h^k - B_h u_h^{k+1}), C_h^t q) = (\frac{i_h u_h^{k+1} - i_h u_h^k}{\delta t}, C_h^t q), \forall q \in M_h. \tag{24}$$

where $f_h^{k+1} = \pi_h f(t^{k+1})$.

With the above operator formulation, we are now in position to establish the following *a priori* estimate :

Lemma 3.1. *The solution of the scheme (23)- (24) is bounded in the following sense:*

$$\|\delta u_h^{n+1}\|_0^2 + \delta t\|B_h u_h^{n+1}\|_0^2 + \delta t \sum_{k=1}^{n} \delta t\|C_h^t \psi_h^{k+1}\|_0^2 \leq C(\|\delta u_h^0\|^2 + \delta_t\|B_h u_h^0\|^2 + \delta t \sum_{k=0}^{n} \|\delta f_h^{k+1}\|_0^2)$$

Proof. The proof in [11] for the semi-discretized consistent splitting scheme makes essential use of the identity $-\Delta u = \nabla\times\nabla\times u - \nabla\nabla\cdot u$ which is not well defined for $u \in X_h$. Therefore, we consider $A_h u - B_h^t B_h u$ as a discrete counterpart of $\nabla\times\nabla\times u$. Then, the proof of this result can proceed essentially the same as in [11].

Applying the operator δ to (23) and adding a zero term to it, $-B_h^t B_h \delta u_h^{k+1} + B_h^t B_h \delta u_h^{k+1}$, we find that

$$\frac{\delta u_h^{k+1} - \delta u_h^k}{\delta t} + A_h \delta u_h^{k+1} - B_h^t B_h \delta u_h^{k+1} + B_h^t B_h u_h^{k+1} + B_h^t \psi_h^k = \delta f_h^{k+1}, \tag{25}$$

where we have set $\psi_h^k = \delta p_h^k - B_h u_h^k$. Thanks to Lemma 2.1, we have

$$(C_h^t \delta\psi_h^{k+1}, C_h^t q) = (\frac{\delta u_h^{k+1} - \delta u_h^k}{\delta t}, B_h^t q). \tag{26}$$

Taking the inner product of (25) with $2\delta t\delta u_h^{k+1}$ and using the identity $2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2$, we derive

$$\begin{aligned}
&\|\delta u_h^{k+1}\|_0^2 - \|\delta u_h^k\|_0^2 + \|\delta^2 u_h^{k+1}\|_0^2 + 2\delta t\|\nabla\delta u_h^{k+1}\|_0^2 - 2\delta t\|B_h \delta u_h^{k+1}\|_0^2\\
&\quad + \delta t(\|B_h u_h^{k+1}\|_0^2 - \|B_h u_h^k\|_0^2) + \delta t\|B_h \delta u_h^{k+1}\|_0^2\\
&\quad + 2\delta t(B_h^t \psi_h^k, \delta u_h^{k+1})\\
&= 2\delta t(\delta f_h^{k+1}, \delta u_h^{k+1}).
\end{aligned} \tag{27}$$

Then, take $q = 2\delta t^2 \psi_h^k$ in (26), we find

$$\delta t^2(\|C_h^t \psi_h^{k+1}\|_0^2 - \|C_h^t \psi_h^k\|_0^2) - \delta t^2\|C_h^t \delta\psi_h^{k+1}\|_0^2 = 2\delta t(\delta u_h^{k+1} - \delta u_h^k, B_h^t \psi_h^k). \tag{28}$$

Next, we take $q = 2\delta t^2 \psi_h^{k+1}$ in (26) and replace $k+1$ by k to obtain

$$2\delta t^2\|C_h^t \psi_h^k\|_0^2 = 2\delta t(\delta u_h^k, B_h^t \psi_h^k). \tag{29}$$

We take $q = \delta t^2 \delta\psi_h^{k+1}$ in (26) again and use the Cauchy-Schwarz inequality to find

$$\delta t^2 \|C_h^t \delta\psi_h^{k+1}\|_0^2 \le \|\delta^2 u_h^{k+1}\|_0^2. \tag{30}$$

Summing up (27)∼(30), and noticing that $\|B_h v_h\|_0 \le \|\nabla v_h\|_0, \forall v_h \in X_h$, we obtain

$$\begin{aligned}&\|\delta u_h^{k+1}\|_0^2 - \|\delta u_h^k\|_0^2 + \delta t(\|B_h u_h^{k+1}\|_0^2 - \|B_h u_h^k\|_0^2)\\ &\quad + \delta t^2(\|C_h^t \psi_h^{k+1}\|_0^2 + \|C_h^t \psi_h^k\|_0^2)\\ &= 2\delta t(\delta f^{k+1}, \delta u_h^{k+1}) \le \delta t\|\delta u_h^{k+1}\|_0^2 + \delta t\|\delta f_h^{k+1}\|_0^2.\end{aligned} \tag{31}$$

Finally, taking the sum of above relation from $k = 0$ to $n \le [T/\delta t] - 1$, we derive

$$\begin{aligned}&\|\delta u_h^{n+1}\|_0^2 + \delta t\|B_h u_h^{n+1}\|_0^2 + \delta t \sum_{k=1}^{n} \delta t \|C_h^t \psi_h^{k+1}\|_0^2\\ &\lesssim \|\delta u_h^0\|_0^2 + \delta t\|B_h u_h^0\|_0^2 + \delta t \sum_{k=0}^{n} \|\delta u_h^{k+1}\|_0^2 + \delta t \sum_{k=0}^{n} \|\delta f_h^{k+1}\|_0^2.\end{aligned} \tag{32}$$

We conclude (25) by applying the discrete Gronwall Lemma to the above. □

3.2. **Error estimates.** In order to simplify the error analysis, instead of comparing directly our numerical solution (u_h^k, p_h^k) with the exact solution $(u(t^k), p(t^k))$, we shall compare (u_h^k, p_h^k) with $(w_h(t^k), q_h(t^k)) \in X_h \times M_h$ where $(w_h(t), q_h(t))$ is the mixed approximation of $(u(t), p(t))$ defined as follows:

$$\begin{cases}(\nabla w_h(t), \nabla v_h) + (B_h^t q_h(t), v_h) = (\nabla u(t), \nabla v_h) - (p(t), \nabla \cdot v_h), \forall v_h \in X_h,\\ (B_h w_h(t), r_h) = 0,\ \forall r_h \in M_h.\end{cases} \tag{33}$$

It is well-known from the regularity properties of the Stokes problem that we have the following error estimates (see, for instance, [7, 10]):

Lemma 3.2. *If $u^{(j)} \in L^\beta(H^{l+1}(\Omega)^d) \cap H_0^1(\Omega)^d)$, $p^{(j)} \in L^\beta(H^l(\Omega))$ for $1 \le \beta \le \infty$ and $j = 0, 1, \cdots$, then*

$$\begin{aligned}&\|u^{(j)} - w_h^{(j)}\|_{L^\beta(L^2(\Omega)^d)} + h\left(\|u^{(j)} - w_h^{(j)}\|_{L^\beta(H^1(\Omega)^d)} + \|p^{(j)} - q_h^{(j)}\|_{L^\beta(L^2(\Omega))}\right)\\ &\lesssim h^{l+1}\left(\|u^{(j)}\|_{L^\beta(H^{l+1}(\Omega)^d)} + \|p^{(j)}\|_{L^\beta(H^l(\Omega))}\right).\end{aligned} \tag{34}$$

Lemma 3.3. *If $u^{(j)} \in L^\beta(H^2(\Omega)^d \cap H_0^1(\Omega)^d)$ and $p^{(j)} \in L^\beta(H^1(\Omega))$ for all $j = 0, 1, \cdots,$ and $1 \le \beta \le \infty$. Then,*

$$\begin{aligned}&\|w_h^{(j)}\|_{L^\beta(W^{0,\infty}(\Omega)^d \cap W^{1,3}(\Omega)^d)} + \|C_h^t q_h^{(j)}\|_{L^\beta(L^2(\Omega))}\\ &\lesssim \|u^{(j)}\|_{L^\beta(H^2(\Omega)^d)} + \|p^{(j)}\|_{L^\beta(H^1(\Omega))}.\end{aligned} \tag{35}$$

For convenience, we denote

$$\begin{cases}w_h^k = w_h(t^k), \quad q_h^k = q_h(t^k), \quad u^k = u(t^k)\\ e_h^k = w_h^k - u_h^k, \quad \varepsilon_h^k = q_h^k - p_h^k,\\ \phi_h^k = \varepsilon_h^k - \varepsilon_h^{k-1} - B_h e_h^k.\end{cases} \tag{36}$$

Let us assume

(H1) $\|e_h^0\|_0 \lesssim \min\,(h^{l+1}, \delta t^{3/2} h^{l-1}), \quad \|\varepsilon_h^0\|_0 \lesssim \delta t h^{l-1}$,

(H2) $u^{(j)} \in L^2(H^{l+1}(\Omega)^d),\ 0 \le j \le 3, \quad u^{(4)} \in L^2(L^2(\Omega)^d)$;
$p^{(j)} \in L^2(H^1(\Omega)),\ j = 1, 2; \quad p \in L^2(H^l(\Omega))$.

Remark 1. If we set $u_h^0 = w_h^0$ and $p_h^0 = q_h^0$, then the hypothesis (H1) is naturally satisfied.

To simplify the analysis, we assume that the solution is sufficiently smooth as specified in (H2). The assumption can be somewhat weakened at the expense of a more complicated analysis.

The main result is the following:

Theorem 3.4. *Assuming (H1-H2), we have*

$$\begin{aligned} &\|u-u_h\|_{\ell^2(H^1(\Omega)^d)} + \|p-p_h\|_{\ell^2(L^2(\Omega))} \lesssim \delta t + h^l, \\ &\|u-u_h\|_{\ell^2(L^2(\Omega)^d)} \lesssim \delta t + h^{l+1}. \end{aligned} \tag{37}$$

The proof of this result will be carried out with the help of a sequence of lemmas which we establish below.

Lemma 3.5. *We define*

$$R_h^{k+1} = \frac{w_h^{k+1} - w_h^k}{\delta t} - \partial_t u^{k+1}.$$

Then, we have the following bounds:

$$\|R_h\|_{l^2(L^2(\Omega)^d)} \lesssim \delta t\|u_{tt}\|_{L^2(L^2)} + h^{l+1}\|u_t\|_{L^2(H^{l+1})}, \tag{38}$$

$$\|\delta R_h\|_{l^2(L^2(\Omega)^d)} \lesssim \delta t^2\|u^{(3)}\|_{L^2(L^2)} + \delta t h^{l+1}\|u_{tt}\|_{L^2(H^{l+1})}, \tag{39}$$

$$\|\delta^2 R_h\|_{l^2(L^2(\Omega)^d)} \lesssim \delta t^2\|u^{(4)}\|_{L^2(L^2)} + \delta t h^{l+1}\|u^{(3)}\|_{L^2(H^{l+1})}. \tag{40}$$

Proof. We rewrite the residue as

$$R_h^{k+1} = \frac{1}{\delta t}\int_{t^k}^{t^{k+1}} \partial_t(w_h(t) - u(t))dt + \frac{u^{k+1}-u^k}{\delta t} - \partial_t u^{k+1}. \tag{41}$$

Thanks to Lemma 3.2 and Cauchy-Schwarz inequality, we can derive (38) from the following two inequalities:

$$\delta t\sum_{k=0}^{K}\frac{1}{\delta t^2}\|\int_{t^k}^{t^{k+1}}\partial_t(w_h-u)dt\|_0^2 \le \sum_{k=0}^{K}\int_{t^k}^{t^{k+1}}\|\partial_t(w_h-u)\|_0^2 dt \lesssim h^{2l+2}\|u_t\|^2_{L^2(H^{l+1})}.$$

$$\begin{aligned} \delta t\sum_{k=0}^{K}\|\frac{u^{k+1}-u^k}{\delta t} - \partial_t u^{k+1}\|_0^2 &= \delta t\sum_{k=0}^{K}\|\frac{1}{\delta t}\int_{t^k}^{t^{k+1}}(t-t_k)u_{tt}(t)dt\|_0^2 \\ &\le \delta t^2\sum_{k=0}^{K}\int_{t^k}^{t^{k+1}}\|u_{tt}\|_0^2 dt \lesssim \delta t^2\|u_{tt}\|^2_{L^2(L^2)}. \end{aligned} \tag{42}$$

We can derive (39) and (40) by using a similar procedure. □

Lemma 3.6. *We have the following estimates:*

$$\|\delta e_h\|_{\ell^\infty(L^2(\Omega)^d)} \lesssim \delta t^{3/2} + \delta t h^{l+1}, \tag{43}$$

$$\|B_h e_h\|_{\ell^\infty(L^2(\Omega)^d)} \lesssim \delta t + \delta t^{1/2} h^{l+1}, \tag{44}$$

and

$$\|\delta^2 e_h\|_{\ell^\infty(L^2(\Omega)^d)} \lesssim \delta t^{5/2} + \delta t^2 h^{l+1}, \tag{45}$$

$$\|B_h \delta e_h\|_{\ell^\infty(L^2(\Omega)^d)} \lesssim \delta t^2 + \delta t^{3/2} h^{l+1}. \tag{46}$$

Proof. Rewriting (33) using the discrete operators and comparing with (6), we find that $(w_h(t), q_h(t))$ satisfies at time $t = t^{k+1}$,

$$\begin{cases} \dfrac{w_h^{k+1} - w_h^k}{\delta t} + A_h w_h^{k+1} + B_h^t q_h^{k+1} = f_h^{k+1} + \tilde{R}_h^{k+1}, \\ B_h w_h^{k+1} = 0. \end{cases} \tag{47}$$

where

$$\tilde{R}_h^{k+1} = \frac{w_h^{k+1} - w_h^k}{\delta t} - \pi_h \partial_t u(t^{k+1}). \tag{48}$$

Subtracting the equation (47) from (23), we find the error equation

$$\frac{e_h^{k+1} - e_h^k}{\delta t} + A_h e_h^{k+1} + B_h^t \varepsilon_h^k = \tilde{R}_h^{k+1} + B_h^t (q_h^k - q_h^{k+1}). \tag{49}$$

On the other hand, by adding some zero terms to (24) and using Lemma 2.1, we can rewrite (24) as

$$(C_h^t \phi_h^{k+1}, C_h^t q) = (C_h^t \delta q_h^{k+1}, C_h^t q) + (\frac{e_h^{k+1} - e_h^k}{\delta t}, B_h^t q). \tag{50}$$

Applying the operator δ to the above two relations, we obtain

$$\begin{aligned} &\frac{\delta e_h^{k+1} - \delta e_h^k}{\delta t} + A_h \delta e_h^{k+1} - B_h^t B_h \delta e_h^{k+1} + B_h^t B_h e_h^{k+1} + B_h^t B_h \phi_h^k \\ &= \delta \tilde{R}_h^{k+1} - B_h^t \delta^2 q_h^{k+1}, \end{aligned} \tag{51}$$

$$(C_h^t \delta \phi_h^{k+1}, C_h^t q) = (C_h^t \delta^2 q_h^{k+1}, C_h^t q) + (\frac{\delta e_h^{k+1} - \delta e_h^k}{\delta t}, B_h^t q). \tag{52}$$

Taking the inner product of (51) with $2\delta t \delta e_h^{k+1}$, we get

$$\begin{aligned} &\|\delta e_h^{k+1}\|_0^2 - \|\delta e_h^k\|_0^2 + \|\delta^2 e_h^{k+1}\|_0^2 + 2\delta t \|\nabla \delta e_h^{k+1}\|_0^2 - 2\delta t \|B_h \delta e_h^{k+1}\|_0^2 \\ &+ \delta t (\|B_h e_h^{k+1}\|_0^2 - \|B_h e_h^k\|_0^2) + \delta t \|B_h \delta e_h^{k+1}\|_0^2 \\ &+ 2\delta t (B_h^t \phi_h^k, \delta e_h^{k+1}) = 2\delta t (\delta \tilde{R}_h^{k+1}, \delta e_h^{k+1}) - 2\delta t (\delta^2 q_h^{k+1}, B_h \delta e_h^{k+1}). \end{aligned} \tag{53}$$

Taking $q = 2\delta t^2 \phi_h^k$ in (52), we find

$$\begin{aligned} &\delta t^2 (\|C_h^t \phi_h^{k+1}\|_0^2 - \|C_h^t \phi_h^k\|_0^2) - \delta t^2 \|C_h^t \delta \phi_h^{k+1}\|_0^2 \\ &= 2\delta t (\delta e_h^{k+1}, B_h^t \phi_h^k) - 2\delta t (\delta e_h^k, B_h^t \phi_h^k) + 2\delta t^2 (C_h^t \delta^2 q_h^{k+1}, C_h^t \phi_h^k). \end{aligned} \tag{54}$$

Taking $q = 2\delta t^2 \phi_h^{k+1}$ in (50), and replacing $k+1$ by k, we derive

$$2\delta t^2 \|C_h^t \phi_h^k\|_0^2 = 2\delta t (\delta e_h^k, B_h^t \phi_h^k) + 2\delta t^2 (C_h^t \delta q_h^k, C_h^t \phi_h^k). \tag{55}$$

Applying δ to (24) and adding the term $i_h \delta w_h^{k+1} - i_h \delta w_h^k$ to the right hand side, we get

$$(C_h^t (\delta p_h^{k+1} - \delta p_h^k - B_h \delta u_h^{k+1}), C_h^t q) = (\frac{\delta e_h^{k+1} - \delta e_h^k}{\delta t}, B_h^t q). \tag{56}$$

We now take $q = (\delta p_h^{k+1} - \delta p_h^k - B_h \delta u_h^{k+1})$ in the above and use the Caughy-Schwarz inequality to get

$$\delta t^2 \|C_h^t (\delta p_h^{k+1} - \delta p_h^k - B_h \delta u_h^{k+1})\|_0^2 \le \|\delta e_h^{k+1} - \delta e_h^k\|_0^2. \tag{57}$$

We then derive from the above that

$$\delta t^2 \|C_h^t \delta \phi_h^{k+1}\|_0^2 \le \|\delta e_h^{k+1} - \delta e_h^k\|_0^2 + \delta t^2 \|C_h^t \delta^2 q_h^{k+1}\|_0^2. \tag{58}$$

Summing up (53)$\sim$ (55) and (58) and dropping some unnecessary positive terms, we obtain

$$\begin{aligned}
&\|\delta e_h^{k+1}\|_0^2 - \|\delta e_h^k\|_0^2 + \delta t(\|B_h e_h^{k+1}\|_0^2 - \|B_h e_h^k\|_0^2) + \delta t\|B_h \delta e_h^{k+1}\|_0^2 \\
&\quad + \delta t^2(\|C_h^t \phi_h^{k+1}\|_0^2 - \|C_h^t \phi_h^k\|_0^2) + 2\delta t^2\|C_h^t \phi_h^k\|_0^2 \\
&\le 2\delta t(\delta \tilde{R}_h^{k+1}, \delta e_h^{k+1}) - 2\delta t(\delta^2 q_h^{k+1}, B_h \delta e_h^{k+1}) \\
&\quad + 2\delta t^2(C_h^t \delta^2 q_h^{k+1}, C_h^t \phi_h^k) + 2\delta t^2(C_h^t \delta q_h^k, C_h^t \phi_h^k) \\
&\quad + \delta t^2\|C_h^t \delta^2 q_h^{k+1}\|_0^2.
\end{aligned} \tag{59}$$

We now estimate the terms on the right hand side by using Lemma 2.1 and Cauchy-Schwarz inequality,

$$2\delta t(\delta \tilde{R}_h^{k+1}, \delta e_h^{k+1}) = 2\delta t(\delta R_h^{k+1}, \delta e_h^{k+1}) \lesssim \delta t\|\delta R_h^{k+1}\|_0^2 + \delta t\|\delta e_h^{k+1}\|_0^2, \tag{60}$$

$$2\delta t(\delta^2 q_h^{k+1}, B_h \delta e_h^{k+1}) \lesssim \delta t\|\delta^2 q_h^{k+1}\|_0^2 + \delta t\|B_h \delta e_h^{k+1}\|_0^2, \tag{61}$$

$$2\delta t^2(C_h^t \delta^2 q_h^{k+1}, C_h^t \phi_h^k) \lesssim \frac{1}{2}\delta t^2\|C_h^t \phi_h^k\|_0^2 + \delta t^2\|C_h^t \delta^2 q_h^{k+1}\|_0^2, \tag{62}$$

$$2\delta t^2(C_h^t \delta q_h^{k+1}, C_h^t \phi_h^k) \lesssim \frac{1}{2}\delta t^2\|C_h^t \phi_h^k\|_0^2 + \delta t^2\|C_h^t \delta q_h^{k+1}\|_0^2. \tag{63}$$

Plugging the above estimates in (59), we find

$$\begin{aligned}
&\|\delta e_h^{k+1}\|_0^2 - \|\delta e_h^k\|_0^2 + \delta t(\|B_h e_h^{k+1}\|_0^2 - \|B_h e_h^k\|_0^2) + \delta t^2(\|C_h^t \phi_h^{k+1}\|_0^2 - \|C_h^t \phi_h^k\|_0^2) \\
&\le \delta t\|\delta R_h^{k+1}\|_0^2 + \delta t\|\delta^2 q_h^{k+1}\|_0^2 + \delta t^2\|C_h^t \delta^2 q_h^{k+1}\|_0^2 + \delta t^2\|C_h^t \delta q_h^{k+1}\|_0^2.
\end{aligned} \tag{64}$$

Thanks to Lemma 3.3 and (H2), we derive easily that

$$\sum_{k=1}^{n} \left(\delta t\|\delta^2 q_h^{k+1}\|_0^2 + \delta t^2\|C_h^t \delta^2 q_h^{k+1}\|_0^2 + \delta t^2\|C_h^t \delta q_h^{k+1}\|_0^2\right) \lesssim \delta t^3.$$

Summing up the above for k from 1 to $n \le [T/\delta t] - 1$, we infer from Lemmas 3.3 and 3.5 that

$$\begin{aligned}
&\|\delta e_h^{n+1}\|_0^2 + \delta t\|B_h e_h^{n+1}\|_0^2 + \delta t^2\|C_h^t \phi_h^{n+1}\|_0^2 \\
&\lesssim \|\delta e_h^1\|_0^2 + \delta t\|B_h e_h^1\|_0^2 + \delta t^2\|C_h^t \phi_h^1\|_0^2 + \delta t^3 + \delta t^2 h^{2l+2}.
\end{aligned} \tag{65}$$

In order to estimate the first three terms on the right hand side, we take the inner product of (49) at $k = 0$ with $2\delta t e_h^1$, and take $q = 2\delta t^2 C_h^t \phi_h^1$ in (50) at $k = 0$. Summing up the two relations, we find

$$\begin{aligned}
&\|e_h^1\|_0^2 + \|\delta e_h^1\|_0^2 + 2\delta t\|\nabla e_h^1\|_0^2 + 2\delta t^2\|C_h^t \phi_h^1\|_0^2 \\
&= \|e_h^0\|_0^2 + 2\delta t(R_h^1, e_h^1) - 2\delta t(B_h^t \delta q_h^1, e_h^1) - 2\delta t(B_h^t \varepsilon_h^0, e_h^1) \\
&\quad + 2\delta t^2(C_h^t \delta q_h^1, C_h^t \phi_h^1) + 2\delta t(e_h^1 - e_h^0, B_h^t \phi_h^1).
\end{aligned} \tag{66}$$

By the hypothesis (H1), we have

$$
\begin{aligned}
2\delta t(R_h^1, e_h^1) &\lesssim \frac{1}{3}\|e_h^1\|_0^2 + \delta t^2\|R_h^1\|_0^2,\\
2\delta t(B_h^t\delta q_h^1, e_h^1) &\lesssim \frac{1}{3}\|e_h^1\|_0^2 + \delta t^2\|B_h^t\delta q_h^1\|_0^2,\\
2\delta t^2(C_h^t\delta q_h^1, C_h^t\phi_h^1) &\lesssim \frac{1}{3}\delta t^2\|C_h^t\phi_h^1\|_0^2 + \delta t^2\|C_h^t\delta q_h^1\|_0^2,\\
2\delta t(\delta e_h^1, B_h^t\phi_h^1) &\leq \frac{4}{3}\delta t^2\|B_h^t\phi_h^1\|_0^2 + \frac{3}{4}\|\delta e_h^1\|_0^2,\\
\|e_h^0\|_0^2 &\lesssim \delta t^3 h^{2l-2},\\
2\delta t(B_h^t\varepsilon_h^0, e_h^1) &\lesssim \delta t\|\nabla e_h^1\|_0^2 + \delta t^3 h^{2l-2}.
\end{aligned} \tag{67}
$$

Notice that Lemma 2.1 implies that $\|B_h^t\phi_h\| \leq C_h^t\phi_h\|$ for all $\phi_h \in M_h$, we can then conclude (43) and (44) from the above and (65).

By applying the operator δ again and repeating the same procedure as above, we can establish (45) and (46). □

For the next estimate, we need to use the discrete inverse Stokes operator : $S_h : X_h' \to X_h$ which is defined in such a way that for all $v_h \in X_h$, $(S_h(v_h), \gamma_h) \in (X_h, M_h)$ is the solution of the following discrete stokes system:

$$
\begin{cases}
A_h^t S_h(v_h) + B_h^t\gamma_h = v_h,\\
B_h S_h(v_h) = 0.
\end{cases} \tag{68}
$$

We recall (cf. [10]) that there exists a constant $c_1 > 0$, s.t.

$$
\|S_h(v_h)\|_1 + \|\gamma_h\|_0 \leq c_1\|v_h\|_{-1}, \tag{69}
$$

and that the linear form $X_h' \to (v_h, S_h(v_h))^{\frac{1}{2}}$ induces a semi-norm on V_h, which we denote by $\|v_h\|_\star = (v_h, S_h(v_h))^{\frac{1}{2}}$, and we have

$$
\|S_h(v_h)\|_1 \leq c\|v_h\|_\star. \tag{70}
$$

Lemma 3.7. *The following estimates hold:*

$$
\|e_h\|_{\ell^2(L^2(\Omega)^d)} \lesssim \delta t + h^{l+1}, \tag{71}
$$

$$
\|\delta e_h\|_{\ell^2(L^2(\Omega)^d)} \lesssim \delta t^2 + \delta t h^{l+1}. \tag{72}
$$

Proof. Taking the inner product of (49) with $2\delta t S_h(e_h^{k+1})$ and noticing that $B_h S_h(e_h^{k+1}) = 0$ and

$$
(A_h e_h^{k+1}, S_h(e_h^{k+1})) = (e_h^{k+1}, A_h^t S_h(e_h^{k+1})) = \|e_h^{k+1}\|_0 - (\gamma_h, B_h e_h^{k+1}).
$$

we obtain

$$
\begin{aligned}
&\|e_h^{k+1}\|_\star^2 - \|e_h^k\|_\star^2 + \|\delta e_h^{k+1}\|_\star^2 + 2\delta t\|e_h^{k+1}\|_0^2 - 2\delta t(\gamma_h, B_h e_h^{k+1})\\
&= 2\delta t(R_h^{k+1}, S_h(e_h^{k+1})).
\end{aligned} \tag{73}
$$

From Lemmas 3.5, 3.6 and (69),

$$
\begin{aligned}
2\delta t(\gamma_h, B_h e_h^{k+1}) &\leq 2\delta t\|\gamma_h\|_0\|B_h e_h^{k+1}\|_0\\
&\leq c_1\delta t\|e_h^{k+1}\|_{-1}\|B_h e_h^{k+1}\|_0\\
&\lesssim \frac{1}{2}\delta t\|e_h^{k+1}\|_0^2 + \delta t^3 + \delta t^2 h^{2l+2},
\end{aligned} \tag{74}
$$

$$
\begin{aligned}
2\delta t(R_h^{k+1}, S_h(e_h^{k+1})) &\le 2\delta t\|S_h(e_h^{k+1})\|_1\|R_h^{k+1}\|_{-1} \\
&\lesssim \frac{1}{2}\delta t\|e_h^{k+1}\|_0^2 + \delta t\|R_h^{k+1}\|_0^2
\end{aligned}
\tag{75}
$$

Taking the summation of (73) for $k=0$ to $n \le [T/\delta t]-1$, Thanks to the above two inequalities, Lemma 3.5 and (H1), we obtain

$$
\begin{aligned}
&\|e_h^{n+1}\|_\star^2 + \sum_{k=1}^{n}\|\delta e_h^{k+1}\|_\star^2 + \delta t\sum_{k=1}^{n}\|e_h^{k+1}\|_0^2 \\
&\lesssim \delta t^2 + \delta t h^{2l+2} + \|e_h^0\|_\star^2 + \|R_h\|_{l^2(L^2(\Omega)^d)}^2 \\
&\lesssim \delta t^2 + h^{2l+2}.
\end{aligned}
\tag{76}
$$

which implies in particular (71).

We can derive (72) in a similar fashion by applying the operator δ to (49) and taking the inner product with with $S_h(\delta e_h^{k+1})$. □

Proof of Theorem 3.4. Since we rewrite the error equations of (49) and (50) as a discrete non-homogeneous stokes system for $(e_h^{k+1}, \varepsilon_h^k) \in X_h \times M_h$

$$
\begin{cases}
A_h e_h^{k+1} + B_h^t \varepsilon_h^k = R_h^{k+1} + B_h^t(q_h^k - q_h^{k+1}) - \dfrac{e_h^{k+1} - e_h^k}{\delta t}, \\
B_h e_h^{k+1} = B_h e_h^{k+1}.
\end{cases}
\tag{77}
$$

Now, the standard result for the discrete non-homogeneous Stokes system leads to

$$
\begin{aligned}
\|e_h^{k+1}\|_1 + \|\varepsilon_h^k\|_0 \le{}& \|R_h^{k+1}\|_{-1} + \|B_h^t(q_h^k - q_h^{k+1})\|_{-1} \\
&+ \frac{1}{\delta t}\|e_h^{k+1} - e_h^k\|_{-1} + \|B_h e_h^{k+1}\|_0.
\end{aligned}
\tag{78}
$$

Thanks to Lemmas 3.3, 3.5, 3.6 and 3.7, we have the following bounds:

$$
\begin{aligned}
\|R_h\|_{l^2(H^{-1}(\Omega)^d)} &\lesssim \|R_h\|_{l^2(L^2(\Omega)^d)} \lesssim \delta t + h^{l+1}, \\
\|B_h^t \delta q_h\|_{l^2(H^{-1}(\Omega)^d)} &\lesssim \|\nabla \delta q_h\|_{l^2(L^2(\Omega)^d)} \lesssim \delta t, \\
\delta t\sum_{k=1}^{K}(1/\delta t^2\|e_h^{k+1} - e_h^k\|_0^2) &\lesssim \delta t^2 + h^{2l+2}.
\end{aligned}
\tag{79}
$$

Then the proof is complete by summing up (78) and using Lemma 3.2. □

4. **Concluding remarks.**

In [11], ample numerical results, with sufficiently fine spatial discretization such that the errors are dominated by that of the time discretization, are presented to show that the consistent splitting schemes lead to optimal error estimates in time for both velocity and pressure. The results in the last section show that the first-order consistent splitting scheme also leads to optimal error estimates in space for both velocity and pressure, provided that the inf-sup condition is satisfied.

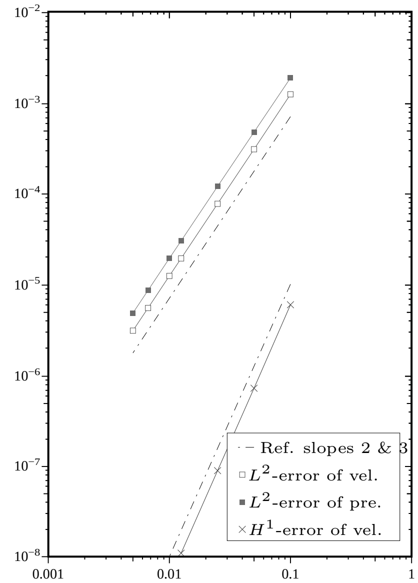

FIGURE 1. Convergence rates using P_2/P_1 finite elements

Now, we present some numerical experiments to verify our error estimates. We set the exact solution of (6) to be

$$\begin{aligned} &u(x,y) = \sin t\,(\pi\sin(2\pi y)\sin^2(\pi x), -\pi\sin(2\pi x)\sin^2(\pi y))^t,\\ &p(x,y) = \sin t\,\cos(\pi x)\sin(\pi y). \end{aligned} \tag{80}$$

and we choose δt sufficiently small so that the errors are dominated by the spatial discretization error. In Figure 1, we plot the errors of the scheme (2-3) with P_2/P_1 finite elements for various mesh size h. Second-order convergence rates are observed for the H^1-errors of the velocity and for L^2-errors of the pressure, while third-order convergence rates are observed for the L^2-errors of the velocity. There results are in full agreement with Theorem 3.4 for $l = 2$.

Although we presented our analysis using the finite element framework, we can also carry out the same procedure for a spectral or spectral element method as long as the *strong* (with c independent of h in (13)) inf-sup condition is satisfied. We recall that there are at least two pairs of spectral approximation spaces that satisfy the *strong* inf-sup condition (13) (cf. [2]). However, the most popular pair $P_N \times P_{N-2}$ (N plays the role of $1/h$) only satisfies a "weaker" inf-sup condition with

$$\inf_{q_h\in M_h}\sup_{v_h\in X_h}\frac{(\nabla\cdot v_h, q_h)}{\|\nabla v_h\|_0} \ge c_h\|q_h\|_0. \tag{81}$$

with $c_h := \beta_N = N^{-\frac{1-d}{2}} \to 0$ as $N \to \infty$ ($d = 2$ or 3 is the dimension; see, for instance, [2]). The stability analysis in Section 3.1 will still carry through with this "weaker" inf-sup condition, however, an error analysis by using the same procedure as in Section 3.2 will lead to error estimates of the form

$$\|e_h^k\|_{\ell^2(H^1(\Omega)^d)} + c_h\|q_h^k\|_{\ell^2(L^2(\Omega)^d)} \lesssim c_h^{-1}(\delta t + h^l). \tag{82}$$

We recall that numerical results in [11] seem to indicate that the term c_h^{-1} should not be present in the above estimate. Thus, how to remove the term c_h^{-1} in (82) is still an open problem.

Acknowledgment. The authors would like to thank Professor Jean-Luc Guermond for stimulating discussions and for providing the finite element numerical results.

REFERENCES

[1] I. Babuška, *The finite element method with Lagrangian multipliers*, Numer. Math., **20** (1972/73), 179–192.

[2] C. Bernardi and Y. Maday, *Uniform inf-sup conditions for the spectral discretization of the Stokes problem*, Math. Models Methods Appl. Sci., **9** (1999), 395–414.

[3] F. Brezzi, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, **8** (1974), 129–151.

[4] A. J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp., **22** (1968), 745-762.

[5] W. E and J-G. Liu, *Projection method I: Convergence and numerical boundary layers* SIAM J. Numer. Anal., **32** (1995), 1017-1057.

[6] W. E and J-G. Liu, *Gauge method for viscous incompressible flows*, Commun. Math. Sci., **1** (2003), 317-332.

[7] V. Girault and P. A. Raviart, "Finite Element Methods for Navier-Stokes Equations," Springer-Verlag, 1986.

[8] K. Goda, *A multistep technique with implicit difference schemes for calculating two- or three-dimensional cavity flows*, J. Comput. Phys., **30** (1979), 76–95.

[9] J. L. Guermond, P. Minev and J. Shen, *An Overview of Projection methods for incompressible flows*, Comput. Methods Appl. Mech. Engrg., **195** (2006), 6011-6045.

[10] J. L. Guermond and L. Quartapelle, *On stability and convergence of projection methods based on pressure Poisson equation*, Internat. J. Numer. Methods Fluids, **26** (1998), 1039–1053.

[11] J. L. Guermond and J. Shen, *A new class of truly consistent splitting schemes for incompressible flows*, J. Comput. Phys., **192** (2003), 262–276.

[12] J. L. Guermond and J. Shen, *Velocity-correction projection methods for incompressible flows*, SIAM J. Numer. Anal., **41** (2003), 112–134.

[13] J. L. Guermond and J. Shen, *On the error estimates of rotational pressure-correction projection methods*, Math. Comp., **73** (2004), 1719-1737.

[14] J. L. Guermond, *Some implementations of projection methods for Navier-Stokes equations*, RAIRO Modél. Math. Anal. Numér., **30** (1996), 637–667.

[15] J. L. Guermond, *Un résultat de convergence à l'ordre deux en temps pour l'approximation des équations de Navier–Stokes par une technique de projection*, Modél. Math. Anal. Numér. (M2AN), **33** (1999), 169–189.

[16] J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization*, SIAM J. Numer. Anal., **19** (1982), 275-311.

[17] J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem. III. Smoothing property and higher order estimates for spatial discretization*, SIAM J. Numer. Anal., **25** (1988), 489-512.

[18] H. Johnson and J-G. Liu, *Accurate, stable and efficient Navier-Stokes solvers based on explicit treatment of the pressure term*, J. Comput. Phys., **199** (2004), 221–259.

[19] G. E. Karniadakis, M. Israeli and S. A. Orszag, *High-order splitting methods for the incompressible Navier-Stokes equations*, J. Comput. Phys., **97** (1991), 414-443.

[20] J. van Kan, *A second-order accurate pressure-correction scheme for viscous incompressible flow*, SIAM J. Sci. Stat. Comput., **7** (1986), 870-891.

[21] G. E. Karniadakis and S. .J. Sherwin, "Spectral/hp Element Methods for CFD," Oxford UniversityPress, 1999.

[22] J. Kim and P. Moin *Application of a fractional-step method to incompressible Navier-Stokes equations*, J. Comput. Phys., **59** (1985), 380-323.

[23] R. H. Nochetto and J-H. Pyo, *Error estimates for semi-discrete gauge methods for the Navier-Stokes equations*, Math. Comp., **74** (2005), 521–542.

[24] S. A. Orszag, M. Israeli and M. Deville, *Boundary conditions for incompressible flows*, J. Sci. Comput., **1** (1986), 75-111.

[25] A. Prohl, “Projection and Quasi-compressibility Methods for Solving the Incompressible Navier-Stokes Equations,” Advances in Numerical Mathematics, B. G. Teubner 1997.

[26] J. Shen, *Projection schemes for the Navier-Stokes equations*, Appl. Math. Let., **5** (1992), 35–37.

[27] L. J. P. Timmermans, P. D. Minev and F. N. Van De Vosse, *An approximate projection scheme for incompressible flow using spectral elements*, Int. J. Numer. Methods Fluids, **22** (1996), 673–688.

[28] R. Temam, *Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires II*, Arch. Rat. Mech. Anal., **33** (1969), 377-385.

Received July 2006; revised March 2007.

E-mail address: `shen@math.purdue.edu`
E-mail address: `xfyang@math.purdue.edu`