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A finite element multigrid preconditioner for Chebyshev–collocation methods

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Abstract

This paper concerns the iterative solution of the linear system arising from the Chebyshev–collocation approximation of second-order elliptic equations and presents an optimal multigrid preconditioner based on alternating line Gauss–Seidel smoothers for the corresponding stiffness matrix of bilinear finite elements on the Chebyshev–Gauss–Lobatto grid.

Keywords: Multigrid; Preconditioning; Finite element; Chebyshev–collocation

1. Introduction

The aim of this paper is to develop an optimal multigrid preconditioner for the spectral-collocation method (cf. [1,7]) for second-order elliptic equations. The derivative matrices of the spectral-collocation method, being usually full and ill-conditioned, are often preconditioned by using the corresponding matrices of the finite difference/finite element methods on the grid formed by the spectral-collocation points (cf., for instance, [2,3,12]). It has been proved in many cases (cf. [5,9–11]) that the aforementioned preconditioners are optimal in the sense that the condition numbers of the preconditioned systems are independent of the discretization parameters. But how to efficiently apply the preconditioner still remains a challenging problem since the grid formed by the spectral-collocation points, containing long-thin elements, is not shape-regular, and hence it is not clear how some standard methods such as multigrid method can be efficiently employed.

An efficient multigrid preconditioner will be presented in this paper for the aforementioned finite element matrix. By using this preconditioner together with some suitable conjugate-gradient like method

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and a fast Fourier transform technique, the overall computational complexity of the spectral-collocation method for second-order elliptic equations will be nearly optimal (up to a logarithmic term).

2. Finite element multigrid preconditioner

Consider the following model elliptic problem:

$$\mathcal{L}u := -\nabla \cdot \big(b(x, y)\nabla u\big) = f, \quad (x, y) \in \Omega = (-1, 1)^2; \qquad u|_{\partial\Omega} = 0. \tag{1}$$

We assume that $f(x, y)$ and $b(x, y)$ are sufficiently smooth, and $0 < \alpha \leqslant b(x, y) \leqslant \beta$ in Ω for some positive constants α and β.

We introduce the Chebyshev–Gauss–Lobatto (CGL) points in $[-1, 1]$:

$$\xi_i = -\cos\frac{i\pi}{N}, \quad i = 0, \ldots, N; \qquad \eta_j = -\cos\frac{j\pi}{M}, \quad j = 0, \ldots, M.$$

Let $P^0_{N,M}$ be the space of polynomials which are of degree less than or equal to N and M, respectively, in the x and y directions, and which satisfy the homogeneous Dirichlet boundary condition on $\overline{\Omega}$. Then, we can write the Chebyshev–collocation approximation for (1) as to find $u_{NM} \in P^0_{N,M}$ such that

$$-\big(\nabla \cdot I_{NM}(b\nabla u_{NM}), v\big)_{N,M} = (f, v)_{N,M} \quad \forall v \in P^0_{N,M}, \tag{2}$$

where I_{NM} is the interpolation operator based on the CGL points, and $(\cdot,\cdot)_{N,M}$ is the discrete inner product associated with the Chebyshev–Gauss–Lobatto quadrature.

Let $\boldsymbol{u}_{NM}$ and $\boldsymbol{f}_{NM}$ be respectively the vectors composed by the values of $u_{NM}(x, y)$ and $f(x, y)$ at the interior CGL points $\{\xi_i, \eta_j\}_{1\leqslant i\leqslant N-1,\ 1\leqslant j\leqslant M-1}$, we can rewrite (2) as a linear system

$$L_{\mathrm{sp}}\boldsymbol{u}_{NM} = W_{\mathrm{sp}}\boldsymbol{f}_{NM}, \tag{3}$$

where L_{sp} is the Chebyshev spectral differentiation matrix associated to the operator $\mathcal{L}$, and W_{sp} is the diagonal matrix formed by the weights of the CGL quadrature. Since the matrix L_{sp} is usually full and ill-conditioned, it is prohibitive to use a direct inversion method or an iterative method without preconditioning. Hence, it is imperative to use an iterative method with a good preconditioner.

In view of the fast spectral–Galerkin Poisson/Helmholtz solver developed recently (cf. Shen [13–15]), a natural and optimal preconditioner is the operator L_{cg} associated to the Chebyshev–Galerkin discretization of the operator "$-\Delta$", which is spectrally equivalent to the original elliptic operator "$-\nabla \cdot (b\nabla)$". This strategy has proven to be very effective if the coefficients $b(x, y)$ varies moderately, i.e., $\max b(x, y)/\min b(x, y)$ is not too large (cf. [15]). However, the iteration process slows down considerably as the ratio $\max b(x, y)/\min b(x, y)$ increases. Hence, a more robust preconditioner is needed for the latter case.

2.1. A finite element preconditioner

Although finite element/finite difference preconditioners have been widely used since the original paper by Orszag [12] and one-dimensional analyses (cf. [5,9]) were available for quite some time, only recently the optimality of the finite element/finite difference preconditioner for the Chebyshev–collocation method in the two-dimensional case was rigorously established (cf. [10,11]) in some cases.

In particular, Kim and Parter [10] proved that the finite element preconditioner for (3), based on the weighted inner product

$$a_\omega(u,v)=\int_\Omega b(x,y)\,\nabla u\cdot\nabla\big(v\,\omega(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y,\quad \omega(x,y)=\big(1-x^2\big)^{-1/2}\big(1-y^2\big)^{-1/2},$$

is optimal. However, from the implementation point of view, it is preferable in practice to use the standard finite element formulation based on the inner product

$$a(u,v)=\int_\Omega b(x,y)\,\nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}y,$$

with which the bilinear finite element approximation to (1) on the CGL grid leads to the linear system

$$A_{\mathrm{fe}}\boldsymbol{u}_{NM}=M_{\mathrm{fe}}\boldsymbol{f}_{NM}, \tag{4}$$

where A_{fe} and M_{fe} are respectively the stiffness and mass matrices associated with standard bilinear finite element approximation to (1) on the CGL grid.

It appears unnatural to precondition the Chebyshev–collocation system (3) without using the (Chebyshev) weighted inner product, indeed, A_{fe} is not a good preconditioner for L_{sp}. However, since both (3) and (4) are legitimate approximation of (1), it is conjectured, and has been confirmed by ample numerical results (cf. [2–4]), that $M_{\mathrm{fe}}^{-1}A_{\mathrm{fe}}$ is an optimal preconditioner for $W_{\mathrm{sp}}^{-1}L_{\mathrm{sp}}$, i.e., (4) provides an optimal preconditioner for (3). Hence, in order to solve (4) efficiently, we seek below an optimal multigrid preconditioner for (4).

2.2. *An optimal convergence result of the one-dimensional multigrid method*

To the best of our knowledge, there are no rigorous error analysis available for multigrid methods in a general non-shape-regular grid. We present below a optimal convergence result on the multigrid preconditioning for the piecewise linear finite element approximation of the one-dimensional model problem

$$-u''=f\quad\text{in }(-1,1);\qquad u(-1)=u(1)=0, \tag{5}$$

on the Chebyshev–Gauss–Lobatto grid which is not shape-regular.

Let $M_k\subset H_0^1(\Omega)$ be the space of piecewise linear finite elements based on the Chebyshev–Gauss–Lobatto grid $\{1-\cos j\pi/2^k\colon\ j=0,1,\dots,2^k\}$. Then, we have a nested sequence

$$M_1\subset M_2\subset\cdots\subset M_J=M,$$

and we can define the hierarchical subspaces $V_1=M_1$, $V_j=(I_j-I_{j-1})M$ for $j=2,\dots,J$ such that $M=V_1\oplus V_2\oplus\cdots\oplus V_J$.

We define $A:M\to M$ by

$$(Au,v)=(u,v)_A:=a(u,v),\quad \forall u,v\in M,$$

and let $A_i : M_i \to M_i$ be the restriction of A on M_i; $Q_i, P_i : M \to M_i$ be the orthogonal projections operators from M to M_i with respect to $(\cdot,\cdot)$ and $(\cdot,\cdot)_A$, respectively. More precisely,

$$(A_i u_i, v_i) = (A u_i, v_i) \quad \forall u_i, v_i \in M_i,$$

$$(Q_i u, v_i) = (u, v_i) \qquad \forall u \in M, v_i \in M_i,$$

$$(P_i u, v_i)_A = (u, v_I)_A \quad \forall u \in M, v_i \in M_i.$$

Now, let $R_i : M_i \to M_i$ be a smoother, and $T_i = R_i Q_i A = R_i A_i P_i$. Then, the error operator for the V-cycle multigrid method is

$$E_J^s = (I - T_1)(I - T_2)\cdots(I - T_J)(I - T_J)\cdots(I - T_1) = E_J^* E_J.$$

Then, we have the following result:

Theorem 1. *If the Chebyshev–Gauss–Lobatto grid is used for the piecewise linear finite element approximation to* (5) *and if the symmetric Gauss–Seidel smoother is used, we have the following estimate for the V-cycle multigrid method*:

$$\|E_J\|_A^2 \leqslant \tfrac{1}{5}.$$

The proof is quite technical and uses special properties of the Chebyshev–Gauss–Lobatto points. We refer to [16] for details.

Remark 1. The above result can be generalized, with a slightly larger error constant, to the grid based on the Gauss–Lobatto points of the Jacobi polynomials, including, in particular, the Legendre–Gauss–Lobatto points. However, we are currently unable to derive a meaningful error estimate for the two-dimensional case, although the numerical results presented below strongly suggest that in the two-dimensional case the multigrid preconditioner for the bilinear finite element approximation of (1) using alternating line Gauss–Seidel smoother is optimal.

2.3. Discussions on the two-dimensional multigrid method

On a shape-regular grid, the finite element multigrid method is known to be optimal for the elliptic problem (1) (see, e.g., Xu [18]). The linear system (4) is derived from a bilinear finite element approximation on the CGL grid. This grid is obviously not shape-regular, with elements having very high aspect ratios near the boundary. In the one-dimensional case, the multigrid method on the CGL grid is also optimal (see Wang [16] for a rigorous analysis). However, in the 2D case, standard multigrid techniques applied to (4) converge very slowly. We will present an optimal multigrid preconditioner for (4) by taking into account the special structure of the CGL grid.

The problem associated with the multigrid method on this grid is reminiscent to the anisotropic problem on shape-regular grid (cf. Wesseling [17], Hackbusch [8]). When the aspect ratios become high, the multigrid method with pointwise Gauss–Seidel smoothers converge slowly. The reason for this phenomenon is that, on an element with high aspect ratio, the points along one direction are much more strongly coupled than the points along the other direction. By “strongly coupled points” we mean the points associated to large (in absolute value) coefficients in the stiffness matrix. Naturally, points that are closer to each other are more strongly coupled. A few treatments have been successfully used to solve this problem, including line smoothing and semi-coarsening, in [8,17].

A general rule of resolving the high aspect ratio is: *points that are strongly coupled should be updated simultaneously*. Since the strongest couplings occur along the boundaries in both the x and y directions, this leads us to choose the alternating line Gauss–Seidel smoother, namely, an x-line Gauss–Seidel followed by a y-line Gauss–Seidel for pre-smoothing and a y-line Gauss–Seidel followed by an x-line Gauss–Seidel for post-smoothing.

3. Numerical results

Since the multigrid algorithm is designed as a preconditioner for the finite element approximation (4) which in turn is an optimal preconditioner for the Chebyshev–collocation approximation of (3), we use the multigrid algorithm to precondition directly the Chebyshev–collocation system (3). In all computations, we use the standard V-cycle multigrid algorithm with the stopping criteria to be that the relative residual is less than 10^{-6}.

Although (1) is a symmetric positive definite problem, the corresponding Chebyshev–collocation system (3) is non-symmetric due to the non-uniform Chebyshev weight. However, (3) is still positive definite. In fact, the eigenvalues of L_{sp} are all real positive (cf. [6]). Hence, we can apply, for example, the preconditioned conjugate gradient squared (PCGS) iterative method.

First of all, we compare the effectiveness of the alternating line Gauss–Seidel (ALGS) smoother with the pointwise Gauss–Seidel (PGS) smoother and x-line Gauss–Seidel (LGS) smoother. Since ALGS in fact performs two line-GS smoothings for each smoothing step (one x-line and one y-line), we also let the PGS and LGS perform two iterations in each pre- and post-smoothing step to ensure a fair comparison. Table 1 displays the multigrid iteration counts with the above three smoothers for solving (4) with $b(x, y) \equiv 1$. The iteration counts in Table 1 clearly indicate that the alternating line Gauss–Seidel smoother produces a robust multigrid method on the CGL grid. The (one-direction) line Gauss–Seidel smoother is better than the pointwise Gauss–Seidel smoother, but neither of them is robust.

Table 1
Iteration counts for the MG method with different smoothers

Grid size	PGS	LGS	ALGS
7×7	4	3	3
15×15	5	4	4
31×31	6	5	4
63×63	10	8	4
127×127	15	11	4
255×255	25	16	5

Table 2
Iteration counts for PCGS: using MG preconditioner with variable $b(x, y)$

$b(x, y)$	1	$10^4(1-x^2)(1-y^2)+1$	$100x^2+y^2+1$	$(1+x^2+y^2)^4$
7×7	3	22	11	9
15×15	4	25	16	8
31×31	3	32	13	7
63×63	3	5	12	6
127×127	2	6	6	6
255×255	2	5	6	4

Next, we use the multigrid preconditioner with alternating line Gauss–Seidel smoother to solve (3) with four different functions $b(x, y)$. In Table 2, we list the PCGS iteration counts with the functions $b(x, y)$ built into the multigrid preconditioner. The results are summarized below:

- For $b(x, y) = 1$ and $(1 + x^2 + y^2)^4$, the PCGS method converges uniformly as expected.
- For $b(x, y) = 10^4(1 - x^2)(1 - y^2) + 1$, where $\max b = 10001$ and $\min b = 1$, the PCGS method converges uniformly as the refinement level gets higher. The faster convergence on finer meshes can be explained by the fact that the multigrid solution gets closer to the spectral solution as the mesh gets finer. This example shows that the finite element multigrid preconditioner is very effective for problems with large variation in $b(x, y)$.
- For $b(x, y) = 100x^2 + y^2 + 1$, which is an anisotropic function and being used here to test the robustness of the method, the result is similar to the high variation case above.

Since the convergence rate of the PCGS iteration is independent of the discretization parameters N and M, and since the matrix–vector product $L_{sp}\boldsymbol{u}_{NM}$ can be evaluated in $O(NM\log(NM))$ operations thanks to the Fast Fourier Transform, the total operation counts for solving the Chebyshev–collocation system (3) is $O(NM\log NM)$ which is quasi-optimal.

4. Concluding remarks

We have presented an optimal finite element multigrid preconditioner for solving the linear system arising from the Chebyshev–collocation approximation of second-order elliptic equations. The complete algorithm has a quasi-optimal computational complexity while providing spectral accuracy. Furthermore, numerical results indicate that it is also very robust.

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