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ERROR ANALYSIS OF FULLY DISCRETE VELOCITY-CORRECTION METHODS FOR INCOMPRESSIBLE FLOWS

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Abstract. A fully discrete version of the velocity-correction method, proposed by Guermond and Shen (2003) for the time-dependent Navier-Stokes equations, is introduced and analyzed. It is shown that, when accounting for space discretization, additional consistency terms, which vanish when space is not discretized, have to be added to establish stability and optimal convergence. Error estimates are derived for both the standard version and the rotational version of the method. These error estimates are consistent with those by Guermond and Shen (2003) as far as time discretiztion is concerned and are optimal in space for finite elements satisfying the inf-sup condition.

1. Introduction

Projection methods, whose original version was introduced by Chorin [3] and Temam [27] in the late 1960s, are widely used to approximate the incompressible time-dependent Navier-Stokes equations. They are designed to overcome the difficulty caused by the incompressibility constraint which couples the velocity and the pressure. We refer to a recent review on this topic [13] where projection schemes are classified into three families: pressure-correction (cf. e.g. [4, 7, 14, 20, 23, 26, 28, 29]), velocity-correction (cf. [12, 18, 19, 22]), and consistent splitting scheme [11, 17, 25] (which is equivalent, in the space continuous case only, to the so-called gauge method [5, 21]).

Velocity-correction schemes (in semi-discretized form) were first introduced in a disguised form in [22, 18], and rigorously analyzed by Guermond and Shen in [12]. The main difference between the velocity-correction methods and the pressure-correction or the consistent-splitting methods is that, in velocity-correction methods, the viscous term is made explicit in the first sub-step and corrected in the second sub-step, whereas in the other methods it is the pressure gradient which is made explicit first and corrected afterward. In addition to convergence proofs on various semi-discretized forms of the velocity-correction scheme, numerical tests on a second-order fully discretized version of the method are also reported in [12]. These tests, using spectral and finite element methods, show that the method is

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stable and yield quasi-optimal results in time and space for the velocity and the pressure. However, to the best of our knowledge, there is no further work in the literature that provides a rigorous stability and error analysis for the fully discretized method, using either finite element or spectral approximation in space.

A rather general strategy for analyzing various two-step projection methods has been devised in [8]. The main ingredient of this theory is to consider two different approximation spaces for the velocity, one for each sub-step. Using the notations from [8], the velocity approximation in the viscous sub-step is chosen in a finite-dimensional space X_h, and that in the projection sub-step is chosen in another finite-dimensional space Y_h which contains X_h. For the special choice of $Y_h = X_h$, there is no essential difference in the analysis between the fully discrete case and the semi-discrete case; that is to say, all the arguments from [12] carry over to the fully discrete situation naturally. However, the situation $X_h = Y_h$ implies that the pressure is computed by solving a Darcy problem in mixed form. In order to compute the pressure by solving a Poisson problem, thus avoiding a possibly awkward Darcy problem, we have to look at situations where $X_h \neq Y_h$. However, if one naively uses the semi-discrete forms of the algorithm using $X_h \neq Y_h$, one observes a subtle inconsistency, especially for the rotational form of the scheme, which makes it very difficult, if not impossible, to prove the stability and optimal convergence of the fully discretized scheme. The primary goal of the present paper is to construct a fully discrete velocity-correction scheme which removes the inconsistency mentioned above. This is done by adding terms that vanish when the space is continuous and when $Y_h = X_h$. A particular instance of the fully discretized method that we propose consists of solving a discrete (standard) Poisson equation for the pressure.

The paper is organized as follows. In §2 we introduce notation and the discrete setting for the space approximation. In §3 we discuss how the velocity-correction algorithm in standard form should be discretized in space and show in particular that naively discretizing the semi-discrete algorithm yields inconsistencies as mentioned above. In §4 we prove stability and convergence for the first-order rotational velocity-correction scheme. In §5 we study the second-order version of the rotational velocity-correction scheme. The two major results of this paper are Theorems 4.1 and 5.1. Concluding remarks are reported in §6.

2. Preliminaries

2.1. **The continuous problem.** Since it is well known that non-linear terms in the Navier-Stokes equations do not affect the formal accuracy of fractional-step projection methods provided they are consistently treated, we henceforth restrict ourselves to the time-dependent Stokes problem:

$$\begin{cases} \partial_t u - \nabla^2 u + \nabla p = f \quad \text{in } \Omega \times [0,T], \\ \operatorname{div} u = 0 \quad \text{in } \Omega \times [0,T], \end{cases} \tag{2.1}$$

supplemented with initial and, for simplicity, homogeneous Dirichlet boundary conditions

$$u|_{t=0} = v_0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.2}$$

In the above problem, $f \in L^2((0,T)\times\Omega)$ is a body force, and Ω is an open bounded domain in R^d ($d = 2$ or 3) with a boundary sufficiently smooth so that the usual

H^2 regularity holds for the steady Stokes problem with homogeneous Dirichlet boundary conditions and a source term in $L^2(\Omega)$. The symbol ∂_t denotes the partial derivative with respect to time. We also use d_t in the rest of the paper to denote derivatives with respect to time.

We denote by $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ the usual Sobolev spaces equipped with the norm $\|\cdot\|_{s,p}$ for $0 \le s \le \infty$, $1 \le p \le \infty$. In particular, we denote the Hilbert spaces $W^{s,2}(\Omega)$ by $H^s(\Omega)$ $(s = 0, \pm 1, \dots)$ with norm $\|\cdot\|_s$ and semi norm $|\cdot|_s$. The norm and inner product of $L^2(\Omega) = H^0(\Omega)$ are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$ respectively.

We shall also make use of the following Hilbert spaces:

$$L^2_{\int=0}(\Omega) = \{q \in L^2(\Omega),\ \textstyle\int_\Omega q = 0\}, \tag{2.3}$$

$$H^1_{\int=0}(\Omega) = \{q \in H^1(\Omega),\ \textstyle\int_\Omega q = 0\}, \tag{2.4}$$

$$H = \{v \in L^2(\Omega)^d,\ \nabla\cdot v = 0,\ v\cdot n|_\Gamma = 0\}. \tag{2.5}$$

In particular, the following Helmholtz decomposition of $L^2(\Omega)^d$ plays an important role for the analysis of projection methods:

$$L^2(\Omega)^d = H \oplus \nabla H^1_{\int=0}(\Omega). \tag{2.6}$$

2.2. **The discrete setting.** Let $\delta t > 0$ be a real number that we henceforth refer to as the time step. We set $t^k = k\delta t$ for $0 \le k \le K = [T/\delta t]$. For every function which is continuous in time, $\phi(t)$, we denote $\phi^k := \phi(t^k)$ and define the difference operator δ, acting on sequences, by $\delta\phi^k := \phi^k - \phi^{k-1}$. Let W be a Banach space; we set $L^p(W) = L^p(0,T;W)$. To account for time sequences we also set $\ell^p(W) := \{w = (w^0, w^1, \dots, w^K),\ w^k \in W,\ 0 \le k \le K, \|\phi\|_{\ell^p(W)} < +\infty\}$ with

$$\|\phi\|_{\ell^p(W)} := (\delta t \sum_{k=0}^{K} \|\phi^k\|_W^p)^{\frac{1}{p}}, \qquad \|\phi\|_{\ell^\infty(W)} := \max_{0\le k\le K}(\|\phi^k\|_W). \tag{2.7}$$

Let $\{X_h\}_{h>0}$, $\{M_h\}_{h>0}$ be two families of conforming approximations of $H_0^1(\Omega)^d$ and $L_0^2(\Omega)$, respectively. The pair (X_h, M_h) is assumed to be compatible in the sense that the following LBB conditions hold uniformly with respect to h:

$$\exists c > 0, \quad \inf_{q_h\in M_h} \sup_{v_h\in X_h} \frac{(\nabla\cdot v_h, q_h)}{\|\nabla v_h\|_0} \ge c\,\|q_h\|_0. \tag{2.8}$$

We henceforth denote by c a generic constant that is independent of the mesh-size h and the time step δt but possibly depends on the data and the solution. Whenever no confusion is possible we use the expression $A \lesssim B$ to say that there exists a generic constant c such that $A \le c\,B$.

The two (families of) spaces X_h and M_h are also assumed to satisfy the following approximation properties: There exists an integer $l > 0$ such that for all $r \in [1, l]$,

$$\inf_{v_h\in X_h}\{\|v - v_h\|_0 + h\|v - v_h\|_1\} \lesssim h^{r+1}\|v\|_{r+1}, \quad \forall v \in H^{r+1}(\Omega)^d \cap H_0^1(\Omega)^d. \tag{2.9}$$

$$\inf_{q_h\in Q_h}\{\|q - q_h\|_0 + h\|q - q_h\|_1\} \lesssim h^r\|q\|_r, \quad \forall q \in H^r(\Omega) \cap L_0^2(\Omega). \tag{2.10}$$

In order to formulate the semi-discrete Stokes problem in a way which is similar to its continuous differential counterpart, we introduce several discrete differential operators as in [8]. We define the discrete Laplace operator, $A_h : X_h \to X_h'$, by

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h), \quad \forall (u_h, v_h) \in X_h \times X_h, \tag{2.11}$$

the discrete divergence operator, $B_h : X_h \to M_h$, and the discrete gradient operator, $B_h^T : M_h \to X_h'$, by

$$(B_h v_h, p_h) = -(\nabla \cdot v_h, p_h) = (v_h, B_h^T p_h), \quad \forall (v_h, p_h) \in X_h \times M_h. \tag{2.12}$$

We also define an extension of the L^2-projection onto X_h, $\pi_h : H^{-1}(\Omega)^d \to X_h'$ such that

$$(\pi_h f, v_h) = (f, v_h), \; \forall v_h \in X_h. \tag{2.13}$$

Using the discrete framework defined above, the time-dependent Stokes problem (2.1) can be semi-discretized as follows: Setting $f_h = \pi_h f$ and $v_{0,h} = \pi_h u_0$, we look for $u_h(t) \in \mathcal{C}^0([0,T]; X_h)$ and $p_h(t) \in L^2((0,T); M_h)$ such that

$$\begin{cases} \dfrac{du_h}{dt} + A_h u_h + B_h^T p_h = f_h, \quad 0 < t \leq T, \\ B_h u_h = 0, \\ u_h|_{t=0} = v_{0,h}. \end{cases} \tag{2.14}$$

It is well known that the above problem admits a unique solution which is stable with respect to the data. Furthermore, since X_h and M_h are convergent and stable approximations of $H_0^1(\Omega)^d$ and $H^1_{\int=0}(\Omega)$, the solution to (2.14) converges in an appropriate sense to that of the continuous problem (2.1). For more details on the above formulation using finite elements we refer to [6, 15, 16].

2.3. **The (X_h, Y_h) pair.** Following Guermond [8], we introduce an additional discrete setting so as to relax the incompressibility constraint and to build a discrete version of the Helmholtz decomposition (2.6). More precisely, we want to decompose each discrete vector field $\tilde{u}_h \in X_h$ into the sum of a discrete-divergence-free vector field u_h plus the discrete-gradient of a scalar field ϕ_h in M_h. There are numerous ways of achieving this decomposition. For instance, we could set $\tilde{u}_h = u_h + B_h^T \phi_h$, with $u_h \in X_h$ and $B_h u_h = 0$. Another possibility could be to set $\tilde{u}_h = u_h + \nabla \phi_h$ where u_h is enforced to be orthogonal to ∇M_h, provided M_h is constructed so that $M_h \subset H^1_{\int=0}(\Omega)$. In this case it is natural to choose u_h to be in $X_h + \nabla M_h$. Even though this alternative may seem odd, it turns out to be optimal and very easy to implement, since it implies solving a discrete Poisson problem using the usual $(\nabla \phi_h, \nabla \psi_h)$ bilinear form.

In order to present a unified analysis for the many possible realizations of the discrete Helmholtz decomposition, we introduce a finite dimensional subspace $Y_h \in L^2(\Omega)^d$. For the sake of simplicity we assume that $X_h \subset Y_h$ and we denote by i_h the continuous injection of X_h into Y_h; the transpose of i_h is the L^2-projection of Y_h onto X_h. Furthermore, we assume that we have at hand an operator $C_h : Y_h \longrightarrow M_h$ which is an extension of B_h, i.e.,

$$C_h i_h = B_h, \qquad i_h^T C_h^T = B_h^T. \tag{2.15}$$

Owing to (2.8), B_h is surjective. C_h being an extension of B_h, this immediately implies that C_h is also surjective and C_h^T is injective. As a result $\|C_h^T q\|_0$ is a norm and, upon setting $H_h = \mathrm{KER} C_h$, the following orthogonal decomposition of Y_h holds:

$$Y_h = H_h \oplus C_h^T(M_h). \tag{2.16}$$

This decomposition is a discrete counterpart of (2.6). Finally, we also assume that A_h and C_h satisfy the following hypotheses:

(2.17)
$$\forall v_h \in X_h,\ \forall v \in [H_0^1(\Omega)\cap H^2(\Omega)]^d, \quad (\|v_h - v\|_1 \lesssim h\|v\|_2) \Rightarrow \|A_h v_h\|_0 \lesssim \|v\|_2,$$

$$\forall q_h \in M_h,\ \forall q \in H^1_{\int=0}(\Omega), \quad (\|q_h - q\|_0 \lesssim h\|q\|_1) \Rightarrow \|C_h^T q_h\|_0 \lesssim \|q\|_1. \tag{2.18}$$

These hypotheses are usually satisfied when X_h, Y_h, and M_h are constructed using finite elements with shape-regular meshes.

Various realizations of Y_h and C_h are described in [8, 10]. An obvious one is $Y_h = X_h$ and $C_h = B_h$. Assuming $M_h \subset H^1_{\int=0}(\Omega)$, another interesting choice consists of setting $Y_h = X_h + \nabla M_h$ and defining C_h such that $(C_h v_h, q_h) = (v_h, \nabla q_h) = (v_h, C_h^T q_h)$, for all $v_h \in Y_h$, $q_h \in M_h$. This particular setting implies that C_h^T is the restriction of ∇ to M_h, i.e., $C_h^T q_h = \nabla q_h$, $\forall q_h \in M_h$. In particular, the bilinear form $(C_h^T q_h, C_h^T r_h)$ reduces to the usual weak form $(\nabla q_h, \nabla r_h)$ associated with the Poisson problem supplemented with Neumann boundary conditions, which is really easy to implement.

3. Fully discretized velocity-correction in standard form

3.1. A naive discretization. Consider for the time being the first-order backward Euler method. The standard velocity-correction scheme proposed in [12] in semi-discrete form is as follows: Set $u^0 = u(t^0)$, then for $k \geq 0$, compute $u^{k+1} \in H$ and $p^{k+1} \in L^2_0(\Omega)$ such that

$$\begin{cases} \dfrac{u^{k+1} - \tilde{u}^k}{\delta t} - \nabla^2 \tilde{u}^k + \nabla p^{k+1} = f(t^{k+1}), \\ \nabla \cdot u^{k+1} = 0, \quad u^{k+1} \cdot n|_\Gamma = 0; \end{cases} \tag{3.1}$$

and then find $\tilde{u}^{k+1} \in H_0^1(\Omega)^d$ such that

$$\frac{\tilde{u}^{k+1} - u^{k+1}}{\delta t} - \nabla^2(\tilde{u}^{k+1} - \tilde{u}^k) = 0, \quad \tilde{u}^{k+1}|_\Gamma = 0. \tag{3.2}$$

A seemingly natural way to discretize the above algorithm in space is as follows: Setting $\tilde{u}_h^0 = \pi_h u_0$ and $f_h^{k+1} = \pi_h f(t^{k+1})$, for $k \geq 1$, compute $(u_h^{k+1}, p_h^{k+1}) \in Y_h \times M_h$ such that

$$\begin{cases} \dfrac{u_h^{k+1} - i_h \tilde{u}_h^k}{\delta t} + i_h A_h \tilde{u}_h^k + C_h^T p_h^{k+1} = i_h f_h^{k+1}, \\ C_h u_h^{k+1} = 0; \end{cases} \tag{3.3}$$

and then compute $\tilde{u}_h^{k+1} \in X_h$ such that

$$\frac{\tilde{u}_h^{k+1} - i_h^T u_h^{k+1}}{\delta t} + A_h \tilde{u}_h^{k+1} - A_h \tilde{u}_h^k = 0. \tag{3.4}$$

Let us now assume that this algorithm converges to a steady state as $k \to \infty$. Then (3.4) yields $\tilde{u}_h = i_h^T u_h$, which in turn implies $B_h \tilde{u}_h = B_h i_h^T u_h$. Therefore, we usually have $B_h \tilde{u}_h \neq 0$ unless $B_h i_h^T u_h = C_h u_h$, which is true only if i_h^T is the identity operator and $B_h = C_h$. Observe that the equality $B_h = C_h$ holds only if $X_h = Y_h$. We then conclude that (3.3)-(3.4) is consistent only if $X_h = Y_h$, which greatly reduces implementation options. As a result, one must find a consistent way to discretize (3.1)-(3.2) in order to use more convenient implementation options for which $X_h \neq Y_h$. This is one of the main goals of the present paper.

3.2. **Consistent discretization.** The above observation led us to consider the following alternative discretization of (3.1)-(3.2):

$$\begin{cases} \dfrac{u_h^{k+1} - i_h\tilde{u}_h^k}{\delta t} + i_h A_h \tilde{u}_h^k + C_h^T p_h^{k+1} + i_h B_h^T p_h^k - C_h^T p_h^k = i_h f_h^{k+1}, \\ C_h u_h^{k+1} = 0, \end{cases} \tag{3.5}$$

and

$$\frac{\tilde{u}_h^{k+1} - i_h^T u_h^{k+1}}{\delta t} + A_h \tilde{u}_h^{k+1} - A_h \tilde{u}_h^k = 0. \tag{3.6}$$

Clearly, the term $i_h B_h^T p_h^k - C_h^T p_h^k$ in (3.5) vanishes when $X_h = Y_h$, implying that the above algorithm is the same as (3.3)-(3.4) when $X_h = Y_h$. Let us observe also that (3.6) can be rewritten in another equivalent form as follows: Applying i_h^T to (3.5) and adding the result to (3.6). Upon noticing that $i_h^T i_h B_h^T p_h^k = i_h^T C_h^T p_h^k$ thanks to (2.15) and the fact that $i_h^T i_h|_{X_h}$ is the identity on X_h, we obtain

$$\frac{\tilde{u}_h^{k+1} - \tilde{u}_h^k}{\delta t} + A_h \tilde{u}_h^{k+1} + B_h^T p_h^{k+1} = f_h^{k+1}, \tag{3.7}$$

which is equivalent to (3.6). To understand why the new algorithm (3.5)-(3.7) (or (3.5)-(3.6), equivalently) is better than (3.3)-(3.4) in general, let us apply i_h to (3.7) at time step t^k and subtract the result from (3.5), giving

$$\frac{u_h^{k+1} - 2i_h\tilde{u}_h^k + i_h\tilde{u}_h^{k-1}}{\delta t} + C_h^T(p_h^{k+1} - p_h^k) = i_h(f_h^{k+1} - f_h^k). \tag{3.8}$$

Assuming that there is a steady state as $k \to \infty$, this equation implies $u_h = i_h\tilde{u}_h$, which in turn yields $0 = C_h u_h = C_h i_h \tilde{u}_h = B_h \tilde{u}_h$, since $C_h i_h = B_h$ by definition of C_h being an extension of B_h. In other words, at steady state we have the desired property $B_h\tilde{u}_h = 0$, which suggests that (3.5)-(3.7) is a consistent way of implementing (3.1)-(3.2). Actually, the above manipulation yields an efficient way to implement (3.5)-(3.7) without computing u_h^{k+1}, which might live in an odd space (think of $Y_h = X_h + \nabla M_h$ for instance). Owing to the constraint $(u^{k+1}, C_h^T r_h) = 0$ for all $r_h \in M_h$, (3.8) can be equivalently rewritten as

$$(C_h^T(p_h^{k+1} - p_h^k), C_h^T r_h) = (f^{k+1} - f^k, B_h^T r_h) + \frac{1}{\delta t}(2\tilde{u}_h^k - \tilde{u}_h^{k-1}, B_h^T r_h). \tag{3.9}$$

Hence the algorithm is simply composed of the two sub-steps (3.9)-(3.7). As a result, choosing Y_h only amounts to selecting a realization of C_h^T with which the user is comfortable. For instance, choosing $Y_h = X_h + \nabla M_h$ implies that (3.9) is a simple discrete Poisson problem using the standard bilinear form $(\nabla ., \nabla .)$.

Remark 3.1. Actually, the algorithm (3.9)-(3.7) is exactly what was proposed in [12] as an equivalent alternative to (3.1)-(3.2) in a semi-discrete setting (see (2.8)-(2.9) in [12]). The Finite Element computations reported in [12] have been done using (3.9)-(3.7). When [12] was written, it was not clear that (2.8)-(2.9) from [12] and (3.1)-(3.2) could yield different fully discrete implementations. One goal of the present paper is to clarify this observation.

Instead of using the Euler scheme, one can use a higher-order method. For instance, using the second-order backward difference formula (BDF2), the fully discrete velocity-correction scheme in standard form takes the following form:

$$\begin{cases} \dfrac{3u_h^{k+1} - 4i_h\tilde{u}_h^k + i_h\tilde{u}_h^{k-1}}{\delta t} + i_hA_h\tilde{u}_h^k + C_h^T p_h^{k+1} + i_hB_h^T p_h^k - C_h^T p_h^k = i_hf_h^{k+1}, \\ C_hu_h^{k+1} = 0, \end{cases} \tag{3.10}$$

and

$$\frac{3\tilde{u}_h^{k+1} - 4\tilde{u}_h^k + \tilde{u}_h^{k-1}}{\delta t} + A_h\tilde{u}_h^{k+1} + B_h^T p_h^{k+1} = f_h^{k+1}. \tag{3.11}$$

We finish this section by stating the following convergence results.

Theorem 3.1. *Let u, p solve* (2.1). *Assume enough regularity is at hand for u and p. Let u_h, p_h solve* (3.5)-(3.7), *then*

$$\|u - \tilde{u}_h\|_{\ell^2(L^2(\Omega)^d)} \quad + \quad \|u - i_h^T u_h\|_{\ell^2(L^2(\Omega)^d)} \lesssim \delta t + h^{l+1}, \tag{3.12}$$

$$\|u - \tilde{u}_h\|_{l^2(H^1(\Omega)^d)} \quad + \quad \|p - p_h\|_{l^2(L^2(\Omega))} \lesssim \delta t + h^l. \tag{3.13}$$

Let u_h, p_h solve (3.10)-(3.11) *and assume the scheme be appropriately initialized. Then*

$$\|u - \tilde{u}_h\|_{\ell^2(L^2(\Omega)^d)} \quad + \quad \|u - i_h^T u_h\|_{\ell^2(L^2(\Omega)^d)} \lesssim \delta t^2 + h^{l+1}, \tag{3.14}$$

$$\|u - \tilde{u}_h\|_{\ell^2(H^1(\Omega)^d)} \quad + \quad \|p - p_h\|_{\ell^2(L^2(\Omega))} \lesssim \delta t + h^l. \tag{3.15}$$

Proof. We omit the details since they are similar to those in the proof of the rotational version of the algorithm which is detailed in the next section. □

Remark 3.2. The estimate (3.15) is one-order suboptimal with respect to δt. The suboptimality is sharp in the sense that it cannot be improved. The origin of this defect is an inconsistent/artificial boundary condition which is enforced by (3.2). This equation implies that at the boundary of the flow domain

$$\nabla^2\tilde{u}^{k+1}\cdot n|_\Gamma = \nabla^2\tilde{u}^k\cdot n|_\Gamma = \ldots = \nabla^2\tilde{u}^0\cdot n|_\Gamma, \tag{3.16}$$

which together with (3.1) in turns gives

$$\left.\frac{\partial p}{\partial n}\right|_\Gamma = (f(t^{k+1}) + \nabla^2\tilde{u}^0)\cdot n|_\Gamma. \tag{3.17}$$

It is obviously an artificial Neumann boundary condition on the pressure. This phenomenon is identical to what is observed for the standard form of the pressure-correction scheme; see e.g. [10, 13, 26]. The accuracy of the scheme is limited to $\mathcal{O}(\delta t)$ by the numerical boundary layer induced by this inconsistent/artificial boundary condition. The $\mathcal{O}(\delta t)$ barrier can be (partially) overcome by considering the rotational form of the method which is discussed in the next section.

4. Fully discretized velocity-correction in rotational form

In this section we focus our attention on the velocity-correction method in rotational form using the first-order Euler scheme. This allows us to concentrate on the main issues by bypassing the technical issues associated with higher-order schemes. This strategy is based on the observation made in [9] that the splitting error (i.e., the difference between the discrete solution and that from the equivalent one-step algorithm where the pressure is implicit and the discrete impressibility constraint is enforced) does not depend on the time stepping. The stability analysis of the BDF2 time stepping is done in the next section for completeness, but all the key ingredients of the method are detailed in the present section using the first-order Euler time stepping.

4.1. **Consistent fully discretization.** Consider the rotational velocity-correction scheme in differential form as introduced in [12]: Set $\tilde{u}^0 = u(t^0)$ and for $k \geq 0$, find $u^{k+1} \in H$, $p^{k+1} \in L^2_0(\Omega)$, and $\tilde{u}^{k+1} \in H^1_0(\Omega)^d$ such that

$$\begin{cases} \dfrac{u^{k+1} - \tilde{u}^k}{\delta t} + \nabla\times\nabla\times\tilde{u}^k + \nabla p^{k+1} = f(t^{k+1}), \\ \nabla \cdot u^{k+1} = 0, \quad u^{k+1} \cdot n|_\Gamma = 0, \end{cases} \tag{4.1}$$

and

$$\frac{\tilde{u}^{k+1} - u^{k+1}}{\delta t} - \nabla^2\tilde{u}^{k+1} - \nabla \times \nabla \times \tilde{u}^k = 0, \quad \tilde{u}^{k+1}|_\Gamma = 0. \tag{4.2}$$

Since $\nabla\times\nabla\times\tilde{u}^k = -\nabla^2\tilde{u}^k + \nabla\nabla\cdot\tilde{u}^k$, a natural approximation of $\nabla\times\nabla\times\tilde{u}^k$ is $i_h A_h \tilde{u}^k_h - C^T_h B_h \tilde{u}^k_h$, leading to the following fully discretized scheme: Set $\tilde{u}^0_h = \pi_h u_0$, then compute $\tilde{u}^{k+1}_h \in X_h$, $p^{k+1}_h \in M_h$, and $u^{k+1}_h \in Y_h$ such that

$$\begin{cases} \dfrac{u^{k+1}_h - i_h\tilde{u}^k_h}{\delta t} + i_h A_h \tilde{u}^k_h - C^T_h B_h \tilde{u}^k_h + C^T_h p^{k+1}_h = i_h f^{k+1}_h, \\ C_h u^{k+1}_h = 0, \end{cases} \tag{4.3}$$

and

$$\frac{\tilde{u}^{k+1}_h - i^T_h u^{k+1}_h}{\delta t} + A_h\tilde{u}^{k+1}_h - A_h\tilde{u}^k_h + B^T_h B_h \tilde{u}^k_h = 0. \tag{4.4}$$

By proceeding as in §3.1, one can show that this naive algorithm is not consistent at steady state.

Inspired by the discussion in §3.2, we now consider the following modified algorithm: Compute $\tilde{u}^{k+1}_h \in X_h$, $p^{k+1}_h \in M_h$, and $u^{k+1}_h \in Y_h$ such that

$$\begin{cases} \dfrac{u^{k+1}_h - i_h\tilde{u}^k_h}{\delta t} + i_h A_h \tilde{u}^k_h - C^T_h B_h \tilde{u}^k_h + C^T_h p^{k+1}_h + i_h B^T_h p^k_h - C^T_h p^k_h = i_h f^{k+1}_h, \\ C_h u^{k+1}_h = 0, \end{cases} \tag{4.5}$$

and

$$\frac{\tilde{u}^{k+1}_h - i^T_h u^{k+1}_h}{\delta t} + A_h\tilde{u}^{k+1}_h - A_h\tilde{u}^k_h + B^T_h B_h \tilde{u}^k_h = 0. \tag{4.6}$$

Again, by proceeding as in §3.2 this algorithm can be rewritten in an entirely equivalent way so as to completely avoid computing the velocity $u^{k+1}_h \in Y_h$. To see this, let us apply i^T_h to (4.5) and add the result to (4.6) to obtain

$$\frac{\tilde{u}^{k+1}_h - \tilde{u}^k_h}{\delta t} + A_h\tilde{u}^{k+1}_h + B^T_h p^{k+1}_h = f^{k+1}_h. \tag{4.7}$$

Note that we used the following properties: $i^T_h i_h|_{X_h}$ is the identity and $C_h i_h = B_h$. Now applying $-i_h$ to (4.7) at time step t^k and adding the result to (4.5) yields

$$\frac{u^{k+1}_h - 2i_h\tilde{u}^k_h + i_h\tilde{u}^{k-1}_h}{\delta t} + C^T_h(p^{k+1}_h - p^k_h - B_h\tilde{u}^k_h) = i_h(f^{k+1}_h - f^{k-1}_h). \tag{4.8}$$

Then, owing to the constraint $C_h u_h^{k+1} = 0$, this problem can be recast into the following form: Solve for $\phi_h^{k+1} \in M_h$ such that

$$(C_h^T \phi_h^{k+1}, C_h^T r_h) = (f^{k+1} - f^k, B_h^T r_h) + \frac{1}{\delta t}(2\tilde{u}_h^k - \tilde{u}_h^{k-1}, B_h^T r_h), \quad \forall r_h \in M_h. \tag{4.9}$$

Then set

$$p_h^{k+1} = \phi_h^{k+1} + p_h^k + B_h \tilde{u}_h^k. \tag{4.10}$$

In conclusion, an efficient way of implementing this algorithm consists of solving (4.9)-(4.10)-(4.7).

Remark 4.1. The algorithm (4.9)-(4.10)-(4.7) is the discrete counterpart of the algorithm (3.6)-(3.7)-(3.8) which was proposed in [12].

4.2. **Error estimates for the first-order rotational scheme.** Let us start by rewriting the algorithm in a way which is better suited for the error analysis. Inspired by the analysis in [12] where it is shown that one has to work with time increments to prove stability, we now construct the algorithm for the time increments of the discrete unknowns.

First, we define

$$\phi_h^{k+1} := \delta p_h^{k+1} - B_h \tilde{u}_h^k, \qquad D_h^t \tilde{u}_h^{k+1} := A_h \delta \tilde{u}_h^{k+1} + B_h^T B_h \tilde{u}_h^k. \tag{4.11}$$

Then we apply the increment operator δ to (4.5), (4.6), and (4.7) to obtain

$$\begin{cases} \delta u_h^{k+1} + \delta t C_h^T \phi_h^{k+1} - \delta t i_h \delta f_h^{k+1} = i_h \delta \tilde{u}_h^k - \delta t i_h D_h^t \tilde{u}_h^k \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \delta t (C_h^T - i_h B_h^T)\phi_h^k, \\ C_h \delta u_h^{k+1} = 0, \end{cases} \tag{4.12}$$

$$\delta \tilde{u}_h^{k+1} + \delta t (D_h^t \tilde{u}_h^{k+1} - D_h^t \tilde{u}_h^k) = i_h^T \delta u_h^{k+1}, \tag{4.13}$$

$$\delta \tilde{u}_h^{k+1} - \delta \tilde{u}_h^k + \delta t D_h^t \tilde{u}_h^{k+1} = -\delta t B_h^T \phi_h^{k+1} + \delta t \delta f_h^{k+1}. \tag{4.14}$$

As usual we are going to compare (u_h^k, p_h^k) with $(w_h(t^k), q_h(t^k)) \in X_h \times M_h$, which is the mixed approximation of $(u(t), p(t))$ defined as follows:

$$\begin{cases} (\nabla w_h(t), \nabla v_h) + (B_h^T q_h(t), v_h) = (\nabla u(t), \nabla v_h) - (p(t), \nabla \cdot v_h), \quad \forall v_h \in X_h, \\ (B_h w_h(t), r_h) = -(\nabla \cdot u(t), r_h), \quad \forall r_h \in M_h. \end{cases} \tag{4.15}$$

From the regularity properties of the Stokes problem, the following error estimates hold [9, 15].

Lemma 4.1. *If $u^{(j)} \in L^\beta(H^{l+1}(\Omega)^d) \cap H_0^1(\Omega)^d$, $p^{(j)} \in L^\beta(H^l(\Omega))$ for $1 \le \beta \le \infty$ and $j = 0, 1, \ldots$, then*

$$\begin{aligned} &\|u^{(j)} - w_h^{(j)}\|_{L^\beta(L^2(\Omega)^d)} + h \left(\|u^{(j)} - w_h^{(j)}\|_{L^\beta(H^1(\Omega)^d)} + \|p^{(j)} - q_h^{(j)}\|_{L^\beta(L^2(\Omega))} \right) \\ &\qquad\qquad\qquad \lesssim h^{l+1} \left(\|u^{(j)}\|_{L^\beta(H^{l+1}(\Omega)^d)} + \|p^{(j)}\|_{L^\beta(H^l(\Omega))} \right). \end{aligned} \tag{4.16}$$

We now rewrite (4.12)-(4.13)-(4.14) using w_h and q_h. Owing to the definition of $(w_h(t), q_h(t))$, the following identity holds at time $t = t^{k+1}$:

$$\begin{cases} w_h^{k+1} + \delta t B_h^T q_h^{k+1} - \delta t f_h^{k+1} - \delta t R_h^{k+1} = w_h^k - \delta t A_h w_h^{k+1}, \\ B_h w_h^{k+1} = 0, \end{cases} \tag{4.17}$$

where we have set $R_h^{k+1} = \frac{1}{\delta t}(w_h^{k+1} - w_h^k) - \pi_h \partial_t u^{k+1} \in X_h$. After applying $i_h\delta$ to (4.17), we obtain

$$\begin{cases} i_h\delta w_h^{k+1} + \delta t i_h B_h^T \delta q_h^{k+1} - \delta t i_h \delta f_h^{k+1} - i_h\delta R_h^{k+1} = i_h\delta w_h^k - \delta t i_h D_h^t w_h^{k+1}, \\ i_h B_h \delta w_h^{k+1} = 0. \end{cases} \tag{4.18}$$

Let us now introduce the following notation to denote various errors:

$$\begin{cases} e_h^{k+1} = i_h w_h^{k+1} - u_h^{k+1}, \quad \tilde{e}_h^{k+1} = w_h^{k+1} - \tilde{u}_h^{k+1}, \quad \tilde{\psi}_h^{k+1} = \delta w_h^{k+1} + \tilde{e}_h^k, \\ \epsilon_h^{k+1} = q_h^{k+1} - p_h^{k+1}, \qquad \varepsilon_h^{k+1} = \delta q_h^{k+1} - \phi_h^{k+1}, \quad \tilde{\phi}_h^{k+1} = \delta q_h^{k+1} - \phi_h^k. \end{cases} \tag{4.19}$$

Subtracting (4.12) from (4.18), we obtain

$$\begin{cases} \delta e_h^{k+1} + \delta t C_h^T \varepsilon_h^{k+1} - \delta t i_h \delta R_h^{k+1} = i_h \delta \tilde{e}_h^k - \delta t i_h D_h^t \tilde{\psi}_h^{k+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad - \delta t i_h B_h^T \tilde{\phi}_h^{k+1} + \delta t C_h^T \tilde{\phi}_h^{k+1}, \\ C_h \delta e_h^{k+1} = 0. \end{cases} \tag{4.20}$$

After applying δ to (4.17) and subtracting (4.14) from it, we obtain

$$\delta\tilde{e}_h^{k+1} - \delta\tilde{e}_h^k + \delta t D_h^t \tilde{e}_h^{k+1} = -\delta t B_h^T \varepsilon_h^{k+1} + \delta t \delta R_h^{k+1}. \tag{4.21}$$

Now adding some zero terms to (4.13), we can rewrite it as

$$\delta\tilde{e}_h^{k+1} + \delta t D_h^t (\tilde{e}_h^{k+1} - \tilde{\psi}_h^{k+1}) = i_h^T \delta e_h^{k+1}. \tag{4.22}$$

The error analysis will be based entirely on the three equations (4.20)-(4.21)-(4.22).

Let us assume that the algorithm is initialized so that the following holds:

$$\begin{cases} \|\tilde{e}_h^0\|_0 \lesssim \min(h^{l+1}, \delta t^2 h^{l-1}), \quad \|\tilde{e}_h^0\|_1 \lesssim \min(h^l, \delta t h^{l-1}), \\ \|A_h \tilde{e}_h^0\|_0 \lesssim \min(h^{l-1}, \delta t h^{l-2}), \quad \|B_h^t B_h \tilde{e}_h^0\|_0 \lesssim \min(h^{l-1}, \delta t h^{l-2}), \\ \|\epsilon_h^0\|_1 \lesssim \min(h^{l-1}, \delta t h^{l-2}), \end{cases} \tag{H1}$$

and the solution to (2.1) satisfies the following regularity hypothesis

$$\begin{aligned} &u, u_t, u_{tt} \in L^2(H^{l+1}(\Omega)^d \cap H_0^1(\Omega)^d), \quad u_{ttt} \in L^2(L^2(\Omega)^d) \\ &p \in L^2(H^l(\Omega)), \quad p_t, p_{tt} \in L^2(H^1(\Omega)). \end{aligned} \tag{H2}$$

Remark 4.2. If we set $\tilde{u}_h^0 = w_h^0$ and $p_h^0 = q_h^0$, then the hypothesis (H1) is naturally satisfied.

We are now in position to establish the first error estimate.

Lemma 4.2. *If the hypotheses* (H1)-(H2) *hold, we have*

$$\|\delta\tilde{e}_h\|_{l^\infty(L^2(\Omega)^d)} + \|\tilde{e}_h - i_h^T e_h\|_{l^\infty(L^2(\Omega)^d)} \lesssim \delta t(\delta t + h^{l+1}), \tag{4.23}$$

$$\|i_h^T \delta e_h - \delta\tilde{e}_h\|_{l^2(L^2(\Omega)^d)} + \|\delta^2 \tilde{e}_h\|_{l^2(L^2(\Omega)^d)} \lesssim \delta t^{5/2} + \delta t^{3/2} h^{l+1}, \tag{4.24}$$

$$\|B_h \tilde{u}_h\|_{l^\infty(L^2(\Omega))} \lesssim \delta t^{1/2}(\delta t + h^{l+1}). \tag{4.25}$$

Proof. Let us first recall a series of standard identities that will be used throughout the paper:

$$\begin{cases} 2(a,b) = |a|^2 + |b|^2 - |a-b|^2, \\ 2(a-b,a) = |a|^2 + |b|^2 - |a-b|^2, \\ 2(a-2b+c,b) = (|a|^2 - |b|^2) - (|b|^2 - |c|^2) - |a-b|^2 - |b-c|^2. \end{cases} \tag{4.26}$$

First, we square (4.20) and, noticing that $i_h^T C_h^T q = B_h^T q, \forall q \in M_h$ and $\|i_h v\|_0 = \|v\|_0, \forall v \in X_h$, we obtain

$$\begin{aligned}\|\delta e_h^{k+1}\|_0^2 &+ \delta t^2\|C_h^T\varepsilon_h^{k+1}\|_0^2 + \delta t^2\|\delta R_h^{k+1}\|_0^2\\ &- 2\delta t(\delta e_h^{k+1} + \delta t C_h^T\varepsilon_h^{k+1}, i_h\delta R_h^{k+1}) = \|\delta\tilde{e}_h^k\|_0^2 + \delta t^2\|D_h^t\tilde{\psi}_h^{k+1}\|_0^2\\ &- 2\delta t(i_h\delta\tilde{e}_h^k, i_h D_h^t\tilde{\psi}_h^{k+1}) + \delta t^2\|C_h^T\tilde{\phi}_h^{k+1}\|_0^2 - \delta t^2\|B_h^T\tilde{\phi}_h^{k+1}\|_0^2.\end{aligned} \tag{4.27}$$

By the identities (4.26), we have

$$\begin{aligned}2\delta t(i_h\delta\tilde{e}_h^k, i_h D_h^t\tilde{\psi}_h^{k+1}) &= 2\delta t(\delta\tilde{e}_h^k, A_h\delta\tilde{\psi}_h^{k+1} + B_h^T B_h\tilde{\psi}_h^k)\\ &= 2\delta t(\delta\tilde{e}_h^k, A_h\delta^2 w_h^{k+1} + A_h\delta\tilde{e}_h^k + B_h^T B_h\tilde{e}_h^{k-1})\\ &= \delta t(2\|\nabla\delta\tilde{e}_h^k\|_0^2 - \|B_h\delta\tilde{e}_h^k\|_0^2) + \delta t(\|B_h\tilde{e}_h^k\|_0^2 - \|B_h\tilde{e}_h^{k-1}\|_0^2)\\ &\quad + 2\delta t(\delta\tilde{e}_h^k, A_h\delta^2 w_h^{k+1}).\end{aligned}$$

By substituting the above equality into (4.27), we infer

$$\begin{aligned}\|\delta e_h^{k+1}\|_0^2 &+ \delta t^2\|C_h^T\varepsilon_h^{k+1}\|_0^2 + \delta t(2\|\nabla\delta\tilde{e}_h^k\|_0^2 - \|B_h\delta\tilde{e}_h^k\|_0^2)\\ &+ \delta t(\|B_h\tilde{e}_h^k\|_0^2 - \|B_h\tilde{e}_h^{k-1}\|_0^2) = \|\delta\tilde{e}_h^k\|_0^2 + \delta t^2\|D_h^t\tilde{\psi}_h^{k+1}\|_0^2\\ &+ \delta t^2\|C_h^T\tilde{\phi}_h^{k+1}\|_0^2 - \delta t^2\|B_h^T\tilde{\phi}_h^{k+1}\|_0^2 + 2\delta t(i_h^T\delta e_h^{k+1}, \delta R_h^{k+1})\\ &+ 2\delta t^2(B_h^T\varepsilon_h^{k+1}, \delta R_h^{k+1}) - \delta t^2\|\delta R_h^{k+1}\|_0^2 - 2\delta t(\delta\tilde{e}_h^k, A_h\delta^2 w_h^{k+1}).\end{aligned} \tag{4.28}$$

Next, we square (4.21) and use (4.26) to obtain

$$\begin{aligned}\|\delta^2\tilde{e}_h^{k+1}\|_0^2 &+ \delta t^2\|D_h^t\tilde{e}_h^{k+1}\|_0^2 + \delta t(\|\nabla\delta\tilde{e}_h^{k+1}\|_0^2 - \|\nabla\delta\tilde{e}_h^k\|_0^2)\\ &+ \delta t\|\nabla\delta^2\tilde{e}_h^{k+1}\|_0^2 + \delta t(\|B_h\tilde{e}_h^{k+1}\|_0^2 - \|B_h\tilde{e}_h^k\|_0^2)\\ &- \delta t(\|B_h\tilde{e}_h^k\|_0^2 - \|B_h\tilde{e}_h^{k-1}\|_0^2) - \delta t(\|B_h\delta\tilde{e}_h^{k+1}\|_0^2 + \|B_h\delta\tilde{e}_h^k\|_0^2)\\ &= \delta t^2\|B_h^T\varepsilon_h^{k+1}\|_0^2 + \delta t^2\|\delta R_h^{k+1}\|_0^2 - 2\delta t^2(B_h^T\varepsilon_h^{k+1}, \delta R_h^{k+1}).\end{aligned} \tag{4.29}$$

Then, we square (4.22) to obtain

$$\begin{aligned}\|\delta\tilde{e}_h^{k+1}\|_0^2 &+ \delta t^2\|D_h^t\tilde{e}_h^{k+1} - D_h^t\tilde{\psi}_h^{k+1}\|_0^2\\ &+ 2\delta t(\delta\tilde{e}_h^{k+1}, D_h^t\tilde{e}_h^{k+1} - D_h^t\tilde{\psi}_h^{k+1}) = \|i_h^T\delta\tilde{e}_h^{k+1}\|_0^2.\end{aligned} \tag{4.30}$$

Notice that

$$\begin{aligned}2\delta t(\delta\tilde{e}_h^{k+1}, D_h^t\tilde{e}_h^{k+1} - D_h^t\tilde{\psi}_h^{k+1}) &= 2\delta t(\delta\tilde{e}_h^{k+1}, D_h^t\delta\tilde{e}_h^{k+1} - D_h^t\delta w_h^{k+1})\\ &= 2\delta t(\nabla\delta\tilde{e}_h^{k+1}, \nabla\delta^2\tilde{e}_h^{k+1}) + 2\delta t(B_h\delta\tilde{e}_h^{k+1}, B_h\delta\tilde{e}_h^k)\\ &\quad - 2\delta t(\delta\tilde{e}_h^{k+1}, D_h^t\delta w_h^{k+1})\\ &= \delta t(\|\nabla\delta\tilde{e}_h^{k+1}\|_0^2 - \|\nabla\delta\tilde{e}_h^k\|_0^2 + \|\nabla\delta^2\tilde{e}_h^{k+1}\|_0^2)\\ &\quad + \delta t(\|B_h\delta\tilde{e}_h^{k+1}\|_0^2 + \|B_h\delta\tilde{e}_h^k\|_0^2 - \|B_h\delta^2\tilde{e}_h^{k+1}\|_0^2)\\ &\quad - 2\delta t(\delta\tilde{e}_h^{k+1}, A_h\delta^2 w_h^{k+1}),\end{aligned}$$

and

$$\delta t^2\|D_h^t\tilde{e}_h^{k+1} - D_h^t\tilde{\psi}_h^{k+1}\|_0^2 = \|i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}\|_0^2,$$

so we can rewrite (4.30) as

$$
\begin{aligned}
&\|\delta\tilde{e}_h^{k+1}\|_0^2 + \|i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}\|_0^2 + \delta t(\|\nabla\delta\tilde{e}_h^{k+1}\|_0^2 - \|\nabla\delta\tilde{e}_h^k\|_0^2) \\
&\qquad + \delta t(\|B_h\delta\tilde{e}_h^{k+1}\|_0^2 + \|B_h\delta\tilde{e}_h^k\|_0^2) + \delta t(\|\nabla\delta^2\tilde{e}_h^{k+1}\|_0^2 - \|B_h\delta^2\tilde{e}_h^{k+1}\|_0^2) \\
&\qquad = \|i_h^T\delta e_h^{k+1}\|_0^2 + 2\delta t(\delta\tilde{e}_h^{k+1}, A_h\delta^2 w_h^{k+1}).
\end{aligned} \tag{4.31}
$$

Now we add (4.28), (4.29), and (4.31). Note that the definition of B_h together with the homogeneous Dirichlet boundary conditions on v_h imply

$$
\|B_h v_h\|_0^2 \le \|\nabla\cdot v_h\|_0^2 \le \|\nabla\cdot v_h\|_0^2 + \|\nabla\times v_h\|_0^2 = \|\nabla v_h\|_0^2, \qquad \forall v_h \in X_h.
$$

Observe, moreover, that $\|i_h^T u\|_0 \le \|u\|_0$, $\forall u \in Y_h$. These two facts then yield

$$
\begin{aligned}
\|\delta\tilde{e}_h^{k+1}\|_0^2 &+ \delta t^2(\|C_h^T\varepsilon_h^{k+1}\|_0^2 - \|B_h^T\varepsilon_h^{k+1}\|_0^2) + \delta t(\|B_h\tilde{e}_h^{k+1}\|_0^2 - \|B_h\tilde{e}_h^k\|_0^2) \\
&+ \delta t^2\|D_h^t\tilde{e}_h^{k+1}\|_0^2 + 2\delta t(\|\nabla\delta\tilde{e}_h^{k+1}\|_0^2 - \|\nabla\delta\tilde{e}_h^k\|_0^2) + \|\delta^2\tilde{e}_h^{k+1}\|_0^2 \\
&+ \|i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}\|_0^2 + \delta t\|\nabla\delta\tilde{e}_h^k\|_0^2 + \delta t\|\nabla\delta^2\tilde{e}_h^{k+1}\|_0^2 \\
&\le \|\delta\tilde{e}_h^k\|_0^2 + \delta t^2(\|C_h^T\tilde{\phi}_h^{k+1}\|_0^2 - \|B_h^T\tilde{\phi}_h^{k+1}\|_0^2) + \delta t^2\|D_h^t\tilde{\psi}_h^{k+1}\|_0^2 \\
&\quad + 2\delta t(\delta^2\tilde{e}_h^{k+1}, A_h\delta^2 w_h^{k+1}) + 2\delta t(\delta R_h^{k+1}, i_h^T\delta e_h^{k+1}).
\end{aligned} \tag{4.32}
$$

We now derive bounds for the last four terms in the right-hand side.

The definition of i_h^T implies

$$
\|i_h i_h^T v_h\|_0^2 + \|v_h - i_h i_h^T v_h\|_0^2 = \|v_h\|_0^2, \qquad \forall v_h \in Y_h.
$$

Hence, from (2.15), we infer

$$
\begin{aligned}
\|C_h^T q_h\|_0^2 - \|B_h^T q_h\|_0^2 &= \|C_h^T q_h\|_0^2 - \|i_h i_h^T C_h^T q_h\|_0^2 = \|C_h^T q_h - i_h i_h^T C_h^T q_h\|_0^2 \\
&= \|(C_h^T - i_h B_h^T) q_h\|_0^2, \qquad \forall q_h \in M_h.
\end{aligned}
$$

Owing to this result together with the definition of $\tilde{\phi}_h$, we deduce that

$$
\begin{aligned}
\|C_h^T\tilde{\phi}_h^{k+1}\|_0^2 - \|B_h^T\tilde{\phi}_h^{k+1}\|_0^2 &= \|(C_h^T - i_h B_h^T)(\delta^2 q_h^{k+1} + \varepsilon_h^k)\|_0^2 \\
&= \|(C_h - i_h B_h^T)\delta^2 q_h^{k+1}\|_0^2 + \|(C_h - i_h B_h^T)\varepsilon_h^k\|_0^2 \\
&\quad + 2((C_h^T - i_h B_h^T)\delta^2 q_h^{k+1}, (C_h^T - i_h B_h^T)\varepsilon_h^k).
\end{aligned}
$$

Then, using (2.18) together with Lemma 4.1, we obtain

$$
\|C_h^T\tilde{\phi}_h^{k+1}\|_0^2 - \|B_h^T\tilde{\phi}_h^{k+1}\|_0^2 \lesssim \delta t^3 + (1+\delta t)\|(C_h - i_h B_h^T)\varepsilon_h^k\|_0^2.
$$

A bound on $\|D_h^t\tilde{\psi}_h^{k+1}\|_0^2$ can be obtained as follows:

$$
\|D_h^t\tilde{\psi}_h^{k+1}\|_0^2 \le (\|D_h^t\delta w_h^{k+1}\|_0 + \|D_h^t\tilde{e}_h^k\|_0)^2 = (\|A_h\delta^2 w_h^{k+1}\|_0 + \|D_h^t\tilde{e}_h^k\|_0)^2.
$$

Then using (2.17) together with Lemma 4.1, we obtain

$$
\|D_h^t\tilde{\psi}_h^{k+1}\|_0^2 \lesssim \delta t^3 + (1+\delta t)\|D_h^t\tilde{e}_h^k\|_0^2.
$$

For the two other terms from the right-hand side, upper bounds can be derived as follows:

$$
\begin{aligned}
2\delta t(\delta R_h^{k+1}, i_h^T\delta e_h^{k+1}) &= 2\delta t(\delta R_h^{k+1}, i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}) + 2\delta t(\delta R_h^{k+1}, \delta\tilde{e}_h^{k+1}) \\
&\le 2\delta t\|\delta R_h^{k+1}\|_0^2 + \delta t\|i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}\|_0^2 + \delta t\|\delta\tilde{e}_h^{k+1}\|_0^2 \\
&\lesssim \delta t(\delta t^2 + \delta t h^{l+1})^2 + \delta t\|i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}\|_0^2 + \delta t\|\delta\tilde{e}_h^{k+1}\|_0^2,
\end{aligned}
$$

$$2\delta t(\delta^2\tilde{e}_h^{k+1}, D_h^t\delta w_h^{k+1}) \le \delta t\|\nabla\delta^2\tilde{e}_h^{k+1}\|_0^2 + \delta t\|\nabla\delta^2 w_h^{k+1}\|_0^2$$
$$\lesssim \delta t\|\nabla\delta^2\tilde{e}_h^{k+1}\|_0^2 + \delta t^5.$$

Substituting all the inequalities above into (4.32), we obtain

$$\begin{aligned}(1-\delta t)\|\delta\tilde{e}_h^{k+1}\|_0^2 &+ \delta t^2\|(C_h - i_hB_h^T)\varepsilon_h^{k+1}\|_0^2 + \delta t\|B_h\tilde{e}_h^{k+1}\|_0^2\\ &+ \delta t^2\|D_h^t\tilde{e}_h^{k+1}\|_0^2 + 2\delta t\|\nabla\delta\tilde{e}_h^{k+1}\|_0^2\\ &+ \|\delta^2\tilde{e}_h^{k+1}\|_0^2 + (1-\delta t)\|i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}\|_0^2\\ &\lesssim \|\delta\tilde{e}_h^k\|_0^2 + \delta t^2(1+\delta t)\|(C_h^T - i_hB_h^T)\varepsilon_h^k\|_0^2 + \delta t\|B_h\tilde{e}_h^k\|_0^2\\ &\quad + (1+\delta t)\delta t^2\|D_h^t\tilde{e}_h^k\|_0^2 + \delta t\|\nabla\delta\tilde{e}_h^k\|_0^2 + \delta t^3(\delta t + h^{l+1})^2.\end{aligned}$$

The discrete Gronwall lemma yields, for all $n \le [T/\delta t] - 1$,

$$\begin{aligned}\|\delta\tilde{e}_h^{n+1}\|_0^2 &+ \delta t^2\|(C_h^T - i_hB_h^T)\varepsilon_h^{n+1}\|_0^2 + \delta t\|B_h\tilde{e}_h^{n+1}\|_0^2 + \delta t^2\|D_h^t\tilde{e}_h^{n+1}\|_0^2\\ &+ \delta t\|\nabla\delta\tilde{e}_h^{k+1}\|_0^2 + \sum_{k=1}^n\left(\|i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}\|_0^2 + \|\delta^2\tilde{e}_h^{k+1}\|_0^2\right)\\ &\lesssim \|\delta\tilde{e}_h^1\|_0^2 + \delta t^2\|(C_h^T - i_hB_h^T)\varepsilon_h^1\|_0^2 + \delta t\|B_h\tilde{e}_h^1\|_0^2\\ &\quad + \delta t^2\|D_h^t\tilde{e}_h^1\|_0^2 + \delta t\|\nabla\delta\tilde{e}_h^1\|_0^2 + \delta t^2(\delta t + h^{l+1})^2.\end{aligned}$$

In order to estimate the initial error terms in the right-hand side of the above inequality we make use of (4.17)-(4.6) at the first time step (i.e., $k = 0$). More precisely, the equations that control e_h^1, $\tilde{e}_h^1$, and ε_h^1 are obtained by subtracting (4.5) from the equation obtained by applying i_h to (4.17), and adding some zero terms to (4.6) as follows:

$$\begin{aligned}e_h^1 + \delta tC_h^T\varepsilon_h^1 &= i_h\tilde{e}_h^0 + \delta ti_hR_h^1 - \delta ti_hA_h\delta w_h^1 - \delta ti_hA_h\tilde{e}_h^0\\ &\quad + \delta tC_h^T\delta q_h^1 - \delta ti_hB_h^T\delta q_h^1 - \delta ti_hB_h^T\epsilon_h^0,\end{aligned} \tag{4.33}$$

$$\tilde{e}_h^1 + \delta tA_h\tilde{e}_h^1 = i_h^Te_h^1 + \delta tA_h\tilde{e}_h^0 - \delta tB_h^TB_h\tilde{e}_h^0 + \delta tA_h\delta w_h^1. \tag{4.34}$$

Taking the square for (4.33) and (4.34), we have

$$\begin{aligned}\|e_h^1\|_0^2 + \delta t^2\|C_h^T\varepsilon_h^1\|_0^2 &\lesssim \|\tilde{e}_h^0\|_0^2 + \delta t^2\|R_h^1\|_0^2 + \delta t^2\|A_h\delta w_h^1\|_0^2 + \delta t^2\|A_h\tilde{e}_h^0\|_0^2\\ &\quad + \delta t^2\|C_h^T\delta q_h^1\|_0^2 + \delta t^2\|B_h^T\delta q_h^1\|_0^2 + \delta t^2\|B_h^T\epsilon_h^0\|_0^2,\end{aligned}$$

and

$$\begin{aligned}\|\tilde{e}_h^1\|_0^2 + \delta t^2\|A_h\tilde{e}_h^1\|_0^2 + \delta t\|\tilde{e}_h^1\|_1^2 &\lesssim \|e_h^1\|_0^2 + \delta t^2\|A_h\delta w_h^1\|_0^2\\ &\quad + \delta t^2\|A_h\tilde{e}_h^0\|_0^2 + \delta t^2\|B_h^TB_h\tilde{e}_h^0\|_0^2.\end{aligned}$$

From the initialization hypothesis (H1), we obtain

$$\|\tilde{e}_h^1\|_0^2 + \|\tilde{e}_h^0\|_0^2 + \delta t\|\tilde{e}_h^1\|_1^2 + \delta t^2\|C_h^T\varepsilon_h^1\|_0^2 + \delta t^2\|A_h\tilde{e}_h^1\|_0^2 \lesssim \delta t^2(\delta t + h^{l+1})^2.$$

Collecting the above results yields the desired bound

$$\begin{aligned}\|\delta\tilde{e}_h^{n+1}\|_0^2 &+ \delta t^2\|D_h^t\tilde{e}_h^{n+1}\|_0^2 + \delta t\|B_h\tilde{e}_h^{n+1}\|_0^2\\ &+ \sum_{k=1}^n(\|i_h^T\delta e_h^{k+1} - \delta\tilde{e}_h^{k+1}\|_0^2 + \|\delta^2\tilde{e}_h^{k+1}\|_0^2) \le \delta t^2(\delta t + h^{l+1})^2.\end{aligned}$$

Finally, the bound on $\tilde{e}_h - i_h^T e_h$ can be obtained from (4.6) as follows:

$$\|\tilde{e}_h^{n+1} - i_h^T e_h^{n+1}\|_0 = \delta t \|D_h^t \tilde{e}_h^{n+1} - A_h \delta w_h^{n+1}\|_0$$
$$\lesssim \delta t \|D_h^t \tilde{e}_h^{n+1}\|_0 + \delta t^2 \lesssim \delta t(\delta t + h^{l+1}).$$

This completes the proof. □

Remark 4.3. The estimate (4.25) is remarkable in the sense that, even though the time stepping scheme is only first-order, the discrete divergence of $\tilde{u}_h$ is $\frac{3}{2}$-order with respect to time. Actually, the $\frac{3}{2}$-order holds also if we replace the first-order backward Euler time stepping with the second-order BDF2 (i.e., $R_h^{k+1} \sim O(\delta t^2 + h^{l+1})$). That the splitting error can be smaller than the consistency error induced by the time stepping has also been observed in [9]. This $\frac{3}{2}$-order on the discrete divergence of $\tilde{u}_h$ is the key reason why the second-order rotational scheme yields better error estimate for velocity in the H^1-norm and the pressure in the L^2-norm than the standard version of the algorithm (compare Theorem 3.1 and Theorem 5.1).

We are now in position to prove the major result of this section:

Theorem 4.1. *Let* u_h, $\tilde{u}_h$, p_h *solve* (4.5)-(4.6) (*or the equivalent algorithm* (4.9)-(4.10)-(4.7)) *and assume* (H1)-(H2), *then the following error estimates hold:*

$$\|u - \tilde{u}_h\|_{\ell^2(L^2(\Omega)^d)} + \|u - i_h^T u_h\|_{\ell^2(L^2(\Omega)^d)} \lesssim \delta t + h^{l+1}, \tag{4.35}$$

$$\|u - \tilde{u}_h\|_{\ell^2(H^1(\Omega)^d)} + \|u - i_h^T u_h\|_{\ell^2(H^1(\Omega)^d)} + \|p - p_h\|_{\ell^2(L^2(\Omega))} \lesssim \delta t + h^l. \tag{4.36}$$

Proof. We reconstruct a non-homogeneous Stokes equation for the errors $\tilde{e}$, ϵ, by subtracting (4.7) from (4.17),

$$\begin{cases} A_h \tilde{e}_h^{k+1} + B_h^T \epsilon_h^{k+1} = R_h^{k+1} - \dfrac{\tilde{e}_h^{k+1} - \tilde{e}_h^k}{\delta t}, \\ B_h \tilde{e}_h^{k+1} = -B_h \tilde{u}_h^{k+1}. \end{cases}$$

Standard stability results on non-homogeneous Stokes problems yield

$$\|\tilde{e}_h^{k+1}\|_1 + \|\epsilon_h^{k+1}\|_0 \le \|R_h^{k+1}\|_{-1} + \frac{1}{\delta t}\|\tilde{e}_h^{k+1} - \tilde{e}_h^k\|_{-1} + \|B_h \tilde{u}_h^{k+1}\|_0.$$

Owing to Lemma 4.2, we have

$$\frac{1}{\delta t^2}\|\tilde{e}_h^{k+1} - \tilde{e}_h^k\|^2_{\ell^2(H^{-1}(\Omega)^d)} \lesssim \frac{1}{\delta t^2}\|\tilde{e}_h^{k+1} - \tilde{e}_h^k\|^2_{\ell^2(L^2(\Omega)^d)} \lesssim \delta t^2 + h^{2l+2},$$
$$\|B_h \tilde{u}_h^{k+1}\|_{\ell^2(L^2(\Omega)^d)} \le \|B_h \tilde{e}_h^{k+1}\|_{\ell^\infty(L^2(\Omega)^d)} \lesssim \delta t^{3/2} + \delta t^{1/2} h^{l+1}.$$

This immediately implies

$$\|\tilde{e}_h\|_{\ell^2(H^1(\Omega)^d)} + \|\epsilon_h\|_{\ell^2(L^2(\Omega))} \lesssim \delta t + h^{l+1}.$$

Moreover, using the Poincaré inequality we infer $\|\tilde{e}\|_{l^2(l^2(\Omega)^d)} \lesssim \delta t + h^{l+1}$, which in turn together with (4.23) yield $\|i_h^T e\|_{l^2(l^2(\Omega)^d)} \lesssim \delta t + h^{l+1}$. The desired results are then consequences of Lemma 4.1. □

Remark 4.4. The ℓ^2 discrete norm in time in the estimates (4.35)-(4.36) can be replaced by the ℓ^∞-norm with a little more regularity asumption on the solution. We refer, e.g., to [10, Theorem 4.1] where such estimates are proven for the fully discrete standard form of the pressure-correction method.

5. Second-order rotational velocity-correction scheme

We now focus our attention on the fully discrete rotational velocity-correction algorithm with BDF2 in time. To shorten the presentation we only give the proof of the stability of the algorithm and we just mention the final convergence result. The technical details are similar to those in the proof of Theorem 4.1 plus a duality argument involving the right inverse of the Stokes operator (details can be found in [9, 24]).

Replacing the Euler time stepping in (4.5) and (4.7) by BDF2 yields the following algorithm: Find $(\tilde{u}_h^{k+1}, p_h^{k+1}, u_h^{k+1}) \in (X_h, M_h, Y_h)$ such that

$$\begin{cases} \dfrac{3u_h^{k+1} - 4i_h\tilde{u}_h^k + i_h\tilde{u}_h^{k-1}}{2\delta t} + i_hA_h\tilde{u}_h^k - C_h^TB_h\tilde{u}_h^k + C_h^Tp_h^{k+1} \\ \qquad\qquad\qquad + (i_hB_h^T - C_h^T)p_h^k = i_hf_h^{k+1}, \\ C_hu_h^{k+1} = 0, \end{cases} \tag{5.1}$$

and

$$\frac{3\tilde{u}_h^{k+1} - 4\tilde{u}_h^k + \tilde{u}_h^{k-1}}{2\delta t} + A_h\tilde{u}_h^{k+1} + B_h^Tp_h^{k+1} = f_h^{k+1}. \tag{5.2}$$

Lemma 5.1. *The solution of the scheme* (5.1)-(5.2) *is bounded in the following sense:*

$$\|\delta\tilde{u}_h\|_{l^\infty(L^2(\Omega)^d)} + \delta t^{\frac{1}{2}}\|B_h\tilde{u}_h\|_{l^\infty(L^2(\Omega))} \leq \|\delta f_h\|_{l^2(L^2(\Omega)^d)}. \tag{5.3}$$

Proof. For simplicity we omit the source term f since it does not affect the stability of the algorithm. We proceed as in the proof of Lemma 4.1.

Applying i_h^T to (5.1) and subtracting the result from (5.2), we obtain

$$\frac{3(\tilde{u}_h^{k+1} - i_h^Tu_h^{k+1})}{2\delta t} + A_h\tilde{u}_h^{k+1} - A_h\tilde{u}_h^k + B_h^TB_h\tilde{u}_h^k = 0. \tag{5.4}$$

After applying the time increment operator δ to (5.1), (5.2) and (5.4), we have

$$\begin{cases} 3\delta u_h^{k+1} + 2\delta tC_h^T\phi_h^{k+1} = i_h(4\delta\tilde{u}_h^k - \delta\tilde{u}_h^{k-1}) \\ \qquad\qquad - 2\delta ti_hD_h^t\tilde{u}_h^k + 2\delta t(C_h^T - i_hB_h^T)\phi_h^k, \\ C_h\delta u_h^{k+1} = 0, \end{cases} \tag{5.5}$$

$$3\delta\tilde{u}_h^{k+1} - 4\delta\tilde{u}_h^k + \delta\tilde{u}_h^{k-1} + 2\delta tD_h^t\tilde{u}_h^{k+1} = -2\delta tB_h^T\phi_h^{k+1}, \tag{5.6}$$

$$3\delta\tilde{u}_h^{k+1} + 2\delta tD_h^t\delta\tilde{u}_h^{k+1} = 3i_h^T\delta u_h^{k+1}. \tag{5.7}$$

The entire stability analysis is based on the equations (5.5)–(5.6)–(5.7) above. In the following, we will square each of them, sum up the results, use the two inequalities

$$\|B_hv_h\|_0 \leq \|\nabla v_h\|_0, \quad \forall v_h \in X_h; \qquad \|i_h^Ty_h\|_0 \leq \|y_h\|_0, \forall y_h \in Y_h, \tag{5.8}$$

and apply the discrete Gronwall lemma to the resulted inequality.

First, we square (5.5) to obtain

$$\begin{aligned} 9\|\delta u_h^{k+1}\|_0^2 + 4\delta t^2\|C_h^T\phi_h^{k+1}\|_0^2 &= \|4\delta\tilde{u}_h^k - \delta\tilde{u}_h^{k-1}\|_0^2 + 4\delta t^2\|D_h^t\tilde{u}_h^k\|_0^2 \\ &- 4\delta t(4\delta\tilde{u}_h^k - \delta\tilde{u}_h^{k-1}, D_h^t\tilde{u}_h^k) + 4\delta t^2\|(C_h^T - i_hB_h^T)\delta\phi_h^k\|_0^2. \end{aligned} \tag{5.9}$$

Note that we used the fact that $(i_h v_h, (C_h^T - i_h B_h^T)q_h) = 0$ for all $v_h \in X_h$ and all $q_h \in M_h$, since $i_h^T i_h$ is the identity and $i_h^T C_h^T = B_h^T$. Then, we square (5.6) to get

$$\|3\delta\tilde{u}_h^{k+1} - 4\delta\tilde{u}_h^k + \delta\tilde{u}_h^{k-1}\|_0^2 + 4\delta t^2 \|D_h^t \tilde{u}_h^{k+1}\|_0^2 + 4\delta t(3\delta\tilde{u}_h^{k+1} - 4\delta\tilde{u}_h^k + \delta\tilde{u}_h^{k-1}, D_h^t \tilde{u}_h^{k+1}) = 4\delta t^2 \|B_h^T \phi_h^{k+1}\|_0^2. \tag{5.10}$$

Finally, squaring (5.7) leads to

$$9\|\delta\tilde{u}_h^{k+1}\|_0^2 + 4\delta t^2 \|D_h^t \delta\tilde{u}_h^{k+1}\|_0^2 + 4\delta t(3\delta\tilde{u}_h^{k+1}, D_h^t \delta\tilde{u}_h^{k+1}) = 9\|i_h^T \delta u_h^{k+1}\|_0^2. \tag{5.11}$$

Now we sum up (5.9)–(5.10)–(5.11) to obtain

$$\begin{aligned} 9\|\delta u_h^{k+1}\|_0^2 &+ 4\delta t^2(\|C_h^T \phi_h^{k+1}\|_0^2 - \|B_h^T \phi_h^{k+1}\|_0^2) \\ &+ 4\delta t^2 \|D_h^t \tilde{u}_h^{k+1}\|_0^2 + 9\|\delta\tilde{u}_h^{k+1}\|_0^2 + I_1 + I_2 \\ &+ 4\delta t^2 \|D_h^t \delta\tilde{u}_h^{k+1}\|_0^2 = 4\delta t^2 \|D_h^t \tilde{u}_h^k\|_0^2 \\ &+ 4\delta t^2 \|(C_h^T - i_h B_h^T)\delta\phi_h^k\|_0^2 + 9\|i_h^T \delta u_h^{k+1}\|_0^2, \end{aligned} \tag{5.12}$$

where, to simplify notation, we have set

$$I_1 := \|3\delta\tilde{u}_h^{k+1} - 4\delta\tilde{u}_h^k + \delta\tilde{u}_h^{k-1}\|_0^2 - \|4\delta\tilde{u}_h^k - \delta\tilde{u}_h^{k-1}\|_0^2,$$

and

$$\begin{aligned} I_2 := 4\delta t[(3\delta\tilde{u}_h^{k+1}, D_h^t \delta\tilde{u}_h^{k+1}) &+ (3\delta\tilde{u}_h^{k+1} - 4\delta\tilde{u}_h^k + \delta\tilde{u}_h^{k-1}, D_h^t \tilde{u}_h^{k+1}) \\ &+ (4\delta\tilde{u}_h^k - \delta\tilde{u}_h^{k-1}, D_h^t \tilde{u}_h^k)]. \end{aligned}$$

Observing that $\|C_h^T \phi_h^{k+1}\|_0^2 - \|B_h^T \phi_h^{k+1}\|_0^2 = \|(C_h^T - i_h B_h^T)\phi_h^{k+1}\|_0^2$ and $\|i_h^T \delta u_h^{k+1}\|_0 \le \|\delta u_h^{k+1}\|_0$, (5.12) can be rewritten as

$$\begin{aligned} 9\|\delta\tilde{u}_h^{k+1}\|_0^2 &+ I_1 + I_2 + 4\delta t^2 \|(C_h^T - i_h B_h^T)\phi_h^{k+1}\|_0^2 + 4\delta t^2 \|D_h^t \tilde{u}_h^{k+1}\|_0^2 \\ &+ 4\delta t^2 \|D_h^t \delta\tilde{u}_h^{k+1}\|_0^2 \le 4\delta t^2 \|D_h^t \tilde{u}_h^k\|_0^2 + 4\delta t^2 \|(C_h^T - i_h B_h^T)\delta\phi_h^k\|_0^2. \end{aligned} \tag{5.13}$$

Now we compute the terms I_1 and I_2. For I_1, we have

$$\begin{aligned} 9\|\delta\tilde{u}_h^{k+1}\|_0^2 + I_1 &= 3\|\delta\tilde{u}_h^{k+1}\|_0^2 + 3\|2\delta\tilde{u}_h^{k+1} - \delta\tilde{u}_h^k\|_0^2 + 3\|\delta^3\tilde{u}_h^{k+1}\|_0^2 \\ &\quad - 3\|\delta\tilde{u}_h^k\|_0^2 - 3\|2\delta\tilde{u}_h^k - \delta\tilde{u}_h^{k-1}\|_0^2. \end{aligned} \tag{5.14}$$

The term I_2 can be simplified as follows:

$$I_2 = 4\delta t \left[(3\delta\tilde{u}_h^{k+1}, D_h^t \tilde{u}_h^{k+1}) + (3\delta\tilde{u}_h^{k+1} - 4\delta\tilde{u}_h^k + \delta\tilde{u}_h^{k-1}, D_h^t \delta\tilde{u}_h^{k+1}) \right].$$

Then, using the identity $2(3a - 4b + c, a - b) = 5(a-b)^2 + (a - 2b + c)^2 - (b-c)^2$ together with the definition of D_h^t, we infer

$$\begin{aligned} \frac{1}{\delta t} I_2 &= 12\|\nabla\delta\tilde{u}_h^{k+1}\|_0^2 + 12(\delta\tilde{u}_h^{k+1}, B_h^T B_h \tilde{u}_h^k) \\ &\quad + 4(3\delta\tilde{u}_h^{k+1} - 4\delta\tilde{u}_h^k + \delta\tilde{u}_h^{k-1}, A_h \delta^2\tilde{u}_h^{k+1} + B_h^T B_h \delta\tilde{u}_h^k) \\ &= 12\|\nabla\delta\tilde{u}_h^{k+1}\|_0^2 + 10\|\nabla\delta^2\tilde{u}_h^{k+1}\|_0^2 + 2\|\nabla\delta^3\tilde{u}_h^{k+1}\|_0^2 - 2\|\nabla\delta^2\tilde{u}_h^k\|_0^2 \\ &\quad + 6\|B_h \tilde{u}_h^{k+1}\|_0^2 - 6\|B_h \tilde{u}_h^k\|_0^2 - 6\|B_h \delta^2\tilde{u}_h^{k+1}\|_0^2 \\ &\quad - 8\|B_h \delta\tilde{u}_h^k\|_0^2 - 2\|B_h \delta^2\tilde{u}_h^k\|_0^2 + 2\|B_h \delta\tilde{u}_h^{k-1}\|_0^2. \end{aligned}$$

Thanks to the fact that $\|B_h v_h\|_0 \le \|\nabla v_h\|_0$ for all $v_h \in X_h$, we deduce

$$\begin{aligned}\frac{1}{\delta t} I_2 \ge{}& 4\|\nabla\delta\tilde{u}_h^{k+1}\|_0^2 + 8(\|\nabla\delta\tilde{u}_h^{k+1}\|_0^2 - \|\nabla\delta\tilde{u}_h^{k}\|_0^2)\\ &+ 4(\|\nabla\delta^2\tilde{u}_h^{k+1}\|_0^2 - \|\nabla\delta^2\tilde{u}_h^{k}\|_0^2) + 6(\|B_h\tilde{u}_h^{k+1}\|_0^2 - \|B_h\tilde{u}_h^{k}\|_0^2).\end{aligned} \tag{5.15}$$

Combining (5.13) with (5.14) and (5.15), we finally obtain

$$\begin{aligned}3\|\delta\tilde{u}_h^{k+1}\|_0^2 &+ 6\|B_h\tilde{u}_h^{k+1}\|_0^2 + 3\|2\delta\tilde{u}_h^{k+1} - \delta\tilde{u}_h^{k}\|_0^2 + 8\|\nabla\delta\tilde{u}_h^{k+1}\|_0^2 + 4\|\nabla\delta^2\tilde{u}_h^{k+1}\|_0^2\\ &+ 4\delta t^2\|(C_h^T - i_h B_h^T)\phi_h^{k+1}\|_0^2 + 4\delta t^2\|D_h^t\tilde{u}_h^{k+1}\|_0^2\\ &+ 3\|\delta^2\tilde{u}_h^{k+1}\|_0^2 + 4\|\nabla\delta\tilde{u}_h^{k+1}\|_0^2 + 4\delta t^2\|D_h^t\delta\tilde{u}_h^{k+1}\|_0^2\\ &\le 3\|\delta\tilde{u}_h^{k}\|_0^2 + 6\|B_h\tilde{u}_h^{k}\|_0^2) + 3\|2\delta\tilde{u}_h^{k} - \delta\tilde{u}_h^{k-1}\|_0^2 + 8\|\nabla\delta\tilde{u}_h^{k}\|_0^2 + 4\|\nabla\delta^2\tilde{u}_h^{k}\|_0^2\\ &+ 4\delta t^2\|(C_h^T - i_h B_h^T)\delta\phi_h^{k}\|_0^2 + 4\delta t^2\|D_h^t\tilde{u}_h^{k}\|_0^2.\end{aligned}$$

The conclusion then follows readily by using the discrete Gronwall lemma. □

Theorem 5.1. *Under appropriate regularity assumptions and initialization hypotheses, the solution to* (5.1)-(5.2) *satisfies*

$$\|u - \tilde{u}_h\|_{\ell^2(L^2(\Omega)^d)} + \|u - i_h^T u_h\|_{\ell^2(L^2(\Omega)^d)} \lesssim \delta t^2 + h^{l+1}, \tag{5.16}$$

$$\|u - \tilde{u}_h\|_{\ell^2(H^1(\Omega)^d)} + \|p - p_h\|_{\ell^2(L^2(\Omega))} \lesssim \delta t^{3/2} + h^l. \tag{5.17}$$

Proof. Proceed as in the proof of Lemma 5.1 and use the right inverse of the discrete stokes operator as in [9, 12, 24]. □

Remark 5.1. Note that the estimate (5.17) is $\frac{1}{2}$-order suboptimal with respect to δt. This phenomenon is also observed for the rotational form of the pressure-correction method. It has been analyzed for the pressure-correction method in [12]. The lack of optimality is related to the smoothness of the boundary. Actually, if the domain is a two-dimensional channel with one periodic direction, it has been shown in [2], using the normal mode analysis, that the rotational pressure-correction method is fully second-order. In the general case, if the boundary of the domain is smooth, say of class $\mathcal{C}^1$, numerical evidences reported in [12] show that the method is also fully second-order. But, if the boundary of the domain is only piecewise $\mathcal{C}^1$, say Ω is a convex rectangle, then the $\delta t^{\frac{1}{2}}$ suboptimality manifests itself in the convergence tests. This tends to confirm that our analysis is sharp under the assumption the domain is such that H^2 regularity holds for the steady Stokes problem supplemented with homogeneous Dirichlet boundary conditions and L^2 right-hand sides. Whether rotational pressure-correction and rotational velocity-correction methods can be modified to yield provable full second-order in any circumstance is still, to our best knowledge, an open problem; see [13] for additional details.

Remark 5.2. Here again we only derived the ℓ^2-in-time estimates. It is possible to obtain ℓ^∞-in-time estimates by asssuming more regularity. We refer for instance to [9, Theorem 4.2] where this type of argument is developed for the standard version the pressure-correction method.

6. Concluding remarks

The results in Theorems 4.1 and 5.1 show that the first-order and second-order rotational velocity-correction yield optimal error estimates in space for both the velocity and the pressure, provided that the inf-sup condition is satisfied. The time estimates are optimal for the velocity in the L^2-norm for both schemes. These estimates are also optimal in the H^1-norm for the first-order time stepping but are suboptimal by a $\delta t^{\frac{1}{2}}$ factor for the BDF2 time stepping. All these results are consistent with the numerical results presented in [12].

The present analysis holds for all types of approximations provided the assumptions (2.8) and (2.17)-(2.18) are satisfied. In particular, these conditions are satisfied by most finite element settings for spectral approximations though the story is slightly different. Although there are at least two pairs of spectral approximation spaces that satisfy the inf-sup condition (2.8) uniformly with respect to the polynomial degree N (cf. [1]), the most popular pair $P_N \times P_{N-2}$ only satisfies a *weaker* inf-sup condition,

$$\inf_{q_h\in M_h}\sup_{v_h\in X_h}\frac{(\nabla\cdot v_h, q_h)}{\|\nabla v_h\|_0}\geq c_h\|q_h\|_0, \tag{6.1}$$

where $c_h := \beta_N = N^{\frac{1-d}{2}} \to 0$ as $N\to\infty$ ($d = 2$ or 3 is the dimension; see, for instance, [1]). Although this does not affect the derivation of δt-estimates, it does introduce difficulties for proving δt^2-estimates on the velocity for the second-order schemes, since the constant c_h comes into play when we apply the right-inverse of the discrete Stokes operator, leading to an estimate of the form

$$\|u-\tilde{u}_h\|_{\ell^2(L^2(\Omega)^d)} \lesssim c_h^{-1}(\delta t^2 + h^{l+1}). \tag{6.2}$$

Numerical tests reported in [11, 12] indicate that the term c_h^{-1} does not affects the accuracy on the velocity and should not be present in (6.2). How to remove the term c_h^{-1} in the above error estimate for spectral approximations is still an open problem.

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