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A Fictitious Domain Spectral Method for Solving the Helmholtz Equation in Exterior Domains

Yiqi Gu[1] · **Jie Shen**[2,3]

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Abstract
We extend the fictitious domain spectral method presented in Gu and Shen (SIAM J Sci Comput 43:A309–A329, 2021) for elliptic PDEs in bounded domains to the Helmhotlz equation in exterior domains. We first reduce the problem in an exterior domain to a bounded domain using the exact Dirichlet-to-Neumann operator. Next, we formulate the reduced problem into an equivalent problem in an annulus by using a fictitious domain approach. Then, we apply the Fourier-spectral method in the radial direction to reduce the problem in an annulus to a sequence of 1-D Bessel-type equations, each with a one-sided open boundary condition that are to be determined by the boundary condition of the original Helmholtz equation. We solve these 1-D Bessel-type equations by the Legendre-spectral method, and determine the open boundary conditions with a least square approach. We derive a wave number explicit error estimate for the special case of a circular obstacle, and provide ample numerical results to show the effectiveness of the proposed method.

Keywords Acoustic scattering · Helmholtz equation · Petrov–Galerkin method · Fictitious domain · Error estimate

Mathematics Subject Classification 65N15 · 65N35 · 65N85

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✉ Jie Shen
shen7@purdue.edu

Yiqi Gu
yiqigu@hku.hk

1 Department of Mathematics, The University of Hong Kong, Pok Fu Lam, Hong Kong

2 Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA

3 Eastern Institute for Advanced Study, Eastern Institute of Technology, Ningbo 315200, Zhejiang, People's Republic of China

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1 Introduction

Time harmonic wave propagations appear in many applications such as wave scattering and transmission, noise reduction, fluid-solid interaction, etc. How to efficiently solve the Helmholtz and Maxwell equations arising from acoustic and electromagnetic scattering problems has been a focus of research in the last few decades, see [2–4, 8, 9, 14, 19, 20, 23–25, 28] and the references therein.

We consider in this paper the acoustic scattering problem governed by the Helmholtz equation:

$$\begin{aligned} & -\Delta u_{\mathrm{H}} - k^2 u_{\mathrm{H}} = f \quad \text{in } \Omega = \mathbb{R}^2 \backslash \Omega_1, \\ & u_{\mathrm{H}} = g \quad \text{on } \partial\Omega_1, \\ & \lim_{r\to\infty} r^{1/2}(\partial_r u_{\mathrm{H}} - \mathrm{i}k u_{\mathrm{H}}) = 0, \end{aligned} \tag{1.1}$$

where $k > 0$ is the wave number, Ω_1 is a simply connected bounded domain in $\mathbb{R}^2$, and $r = \sqrt{x^2 + y^2}$. The difficulty caused by the unboundedness of the domain is usually dealt with by domain truncation. Specifically, one first introduces a large domain D_1 of simple geometry that encloses Ω_1 and redefine the problem in the bounded domain $\Omega := D_1 \cap (\mathbb{R}^2 \backslash \Omega_1)$ with the Dirichlet-to-Neumann (DtN) boundary condition on ∂D_1. For convenience, D_1 can be chosen as a simple geometry, such as an open disk in 2-D case or a ball in 3-D case.

In the context of spectral methods, if the obstacle is star-shaped, the solution can be approximated by a transformed field expansion approach [12, 26, 27], which requires solving a sequence of Helmholtz equations in the transformed regular domain. However, the applicability and effectiveness of this approach depend heavily on the shape and regularity of the obstacles. In this work, we adopt an entirely different fictitious domain approach proposed in [16] for elliptic PDEs in two-dimensional complex geometries. Since the forcing function f can be smoothly extended into the obstacle [5, 6], instead of transforming the complex domain into a regular domain, we enclose the original domain Ω with an outside circle and an inside circle, and formulate an extended problem in the annulus, see Fig. 1. At the outside circle, we still use the exact DtN boundary condition, but we determine the boundary condition at the inside circle to enforce the boundary condition at the obstacle. Under the polar coordinates, taking the Fourier expansion of the solution leads to a sequence of 1-D Bessel-type equations with undetermined boundary conditions at the left end. The key step in our agorithm is to determine these boundary conditions such that the boundary condition at the obtacle is satisfied. Once the boundary conditions at the left end are determined, the sequence of 1-D Bessel-type equations can be efficiently solved by a standard spectral-Galerkin method. The algorithm described above is relatively easy to implement compared with the algorithms based on the transformed field expansion approach [26].

Analysis for the proposed method is highly nontrivial. Unlike usual Galerkin approximations of the Helmholtz equations, our method leads to an unusual Petrov–Galerkin formulation in which the “boundary" condition of the trial space is given at the original boundary of the obstacle, which is not part of the boundary of the computational domain. Thus, the well-posedness of this formulation can not be easily established with a usual procedure. In this paper, we focus on the analysis for the special case when the obstacle is a disk. We develop the a priori estimates for the extended Helmholtz problem, and further establish the error estimates with explicit dependence on the wave number for our method, similar to previous results established for different type of methods in [7, 11, 13, 18, 21, 29]. In particular, the error estimates show that our method is not plaqued by the polution effect suffered by low-order methods [1] and it can converge exponentially as soon as the discretization is fine enough when the solution is smooth.

The rest of the paper is organized as follows. In Sect. 2, we describe the fictitious domain method with an annular embedding and formulate the extended problem. In Sect. 3, we describe the dimension-reduction process and introduce a practical algorithm using spectral-Galerkin method for the derived 1-D equations. In Sect. 4, we investigate the a priori estimates of the proposed method for a special case. In Sect. 5, we perform rigorous error analysis following the a priori estimates. In Sect. 6, we present several numerical examples to demonstrate the effectiveness of this method. Some concluding remarks are given in the last section.

2 Problem Formulation

We first introduce some notations, followed by a description the acoustic scattering problem. Then, we formulate an equivalent extended problem and describe a Petrov–Galerkin formulation.

2.1 Notations

We present below some of the notations to be used throughout this paper. Denote by $w^k(r) = r^k$ the weight functions associated with the polar transform, and define the L^2 weighted inner product,

$$(\hat{u}, \hat{v})_{w^k,I'} = \int_{I'} \hat{u}(r)\overline{\hat{v}(r)}r^k \mathrm{d}r, \tag{2.1}$$

for any integer k and any interval $I' \subset [0, +\infty)$. Also, define the L^2 and H^1 weighted spaces

$$L^2_{w^k}(I') = \{\hat{u}(r) : \|\hat{u}\|_{w^k,I'} := \sqrt{(\hat{u}, \hat{u})_{w^k,I'}} < \infty\}, \tag{2.2}$$

$$H^1_{w^k}(I') = \{\hat{u}(r) : \|\hat{u}\|_{1,w^k,I'} := \sqrt{(\hat{u}, \hat{u})_{w^k,I'} + (\partial_r\hat{u}, \partial_r\hat{u})_{w^k,I'}} < \infty\}. \tag{2.3}$$

Similarly, for any domain $\Omega' \subset [0, +\infty) \times [0, 2\pi)$, define the L^2 weighted inner product,

$$(u, v)_{w^k,\Omega'} = \int_{\Omega'} u\overline{v}r^k \mathrm{d}r\mathrm{d}\theta, \tag{2.4}$$

and the L^2 weighted space

$$L^2_{w^k}(\Omega') = \{u(r, \theta) : \|u\|_{w^k,\Omega'} := \sqrt{(u, u)_{w^k,\Omega'}} < \infty\}. \tag{2.5}$$

We will omit the subscript or superscript when $k = 0$, and these spaces reduce to the usual non-weighted spaces.

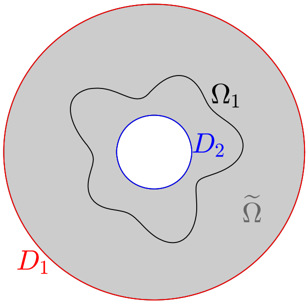
We also denote the line integral over any closed curve $\Gamma \in \mathbb{R}^2$ by

$$(u, v)_\Gamma := \int_\Gamma u\overline{v}\mathrm{d}s. \tag{2.6}$$

2.2 The Extended Problem

As in [26], we first reformulate (1.1) into a problem in a finite domain by the DtN mapping. Indeed, by a classical argument of separation of variables, the solution of (1.1) with $f = 0$

Fig. 1 An illustrative figure for the domain enclosing



for $r \geq r_0$ (b is sufficiently large that Ω_1 is enclosed by $D_1 := \{(r,\theta) : r < b\}$) can be expressed as

$$u_{\mathrm{H}}(r,\theta) = \sum_{m=-\infty}^{\infty} a_m H_m^{(1)}(kr)e^{\mathrm{i}m\theta}, \tag{2.7}$$

where $H_m^{(1)}(kr)$ is the m-th order Hankel function of the first kind. If $u_{\mathrm{H}}(b,\theta)$ is known and

$$\Psi(\theta) := u_{\mathrm{H}}(b,\theta) = \sum_{m=-\infty}^{\infty} \hat{\Psi}_m e^{\mathrm{i}m\theta}, \tag{2.8}$$

then the DtN map T is given by

$$T(u_{\mathrm{H}}) = -\sum_{m=-\infty}^{\infty} k \frac{\partial_z H_m^{(1)}(kr_0)}{H_m^{(1)}(kr_0)} \hat{\Psi}_m e^{\mathrm{i}m\theta}. \tag{2.9}$$

Hence, (1.1) with $\mathrm{Supp}\{f\} \subset D_1$ is equivalent to

$$\begin{aligned} -\Delta u_{\mathrm{H}} - k^2 u_{\mathrm{H}} &= f \quad \text{in } \Omega, \\ u_{\mathrm{H}} &= g \quad \text{on } \partial\Omega_1, \\ \frac{\partial u_{\mathrm{H}}}{\partial r} + T(u_{\mathrm{H}}) &= 0 \quad \text{on } \partial D_1, \end{aligned} \tag{2.10}$$

where $\Omega = D_1 \cap (\mathbb{R}^2 \backslash \Omega_1)$ is the problem domain.

One approach to deal with the possible complex geometry of Ω is the fictitious domain method, which encloses Ω with a standard domain and solve the problem in the new standard domain. For simplicity, we assume $(x, y) = (0, 0) \in \Omega_1$ and let $D_2 := \{(r,\theta) : r < a\}$ be a small disk such that $D_2 \subset \Omega_1$, and denote $\widetilde{\Omega} := \{(r,\theta) : a < r < b\}$ which is an annulus (see Fig. 1). Assume f is smoothly extended from Ω to $\widetilde{\Omega}$ (the extension is still denoted as f) [5, 6], then instead of solving (2.10), we solve the following extended problem:

$$\begin{aligned} -\Delta u - k^2 u &= f \quad \text{in } \widetilde{\Omega}, \\ u &= g \quad \text{on } \partial\Omega_1, \\ \frac{\partial u}{\partial r} + T(u) &= 0 \quad \text{on } \partial D_1. \end{aligned} \tag{2.11}$$

It is clear that $u_{\mathrm{H}} = u|_{\Omega}$, the restriction of solution to the above extended problem to Ω, is the solution of (2.10).

In order to study the well-posedness of the above problem, we define the trial and test spaces as

$$X = \left\{ u = \sum_{m=-\infty}^{\infty} \hat{u}^m(r)e^{\mathrm{i}m\theta} \,:\, \hat{u}^m \in H_w^1(I) \cap L_{w^{-1}}^2(I),\; u|_{\partial\Omega_1} = g \right\}, \tag{2.12}$$

and

$$Y = \left\{ v = \sum_{m=-\infty}^{\infty} \hat{v}^m(r)e^{\mathrm{i}m\theta} \,:\, \hat{v}^m \in H_w^1(I) \cap L_{w^{-1}}^2(I),\; v(a,\theta) = 0\ \forall\theta \right\}. \tag{2.13}$$

We also define the sesquilinear form $B : X \times Y \to \mathbb{C}$ by

$$B(u,v) := (\partial_r u, \partial_r v)_{w,\widetilde{\Omega}} + (\partial_\theta u, \partial_\theta v)_{w^{-1},\widetilde{\Omega}} - k^2(u,v)_{w^{-1},\widetilde{\Omega}} + b(T(u),v)_{\partial D_1}. \tag{2.14}$$

Then a Petrov–Galerkin formulation of (2.11) is given by:

$$\begin{cases} \text{given } f \in L_w^2(\widetilde{\Omega}),\ \text{find } u \in X \text{ such that} \\ B(u,v) = (f,v)_{w,\widetilde{\Omega}}, \quad \forall v \in Y, \end{cases} \tag{2.15}$$

where the weight function $w(r) = r$.

3 Numerical Algorithms

We first present a conceptual algorithm without spatial discretization for the extended problem based on the DtN mapping and Fourier expansion in the azimuthal direction, followed by a practical algorithm using the spectral-Galerkin method.

3.1 An Algorithm Without Spatial Discretization

Similarly as in our previous work [16], we apply dimension reduction on the problem (2.11) by expanding the equation with Fourier series as follows:

$$f = \sum_{m=-\infty}^{\infty} \hat{f}^m(r)e^{\mathrm{i}m\theta} \tag{3.1}$$

and

$$u = \sum_{m=-\infty}^{\infty} \hat{u}^m(r)e^{\mathrm{i}m\theta}. \tag{3.2}$$

Plugging the above expressions in (2.11), we derive a sequence of the 1-D Bessel-type equations for $\hat{u}^m$:

$$\begin{aligned} -\frac{1}{r}\partial_r(r\partial_r\hat{u}^m) + \frac{m^2}{r^2}\hat{u}^m - k^2\hat{u}^m = \hat{f}^m, \quad r \in I := (a,b), \\ \partial_r\hat{u}^m(b) - kD_{m,k}\hat{u}^m(b) = 0, \end{aligned} \tag{3.3}$$

where $D_{m,k} := \frac{\partial_z H_m^{(1)}(kb)}{H_m^{(1)}(kb)}$.

Note that the equation (3.3) is underdetermined since no boundary condition is specified at $r = a$. To ensure the well-posedness of (3.3), we add an artificial boundary condition $\hat{u}^m(a) = t^m$ where t^m is to be determined.

We first figure out the relation between $\hat{u}^m$ and t^m, which is given by the following result.

Lemma 3.1 *Let ϕ^m be the solution to*

$$\begin{aligned} &-\frac{1}{r}\partial_r(r\partial_r\phi^m)+\frac{m^2}{r^2}\phi^m-k^2\phi^m=0, \quad r\in I,\\ &\phi^m(a)=1, \quad \partial_r\phi^m(b)-kD_{m,k}\phi^m(b)=0, \end{aligned} \tag{3.4}$$

and ψ^m be the solution to

$$\begin{aligned} &-\frac{1}{r}\partial_r(r\partial_r\psi^m)+\frac{m^2}{r^2}\psi^m-k^2\psi^m=\hat{f}^m, \quad r\in I,\\ &\psi^m(a)=0, \quad \partial_r\psi^m(b)-kD_{m,k}\psi^m(b)=0. \end{aligned} \tag{3.5}$$

Then the solution to (3.3) *is given by*

$$\hat{u}^m=t^m\phi^m+\psi^m. \tag{3.6}$$

Lemma 3.1 can be trivially verified.

Therefore, the solution to (2.11) is given by

$$u(r,\theta)=\sum_{m=-\infty}^{\infty}(t^m\phi^m(r)+\psi^m(r))e^{\mathrm{i}m\theta}, \tag{3.7}$$

where $\{t_m\}$ can be determined by the boundary condition $u=g$ on $\partial\Omega_1$, namely

$$\sum_{m=-\infty}^{\infty}(t^m\phi^m(r)+\psi^m(r))e^{\mathrm{i}m\theta}=g \quad \text{on } \partial\Omega_1. \tag{3.8}$$

Proposition 3.1 *Let u, given by* (3.7)*, be the solution of* (2.11)*. Then u is also a solution of* (2.15).

Proof Let u, given by (3.7), be the solution of (2.11). It is clear that $\hat{u}^m=t^m\phi^m+\psi^m$ is in $H^1_w(I)\cap L^2_{w^{-1}}(I)$. Also, (3.8) implies $u=g$ on $\partial\Omega_1$, so $u\in X$. For any $v=\sum_{k=-\infty}^{\infty}\hat{v}^k(r)e^{\mathrm{i}k\theta}\in Y$, multiplying $\hat{v}^k$ on both sides of (3.3) and using integration by parts lead to

$$(\partial_r\hat{u}^m,\partial_r\hat{v}^k)_{w,I}+m^2(\hat{u}^m,\hat{v}^k)_{w^{-1},I}-k^2(\hat{u}^m,\hat{v}^k)_{w,I}-kbD_{m,k}\hat{u}^m(b)\overline{\hat{v}^k(b)}=(\hat{f}^m,\hat{v}^k)_{w,I}.$$

Multiplying the above by $e^{\mathrm{i}m\theta}e^{-\mathrm{i}k\theta}$, integrating over θ, and summing up the results for all m leads to $B(u,\hat{v}^ke^{\mathrm{i}k\theta})=(f,\hat{v}^ke^{\mathrm{i}k\theta})_{w,\widetilde{\Omega}}$ for all k. Hence, u is also a solution of (2.15). □

3.2 A Practical Algorithm

We develop below a practical algorithm for finding an approximation to $\{t_m\}$ through (3.8), which leads to an approximation to u given in (3.7).

We first solve the 1-D equations (3.4) and (3.5) by the spectral-Galerkin method. Specifically, define the space $\hat{W}_\sigma=\{u\in H^1_w(I)\cap L^2_{w^{-1}}(I): u(a)=\sigma\}$ for any $\sigma\in\mathbb{C}$, and define the sesquilinear form $\hat{B}_m:\left(H^1_w(I)\cap L^2_{w^{-1}}(I)\right)\times\left(H^1_w(I)\cap L^2_{w^{-1}}(I)\right)\to\mathbb{C}$ by

$$\hat{B}_m(\hat{u},\hat{v}):=(\partial_r\hat{u},\partial_r\hat{v})_{w,I}+m^2(\hat{u},\hat{v})_{w^{-1},I}-k^2(\hat{u},\hat{v})_{w,I}-kbD_{m,k}\hat{u}(b)\overline{\hat{v}(b)}. \tag{3.9}$$

Then multiplying $\hat{v} \in \hat{W}_0$ to both sides of (3.4) and using integration by parts lead to the following variational formulation of (3.4):

$$\begin{cases} \text{find } \phi^m \in \hat{W}_1 \text{ such that} \\ \hat{B}_m(\phi^m, \hat{v}) = 0, \quad \forall \hat{v} \in \hat{W}_0. \end{cases} \tag{3.10}$$

Similarly, the variational formulation of (3.5) is given by

$$\begin{cases} \text{find } \psi^m \in \hat{W}_0 \text{ such that} \\ \hat{B}_m(\psi^m, \hat{v}) = (\hat{f}^m, \hat{v})_{w,I}, \quad \forall \hat{v} \in \hat{W}_0. \end{cases} \tag{3.11}$$

Let P_N be the space of all complex polynomials of degree at most N on I. Denote $\hat{W}_{\sigma,N} = \hat{W}_\sigma \cap P_N$, then the spectral-Galerkin methods for (3.10) and (3.11) are given by

$$\begin{cases} \text{find } \phi_N^m \in \hat{W}_{1,N} \text{ such that} \\ \hat{B}_m(\phi_N^m, \hat{v}_N) = 0, \quad \forall \hat{v}_N \in \hat{W}_{0,N} \end{cases} \tag{3.12}$$

and

$$\begin{cases} \text{find } \psi_N^m \in \hat{W}_{0,N} \text{ such that} \\ \hat{B}_m(\psi_N^m, \hat{v}_N) = (\hat{f}^m, \hat{v}_N)_{w,I}, \quad \forall \hat{v}_N \in \hat{W}_{0,N}. \end{cases} \tag{3.13}$$

By Lemma 3.1, after obtaining ϕ_N^m and ψ_N^m, the approximation to $\hat{u}^m$ can be constructed as

$$\hat{u}_N^m = t^m \phi_N^m + \psi_N^m. \tag{3.14}$$

Finally, we take the truncated expansion

$$u_{MN} := \sum_{m=-M}^{M} \hat{u}_N^m(r) e^{\mathrm{i}m\theta} = \sum_{m=-M}^{M} \left(t^m \phi_N^m + \psi_N^m\right) e^{\mathrm{i}m\theta} \tag{3.15}$$

as the approximate solution of the extended problem (2.11). Here the critical step is to determine $\{t^m\}_{m=-M}^{M}$ subject to the interior boundary condition $u = g$ on $\partial\Omega_1$. One straightforward way is to prescribe J collocation nodes $\{(\hat{r}_j, \hat{\theta}_j)\}_{j=1}^{K}$ on $\partial\Omega_1$, and enforce

$$u_{MN}(\hat{r}_j, \hat{\theta}_j) \approx g(\hat{\theta}_j), \quad j = 1, \cdots, J, \tag{3.16}$$

or

$$\sum_{m=-M}^{M} \left(t^m \phi_N^m(\hat{r}_j) + \psi_N^m(\hat{r}_j)\right) e^{\mathrm{i}m\hat{\theta}_j} \approx g(\hat{\theta}_j), \quad j = 1, \cdots, J. \tag{3.17}$$

For $J > M$, (3.17) corresponds to the following least square problem:

$$\min_{t^m} \sum_{j=1}^{J} \left| \sum_{m=-M}^{M} \left(t^m \phi_N^m(\hat{r}_j) + \psi_N^m(\hat{r}_j)\right) e^{\mathrm{i}m\hat{\theta}_j} - g(\hat{\theta}_j) \right|^2, \tag{3.18}$$

In practical implementation, we empirically choose J between $4M$ and $8M$ which has the best conditioning-efficiency balance on (3.18). Another discretization is taking the projection of the residue onto the finite-dimensional Fourier subspace and setting it zero. Specifically, suppose Ω_1 is characterized by the curve $r = \rho(\theta)$, then we enforce

$$\left(u_{MN}(\rho(\theta), \theta) - g(\theta), e^{\mathrm{i}l\theta}\right)_{[0,2\pi)} = 0, \tag{3.19}$$

or

$$\left(\sum_{m=-M}^{M} \left(t^m \phi_N^m(\rho(\theta)) + \psi_N^m(\rho(\theta)) \right) e^{\mathrm{i}m\theta} - g(\theta), e^{\mathrm{i}l\theta} \right)_{[0,2\pi)} = 0, \tag{3.20}$$

for $l = -M, \cdots, M$, where $(f, g)_{[0,2\pi)} := \int_0^{2\pi} f \bar{g} \, \mathrm{d}\theta$.

To sum up, the following algorithm results from the above discussion.

Algorithm 3.1 *Given M (number of nodes in θ direction) and N (number of nodes in the r direction), we find an approximate solution u_{MN} of the extended problem (2.11) as follows.*

Step 1. *Perform the domain embedding $\Omega \subset \widetilde{\Omega}$ and extend f from Ω to $\widetilde{\Omega}$ smoothly;*
Step 2. *Compute the truncated Fourier expansion of $f(r, \theta)$ with respect to θ, obtaining an approximation to (3.1);*
Step 3. *Solve (3.4) and (3.5) using the spectral-Galerkin formulation (3.12) and (3.13) with degree of freedom N, obtaining approximate solutions $\{\phi_N^m\}$ and $\{\psi_N^m\}$, respectively;*
Step 4. *Determine $\{t^m\}$ through the least square problem (3.18) or the linear system (3.20);*
Step 5. *Compute u_{MN} by (3.15).*

We remark that in Step 4 the least square problem (3.18) with unknowns $\{t^m\}$ is an overdetermined linear system with a $J \times (2M + 1)$ dense matrix. One can solve (3.18) by standard algorithms such as QR factorization or SVD [30]. If the size M and J are moderately large, the collocation nodes $\{(\hat{r}_j, \hat{\theta}_j)\}$ will be densely distributed on $\partial\Omega_1$ such that any adjacent rows of the matrix will be nearly parallel. Consequently, the problem (3.18) will be close to rank-deficient. In this case, the truncated SVD solution is recommended [15]. More precisely, denoting the problem as $\min_{\boldsymbol{t}} \|\boldsymbol{A}\boldsymbol{t} - \boldsymbol{b}\|_2$ and choosing a truncation number $\kappa > 0$, then the approximate solution is given by $\boldsymbol{t}_\kappa = \sum_{i=1}^{\kappa} (\boldsymbol{u}_i^\top \boldsymbol{b}/\sigma_i)\boldsymbol{v}_i$, where $\boldsymbol{U}^\top \boldsymbol{A}\boldsymbol{V} = \boldsymbol{\Sigma}$ is the SVD of $\boldsymbol{A}$ with column partitions $\boldsymbol{U} = [\boldsymbol{u}_1 \cdots \boldsymbol{u}_J]$ and $\boldsymbol{V} = \left[\boldsymbol{v}_1 \cdots \boldsymbol{v}_{2M+1}\right]$, and $\sigma_1, \cdots, \sigma_\kappa$ are the κ largest singular values. This approach can also be applied to the linear system (3.20), although it can be solved by other standard linear solvers. Usually, the linear system (3.20) is better-conditioned than (3.18) if Ω_1 is closer to a circle, namely, if $\rho(\theta) \approx \rho$ which is a constant. However, if the domain geometry is more complex, (3.18) may become better-conditioned than (3.20).

A similar argument presented in [16] implies that the complexity of Algorithm 3.1 is $O(M^3)$ with a small constant included in $O(\cdot)$, provided that $N = O(M)$. Hence the complexity is essentially the same order as the spectral-Galerkin method for the Helmholtz equation with a disk obstacle.

In order to establish the well-posedness and error estimates for the above algorithm, we formulate it as a Petrov–Galerkin method below. We set

$$\begin{aligned} X_{MN} &= \left\{ u_{MN} = \sum_{m=-M}^{M} \hat{u}_N^m(r) e^{\mathrm{i}m\theta} : \hat{u}_N^m \in P_N, \left(u_{MN}(\rho(\theta), \theta) - g(\theta), e^{\mathrm{i}m\theta} \right)_{[0,2\pi)} = 0, |m| \le M \right\}, \\ Y_{MN} &= \left\{ v_{MN} = \sum_{m=-M}^{M} \hat{v}_N^m(r) e^{\mathrm{i}m\theta} : \hat{v}_N^m \in P_N, \left(v_{MN}(a, \theta), e^{\mathrm{i}m\theta} \right)_{[0,2\pi)} = 0, |m| \le M \right\}. \end{aligned} \tag{3.21}$$

Proposition 3.2 *Let u_{MN} be given by (3.15) with $\{t_m\}$ determined from (3.20), then u_{MN} is a solution of the following Petrov–Galerkin method: find $u_{MN} \in X_{MN}$ such that*

$$B(u_{MN}, v_{MN}) = (f, v_{MN})_{w,\widetilde{\Omega}}, \quad \forall v_{MN} \in Y_{MN}. \tag{3.22}$$

Proof Let u_{MN} be given by (3.15) with $\{t_m\}$ determined from (3.20). We have $\hat{u}_N^m \in P_N$, and (3.20) implies that $u_{MN} \in X_{MN}$. On the other hand, for any $v_{MN} = \sum_{k=-M}^{M} \hat{v}_N^k(r)e^{\mathrm{i}k\theta} \in Y_{MN}$ with $\hat{v}_N^k \in \hat{W}_{0,N}$, we derive from (3.12), (3.13) and (3.14) that

$$\hat{B}_m(\hat{u}_N^m, \hat{v}_N^k) = (\hat{f}^m, \hat{v}_N^k)_{w,I}, \quad \forall |k| \le M. \tag{3.23}$$

Multiplying the above by $e^{\mathrm{i}m\theta}e^{-\mathrm{i}k\theta}$, integrating over θ, and summing up the results for all $|m| \le M$ leads to $B(u_{MN}, \hat{v}^k e^{\mathrm{i}k\theta}) = (f, \hat{v}^k e^{\mathrm{i}k\theta})_{w,\widetilde{\Omega}}$ for all $|k| \le M$. Hence, u_{MN} is also a solution of (3.22). □

Remark 3.1 If $\{t_m\}$ are determined through (3.18), we can also formulate it into a similar Petrov–Galerkin formulation. However, its error analysis is much more difficult and will not be considered in this paper.

4 A Priori Estimates

We only consider the special case that Ω_1 is a disk with radius ρ, and a, b are chosen such that $0 < a < \rho < b$. Besides, we assume the boundary value is homogeneous, i.e. $g = 0$. For nonhomogeneous boundary values on $\partial\Omega_1$, the original problem (2.11) can be converted to a new one with homogeneous boundary condition by subtracting an suitable lifting function from u. We assume that $k \ge k_0$ for some positive k_0 throughout the analysis.

Before carrying out an error analysis for the problem (2.11), we establish some *a priori* estimates for the solution of (2.11). These estimates are essential for the error analysis in the next section.

First, we define the 1-D trial space

$$\hat{X} = \{\hat{u} \in H_w^1(I) \cap L_{w^{-1}}^2(I) : \hat{u}(\rho) = 0\}, \tag{4.1}$$

and test space

$$\hat{Y} = \{\hat{v} \in H_w^1(I) \cap L_{w^{-1}}^2(I) : \hat{v}(a) = 0\}. \tag{4.2}$$

Let u be the solution of (2.11), and hence u is a solution of (2.15) (Proposition 3.1). Recall that we can write $u = \sum_{m=-\infty}^{\infty} \hat{u}^m(r)e^{\mathrm{i}m\theta}$ and $f = \sum_{m=-\infty}^{\infty} \hat{f}^m(r)e^{\mathrm{i}m\theta}$. Since $u = 0$ on $\partial\Omega_1$, $\hat{u}^m(\rho) = 0$ for all m, so $\hat{u}^m \in \hat{X}$. For any $\hat{v} \in \hat{Y}$, taking $v = \hat{v}e^{\mathrm{i}m\theta}$ in (2.15), we can derive that the coefficient $\hat{u}^m$ is a solution of the following 1-D variational problem: given $\hat{f}^m \in L_w^2(I)$, find $\hat{u} \in \hat{X}$ such that

$$\hat{B}_m(\hat{u}, \hat{v}) = (\hat{f}^m, \hat{v})_{w,I}, \quad \forall \hat{v} \in \hat{Y}, \tag{4.3}$$

where $\hat{B}_m$ is defined in (3.9).

Lemma 4.1 *For all $\hat{u} \in \hat{X}$, we have*

$$|\hat{u}(a)|^2 \le c_0(a, \rho; m) \left(\|\partial_r \hat{u}\|_{w,I}^2 + m^2 \|\hat{u}\|_{w^{-1},I}^2 \right), \tag{4.4}$$

where

$$c_0(a, \rho; m) := \begin{cases} |m|^{-1}(\rho^{2|m|} - a^{2|m|})(\rho^{2|m|} + a^{2|m|})^{-1}, & m \ne 0, \\ \ln\rho - \ln a, & m = 0. \end{cases} \tag{4.5}$$

Proof If $m \neq 0$, consider the minimization problem

$$\inf_{\hat{u}} \int_a^\rho |\partial_r \hat{u}|^2 r + m^2 |\hat{u}|^2 r^{-1} \mathrm{d}r \quad \text{s.t.} \quad \hat{u}(a) = \delta,\ \hat{u}(\rho) = 0. \tag{4.6}$$

The Euler-Lagrange equation of (4.6) is given by

$$-\partial_r (r \partial_r \hat{u}) + \frac{m^2}{r} \hat{u} = 0, \quad \hat{u}(a) = \delta,\ \hat{u}(\rho) = 0, \tag{4.7}$$

whose solution is $\hat{u}_0(r) = (\rho^m r^{-m} - \rho^{-m} r^m)(\rho^m a^{-m} - \rho^{-m} a^m)^{-1}\delta$. So for any function $\hat{u}$ satisfying $\hat{u}(a) = \delta$, $\hat{u}(\rho) = 0$, we have

$$\begin{aligned} \|\partial_r \hat{u}\|^2_{w,[a,\rho]} + m^2 \|\hat{u}\|^2_{w^{-1},[a,\rho]} &\geq \int_a^\rho |\partial_r \hat{u}_0|^2 r + m^2 |\hat{u}_0|^2 r^{-1} \mathrm{d}r \\ &= m(\rho^{2m} - a^{2m})^{-1}(\rho^{2m} + a^{2m})|\delta|^2 \\ &= |m|(\rho^{2|m|} - a^{2|m|})^{-1}(\rho^{2|m|} + a^{2|m|})|\delta|^2. \end{aligned} \tag{4.8}$$

If $m = 0$, by similar argument, the Euler-Lagrange equation has the solution $\hat{u}_0(r) = (\ln r - \ln \rho)(\ln a - \ln \rho)^{-1}\delta$. So for any function $\hat{u}$ satisfying $\hat{u}(a) = \delta$, $\hat{u}(\rho) = 0$, we have

$$\|\partial_r \hat{u}\|^2_{w,[a,\rho]} \geq \int_a^\rho |\partial_r \hat{u}_0|^2 r \mathrm{d}r = (\ln \rho - \ln a)^{-1} |\delta|^2. \tag{4.9}$$

Using $\hat{u}(a) = \delta$ in (4.8) and (4.9) leads to (4.4). □

Note that the constant c_0 satisfies $0 \leq c_0 < |m|^{-1}$ if $m \neq 0$, and $c_0 \to 0$ as $\rho \to a$. Before giving the a priori estimates, we revisit the 1-D equation (3.4), whose solution ϕ^m can be explicitly formulated as

$$\phi^m(r) = \frac{H^{(1)}_{|m|}(kr)}{H^{(1)}_{|m|}(ka)}. \tag{4.10}$$

We define two constants,

$$c_1(a, b; m, k) := \partial_r \phi^m(a) = k \frac{\partial_z H^{(1)}_{|m|}(ka)}{H^{(1)}_{|m|}(ka)}, \tag{4.11}$$

and

$$c_2(a, b; m, k) := \|\phi^m\|^2_{w,I} = \int_a^b \left| \frac{H^{(1)}_{|m|}(kr)}{H^{(1)}_{|m|}(ka)} \right|^2 r \mathrm{d}r. \tag{4.12}$$

By the property of Hankel functions, we can derive that $|c_1|$ has a linear growth with m (see Fig. 2), namely

$$|c_1| \leq c_{11}(a, b; k)|m| + c_{12}(a, b; k), \quad \forall m, \tag{4.13}$$

for some real constants $c_{11}, c_{12} > 0$, which only depend on a, b and k. We define

$$c_{13}(a, b; k) := \inf_{c_{11}, c_{12} \text{ satisfying } (4.13)} \{c_{11} + c_{12}\}. \tag{4.14}$$

Moreover, c_2 is a decreasing function of m (see Fig. 3), so we have

$$0 < c_2(a, b; m, k) \leq c_2(a, b; 0, k), \tag{4.15}$$

in which the bound is independent of m.

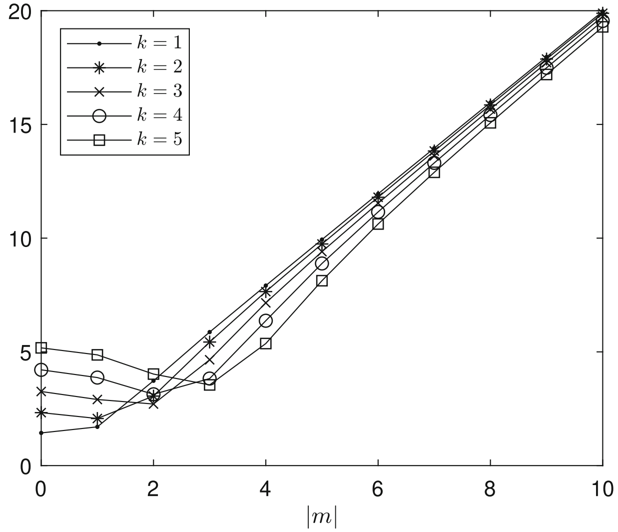


Fig. 2 $|c_1| := |\partial_r \phi^m(a)|$ v.s. $|m|$ for various k

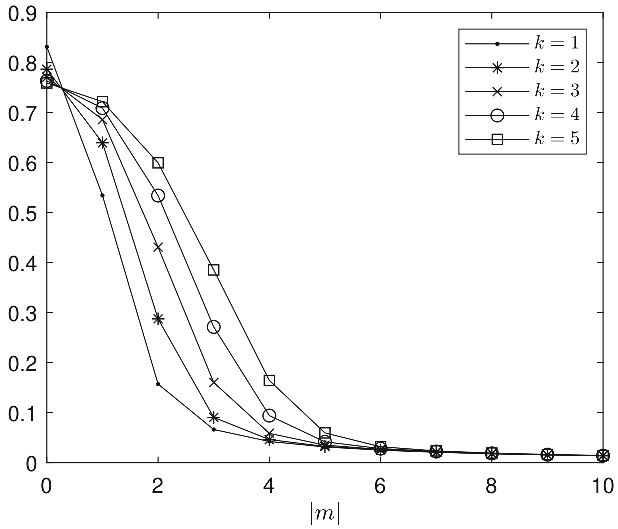


Fig. 3 $c_2 := \|\phi^m\|_{w,I}^2$ v.s. $|m|$ for various k

Let $\hat{u} \in \hat{X}$, by mean-value theorem, there exists some $\xi_{\hat{u}} \in (a, b)$ such that

$$\int_a^b \left(2 - \frac{a}{r}\right) |\hat{u}|^2 r \mathrm{d}r = \left(2 - \frac{a}{\xi_{\hat{u}}}\right) \|\hat{u}\|_{w,I}^2. \tag{4.16}$$

We define

$$\lambda^*(a,b,m,k) := \begin{cases} 0, & \text{if } \mathrm{Re}(c_1) \geq 0, \\ \left(\frac{\max(ac_{13}-1+(2-a/\xi_{\hat{u}})^{-1},0)}{ac_{13}+1-(2-a/\xi_{\hat{u}})^{-1}}\right)^{\frac{1}{2|m|}}, & \text{if } \mathrm{Re}(c_1) < 0, \text{ and } m \neq 0, \\ \exp\left(\frac{(2-a/\xi_{\hat{u}})^{-1}-1}{ac_{13}}\right), & \text{if } \mathrm{Re}(c_1) < 0, \text{ and } m = 0. \end{cases} \tag{4.17}$$

Now we are ready to state the a priori estimate for the 1-D variational formulation (4.3).

Theorem 4.2 *Suppose $k > k_0$ for some $k_0 > 0$. Let $\hat{u}$ be the solution of (4.3). If $\rho \in (a,b)$ satisfies $\frac{a}{\rho} > \lambda^*$, then*

$$\|\partial_r u\|_{w,I} + |m|\|\hat{u}\|_{w^{-1},I} + k\|\hat{u}\|_{w,I} \leq CC_{m,k}\|\hat{f}\|_{w,I}, \tag{4.18}$$

where

$$C_{m,k} = \begin{cases} k^{\frac{4}{3}}, & \text{if } |m| \leq kb, \\ 1, & \text{if } |m| > kb, \end{cases} \tag{4.19}$$

and C is a constant only depending on a, b, ρ, k_0, $\hat{u}$.

Proof By simple calculation, it can be verified using $\frac{a}{\rho} > \lambda^*$, (4.13) and (4.14) that

$$1 + ac_0 \min(\mathrm{Re}(c_1), 0) > (2 - a/\xi_{\hat{u}})^{-1} > 0. \tag{4.20}$$

First, we take $\hat{v} = \hat{u} - \hat{u}(a)\overline{\phi^m} \in \hat{Y}$ in (4.3), then the left hand side

$$\begin{aligned} \hat{B}_m(\hat{u},\hat{v}) &= \hat{B}_m(\hat{u},\hat{u}) - \overline{\hat{u}(a)}\hat{B}_m(\hat{u},\overline{\phi^m}) \\ &= \|\partial_r\hat{u}\|^2_{w,I} + m^2\|\hat{u}\|^2_{w^{-1},I} - k^2\|\hat{u}\|^2_{w,I} - kbD_{m,k}|\hat{u}(b)|^2 \\ &\quad -\overline{\hat{u}(a)}\left[(\partial_r\hat{u},\partial_r\overline{\phi^m})_{w,I}\right. \\ &\quad \left.+m^2(\hat{u},\overline{\phi^m})_{w^{-1},I} - k^2(\hat{u},\overline{\phi^m})_{w,I} - kbD_{m,k}\hat{u}(b)\phi^m(b)\right]. \end{aligned} \tag{4.21}$$

Note that using integration by parts,

$$\begin{aligned} 0 &= \int_I \hat{u}\left(-\partial_r(r\partial_r\phi^m) + \frac{m^2}{r}\phi^m - rk^2\phi^m\right)\mathrm{d}r \\ &= (\partial_r\hat{u},\partial_r\overline{\phi^m})_{w,I} + m^2(\hat{u},\overline{\phi^m})_{w^{-1},I} - k^2(\hat{u},\overline{\phi^m})_{w,I} \\ &\quad -kbD_{m,k}\hat{u}(b)\phi^m(b) + a\hat{u}(a)\partial_r\phi^m(a), \end{aligned} \tag{4.22}$$

so it follows from (4.21) that

$$\begin{aligned} \hat{B}_m(\hat{u},\hat{v}) &= \|\partial_r\hat{u}\|^2_{w,I} + m^2\|\hat{u}\|^2_{w^{-1},I} - k^2\|\hat{u}\|^2_{w,I} \\ &\qquad -kbD_{m,k}|\hat{u}(b)|^2 + a|\hat{u}(a)|^2c_1(a,b;m,k) \\ &= (\hat{f}^m, \hat{u} - \hat{u}(a)\overline{\phi^m})_{w,I}. \end{aligned} \tag{4.23}$$

Define $c_3(a,b,\rho;m,k) := 1 + ac_0\min(\mathrm{Re}(c_1),0)$. By (4.20), we have $c_3 > (2 - a/\xi_{\hat{u}})^{-1} > 0$. Then using (4.4), the real part of (4.23) leads to

$$\begin{aligned} &c_3\left(\|\partial_r\hat{u}\|^2_{w,I} + m^2\|\hat{u}\|^2_{w^{-1},I}\right) - k^2\|\hat{u}\|^2_{w,I} - kb\mathrm{Re}(D_{m,k})|\hat{u}(b)|^2 \\ &\leq \mathrm{Re}(\hat{f},\hat{u})_{w,I} + \mathrm{Re}(\hat{f},\hat{u}(a)\phi^m)_{w,I}. \end{aligned} \tag{4.24}$$

We use the inequality $AB \le \varepsilon A^2 + \frac{B^2}{4\varepsilon}$ for all $A, B, \varepsilon > 0$ repeatedly in the following. Due to the fact that $\mathrm{Re}(D_{m,k}) < 0$ (see (2.34b) of [29]), we obtain

$$c_3\left(\|\partial_r\hat{u}\|^2_{w,I} + m^2\|\hat{u}\|^2_{w^{-1},I}\right) \le k^2\|\hat{u}\|^2_{w,I} + \varepsilon_1 k^2\|\hat{u}\|^2_{w,I} + \frac{1}{4\varepsilon_1 k^2}\|\hat{f}\|^2_{w,I} + \varepsilon_1'|\hat{u}(a)|^2\|\phi^m\|^2_{w,I} + \frac{1}{4\varepsilon_1'}\|\hat{f}\|^2_{w,I} \tag{4.25}$$

for all $\varepsilon_1, \varepsilon_1' > 0$.

Define $c_4(a, b, \rho; m, k) := c_3 - \varepsilon_1' c_0 c_2$. Since c_0 and c_2 are both uniformly bounded independent of m and k, we can choose ε_1' sufficiently small (independent of m and k) such that $(2 - a/\xi_{\hat{u}})^{-1} < c_4 < c_3 \le 1$. Then by Lemma 4.1, it follows that

$$c_4\left(\|\partial_r\hat{u}\|^2_{w,I} + m^2\|\hat{u}\|^2_{w^{-1},I}\right) \le (1+\varepsilon_1)k^2\|\hat{u}\|^2_{w,I} + \left(\frac{1}{4\varepsilon_1 k^2} + \frac{1}{4\varepsilon_1'}\right)\|\hat{f}\|^2_{w,I}. \tag{4.26}$$

It remains to bound $\|\hat{u}\|^2_{w,I}$. Using standard regularity argument, we can easily verify that if $f \in L^2_w(I)$, the weak solution $\hat{u}$ of (4.3) satisfies $(r-a)\partial_r\hat{u} \in \hat{Y}$. Taking $\hat{v} = (r-a)\partial_r\hat{u}$ in (4.3), using the identity $(\hat{u}, \hat{v})_{w,I} + (\hat{v}, \hat{u})_{w,I} = 2\mathrm{Re}(\hat{u}, \hat{v})_{w,I}$ and integration by parts, we obtain

$$2\mathrm{Re}\left(\partial_r\hat{u}, \partial_r\left((r-a)\partial_r\hat{u}\right)\right)_{w,I} = a\int_a^b |\partial_r\hat{u}|^2 \mathrm{d}r + (b-a)b|\partial_r\hat{u}(b)|^2; \tag{4.27}$$

$$2\mathrm{Re}\left(\hat{u}, (r-a)\partial_r\hat{u}\right)_{w^{-1},I} = -a\int_a^b |\hat{u}|^2 r^{-2}\mathrm{d}r + \frac{b-a}{b}|\hat{u}(b)|^2; \tag{4.28}$$

$$2\mathrm{Re}\left(\hat{u}, (r-a)\partial_r\hat{u}\right)_{w,I} = -2\int_a^b |\hat{u}|^2 r\mathrm{d}r + (b-a)b|\hat{u}(b)|^2 + a\int_a^b |\hat{u}|^2\mathrm{d}r; \tag{4.29}$$

Then the real part of (4.3) leads to

$$\begin{aligned} a\|\partial_r\hat{u}\|^2_I + (b-a)b|\partial_r\hat{u}(b)|^2 + m^2\frac{b-a}{b}|\hat{u}(b)|^2 + k^2\int_a^b (2-\frac{a}{r})|\hat{u}|^2 r\mathrm{d}r \\ \le am^2\|\hat{u}\|^2_{w^{-2},I} + k^2(b-a)b|\hat{u}(b)|^2 + 2\left|\mathrm{Re}(\hat{f}, (r-a)\partial_r\hat{u})_{w,I}\right| \\ \le am^2\|\hat{u}\|^2_{w^{-2},I} + k^2(b-a)b|\hat{u}(b)|^2 + (b-a)^2\|\hat{f}\|^2_{w,I} + \|\partial_r\hat{u}\|^2_{w,I}. \end{aligned} \tag{4.30}$$

Note that $a\|\hat{u}\|^2_{w^{-2},I} \le \|\hat{u}\|^2_{w^{-1},I}$, so it follows from (4.26) and (4.30) that

$$\begin{aligned} a\|\partial_r\hat{u}\|^2_I + (b-a)b|\partial_r\hat{u}(b)|^2 + m^2\frac{b-a}{b}|\hat{u}(b)|^2 + k^2(2-\frac{a}{\xi_{\hat{u}}})\|\hat{u}\|^2_{w,I} \\ \le c_4^{-1}(1+\varepsilon_1)k^2\|\hat{u}\|^2_{w,I} + k^2(b-a)b|\hat{u}(b)|^2 \\ + \left[c_4^{-1}(\frac{1}{4\varepsilon_1 k^2} + \frac{1}{4\varepsilon_1'}) + (b-a)^2\right]\|\hat{f}\|^2_{w,I}, \end{aligned} \tag{4.31}$$

where $\xi_{\hat{u}}$ is defined in (4.16). We define $c_5(a, b, \rho; m, k; \hat{u}) := 2 - \frac{a}{\xi_{\hat{u}}} - c_4^{-1}(1+\varepsilon_1)$. Using the fact $c_4 > (2 - a/\xi_{\hat{u}})^{-1}$, we can choose ε_1 small enough (independent of m and k) such that $\frac{1}{2}(2 - a/\xi_{\hat{u}} - c_4^{-1}) < c_5 < 2 - a/\xi_{\hat{u}}$. Then it follows from (4.31) that

$$m^2\frac{b-a}{b}|\hat{u}(b)|^2+k^2c_5\|\hat{u}\|^2_{w,I}$$
$$\leq k^2(b-a)b|\hat{u}(b)|^2+\left[c_4^{-1}(\frac{1}{4\varepsilon_1 k^2}+\frac{1}{4\varepsilon_1'})+(b-a)^2\right]\|\hat{f}\|^2_{w,I}. \tag{4.32}$$

To bound $k^2(b-a)b|\hat{u}(b)|^2$, we consider $|m|>kb$ and $|m|\leq kb$ separately.

(i) $|m|>kb$: In this case, $\frac{m^2}{b}>k^2b$, and hence the term $k^2(b-a)b|\hat{u}(b)|^2$ is absorbed by $m^2\frac{b-a}{b}|\hat{u}(b)|^2$ in (4.32), namely,

$$k^2\|\hat{u}\|^2_{w,I}\leq c_5^{-1}\left[c_4^{-1}(\frac{1}{4\varepsilon_1 k^2}+\frac{1}{4\varepsilon_1'})+(b-a)^2\right]\|\hat{f}\|^2_{w,I}. \tag{4.33}$$

(ii) $|m|\leq kb$: We take $\hat{v}=\hat{u}-\hat{u}(a)\phi^m$ in (4.3) and consider the imaginary part. By using (4.23) we have

$$kb|\mathrm{Im}(D_{m,k})||\hat{u}(b)|^2\leq|\mathrm{Im}(\hat{f},\hat{u})_{w,I}|+|\mathrm{Im}(\hat{f},\hat{u}(a)\phi^m)_{w,I}|. \tag{4.34}$$

So by Lemma 4.1, it follows from (4.34) that

$$\begin{aligned}
&k^2b|\hat{u}(b)|^2\\
&\leq k^2\varepsilon_3\|\hat{u}\|^2_{w,I}+\varepsilon_3'\|\phi^m\|^2_{w,I}|\hat{u}(a)|^2\\
&\quad+\left(\frac{1}{4\varepsilon_3|\mathrm{Im}(D_{m,k})|^2}+\frac{k^2}{4\varepsilon_3'|\mathrm{Im}(D_{m,k})|^2}\right)\|\hat{f}\|^2_{w,I}\\
&\leq k^2\varepsilon_3\|\hat{u}\|^2_{w,I}+\varepsilon_3'c_0c_2(\|\partial_r\hat{u}\|^2_{w,I}+m^2\|\hat{u}\|^2_{w^{-1},I})\\
&\quad+\left(\frac{1}{4\varepsilon_3|\mathrm{Im}(D_{m,k})|^2}+\frac{k^2}{4\varepsilon_3'|\mathrm{Im}(D_{m,k})|^2}\right)\|\hat{f}\|^2_{w,I}
\end{aligned} \tag{4.35}$$

for all $\varepsilon_3,\varepsilon_3'>0$. By choosing ε_3 small enough (independent of m and k) such that $c_5-(b-a)\varepsilon_3>0$ and using (4.35) in (4.32), we can bound $\|\hat{u}\|^2_{w,I}$ as follows

$$\begin{aligned}
&k^2(c_5-(b-a)\varepsilon_3)\|\hat{u}\|^2_{w,I}\leq(b-a)\varepsilon_3'c_0c_2\left(\|\partial_r\hat{u}\|^2_{w,I}+m^2\|\hat{u}\|^2_{w^{-1},I}\right)\\
&+\left[c_4^{-1}(\frac{1}{4\varepsilon_1k^2}+\frac{1}{4\varepsilon_1'})+(b-a)^2+\frac{b-a}{4\varepsilon_3|\mathrm{Im}(D_{m,k})|^2}+\frac{k^2(b-a)}{4\varepsilon_3'|\mathrm{Im}(D_{m,k})|^2}\right]\|\hat{f}\|^2_{w,I}.
\end{aligned} \tag{4.36}$$

Due to the fact that if $|m|\leq kb$, $\mathrm{Im}(D_{m,k})\geq c(kb)^{-1/3}$ for some constant c only depending on a (see (2.35) of [29]), we find

$$\begin{aligned}
&k^2\|\hat{u}\|^2_{w,I}\leq(c_5-(b-a)\varepsilon_3)^{-1}(b-a)\varepsilon_3'c_0c_2\left(\|\partial_r\hat{u}\|^2_{w,I}+m^2\|\hat{u}\|^2_{w^{-1},I}\right)\\
&+(c_5-(b-a)\varepsilon_3)^{-1}\left[c_4^{-1}(\frac{1}{4\varepsilon_1k^2}+\frac{1}{4\varepsilon_1'})+(b-a)^2+c(b-a)k^{\frac{2}{3}}b^{\frac{2}{3}}(\frac{1}{\varepsilon_3}+\frac{k^2}{\varepsilon_3'})\right]\|\hat{f}\|^2_{w,I}.
\end{aligned} \tag{4.37}$$

Finally, note that c_4 and c_5 are both bounded above and below independent of m and k. Combining (4.26), (4.33) and (4.37) leads to the desired result. □

Remark 4.1 Theorem 4.2 implies that the H^1 norm of $\hat{u}$ is bounded by the $\|\hat{f}\|_{w,I}$. The hypothesis $a/\rho>\lambda^*$ implies that if $c_1\geq 0$ or $ac_{13}-(1-(2-a/\xi_{\hat{u}})^{-1})\leq 0$, then the result is valid for all $a<\rho<b$. Otherwise, the result is valid when $\rho\in(a,a/\lambda^*)$, noting that

$0 < \lambda^* < 1$. The latter case requires that ρ should be sufficiently close to a. In practice, the shape of the obstacle is given by the problem, while the radius a of the smaller disk D_1 is set from the method. It is preferable to set D_1 close to Ω_1 to minimize the cost and to ensure that the a priori estimate holds.

The a priori estimate for the original 2-D variational formulation can be directly deduced using Theorem 4.2.

Theorem 4.3 *Suppose $k > k_0$ for some $k_0 > 0$. Let u be the solution of (2.15). If $\rho \in (a, b)$ satisfies $\frac{a}{\rho} > \lambda^*$, then*

$$\|\nabla u\|_{\widetilde{\Omega}} + k\|u\|_{\widetilde{\Omega}} \le Ck^{\frac{4}{3}}\|f\|_{\widetilde{\Omega}}, \tag{4.38}$$

where C is a constant only depending on a, b, ρ, k_0, u.

Proof Thanks to the orthogonality of the Fourier basis, it follows Theorem 4.2 that

$$\begin{aligned}\|\nabla u\|^2_{\widetilde{\Omega}} + k^2\|u\|^2_{\widetilde{\Omega}} &= \sum_{m=-\infty}^{\infty} 2\pi\left(\|\partial_r \hat{u}_m\|^2_{w,I} + m^2\|\hat{u}_m\|^2_{w^{-1},I} + k^2\|\hat{u}_m\|^2_{w,I}\right)\\ &\le \sum_{m=-\infty}^{\infty} CC^2_{m,k}\|\hat{f}\|^2_{w,I} \le Ck^{\frac{8}{3}}\|f\|^2_{\widetilde{\Omega}},\end{aligned}$$

which completes the proof. □

5 Error Estimates

In this section, we will carry out an error analysis for the spectral-Galerkin method (3.22). We still assume that Ω_1 is a disk with radius ρ ($a < \rho < b$) and $g = 0$.

5.1 Analysis of the 1-D Scheme

Let us first investigate the error between the 1-D solutions $\hat{u}^m$ given by (3.6) and $\hat{u}^m_N$ given by (3.14).

We define $\hat{X}_N = \hat{X} \cap P_N$, $\hat{Y}_N = \hat{Y} \cap P_N$, and introduce two projections as follows,

$$\pi^0_N : \hat{Y} \to \hat{Y}_N \quad \text{s.t. } (\hat{v} - \pi^0_N\hat{v}, \hat{v}_N) = 0,\ \forall \hat{v} \in \hat{Y}, \hat{v}_N \in \hat{Y}_N, \tag{5.1}$$

$$\pi^1_N : \hat{X} \to \hat{X}_N \quad \text{s.t. } (\partial(\hat{u} - \pi^1_N\hat{u}), \partial\hat{u}_N) = 0,\ \forall \hat{u} \in \hat{X}, \hat{u}_N \in \hat{X}_N. \tag{5.2}$$

Also, for $\sigma, s \in \mathbb{N}$ and $\sigma \le s$, we introduce

$$B^s_\sigma := \left\{\hat{u} \in L^2(I) : \ [(r-a)(b-r)]^{\frac{l-\sigma}{2}}\,\partial^l_r\hat{u} \in L^2(I),\ \sigma \le l \le s\right\} \tag{5.3}$$

with the seminorm

$$|\hat{u}|_{B^s_\sigma} = \|\,[(r-a)(b-r)]^{\frac{s-\sigma}{2}}\,\partial^s_r\hat{u}\|_I. \tag{5.4}$$

For these two projections, we have the following result [17].

Lemma 5.1 *Let $\sigma = 0$ or 1, for any $\hat{u} \in \hat{Y} \cap B^s_\sigma$ with $s \ge \sigma$ and $s \in \mathbb{N}$, it satisfies*

$$\|\partial_{rr}(\pi^\sigma_N\hat{u} - \hat{u})\|_I + N\|\partial_r(\pi^\sigma_N\hat{u} - \hat{u})\|_I + N^2\|\pi^\sigma_N\hat{u} - \hat{u}\|_I \le C_L N^{2-s}|\hat{u}|_{B^s_\sigma}, \tag{5.5}$$

where C_L is a constant determined by a and b.

Lemma 5.1 is a direct consequence of the Legendre polynomial approximation property (with a scaling from $[-1, 1]$ to $[a, b]$ and a extension to complex functions) that can be found in [17].

In Sect. 4, we characterized $\hat{u}^m$ as a solution of the 1-D variational problem (4.3). Similarly, since Ω_1 is a disk with radius ρ, the condition (3.19) implies $\hat{u}_N^m(\rho) = 0$ for all m, so $\hat{u}_N^m \in \hat{X}_N$. Together with (3.12)-(3.13), we can derive that $\hat{u}_N^m$ is a solution of the following 1-D variational problem

$$\begin{cases} \text{given } \hat{f}^m \in L^2_w(I), \text{ find } \hat{u}_N \in \hat{X}_N \text{ such that} \\ \hat{B}_m(\hat{u}_N, \hat{v}_N) = (\hat{f}^m, \hat{v}_N)_{w,I}, \quad \forall \hat{v}_N \in \hat{Y}_N, \end{cases} \tag{5.6}$$

where $\hat{B}_m$ is defined in (3.9).

Recall ϕ^m is the solution of (3.4), we define $\varphi_N^m = \pi_N^0(\phi^m - 1) + 1$, then $\varphi_N^m(a) = 1$. Also, let s be any integer such that $|\phi^m|_{B_0^s} < \infty$, then by Lemma 5.1,

$$\begin{aligned} \|I_0\|_I^2 &:= \left\| \partial_r(r\partial_r\varphi_N^m) + \frac{m^2}{r}\varphi_N^m - rk^2\varphi_N^m \right\|_I^2 \\ &\le C\left(\|\partial_{rr}(\varphi_N^m - \phi^m)\|_I^2 + \|\partial_r(\varphi_N^m - \phi^m)\|_I^2 + (m^4 + k^4)\|\varphi_N^m - \phi^m\|_I^2 \right) \\ &\le C\left(N^4 + m^4 + k^4\right) N^{-2s} |\phi^m|_{B_0^s}^2, \end{aligned} \tag{5.7}$$

where C only depends on a and b. Then by Sobolev inequality,

$$\begin{aligned} \left|\partial_r\left(\varphi_N^m(a) - \phi^m(a)\right)\right| &\le \left(\frac{1}{b-a} + 2\right)^{\frac{1}{2}} \|\partial_r(\varphi_N^m - \phi^m)\|_I^{\frac{1}{2}} \cdot \|\partial_r(\varphi_N^m - \phi^m)\|_{1,I}^{\frac{1}{2}} \\ &\le C_L\left(\frac{2}{b-a} + 4\right)^{\frac{1}{2}} N^{\frac{3}{2}-s} |\phi^m|_{B_0^s}, \end{aligned} \tag{5.8}$$

namely,

$$|\partial_r\varphi_N^m(a)| \le |c_1| + C_L\left(\frac{2}{b-a} + 4\right)^{\frac{1}{2}} N^{\frac{3}{2}-s} |\phi^m|_{B_0^s}, \tag{5.9}$$

where C_L is the constant in Lemma 5.1 and c_1 is defined in (4.11).

Now we are ready to establish the following error estimate.

Theorem 5.2 *Let $\hat{u}$ and $\hat{u}_N$ be the solutions of 1-D variational problems (4.3) and (5.6), respectively. Let ϕ^m be the function defined by (4.10). Suppose $\phi^m \in B_0^s$ and $\hat{u} \in B_1^{s'}$ for some integers $s > \frac{3}{2}$ and $s' \ge 2$, then there exists some $\xi \in (a, b)$ such that the following statement is true: if*

$$\frac{a}{\rho} > \lambda^{**}(a, b, m, k) := \begin{cases} \left(\frac{\max(ac_{13} - 1 + (2 - a/\xi)^{-1}, 0)}{ac_{13} + 1 - (2 - a/\xi)^{-1}} \right)^{\frac{1}{2|m|}}, & \text{if } m \ne 0, \\ \exp\left(\frac{(2 - a/\xi)^{-1} - 1}{ac_{13}}\right), & \text{if } m = 0, \end{cases} \tag{5.10}$$

there is some constant C only depending on $a, b, \rho, \hat{u}$ such that

$$\begin{aligned} &\|\partial_r(\hat{u} - \hat{u}_N)\|_{w,I} + |m|\|\hat{u} - \hat{u}_N\|_{w^{-1},I} \\ &\quad + \left(k - C(|m|+1)^{-\frac{1}{2}}(N^2 + m^2 + k^2)N^{-2s}|\phi^m|_{B_0^s}\right)\|\hat{u} - \hat{u}_N\|_{w,I} \\ &\le C\left(C_{m,k}^{(1)} N^{-s'} + C_{m,k}^{(2)} N^{\frac{1}{2}-s'} + N^{2-s'}\right)|\hat{u}|_{B_1^{s'}}, \end{aligned} \tag{5.11}$$

where

$$C^{(1)}_{m,k} := \begin{cases} k + |m| + k^{\frac{1}{3}}(k^2 + m^2), & \textit{if } |m| \le kb, \\ k + |m|, & \textit{if } |m| > kb, \end{cases} \tag{5.12}$$

$$C^{(2)}_{m,k} := \begin{cases} k + |m| + 1, & \textit{if } |m| \le kb, \\ |m| + 1, & \textit{if } |m| > kb, \end{cases} \tag{5.13}$$

whenever N is sufficiently large such that

$$C_L(\frac{2}{b-a} + 4)^{\frac{1}{2}} N^{\frac{3}{2}-s} |\phi^m|_{B^s_0} \le C|c_1|. \tag{5.14}$$

In particular, if N is large enough such that

$$C(|m|+1)^{-\frac{1}{2}}(N^2 + m^2 + k^2)N^{-2s}|\phi^m|_{B^s_0} < (1-\varepsilon)k, \tag{5.15}$$

for some $0 < \varepsilon < 1$, then

$$\begin{aligned} &\|\partial_r(\hat{u} - \hat{u}_N)\|_{w,I} + |m|\|\hat{u} - \hat{u}_N\|_{w^{-1},I} + \varepsilon k\|\hat{u} - \hat{u}_N\|_{w,I} \\ &\le C\left(C^{(1)}_{m,k} N^{-s'} + C^{(2)}_{m,k} N^{\frac{1}{2}-s'} + N^{2-s'}\right)|\hat{u}|_{B^{s'}_1}. \end{aligned} \tag{5.16}$$

Proof By simple calculation, it can be verified using $a/\rho > \lambda^{**}$ that $1 - ac_0|c_1| > (2 - a/\xi)^{-1}$. So there exist some $\mu_1, \mu_2 \in \mathbb{R}^+$ satisfying

$$1 - \mu_1 - (1+\mu_2)ac_0|c_1| > (2 - a/\xi)^{-1}. \tag{5.17}$$

Also, let N be sufficiently large such that

$$C_L(\frac{2}{b-a} + 4)^{\frac{1}{2}} N^{\frac{3}{2}-s} |\phi^m|_{B^s_0} \le \frac{\mu_2}{3}|c_1|, \tag{5.18}$$

then by (5.9) it holds that

$$|\partial_r \varphi^m_N(a)| \le (1 + \frac{\mu_2}{3})|c_1|. \tag{5.19}$$

Let $e_N = \hat{u}_N - \pi^1_N \hat{u}$ and $\tilde{e}_N = \hat{u} - \pi^1_N \hat{u}$, then

$$\hat{B}_m(e_N, \hat{v}_N) = \hat{B}_m(\tilde{e}_N, \hat{v}_N), \quad \forall \hat{v}_N \in \hat{Y}_N. \tag{5.20}$$

We first take $\hat{v}_N = e_N - e_N(a)\overline{\varphi^m_N} \in \hat{Y}_N$ in (5.20). Using the inequality

$$\hat{B}_m(\hat{u}, \hat{v}) = \int_I \hat{u} \cdot \overline{\left(\partial_r(r\partial_r \hat{v}) + \frac{m^2}{r}\hat{v} - rk^2\hat{v}\right)} \mathrm{d}r - a\hat{u}(a)\partial_r \overline{\hat{v}(a)}, \tag{5.21}$$

we obtain

$$\begin{aligned} \hat{B}_m(e_N, \hat{v}_N) &= \hat{B}_m(e_N, e_N) - \hat{B}_m(e_N, e_N(a)\overline{\varphi^m_N}) = \|\partial_r e_N\|^2_{w,I} + m^2\|e_N\|^2_{w^{-1},I} \\ &-k^2\|e_N\|^2_{w,I} - kbD_{m,k}|e_N(b)|^2 - \overline{e_N(a)}(e_N, I_0) + a|e_N(a)|^2 \partial_r \varphi^m_N(a), \end{aligned} \tag{5.22}$$

where I_0 is defined in (5.7). Similarly,

$$\begin{aligned} \hat{B}_m(\tilde{e}_N, \hat{v}_N) &= m^2(\tilde{e}_N, e_N)_{w^{-1},I} - k^2(\tilde{e}_N, e_N)_{w,I} - kbD_{m,k}\tilde{e}_N(b)\overline{e_N(b)} - \overline{e_N(a)}(\tilde{e}_N, I_0) \\ &+a\overline{e_N(a)}\tilde{e}_N(a)\partial_r \varphi^m_N(a). \end{aligned} \tag{5.23}$$

Therefore, equating (5.22) and (5.23), and taking the real part lead to

$$\begin{aligned}&\|\partial_r e_N\|^2_{w,I} + m^2\|e_N\|^2_{w^{-1},I} - k^2\|e_N\|^2_{w,I} - kbD_{m,k}|e_N(b)|^2 + a|e_N(a)|^2\mathrm{Re}(\partial_r\varphi_N^m(a))\\ &= m^2\mathrm{Re}((\tilde e_N, e_N)_{w^{-1},I}) - k^2\mathrm{Re}((\tilde e_N, e_N)_{w,I}) - kb\mathrm{Re}(D_{m,k}\tilde e_N(b)\overline{e_N(b)})\\ &\quad + a\mathrm{Re}(\overline{e_N(a)}\tilde e_N(a)\partial_r\varphi_N^m(a)) + \mathrm{Re}(\overline{e_N(a)}(\hat u_N - \hat u, I_0)).\end{aligned} \tag{5.24}$$

We bound the terms on the righthand side of (5.24) as follows

$$\mathrm{Re}((\tilde e_N, e_N)_{w^{-1},I}) \le \frac{1}{4\mu_1}\|\tilde e_N\|^2_{w^{-1},I} + \mu_1\|e_N\|^2_{w^{-1},I}, \tag{5.25}$$

$$-\mathrm{Re}((\tilde e_N, e_N)_{w,I}) \le \varepsilon_4\|e_N\|^2_{w,I} + \frac{1}{4\varepsilon_4}\|\tilde e_N\|^2_{w,I}, \tag{5.26}$$

$$-\mathrm{Re}(D_{m,k}\tilde e_N(b)\overline{e_N(b)}) \le |\mathrm{Re}(D_{m,k})|(\frac{1}{2}|e_N(b)|^2 + \frac{1}{2}|\tilde e_N(b)|^2), \tag{5.27}$$

$$\mathrm{Re}(\overline{e_N(a)}\tilde e_N(a)\partial_r\varphi_N^m(a)) \le \frac{\mu_2 a|c_1|}{3}|e_N(a)|^2 + \frac{(3+\mu_2)^2 a|c_1|}{12\mu_2}|\tilde e_N(a)|, \tag{5.28}$$

$$\mathrm{Re}(\overline{e_N(a)}(\hat u_N - \hat u, I_0)) \le \frac{\mu_2 a|c_1|}{3}|e_N(a)|^2 + \frac{3}{4\mu_2 a|c_1|}\|\hat u_N - \hat u\|^2_I\|I_0\|^2_I, \tag{5.29}$$

where $\varepsilon_4 > 0$ and the inequality (5.19) is used for getting (5.28). Then using Lemma 4.1, it follows (5.24) that

$$c_6(\|\partial_r e_N\|^2_{w,I} + m^2\|e_N\|^2_{w^{-1},I}) \le (1+\varepsilon_4)k^2\|e_N\|^2_{w,I} + I_1, \tag{5.30}$$

where $c_6 := 1 - \mu_1 - (1+\mu_2)ac_0|c_1|$ and

$$\begin{aligned}I_1 \ :=\ & \frac{k^2}{4\varepsilon_4}\|\tilde e_N\|^2_{w,I} + \frac{kb}{2}|\mathrm{Re}(D_{m,k})||\tilde e_N(b)|^2 + \frac{(3+\mu_2)^2 a|c_1|}{12\mu_2}|\tilde e_N(a)|^2\\ &+ \frac{m^2}{4\mu_1}\|\tilde e_N\|^2_{w^{-1},I} + \frac{3}{4\mu_2 a|c_1|}\|\hat u_N - \hat u\|^2_I\|I_0\|^2_I.\end{aligned} \tag{5.31}$$

Note that (5.17) implies

$$c_6 > (2 - a/\xi)^{-1} > 0. \tag{5.32}$$

Next, we bound $k^2\|e_N\|^2_{w,I}$ by taking $\hat v = 2(r-a)\partial_r e_N \in \hat Y_N$ in (5.20). By similar arguments as in (4.27), we obtain

$$\begin{aligned}\mathrm{Re}(\hat B_m(e_N, \hat v)) &= a\|\partial_r e_N\|^2_I + (b-a)b|\partial_r e_N(b)|^2 - am^2\|e_N\|^2_{w^{-2},I} + \frac{m^2(b-a)}{b}|e_N(b)|^2\\ &+ k^2\int_a^b (2 - \frac{a}{r})|e_N|^2 r\mathrm{d}r - k^2(b-a)b|e_N(b)|^2 - 2kb(b-a)\mathrm{Re}(D_{m,k}e_N(b)\partial_r\overline{e_N(b)}),\end{aligned} \tag{5.33}$$

and

$$\begin{aligned}&\mathrm{Re}(\hat B_m(\tilde e_N, \hat v))\\ &= -2\mathrm{Re}\left(\int_a^b (r-a)\partial_r(r\partial_r\tilde e_N)\overline{\partial_r e_N}\mathrm{d}r\right) + 2b(b-a)\mathrm{Re}(\partial_r\tilde e_N(b)\overline{\partial_r e_N(b)})\\ &\quad + 2m^2\mathrm{Re}((\tilde e_N, (r-a)\partial_r e_N)_{w^{-1},I}) - 2k^2\mathrm{Re}((\tilde e_N, (r-a)\partial_r e_N)_{w,I})\\ &\quad - 2kb(b-a)\mathrm{Re}(D_{m,k}\tilde e_N(b)\partial_r\overline{e_N(b)}) \le \left(\frac{2(b-a)^2}{a}\|\partial_r(r\partial_r\tilde e_N)\|^2_I + \frac{a}{2}\|\partial_r e_N\|^2_I\right)\end{aligned}$$

$$+b(b-a)\left(2k^2|Re(D_{m,k})|^2|\tilde{e}_N(b)|^2+2|\partial_r\tilde{e}_N(b)|^2+|\partial e_N(b)|^2\right)$$
$$+\left(\frac{8(b-a)^2m^4}{a^3}\|\tilde{e}_N\|_I^2+\frac{a}{2}\|\partial_r e_N\|_I^2\right)+b(b-a)^2k^4\|\tilde{e}_N\|_I^2+\|\partial_r e_N\|_{w,I}^2. \quad (5.34)$$

Equating (5.33)-(5.34) and using (5.30) lead to

$$m^2\frac{b-a}{b}|e_N(b)|^2+c_7k^2\|e_N\|_{w,I}^2\le c_6^{-1}I_1+I_2+k^2b(b-a)|e_N(b)|^2, \quad (5.35)$$

where $c_7:=2-a/\xi-c_6^{-1}(1+\varepsilon_4)$ with some $\xi\in(a,b)$ satisfying the mean value theorem

$$\int_a^b(2-\frac{a}{r})|e_N|^2r\mathrm{d}r=(2-\frac{a}{\xi})\|e_N\|_{w,I}^2, \quad (5.36)$$

and

$$I_2:=\frac{2(b-a)^2}{a}\|\partial_r(r\partial_r\tilde{e}_N)\|_I^2+b(b-a)(2k^2|Re(D_{m,k})|^2|\tilde{e}_N(b)|^2+2|\partial_r\tilde{e}_N(b)|^2)$$
$$+\frac{8(b-a)^2m^4}{a^3}\|\tilde{e}_N\|_I^2+b(b-a)^2k^4\|\tilde{e}_N\|_I^2. \quad (5.37)$$

Note that (5.32) implies $c_6^{-1}<2-a/\xi$, so we can always choose ε_4 small enough such that $c_7>0$.

For the term $k^2b(b-a)|e_N(b)|^2$ in (5.35), we can use the same argument as in the proof of Theorem 4.2 to deal with $|m|>kb$ and $|m|\le kb$ separately, and obtain

$$\|\partial_r e_N\|_{w,I}^2+m^2\|e_N\|_{w^{-1},I}^2+k^2\|e_N\|_{w,I}^2\lesssim I_1+I_2+I_3, \quad (5.38)$$

with

$$I_3:=\begin{cases}k^{\frac{2}{3}}(k^4+m^4)\|\tilde{e}_N\|_I^2+k^2|\tilde{e}_N(b)|^2, & \text{if } |m|\le kb,\\ 0, & \text{if } |m|>kb,\end{cases} \quad (5.39)$$

where $A\lesssim B$ means $A\le CB$ for some constant C only depending on $a,b,\rho,\hat{u},\mu_1,\mu_2$. Therefore, by the triangle inequality and (5.38),

$$\begin{aligned}&\|\partial_r(\hat{u}-\hat{u}_N)\|_{w,I}^2+m^2\|\hat{u}-\hat{u}_N\|_{w^{-1},I}^2+k^2\|\hat{u}-\hat{u}_N\|_{w,I}^2\\&\le 2\Big(\|\partial_r e_N\|_{w,I}^2+m^2\|e_N\|_{w^{-1},I}^2+k^2\|e_N\|_{w,I}^2\\&\quad+\|\partial_r\tilde{e}_N\|_{w,I}^2+m^2\|\tilde{e}_N\|_{w^{-1},I}^2+k^2\|\tilde{e}_N\|_{w,I}^2\Big)\\&\lesssim\tilde{C}_{m,k}^{(1)}\|\tilde{e}_N\|_I^2+\tilde{C}_{m,k}^{(2)}|\tilde{e}_N(b)|^2\\&\quad+(|m|+1)|\tilde{e}_N(a)|^2+(|m|+1)^{-1}\|\hat{u}_N-\hat{u}\|_I^2\|I_0\|_I^2\\&\quad+\|\partial_r\tilde{e}_N\|_I^2+\|\partial_{rr}\tilde{e}_N\|_I^2+|\partial_r\tilde{e}_N(b)|^2,\end{aligned} \quad (5.40)$$

with

$$\tilde{C}_{m,k}^{(1)}:=\begin{cases}k^2+m^2+k^{\frac{2}{3}}(k^4+m^4), & \text{if } |m|\le kb,\\ k^2+m^2, & \text{if } |m|>kb,\end{cases} \quad (5.41)$$

and

$$\tilde{C}_{m,k}^{(2)}:=\begin{cases}k^2+m^2+1, & \text{if } |m|\le kb,\\ m^2+1, & \text{if } |m|>kb.\end{cases} \quad (5.42)$$

Using (5.7), we have

$$\begin{aligned}
&\|\partial_r(\hat{u}-\hat{u}_N)\|^2_{w,I}+m^2\|\hat{u}-\hat{u}_N\|^2_{w^{-1},I}+k^2\|\hat{u}-\hat{u}_N\|^2_{w,I}\\
&\lesssim \tilde{C}^{(1)}_{m,k}\|\tilde{e}_N\|^2_I+\tilde{C}^{(2)}_{m,k}|\tilde{e}_N(b)|^2+(|m|+1)|\tilde{e}_N(a)|^2\\
&\quad+\|\partial_r\tilde{e}_N\|^2_I+\|\partial_{rr}\tilde{e}_N\|^2_I+|\partial_r\tilde{e}_N(b)|^2\\
&\quad+(|m|+1)^{-1}(N^4+m^4+k^4)N^{-2s}|\phi^m|^2_{B^s_0}\|\hat{u}-\hat{u}_N\|^2_{w,I}
\end{aligned} \tag{5.43}$$

Finally, by the Sobolev inequality and Lemma 5.1, it holds that if $\hat{u}\in B^{s'}_1$,

$$|\partial_r\tilde{e}_N(r)|\le(\frac{1}{b-a}+2)^{\frac{1}{2}}\|\partial_r\tilde{e}_N\|^{\frac{1}{2}}_I\|\partial_r\tilde{e}_N\|^{\frac{1}{2}}_{1,I}\le N^{\frac{3}{2}-s}|\hat{u}|_{B^{s'}_1}, \tag{5.44}$$

$$|\tilde{e}_N(r)|\le(\frac{1}{b-a}+2)^{\frac{1}{2}}\|\partial_r\tilde{e}_N\|^{\frac{1}{2}}_I\|\partial_r\tilde{e}_N\|^{\frac{1}{2}}_{1,I}\le N^{\frac{1}{2}-s}|\hat{u}|_{B^{s'}_1}, \tag{5.45}$$

for all $r\in[a,b]$. Then using Lemma 5.1 again, it follows (5.43) that

$$\begin{aligned}
&\|\partial_r(\hat{u}-\hat{u}_N)\|^2_{w,I}+m^2\|\hat{u}-\hat{u}_N\|^2_{w^{-1},I}+k^2\|\hat{u}-\hat{u}_N\|^2_{w,I}\\
&\lesssim\left(\tilde{C}^{(1)}_{m,k}N^{-2s'}+\tilde{C}^{(2)}_{m,k}N^{1-2s'}+N^{4-2s'}\right)|\hat{u}|^2_{B^{s'}_1}\\
&\quad+(|m|+1)^{-1}(N^4+m^4+k^4)N^{-2s}|\phi^m|^2_{B^s_0}\|\hat{u}-\hat{u}_N\|^2_{w,I},
\end{aligned} \tag{5.46}$$

which leads to the desired result. □

5.2 Analysis of the 2-D Case

Now let us describe the error between the 2-D solutions u given by (3.7) and u_{MN} given by (3.15). We need to introduce the following space

$$\mathcal{H}^{s',s''}=\left\{u=\sum_{m=-\infty}^{\infty}\hat{u}^m(r)e^{\mathrm{i}m\theta}\ :\ \hat{u}^m\in B^{s'}_1\cap H^1_w(I)\cap L^2_{w^{-1}}(I)\cap L^2_w(I)\right\}, \tag{5.47}$$

with the norm

$$\begin{aligned}
\|u\|_{\mathcal{H}^{s',s''}}:=\sum_{m=-\infty}^{\infty}\Big[&|\hat{u}^m|^2_{B^{s'}_1}+(1+m^2)^{s''-1}\|\partial_r\hat{u}_m\|^2_{w,I}+(1+m^2)^{s''}(\|\hat{u}_m\|^2_{w^{-1},I}\\
&+\|\hat{u}_m\|^2_{w,I})\Big]<\infty.
\end{aligned} \tag{5.48}$$

Thanks to Theorem 5.2, we can prove the following result by using the same argument as in the proof of in [29, Theorem 4.3]:

Theorem 5.3 *Let u and u_N be the solutions determined by (3.7) and (3.15), respectively. Suppose $u\in\mathcal{H}^{s',s''}$ for some integers $s'\ge 2$ and $s''\ge 1$, then there exists some $\xi\in(a,b)$ such that the following statement is true: if $\frac{a}{\rho}>\lambda^{**}$, then there is some constant C only depending a,b,ρ,u such that for any $0<\varepsilon<1$,*

$$\begin{aligned}
&\|\nabla(u-u_{MN})\|_{\tilde{\Omega}}+\varepsilon k\|u-u_{MN}\|_{\tilde{\Omega}}\\
&\le C\left(C^{(1)}_{M,k}N^{-s'}+C^{(2)}_{M,k}N^{\frac{1}{2}-s'}+N^{2-s'}+(1+\varepsilon kM^{-1})M^{1-s''}\right)\|u\|_{\mathcal{H}^{s',s''}},
\end{aligned} \tag{5.49}$$

whenever $N > N_0$ for some $N_0 > 0$, which only depends on $a, b, \rho, u, \varepsilon$. Here λ^{**}, $C^{(1)}_{M,k}$ and $C^{(2)}_{M,k}$ are defined in Theorem 5.2.

We remark that if u is sufficiently smooth, the regularity index s' and s'' can be arbitrarily large which implies the spectral convergence of the method.

6 Numerical Results

We present below several numerical examples to illustrate the accuracy of our algorithm.

6.1 Accuracy Test

In this example, we investigate the accuracy of the proposed method by prescribing an explicit solution. We set f in (2.10) as

$$f = \hat{f}^k_{-5}(r,\theta) + \hat{f}^k_{10}(r,\theta) + \hat{f}^k_{20}(r,\theta), \tag{6.1}$$

where

$$\hat{f}^k_j(r,\theta) = \left(-k^2 H^{(1)}_j(kr) + \frac{2k(j+1)}{r} H^{(1)}_{j+1}(kr) - k^2 H^{(1)}_{j+2}(kr)\right) e^{\mathrm{i}j\theta}. \tag{6.2}$$

Then the exact solution is given by

$$u = \hat{u}^k_{-5}(r,\theta) + \hat{u}^k_{10}(r,\theta) + \hat{u}^k_{20}(r,\theta) \tag{6.3}$$

where

$$\hat{u}^k_j(r,\theta) = H^{(1)}_j(kr)e^{\mathrm{i}j\theta}, \tag{6.4}$$

First, the shape of the obstacle is chosen as the following smooth curve

$$\Omega_1 = \{(r,\theta) : r < \rho(\theta) := 1 + 0.2\cos(5\theta)\}, \tag{6.5}$$

and the enclosing annulus is chosen as

$$\widetilde{\Omega} = \{(r,\theta) : 0.8 < r < 1.5\}. \tag{6.6}$$

We implement Algorithm 3.1 for $k = 10, 50, 100$ and $M = 5, 10, \cdots, 100$. The ODEs (3.4) and (3.5) are solved by the spectral-Galerkin methods (3.12) and (3.13), respectively, with the degree of freedom $N = M$. The Legendre polynomials are employed to construct basis functions for the space $\hat{W}_{\sigma,N}$. Also, we take the least square formulation (3.18) to determine $\{t^m\}$, and the number of collocation nodes on $\partial\Omega_1$ is set as $J = 4M$. The problem domains and collocation nodes for $M = 10$ are shown in Fig. 4. The L^2 and L^∞ relative errors, defined as

$$\|u - u_{MN}\|_{L^2(\Omega)}/\|u\|_{L^2(\Omega)} \text{ and } \|u - u_{MN}\|_{L^\infty(\Omega)}/\|u\|_{L^\infty(\Omega)}, \tag{6.7}$$

respectively, are shown in Fig. 6 for various M and k. To investigate the conditioning of the least square system (3.18), we list the condition numbers of the system for various M and k in Fig. 5. We observe that while the condition number increases as M, it surprisingly decreases with k. It can be observed from Fig. 6 that, for all values of K considered, the errors start to decrease rapidly as soon as M is sufficiently large, i.e. $M = O(K)$, and eventually converge exponentially.

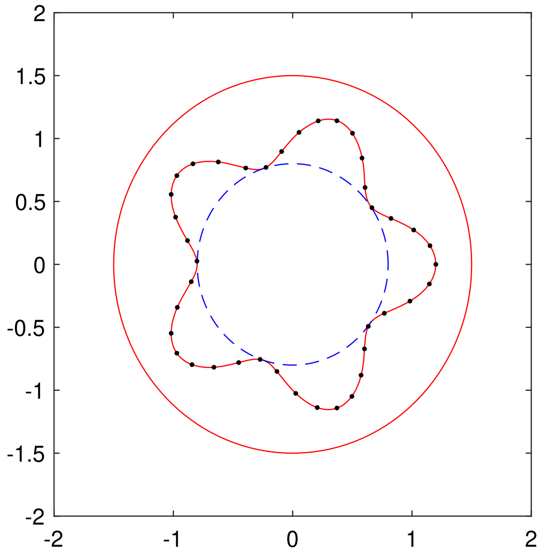


Fig. 4 The original domain, embedding annulus and collocation nodes for $M = 10$ in the first accuracy test

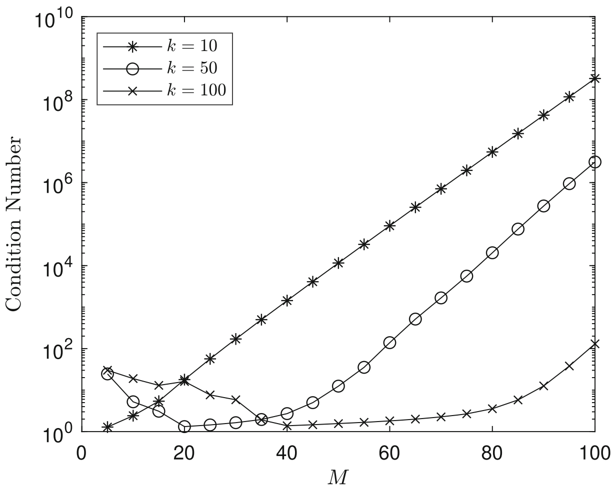


Fig. 5 The condition number of the least square system (3.18) versus M for $k = 10, 50, 100$ in the first accuracy test

Second, we choose the obstacle to be an equilateral pentagon with vertices

$$\left(-r_0 \sin(\frac{2j\pi}{5}), r_0 \cos(\frac{2j\pi}{5} + \frac{\pi}{2})\right), \quad j = 0, \cdots, 4 \tag{6.8}$$

where $r_0 = 0.7\sqrt{2}$, and the enclosing annulus is chosen as

$$\widetilde{\Omega} = \{(r, \theta) : 0.7 < r < 2\}. \tag{6.9}$$

The domains, condition numbers and error curves are shown in Figs. 7, 8 and 9. The behaviors of L^2 and L^∞ error decays are similar to the preceding case.

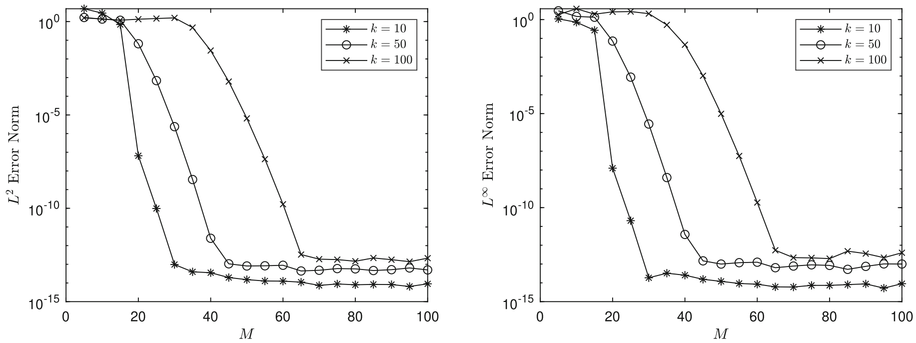


Fig. 6 L^2 and L^∞ solution errors versus M for $k = 10, 50, 100$ in the first accuracy test

Fig. 7 The original domain, embedding annulus and collocation nodes for $M = 10$ in the second accuracy test

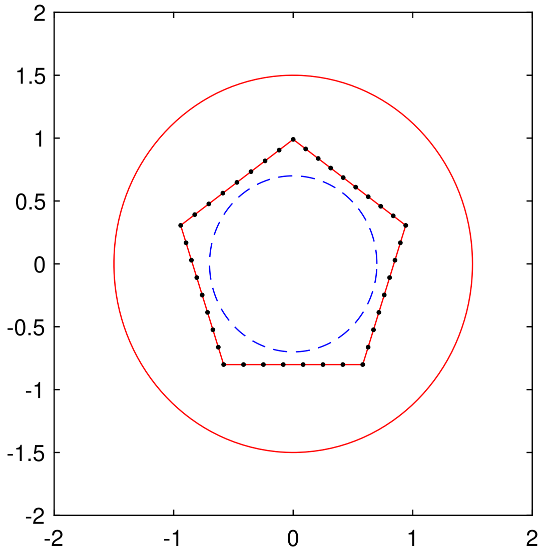
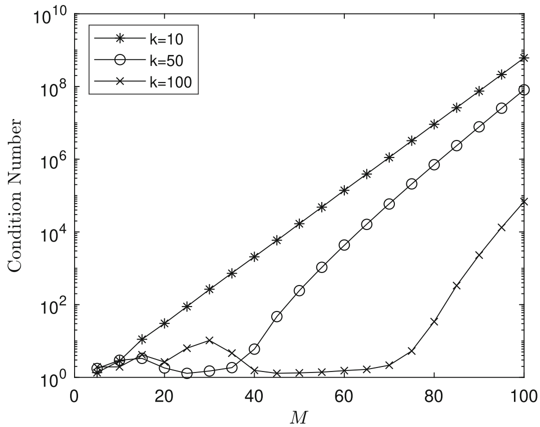


Fig. 8 The condition number of the least square system (3.18) versus M for $k = 10, 50, 100$ in the second accuracy test



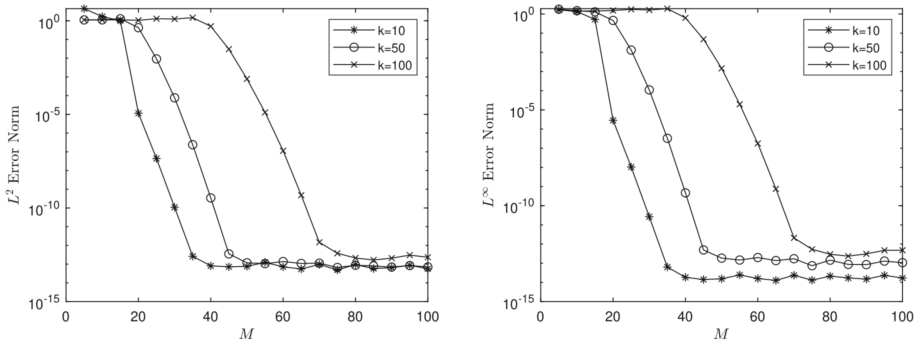


Fig. 9 L^2 and L^∞ solution errors versus M for $k = 10, 50, 100$ in the second accuracy test

6.2 Plane-Wave Scattering

In this example, we simulate the plane-wave scattering problem from a smooth bounded obstacle, where f vanishes in the problem (2.10). The obstacle is characterized by

$$\Omega_1 = \{(r, \theta) : r < \rho(\theta) := 1 + 0.2\cos(4\theta)\}. \tag{6.10}$$

We consider a pressure release (acoustics) or perfectly conducting (TE in electromagnetics) surface, which is given by

$$g(\theta) = -\exp(\mathrm{i}k\rho(\theta)\cos(\theta)). \tag{6.11}$$

Different from the preceding example where we evaluate errors in the entire domain, we compute the surface current on the "near field" (at $r = \rho(\theta)$) [26], which is defined as

$$\nu(\theta) = \rho(\theta)\partial_r u(\rho(\theta), \theta) - \frac{\rho'(\theta)}{\rho(\theta)}\partial_\theta u(\rho(\theta), \theta). \tag{6.12}$$

In this scenario, no explicit solutions are available for this complex geometry, so we use the high-order integral equation (IE) method [10, 22] to provide high-accuracy solutions for error evaluation. We use 1024 discretization points to guarantee a "well-resolved" IE solution.

We consider $k = 10, 50, 100$, and use our algorithm with M varying from $M = 20$ to 300. In this experiment, the enclosing annulus is chosen as

$$\widetilde{\Omega} = \{(r, \theta) : 0.8 < r < 1.4\}. \tag{6.13}$$

Note that when $f = 0$, the solution ψ^m of the 1-D equation (3.5) are identically zero for all m. Hence we only need to compute ϕ^m from the 1-D equation (3.4). For all M, we solve (3.4) by the spectral-Galerkin formulation (3.12) with fixed $N = 80$. To obtain higher accuracy, we set denser collocation nodes where the boundary is concave. We report the domain, condition number and error curves in Figs. 10, 11 and 12. We observe that for this problem, the condition numbers for all k are essentially identical, and the errors still decrease exponentially although the error curves flat out due to the ill conditioning.

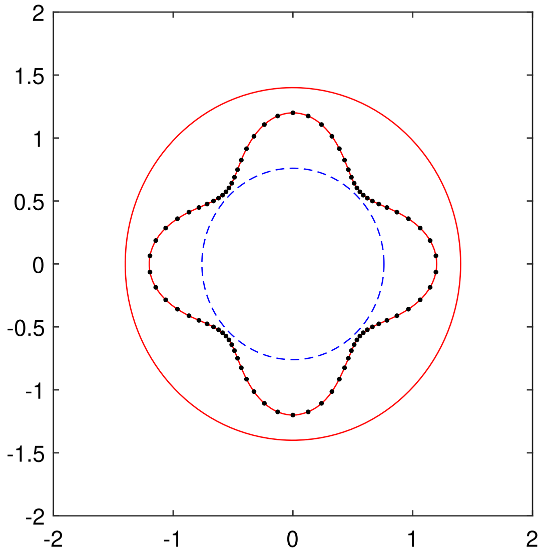


Fig. 10 The original domain, embedding annulus and collocation nodes for $M = 20$ in the plane-wave scattering

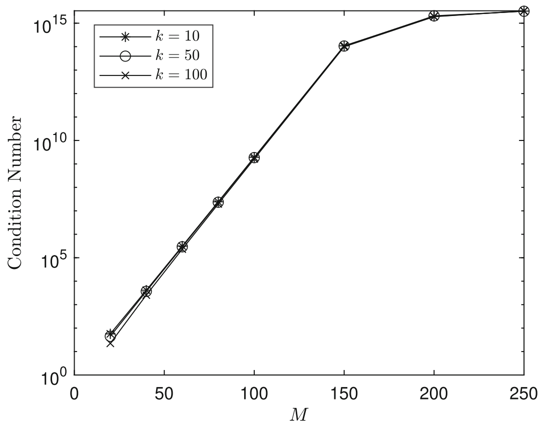


Fig. 11 The condition number of the least square system (3.18) versus M for $k = 10, 50, 100$ in the plane-wave scattering

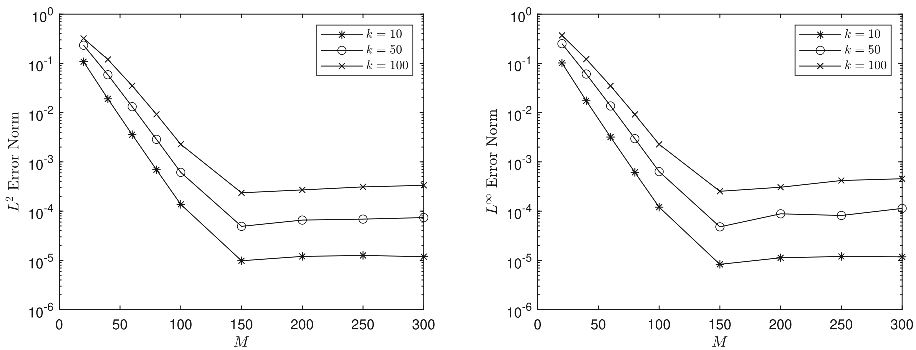


Fig. 12 L^2 and L^∞ errors of the surface current (6.12) versus M for $k = 10, 50, 100$ in the plane-wave scattering

7 Concluding Remarks

In this paper, we developed an efficient spectral method to solve the Helmholtz equation in exterior domains. Using a fictitious domain approach, we embedded the original complex domain in an annulus, and formulated a corresponding extended problem. Assuming the boundary condition at the inner annulus is known, the 2-D equation can be decomposed into a sequence of 1-D equations using Fourier expansion that can be solved by standard spectral-Galerkin methods. Hence, the key for our algorithm was to determine the boundary condition at the inner annulus from the original Dirichlet boundary condition by a least square formulation.

The proposed algorithm is relatively easy to implement, with essentially the same order of computational complexity as the spectral method for elliptic equations in the same domain [16]. We also presented numerical results to show that our algorithm can achieve exponential convergence when the solution is smooth, even for non-smooth polygonal obstacles. However, the associated least square system to determine the boundary conditions at the inner artificial boundary is ill-conditioned and may prevent us from using very fine resolution when the wave number becomes very large. How to improve the current approach for solving the least square system requires further investigation.

We established the well-posedness of the new formulation and carried out error analysis for the special case when the obstacle is a disk. How to extend the analysis to the general case is challenging and still under investigation.

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Data Availability No data is used and generated in this manuscript.

Declarations

Conflict of interest The authors have no relevant financial interest to disclose.

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