ESAIM: M2AN 60 (2026) 517–540
https://doi.org/10.1051/m2an/2026012

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

NUMERICAL ANALYSIS OF A SEMI-IMPLICIT EULER SCHEME FOR THE KELLER-SEGEL MODEL

Xueling Huang[1], Olivier Goubet[2,*] and Jie Shen[3]

Abstract. We study the properties of a semi-implicit Euler scheme widely used for time discretization of Keller-Segel equations in both parabolic-elliptic and parabolic-parabolic forms. We assume the initial mass of the cell density is sufficiently small to ensure that solutions of the continuous Keller-Segel equations exist globally in time. We prove that this linear, decoupled, first-order scheme preserves key properties of the Keller-Segel model at the semi-discrete level, including mass conservation, positivity preservation, and energy dissipation. We also establish optimal error estimates in L^p-norm $(1 < p < \infty)$ for the scheme.

Mathematics Subject Classification. 92C17, 65M12, 35K20, 35B09.

Received February 7, 2025. Accepted January 27, 2026.

1. Introduction

Chemotaxis refers to the directional migration of cells along the concentration gradient of a specific chemoattractant. In the 1970s, Keller [25] and Segel [26] established a classical mathematical model for chemotaxis, building upon the earlier work of Patlak [33]. In this paper we focus on the general dimensionless Keller-Segel (KS) model [32]

$$\frac{\partial \rho}{\partial t} = \Delta\rho - \chi\nabla\cdot(\rho\nabla c), \qquad \boldsymbol{x}\in\Omega,\ t>0, \tag{1}$$

$$\tau\frac{\partial c}{\partial t} = \Delta c - \alpha c + \gamma\rho, \qquad \boldsymbol{x}\in\Omega,\ t>0, \tag{2}$$

subjected to suitable initial condition

$$\rho|_{t=0} = \rho_0 \geq 0, \quad \tau c|_{t=0} = \tau c_0 \geq 0, \quad \text{and} \quad \rho_0, c_0 \not\equiv 0, \tag{3}$$

in a bounded domain $\Omega \subset \mathbf{R}^2$ with smooth boundary $\partial\Omega$. We consider the homogeneous Neumann boundary conditions for the cell density $\rho = \rho(\boldsymbol{x}, t)$ and the concentration of chemoattractants $c = c(\boldsymbol{x}, t)$

$$\frac{\partial \rho}{\partial \boldsymbol{n}} = \frac{\partial c}{\partial \boldsymbol{n}} = 0, \quad \boldsymbol{x}\in\partial\Omega,\ t>0, \tag{4}$$

Keywords and phrases. Keller-Segel equations, mass conservation, positivity preserving, energy dissipation, error estimates.

[1] School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China.
[2] Laboratoire Paul Painlevé UMR 8524 U-Lille CNRS et INRIA, Lille 59000, France.
[3] School of Mathematical Science, Eastern Institute of Technology, Ningbo, Zhejiang 315200, China.
*Corresponding author: olivier.goubet@univ-lille.fr

where $\boldsymbol{n}$ is the outward unit-normal to the boundary $\partial\Omega$. The parameter $\chi > 0$ is the sensitivity of cells to the chemoattractant, $\alpha, \gamma > 0$ represents the consumption and production rate of chemoattractant, respectively. The model is a parabolic-parabolic system when $\tau > 0$, and a parabolic-elliptic system when $\tau = 0$, which means that chemoattractant diffusion is much faster than cell diffusion. Note that if $\tau = 0$, only the initial data ρ_0 is needed in the condition (3).

There have been many mathematical studies on the well-posedness of the KS model (see, *e.g.*, [2, 17, 19] and references therein). In a two-dimensional (2D) bounded domain Ω, solutions of the KS model (1)–(4) exist globally in time under the condition

$$M = \int_\Omega \rho_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} < \frac{\pi^*}{\chi\gamma}, \tag{5}$$

where

$$\pi^* = \begin{cases} 8\pi, & \text{if } \Omega = \left\{\boldsymbol{x} \in \boldsymbol{R}^2; |\boldsymbol{x}| < L\right\} \text{ and } (\rho_0, c_0) \text{ is radial in } \boldsymbol{x}, \text{with } L \text{ is the radius of the domain,} \\ 4\pi, & \text{otherwise.} \end{cases}$$

Otherwise, solutions can blow up and be unbounded in $L^\infty(\Omega)$ [3–5, 9, 19, 30–32]. The KS equations possess important properties, including mass conservation, positivity of the cell density, and energy dissipation (see, *e.g.*, [9, 14, 27, 32, 35] and references therein). A good numerical method should preserve these properties at the discrete level to the greatest extent possible, and its implementation should be simple and amenable to rigorous numerical analysis.

Over the years, various numerical schemes have been developed for the KS model (1)–(4) (see, *e.g.*, [1,7,15,34] and references therein). Since the model is nonlinear and involves two coupled variables ρ and c, for simplicity and efficiency, one favors a proper time-discrete scheme that is linear and decoupled. The following semi-discrete scheme in time is commonly used in practical computations [15, 35]:

$$\frac{\rho^{n+1} - \rho^n}{\delta t} = \Delta\rho^{n+1} - \chi\nabla\cdot\left(\rho^{n+1}\nabla c^n\right), \tag{6}$$

$$\tau\frac{c^{n+1} - c^n}{\delta t} = \Delta c^{n+1} - \alpha c^{n+1} + \gamma\rho^{n+1}, \tag{7}$$

with the homogeneous Neumann boundary conditions, whre δt denotes the time step, ρ^n and c^n are approximations of ρ and c at the nth time step. At each time step, as long as $(\rho^n,\ c^n)$, $j = 0, 1, 2, \ldots, n$ are given, it is efficient to solve ρ^{n+1} from (6) and c^{n+1} from (7) since they are both linear elliptic equations. As time derivatives are approximated by the stable backward Euler method, linear terms are implicitly approximated and nonlinear terms are semi-implicitly approximated. There is sufficient numerical evidence that, with proper spatial discretization, this semi-implicit Euler scheme appears to preserve essential properties of the KS equations, including mass conservation, energy dissipation and positivity preservation [15, 34, 35]. However, it has not been rigorously proven that this scheme, at the semi-discrete level, can preserve all the above properties.

First of all, it is easy to obtain the discrete mass conservation by integrating (6) over the domain Ω with the homogeneous Neumann boundary conditions for the cell density.

For energy dissipation, it was first shown in [35] that the semi-implicit Euler scheme (6) and (7) sayisfies the energy dissipation property. It is also worth noting that Liu *et al.* [28] reformulated the density equation (1) with $\chi = 1$ as $\frac{\partial\rho}{\partial t} = \nabla\cdot\left(e^c\nabla\left(\frac{\rho}{e^c}\right)\right)$, and pointed out the equivalence between the following semi-discrete equation

$$\frac{\rho^{n+1} - \rho^n}{\delta t} = \nabla\cdot\left(e^{c^n}\nabla\left(\frac{\rho^{n+1}}{e^{c^n}}\right)\right),$$

and the semi-implicit Euler equation (6). However, they only performed stability analysis for a related coupled nonlinear scheme where c^n in (6) is replaced by c^{n+1}.

Regarding the positivity of the numerical solution ρ^n, there is a general consensus among researchers that this scheme satisfies the maximum principle. However, to the best of our knowledge, no rigorous proof of this fact exists.

In addition, to the best of our knowledge, there is no error analysis for the semi-implicit Euler scheme (6) and (7). Moreover, as noted in [28, 35], straightforward space discretizations of this semi-discrete (in time) scheme can easily destroy solution positivity and lead to instability.

Comment 1.1. A natural idea is to perform spatial discretization based on the semi-discrete scheme (6) and (7) to construct fully discrete schemes which can preserve the properties (mass conservation, positivity preservation, and energy dissipation) at the fully discrete level. Unfortunately, the authors in [28, 35] pointed out that straightforward space discretizations of this semi-discrete scheme can easily destroy the positivity of the solution and trigger instability. Therefore, additional correction techniques are often added when performing spatial discretization. For example, finite difference and finite element methods are often combined with upwind techniques [28, 34, 35], flux correction methods [7, 13, 23, 37], total variation diminishing methods [11], among others. A major drawback is that energy stability is often lost in such approaches. Zhou and Saito [39, 40] proposed special finite volume schemes that satisfy both positivity preservation and energy dissipation, and derived error estimates. However, their implementation is usually not simple or flexible enough. Very recently, fully discrete schemes which preserve all the essential properties have been constructed using finite element discretizations [15], finite difference methods [29], discontinuous Galerkin methods [1], finite volume methods [16], and any Galerkin-type spatial discretizations with consistent discrete integration by parts [20–22]. However, these works do not provide rigorous error analysis for the proposed numerical schemes.

Comment 1.2. Another interesting scheme is proposed in [36], where the authors construct a semi-discrete (in time) nonlinear scheme based on the Wasserstein gradient flow structure. They show that this scheme unconditionally preserves all essential properties of the KS equations, and the nonlinear equation is uniquely solvable at each time step. A rigorous error analysis is carried out in [8] for a fully discrete scheme based on this time discretization. A distinct advantage of this scheme is that its properties can be easily extended to various spatial discretizations. However, a major drawback is that it requires solving a coupled nonlinear system at each time step.

In summary, a comprehensive and rigorous numerical analysis of the semi-implicit Euler scheme (6) and (7) will not only enable us to fully understand this semi-discrete scheme, but also provide a basis for further error analysis of fully discrete schemes combined with this semi-discretization. The main purpose of this paper is to conduct a rigorous numerical analysis for the semi-implicit Euler scheme (6) and (7). The main contributions of this paper include, assuming that the mass M is sufficiently small, *i.e.*,

$$M < \min\left\{\frac{\pi^*}{\chi\gamma}, \frac{1}{(2+\tau)\gamma\chi C_{gn}}\right\}, \tag{8}$$

where C_{gn} is the Gagliardo-Nirenberg constant in Remark 1.1,

- proving the well-posedness and properties of the scheme (6) and (7), including mass conservation, positivity preservation, and energy dissipation;
- obtaining L^q-bounds ($1 < q \leq \infty$) for the numerical solution of (6) and (7) if (8) is valid; and,
- deriving optimal error estimates in L^p-norm ($1 < p < \infty$) if (8) is valid.

Remark 1.1. The Gagliardo-Nirenberg inequality we have in mind is for any $u \in H^1(\Omega)$

$$\|u\|_{L^4}^4 \leq C_{gn}\|u\|_{L^2}^2\Big(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\Big),$$

where C_{gn} is a numerical constant that depends on Ω [12].

The rest of the paper is organized as follows. In Section 2, we introduce and prove basic properties of the KS model (1)–(4), including mass conservation, positivity preservation, and energy dissipation. In Section 3, a series of bounds for the cell density ρ are estabilished. In Section 4, we analyze the semi-implicit Euler scheme (6) and (7) with respect to well-posedness, mass conservation, positivity preservation and energy dissipation. In Section 5, L^q-bounds $(1 < q \leq \infty)$ of ρ^{n+1} for all n are derived under the assumption (8). In Section 6, we derive optimal error estimates in L^p-norm $(1 < p < \infty)$ between the exact solutions $(\rho,\ c)$ and semi-discrete numerical solutions $(\rho^{n+1},\ c^{n+1})$. We end the paper with some concluding remarks in Section 7.

We now introduce some notations. Throughout the paper, $W^{m,p} = W^{m,p}(\Omega)$ denotes a Sobolev space, and $H^m = W^{m,2}$ denotes a Hilbert space with inner product $(\cdot,\cdot)_{H^m}$ and norm $\|\cdot\|_{H^m}$. Additionally, $(\cdot,\cdot)$ denotes the L^2 inner product and $\|\cdot\|_{L^p}$ denotes the L^p-norm on the domain Ω. We denote by $\mathcal{L}(E,F)$ the space of bounded linear operators from the normed space E into F. Moreover, we shall use boldface letters to denote vectors and vector spaces. We denote by K a generic positive constant that depends on the parameters $\chi, \alpha, \gamma, \tau$ of the equations, and by K^0 a positive constant that also depends on the initial data. The constants K and K^0 are independent of discretization parameters, and may change value from line to line without explicit mention. We will frequently use various continuous and discrete Gronwall lemmas. In addition to the classical Gronwall lemmas (see, for instance, [10]), we also need the uniform Gronwall lemma from [38] as follows:

Lemma 1.1. *Consider $y(t), g(t), h(t)$ nonnegative functions such that*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leq g(t)y(t) + h(t),$$

and for any $t > 0$, there exist constants $k_1, k_2, k_3 > 0$ such that

$$\int_t^{t+1} y(s)\mathrm{d}s \leq k_1,\ \int_t^{t+1} g(s)\mathrm{d}s \leq k_2,\ \int_t^{t+1} h(s)\mathrm{d}s \leq k_3.$$

Then for any $t \geq 1$ we have $y(t) \leq (k_1 + k_3)\exp(k_2)$.

2. Properties of the KS model

There have been a number of numerous studies on the well-posedness of the initial-boundary value problem (1)–(4) [3, 4, 9, 19, 30–32]. Under the assumption (5) that the solutions of the KS model (1)–(4) exist globally in time, we present the basic properties of the KS model, and provide detailed proofs that will be useful later in deriving similar properties for the scheme (6) and (7).

Theorem 2.1. *The KS model (1)–(4) satisfies the following properties:*

– *Mass conservation:*

$$M := \int_\Omega \rho(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_\Omega \rho_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{9}$$

– *Positivity preservation: if $\rho_0(\boldsymbol{x}), c_0(\boldsymbol{x}) \geq 0$ and $\rho_0(\boldsymbol{x}), c_0(\boldsymbol{x}) \not\equiv 0$, then $\rho(\boldsymbol{x},t) \geq 0$ and $c(\boldsymbol{x},t) \geq 0$; if $\rho_0(\boldsymbol{x}), c_0(\boldsymbol{x}) > 0$, then $\rho(\boldsymbol{x},t) > 0$ and $c(\boldsymbol{x},t) > 0$.*

– *Energy dissipation:*

$$\frac{\mathrm{d}E_{\mathrm{tot}}(\rho,c)}{\mathrm{d}t} = -\int_\Omega \rho|\nabla(\log\rho - \chi c)|^2\mathrm{d}\boldsymbol{x} - \frac{\tau\chi}{\gamma}\int_\Omega\left(\frac{\partial c}{\partial t}\right)^2 \leq 0, \tag{10}$$

where the free energy of the model is defined by

$$E_{\mathrm{tot}}(\rho,c) = \int_\Omega\left(f(\rho) - \chi\rho c + \frac{\chi}{2\gamma}|\nabla c|^2 + \frac{\alpha\chi}{2\gamma}c^2\right)\mathrm{d}\boldsymbol{x}, \tag{11}$$

with $f(\rho) = \rho\log\rho - \rho$.

Proof. Integrating the equation (1) over Ω, we deduce the mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}=0,$$

with the homogeneous Neumann boundary condition for ρ.

Set $\rho_+ := \sup\{\rho, 0\}$, $\rho_- := \sup\{-\rho, 0\}$, then $\rho = \rho_+ - \rho_-$ and $|\rho| = \rho_+ + \rho_-$. Multiplying both sides of the equation (1) by the sign function $\mathrm{sgn}\rho$, we have:

$$\frac{\partial\rho}{\partial t}\mathrm{sgn}\rho = \Delta\rho\,\mathrm{sgn}\rho - \chi\nabla\cdot(\rho\nabla c)\mathrm{sgn}\rho.$$

Recalling that $\mathrm{sgn}\xi = \frac{\xi}{|\xi|}$ and $\xi\mathrm{sgn}\xi = |\xi|$, we have

$$\frac{\partial\rho}{\partial t}\mathrm{sgn}\rho = \frac{\partial|\rho|}{\partial t},\quad \nabla\cdot(\rho\nabla c)\mathrm{sgn}\rho = \nabla\cdot(|\rho|\nabla c).$$

By Kato's inequality [24], we know that $\Delta\rho\,\mathrm{sgn}\rho \le \Delta|\rho|$ in $D'(\Omega)$. Then, we have

$$\frac{\partial|\rho|}{\partial t} \le \Delta|\rho| - \chi\nabla\cdot(|\rho|\nabla c).$$

Integrating both sides of the inequality over the region Ω leads to

$$\frac{d}{\mathrm{d}t}\int_{\Omega}|\rho|\mathrm{d}\boldsymbol{x} = \frac{d}{\mathrm{d}t}\int_{\Omega}(\rho_+ + \rho_-)\mathrm{d}\boldsymbol{x} \le 0, \tag{12}$$

with the homogeneous Neumann boundary condition. On the other hand, by the mass conservation, we have

$$\frac{d}{\mathrm{d}t}\int_{\Omega}\rho\mathrm{d}\boldsymbol{x} = \frac{d}{\mathrm{d}t}\int_{\Omega}(\rho_+ - \rho_-)\mathrm{d}\boldsymbol{x} = 0.$$

Subtracting the above from (12), we derive

$$\frac{d}{\mathrm{d}t}\int_{\Omega}\rho_-\mathrm{d}\boldsymbol{x} \le 0.$$

Therefore, we have $\rho_- = 0$ if $\rho_{0_-} = 0$, which implies that $\rho(\boldsymbol{x},t) \ge 0$ if the initial data $\rho_0 \ge 0$.

Moreover, if $\rho_0 > 0$ we can prove that $\rho(\boldsymbol{x},t) > 0$ as follows. Fix $\Omega\times[0,T]$ a given cylinder. Fix $\epsilon > 0$ small enough such that $\rho_0 > \epsilon$. Consider the function $v(\boldsymbol{x},t) = \rho(\boldsymbol{x},t) - \epsilon\exp(-\lambda t)$. We have

$$\frac{\partial v}{\partial t} - \Delta v + \chi\nabla\cdot(v\nabla c) = \epsilon\exp(-\lambda t)(\lambda - \chi\Delta c). \tag{13}$$

Using the basic theory about linear elliptic and linear parabolic equations from Evans' book [12], we carry out the following analysis. In the parabolic-elliptic case $\tau = 0$, we have that $-\Delta c \ge -\alpha c$. Moreover, we know ρ remains bounded in $L^\infty(\Omega)$ (see Sect. 3 below). Then c is smooth and the right hand side of (13) is bounded from below by $\varepsilon\exp(-\lambda t)(\lambda - \chi\alpha\|c\|_{L^\infty})$. In the parabolic-parabolic case $\tau > 0$, assuming that c_0 is smooth enough, we have that $\|-\Delta c\|_{L^\infty}$ is bounded and the right hand side of (13) is bounded from below by $\varepsilon\exp(-\lambda t)(\lambda - \chi\|-\Delta c\|_{L^\infty})$. Choosing λ large enough such that the right hand side of this inequality is non negative and proceeding as above, we have that $\rho(\boldsymbol{x},t)$ is bounded below by a positive function on $\Omega\times[0,T]$.

Next we prove the positivity of c in the case $\tau = 0$. We derive from (2) that

$$c = \gamma(-\Delta + \alpha I)^{-1}\rho,$$

where I is the identity matrix and the operator $-\Delta + \alpha I$ is positive [12]. Therefore, we derive the positivity of $c(\boldsymbol{x}, t)$ since we already showed the positivity of $\rho(\boldsymbol{x}, t)$. In the case $\tau > 0$ we can use an analogous argument. In fact the parabolic equation is the limit of the sequence of following elliptic equations

$$\tau \frac{c^{n+1} - c^n}{\delta t} = \Delta c^{n+1} - \alpha c^{n+1} + \gamma \rho^{n+1}, \quad \forall n \in \mathbb{N},$$

which satisfy the maximum principle. Then the limit parabolic equation also satisfies the maximum principle.

Now we consider the energy dissipation. Since $\Delta \rho = \nabla \cdot (\rho \nabla \log \rho)$, we rewrite (1) as

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla \log \rho) - \chi \nabla \cdot (\rho \nabla c) = \nabla \cdot (\rho \nabla (\log \rho - \chi c)). \tag{14}$$

According to the free energy (11), we know the energy derivative $\frac{\delta E_{\text{tot}}}{\delta \rho} = \log \rho - \chi c$ and $\frac{\delta E_{\text{tot}}}{\delta c} = -\frac{\chi}{\gamma} \Delta c + \frac{\alpha \chi}{\gamma} c - \chi \rho = -\frac{\tau \chi}{\gamma} \frac{\partial c}{\partial t}$. Taking the inner product of (14) with $\frac{\delta E_{\text{tot}}}{\delta \rho}$ and (2) with $\frac{\chi}{\gamma} \frac{\partial c}{\partial t}$, then applying integration by parts and noting that the boundary terms vanish under the Neumann boundary conditions, we obtain the energy dissipation law (10). □

3. Bounds of the exact solution

In this section, under the assumption (5) which ensures the global existence of solutions in time [3, 4, 9, 18, 19, 30–32], we derive L^q-bounds $(1 < q \leq \infty)$ for the exact solution ρ of the KS model (1)–(4).

First, we recall the lemma concerning the energy bound from reference [32] and the references therein. This lemma plays a key role in estabilishing the global well-posedness of the KS equations.

Lemma 3.1. *If the mass of cells $M < \frac{\pi^*}{\chi \gamma}$, there exist positive constants $K_{\rho c}^0$ and K_e^0 such that*

$$\int_\Omega \rho c \mathrm{d}\boldsymbol{x} \leq K_{\rho c}^0 \quad \text{and} \quad |E_{\text{tot}}(\rho, c)| \leq K_e^0. \tag{15}$$

Proof. We sketch the proof following [32]. We drop the term $-\rho$ in the energy due to mass conservation. For $\delta > 0$ to be chosen later the modified energy reads

$$E_{\text{tot}}(t) = \int_\Omega \left(-\rho \log \left(\frac{\exp((\chi + \delta) c)}{\rho} \right) \right) \mathrm{d}\boldsymbol{x} + \int_\Omega \left(\delta \rho c + \frac{\chi}{2\gamma} |\nabla c|^2 + \frac{\alpha \chi}{2\gamma} c^2 \right) \mathrm{d}\boldsymbol{x}.$$

By Jensen's inequality (and mass conservation), we have

$$\int_\Omega \left(-\frac{\rho}{M} \log \left(\frac{\exp((\chi + \delta) c)}{\rho} \right) \right) \mathrm{d}\boldsymbol{x} \geq -\log \left(\frac{1}{M} \int_\Omega \exp((\chi + \delta) c) \right) \mathrm{d}\boldsymbol{x}.$$

Applying Trudinger-Moser inquality, for any small $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$-\log \left(\frac{1}{M} \int_\Omega \exp((\chi + \delta) c) \right) \mathrm{d}\boldsymbol{x} \geq -\log \frac{C_\varepsilon}{M} - \left(\frac{1}{2\pi^*} + \varepsilon \right) (\chi + \delta)^2 \|\nabla c\|_{L^2}^2 - \frac{2(\chi + \delta)}{|\Omega|} \|c\|_{L^1}.$$

In the parabolic-elliptic case $\tau = 0$ we have $\|c\|_{L^1} = \frac{\gamma}{\alpha} M$ and in the parabolic-parabolic case $\tau > 0$, integrating (2) in space and in time $\|c\|_{L^1} \leq \max(\|c_0\|_{L^1}, \frac{\gamma}{\alpha} M)$. We may now chose δ, ε small enough such that the modified energy satisfies $E_{\text{tot}}(\rho, c) \geq K(\int_\Omega \rho c \mathrm{d}\boldsymbol{x} + \|c\|_{H^1}^2) - K^0$ and then bounded from below. Since $E_{\text{tot}}(\rho(t), c(t)) \leq E_{\text{tot}}(\rho_0, c_0)$ the conclusion follows promptly. □

Next, we derive below a series of bound results about the exact solution ρ in $L^q(\Omega)$ $(1 < q \leq \infty)$ for the KS model (1)–(4).

Lemma 3.2. *For $T \in (0,\infty)$, if the small mass condition (8) holds, there exists a positive constant K^0 such that $\int_0^T \int_\Omega \rho(\boldsymbol{x},t)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \le K^0$.*

Proof. Replacing u with $\sqrt{\rho}$ in the Gagliardo-Nirenberg inequality in Remark 1.1 yields

$$\|\sqrt{\rho}\|_{L^4}^4 \le C_{gn}\|\sqrt{\rho}\|_{L^2}^2\Big(\|\sqrt{\rho}\|_{L^2}^2 + \|\nabla\sqrt{\rho}\|_{L^2}^2\Big),$$

i.e.,

$$\|\rho\|_{L^2}^2 = \|\sqrt{\rho}\|_{L^4}^4 \le C_{gn}M(M + \|\nabla\sqrt{\rho}\|_{L^2}^2) = C_{gn}M\bigg(M + \int_\Omega \frac{|\nabla\rho|^2}{\rho}\mathrm{d}\boldsymbol{x}\bigg). \tag{16}$$

Therefore, bounding the L^2 norm of ρ requires bounding the L^2 norm of $\sqrt{\rho}^{-1}\nabla\rho$.

For this purpose, we expand

$$\int_\Omega \rho|\nabla(\log\rho - \chi c)|^2\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{|\nabla\rho|^2}{\rho}\mathrm{d}\boldsymbol{x} + \chi^2\int_\Omega \rho|\nabla c|^2\mathrm{d}\boldsymbol{x} - 2\chi\int_\Omega \nabla\rho\nabla c\mathrm{d}\boldsymbol{x}. \tag{17}$$

According to the equation (2) for c, we have

$$\Delta c = \tau\frac{\partial c}{\partial t} + \alpha c - \gamma\rho.$$

Combining with Green formula and the non-negativity of ρ and c, we can estimate the last term in (17) as follows

$$\begin{aligned} -2\chi\int_\Omega \nabla\rho\nabla c\mathrm{d}\boldsymbol{x} &= 2\chi\int_\Omega \rho\Delta c\mathrm{d}\boldsymbol{x} \\ &= 2\chi\int_\Omega \rho\Big(\tau\frac{\partial c}{\partial t} + \alpha c - \gamma\rho\Big)\mathrm{d}\boldsymbol{x} \\ &\ge 2\tau\chi\int_\Omega \rho\frac{\partial c}{\partial t}\mathrm{d}\boldsymbol{x} - 2\chi\gamma\|\rho\|_{L^2}^2. \end{aligned}$$

By Young's inequality, we know that

$$2\tau\chi\int_\Omega \rho\frac{\partial c}{\partial t}\mathrm{d}\boldsymbol{x} \ge -\bigg|2\tau\chi\int_\Omega \rho\frac{\partial c}{\partial t}\mathrm{d}\boldsymbol{x}\bigg| \ge -\frac{\tau\chi}{\gamma}\int_\Omega \Big(\frac{\partial c}{\partial t}\Big)^2\mathrm{d}\boldsymbol{x} - \tau\chi\gamma\|\rho\|_{L^2}^2.$$

Therefore, (17) can be estimated as

$$\begin{aligned} &\int_\Omega \rho|\nabla(\log\rho - \chi c)|^2\mathrm{d}\boldsymbol{x} + \frac{\tau\chi}{\gamma}\int_\Omega \Big(\frac{\partial c}{\partial t}\Big)^2\mathrm{d}\boldsymbol{x} + (2+\tau)\chi\gamma C_{gn}M^2 \\ &\qquad \ge (1 - (2+\tau)\chi\gamma C_{gn}M)\int_\Omega \frac{|\nabla\rho|^2}{\rho}\mathrm{d}\boldsymbol{x} + \chi^2\int_\Omega \rho|\nabla c|^2\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{18}$$

Integrating (10) between 0 and t leads to

$$\int_0^t \bigg(\int_\Omega \rho|\nabla(\log\rho - \chi c)|^2\mathrm{d}\boldsymbol{x} + \frac{\tau\chi}{\gamma}\int_\Omega \Big(\frac{\partial c}{\partial t}\Big)^2\bigg) \le E_{\mathrm{tot}}(\rho_0, c_0) - E_{\mathrm{tot}}(\rho(t), c(t)). \tag{19}$$

Applying Lemma 3.1 yields $-E_{\mathrm{tot}}(\rho(t), c(t)) \le \gamma K_{\rho c}^0$. Under the small mass condition

$$M < \min\Big\{\frac{\pi^*}{\chi\gamma}, \frac{1}{(2+\tau)\gamma\chi C_{gn}}\Big\},$$

as well as gathering (16), (18) and (19) concludes the proof of the lemma. ∎

Lemma 3.3. *Assume the small mass condition (8) is satisfied. If the initial data $\rho_0 \in L^2(\Omega)$, then $\rho(\boldsymbol{x},t) \in L^\infty\big(0,\infty;L^2(\Omega)\big)$.*

Proof. Taking the scalar product of (1) with ρ leads to

$$\left(\frac{\partial \rho}{\partial t},\rho\right) - (\Delta\rho,\rho) = -\chi(\nabla\cdot(\rho\nabla c).\rho), \tag{20}$$

Using Green's formula, the equation (2) for c, and the non-negativity of c, we have

$$\left(\frac{\partial \rho}{\partial t},\rho\right) = \frac{1}{2}\frac{d}{\mathrm{d}t}\|\rho\|_{L^2}^2, \quad -(\Delta\rho,\rho) = \|\nabla\rho\|_{L^2}^2, \tag{21}$$

and

$$\begin{aligned} -\chi(\nabla\cdot(\rho\nabla c),\rho) = \chi(\rho\nabla c,\nabla\rho) &= \frac{\chi}{2}\big(\rho^2,-\Delta c\big) \\ &= \frac{\chi}{2}\left(\rho^2,\gamma\rho-\tau\frac{\partial c}{\partial t}-\alpha c\right) \\ &\le \frac{\chi\gamma}{2}\big(\rho^2,\rho\big) - \frac{\tau\chi}{2}\left(\rho^2,\frac{\partial c}{\partial t}\right). \end{aligned} \tag{22}$$

For the two terms on the right-hand side of the inequality (22), we apply Cauchy-Schwarz inequality, Gagliardo-Nirenberg inequality and the ε-Young's inequality in turn to derive

$$\begin{aligned} \frac{\chi\gamma}{2}\big(\rho^2,\rho\big) &\le \frac{\chi\gamma}{2}\|\rho^2\|_{L^2}\|\rho\|_{L^2} \\ &\le \frac{\chi\gamma C_{gn}}{2}\|\rho\|_{L^2}^2(\|\nabla\rho\|_{L^2}+\|\rho\|_{L^2}) \\ &\le \frac{\chi\gamma C_{gn}}{2}\left(\frac{1}{4\varepsilon}\|\rho\|_{L^2}^4+\varepsilon\|\nabla\rho\|_{L^2}^2+\frac{1}{2}\|\rho\|_{L^2}^4+\frac{1}{2}\|\rho\|_{L^2}^2\right) \\ &= \frac{\chi\gamma C_{gn}}{2}\left(\frac{1}{2}+\left(\frac{1}{4\varepsilon}+\frac{1}{2}\right)\|\rho\|_{L^2}^2\right)\|\rho\|_{L^2}^2+\frac{\chi\gamma C_{gn}}{2}\varepsilon\|\nabla\rho\|_{L^2}^2, \end{aligned} \tag{23}$$

and

$$\begin{aligned} -\frac{\tau\chi}{2}\left(\rho^2,\frac{\partial c}{\partial t}\right) \le \left|-\frac{\tau\chi}{2}\left(\rho^2,\frac{\partial c}{\partial t}\right)\right| &\le \frac{\tau\chi}{2}\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\|\rho^2\|_{L^2} \\ &\le \frac{\tau\chi C_{gn}}{2}\left\|\frac{\partial c}{\partial t}\right\|_{L^2}(\|\nabla\rho\|_{L^2}\|\rho\|_{L^2}+\|\rho\|_{L^2}^2) \\ &\le \frac{\tau\chi C_{gn}}{2}\left(\frac{1}{4\varepsilon}\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2\|\rho\|_{L^2}^2+\varepsilon\|\nabla\rho\|_{L^2}^2+\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\|\rho\|_{L^2}^2\right) \\ &= \frac{\tau\chi C_{gn}}{2}\left(\frac{1}{4\varepsilon}\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2+\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\right)\|\rho\|_{L^2}^2+\frac{\tau\chi C_{gn}}{2}\varepsilon\|\nabla\rho\|_{L^2}^2. \end{aligned} \tag{24}$$

We substitute (21)–(24) into (20) to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{\mathrm{d}t}\|\rho\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2 \\ &\quad\le \frac{\chi C_{gn}}{2}\left(\frac{\gamma}{2}+\left(\frac{\gamma}{4\varepsilon}+\frac{\gamma}{2}\right)\|\rho\|_{L^2}^2+\frac{\tau}{4\varepsilon}\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2+\tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\right)\|\rho\|_{L^2}^2+\frac{\chi C_{gn}\varepsilon}{2}(\gamma+\tau)\|\nabla\rho\|_{L^2}^2. \end{aligned}$$

As long as ε is small enough so that $\frac{\chi C_{gn}\varepsilon}{2}(\gamma+\tau)\le 1$, we have

$$\frac{d}{\mathrm{d}t}\|\rho\|_{L^2}^2 \le K\left(1+\|\rho\|_{L^2}^2+\tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2+\tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\right)\|\rho\|_{L^2}^2,$$

by integrating $\frac{1}{\varepsilon}$ into the constant K. Due to Lemma 3.2, there exists a constant K^0 such that $\int_t^{t+1}\|\rho\|_{L^2}^2\le K^0$, Additionally, by the energy bound (15) in Lemma 3.1 and the energy dissipation (10), there exists a positive constant K_c^0 such that $\int_t^{t+1}\|\frac{\partial c}{\partial t}\|_{L^2}^2\le K_c^0$. Then, we have

$$\int_t^{t+1} K\left(1+\|\rho\|_{L^2}^2+\tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2+\tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\right)\le K(1+K^0+\tau K_c^0).$$

For $t\ge 1$, applying the uniform Gronwall Lemma 1.1 leads to a bound $\|\rho\|_{L^2}^2\le K^0\exp(K(1+K^0+\tau K_c^0))$. For $t<1$, we apply the classical Gronwall lemma to obtain $\|\rho\|_{L^2}^2\le\widetilde{K^0}$, where

$$\widetilde{K^0}:=\|\rho_0\|_{L^2}^2\exp\left(\int_0^1 K\left(1+\|\rho\|_{L^2}^2+\tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2+\tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\right)\mathrm{d}t\right)=\|\rho_0\|_{L^2}^2\exp\big(K(1+K^0+\tau K_c^0)\big).$$

As a result, we have $\|\rho\|_{L^2}^2\le\min\{K^0,\|\rho_0\|_{L^2}^2\}\exp\big(K(1+K^0+\tau K_c^0)\big)$. □

Lemma 3.4. *Assume the small mass condition (8) is satisfied. For any $1<q\le\infty$ and $T\in(0,\infty)$, if initial data $\rho_0\in L^q(\Omega)$, then $\rho(\boldsymbol{x},t)\in L^\infty(0,T;L^q(\Omega))$.*

Remark 3.1. We emphasize here that for $q>2$ the upper bounds are not uniform in T.

Proof. We begin with a finite $q\in(0,\infty)$. Taking the scalar product of (1) with ρ^{2q-1} leads to

$$\left(\frac{\partial\rho}{\partial t},\rho^{2q-1}\right)-\big(\Delta\rho,\rho^{2q-1}\big)=-\chi\big(\nabla\cdot(\rho\nabla c),\rho^{2q-1}\big),\tag{25}$$

where

$$\left(\frac{\partial\rho}{\partial t},\rho^{2q-1}\right)=\frac{1}{2q}\frac{d}{\mathrm{d}t}\|\rho^q\|_{L^2}^2.\tag{26}$$

$$-\big(\Delta\rho,\rho^{2q-1}\big)=\big(\nabla\rho,\nabla\rho^{2q-1}\big)=(2q-1)\|\rho^{q-1}\nabla\rho\|_{L^2}^2=\frac{2q-1}{q^2}\|\nabla(\rho^q)\|_{L^2}^2.\tag{27}$$

Besides, proceeding as above

$$-\chi\big(\nabla\cdot(\rho\nabla c),\rho^{2q-1}\big)=\chi\big(\rho\nabla c,\nabla\rho^{2q-1}\big)=\frac{(2q-1)\chi}{2q}\big(\nabla(\rho^{2q}),\nabla c\big)=\frac{(2q-1)\chi}{2q}\big(\rho^{2q},-\Delta c\big).\tag{28}$$

Using the equation (2) for c and the non-negativity of c, we have

$$\big(\rho^{2q},-\Delta c\big)=\left(\rho^{2q},\gamma\rho-\tau\frac{\partial c}{\partial t}-\alpha c\right)\le\gamma\big(\rho^{2q},\rho\big)-\tau\left(\rho^{2q},\frac{\partial c}{\partial t}\right).\tag{29}$$

Applying Cauchy-Schwarz inequality, Gagliardo-Nirenberg inequality and the ε-Young's inequality in turn, we derive

$$\gamma\big(\rho^{2q},\rho\big)\le\gamma\|\rho\|_{L^2}\big\|\rho^{2q}\big\|_{L^2}$$

$$
\begin{aligned}
&\le \gamma\|\rho\|_{L^2} C_{gn}\|\rho^q\|_{L^2}(\|\nabla(\rho^q)\|_{L^2} + \|\rho^q\|_{L^2}) \\
&\le \frac{\gamma C_{gn}}{4\varepsilon}\|\rho\|_{L^2}^2\|\rho^q\|_{L^2}^2 + \gamma C_{gn}\varepsilon\|\nabla(\rho^q)\|_{L^2}^2 + \gamma C_{gn}\|\rho\|_{L^2}\|\rho^q\|_{L^2}^2 \\
&\le \gamma C_{gn}\left(\frac{K^0}{4\varepsilon} + (K^0)^{\frac{1}{2}}\right)\|\rho^q\|_{L^2}^2 + \gamma C_{gn}\varepsilon\|\nabla(\rho^q)\|_{L^2}^2,
\end{aligned}
\tag{30}
$$

where $\|\rho\|_{L^2}^2 \le K^0$ is known by Lemma 3.3. Similarly,

$$
\begin{aligned}
-\tau\left(\rho^{2q}, \frac{\partial c}{\partial t}\right) &\le \left|-\tau\left(\rho^{2q}, \frac{\partial c}{\partial t}\right)\right| \le \tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\|\rho^{2q}\|_{L^2} \\
&\le \tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2} C_{gn}\|\rho^q\|_{L^2}(\|\nabla(\rho^q)\|_{L^2} + \|\rho^q\|_{L^2}) \\
&\le \tau C_{gn}\left(\frac{1}{4\varepsilon}\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2 + \left\|\frac{\partial c}{\partial t}\right\|_{L^2}\right)\|\rho^q\|_{L^2}^2 + \tau C_{gn}\varepsilon\|\nabla(\rho^q)\|_{L^2}^2.
\end{aligned}
\tag{31}
$$

Substituting (26)–(31) into (25), we obtain

$$
\begin{aligned}
&\frac{1}{2q}\frac{d}{\mathrm{d}t}\|\rho^q\|_{L^2}^2 + \frac{2q-1}{q^2}\|\nabla(\rho^q)\|_{L^2}^2 \\
&\quad\le \frac{(2q-1)\chi C_{gn}}{2q}\left(\frac{\gamma}{4\varepsilon}K^0 + \gamma(K^0)^{\frac{1}{2}} + \frac{\tau}{4\varepsilon}\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2 + \tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\right)\|\rho^q\|_{L^2}^2 \\
&\qquad + \frac{(2q-1)\chi C_{gn}}{2q}(\gamma+\tau)\varepsilon\|\nabla(\rho^q)\|_{L^2}^2.
\end{aligned}
$$

As long as ε is small enough so that $\frac{\chi q C_{gn}}{2}(\gamma+\tau)\varepsilon \le 1$, we have

$$
\frac{d}{\mathrm{d}t}\|\rho^q\|_{L^2}^2 = \frac{d}{\mathrm{d}t}\|\rho\|_{L^{2q}}^{2q} \le Kq\left(\frac{K^0}{\varepsilon} + (K^0)^{\frac{1}{2}} + \frac{\tau}{\varepsilon}\left\|\frac{\partial c}{\partial t}\right\|_{L^2}^2 + \tau\left\|\frac{\partial c}{\partial t}\right\|_{L^2}\right)\|\rho\|_{L^{2q}}^{2q}.
$$

According to Gronwall lemma and Lemma 3.3, we have

$$
\|\rho\|_{L^{2q}}^{2q} \le \|\rho_0\|_{L^{2q}}^{2q}\exp\big(Kq(K^0T + \tau K^0)\big) < \infty, \quad 1 < q < \infty,
$$

with $\frac{1}{\varepsilon}$ being absorbed by the constant K^0.

Finally, we obtain the L^∞-norm estimate by $\|\rho\|_{L^{2q}} \to \|\rho\|_{L^\infty}$ when $q \to \infty$. □

4. Properties of the semi-discrete scheme

Let δt be the time step of the semi-discrete scheme (6) and (7), and let $N_t \in \mathbb{N}^+$ such that $\delta t N_t = T < \infty$. We first address the well-posedness of the scheme.

Lemma 4.1. *There exists a positive constant K^0 such that if $\delta t K^0 \le 1$, then for any $(\rho^n, c^n) \in H^1(\Omega)\times H^1(\Omega)$, the semi-discrete scheme (6) and (7) admits a unique solution $(\rho^{n+1}, c^{n+1}) \in H^1(\Omega)\times H^1(\Omega)$.*

Proof. The scheme is implemented in two steps: first solve equation (6), then solve equation (7). Solving equation (7) is straightforward. To solve equation (6), we apply Lax-Milgram theorem to $B(\rho^{n+1}, v) = \int_\Omega \rho^n v\mathrm{d}\boldsymbol{x}$ for any $v \in H^1(\Omega)$, where the bilinear form B reads

$$
B(\rho, v) = \int_\Omega \rho v\mathrm{d}\boldsymbol{x} + \delta t\int_\Omega \nabla\rho\cdot\nabla v\mathrm{d}\boldsymbol{x} - \chi\delta t\int_\Omega \rho\nabla c^n\cdot\nabla v\mathrm{d}\boldsymbol{x}.
$$

The main issue is the coercivity of B. Indeed

$$\left|\chi\delta t\int_\Omega \rho\nabla c^n\cdot\nabla\rho \mathrm{d}\boldsymbol{x}\right| \le \chi\delta t\|\rho\|_{L^4}\|\nabla c^n\|_{L^4}\|\nabla\rho\|_{L^2}.$$

Using Gagliardo-Nirenberg inequality, we have

$$\|\rho\|_{L^4} \le C_{gn}^{\frac{1}{4}}\Big(\|\rho\|_{L^2}+\|\rho\|_{L^2}^{\frac{1}{2}}\|\nabla\rho\|_{L^2}^{\frac{1}{2}}\Big).$$

Then, we obtain the coercivity if $\delta t\|\nabla c^n\|_{L^4}^4$ is small enough. Now we claim the lemma below.

Lemma 4.2. *The exists a positive constant K such that*

$$\max_{k\le n}\|\nabla c^n\|_{L^4} \le K\max_{k\le n}\|\nabla\rho^n\|_{L^4}.$$

This lemma is true in the parabolic-elliptic case and the parabolic-parabolic case. Additionally, its proof is similar to the proof of Lemma 6.1 in Section 6. Here we omit it.

Applying Lemma 5.3 concludes the proof. □

Theorem 4.1. *For any $n\in\{0,1,2,\ldots,N_t-1\}$, the semi-discrete scheme (6) and (7) satisfies the following properties:*

- *Mass conservation: $\int_\Omega \rho^{n+1}\mathrm{d}\boldsymbol{x} = \int_\Omega \rho^n\mathrm{d}\boldsymbol{x}$.*
- *Positivity preservation: if $\rho^n, c^n\ge 0$, then $\rho^{n+1}\ge 0$ and $c^{n+1}\ge 0$; if $\rho^n, c^n>0$, then $\rho^{n+1}>0$ and $c^{n+1}>0$.*
- *Energy dissipation:*

$$\begin{aligned}E_{\mathrm{tot}}(\rho^{n+1},c^{n+1})-E_{\mathrm{tot}}(\rho^n,c^n)+\delta t\int_\Omega \rho^{n+1}|\nabla\big(\log\rho^{n+1}-\chi c^n\big)|^2\mathrm{d}\boldsymbol{x}\\ +\left(\frac{\chi\tau}{\gamma\delta t}+\frac{\chi\alpha}{2\gamma}\right)\|c^{n+1}-c^n\|_{L^2}^2+\frac{\chi}{2\gamma}\|\nabla c^{n+1}-\nabla c^n\|_{L^2}^2\le 0,\end{aligned} \tag{32}$$

where

$$E_{\mathrm{tot}}(\rho^{n+1},c^{n+1}) = \int_\Omega\left(f\big(\rho^{n+1}\big)-\chi\rho^{n+1}c^{n+1}+\frac{\chi}{2\gamma}|\nabla c^{n+1}|^2+\frac{\alpha\chi}{2\gamma}\big(c^{n+1}\big)^2\right)\mathrm{d}\boldsymbol{x}, \tag{33}$$

with $f(\rho^{n+1})=\rho^{n+1}\log\rho^{n+1}-\rho^{n+1}$.

Proof. Integrating the equation (6) over the domain Ω, we deduce

$$\int_\Omega \rho^{n+1}\mathrm{d}\boldsymbol{x} = \int_\Omega \rho^n\mathrm{d}\boldsymbol{x},$$

with the homogeneous Neumann boundary condition for ρ^{n+1}.

Let $\rho_+^n := \sup\{\rho^n,0\}$, $\rho_-^n := \sup\{-\rho^n,0\}$, then $\rho^n=\rho_+^n-\rho_-^n$ and $|\rho^n|=\rho_+^n+\rho_-^n$. Multiplying both sides of equation (6) by the sign function $\mathrm{sgn}\rho^{n+1}$, we have

$$\frac{\rho^{n+1}\mathrm{sgn}\rho^{n+1}-\rho^n\mathrm{sgn}\rho^{n+1}}{\delta t} = \Delta\rho^{n+1}\mathrm{sgn}\rho^{n+1}-\chi\nabla\cdot\big(\rho^{n+1}\nabla c^n\big)\mathrm{sgn}\rho^{n+1}.$$

Using the property of symbolic function and Kato's inequality, we have

$$|\rho^{n+1}|\le|\rho^n|+\delta t\Delta|\rho^{n+1}|-\delta t\chi\nabla\cdot(|\rho^{n+1}|\nabla c^n).$$

Integrating both sides of the inequality over the domain Ω, we obtain

$$\int_\Omega |\rho^{n+1}|\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\rho_+^{n+1} + \rho_-^{n+1}\big)\mathrm{d}\boldsymbol{x} \le \int_\Omega |\rho^n|\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\rho_+^n + \rho_-^n\big)\mathrm{d}\boldsymbol{x}, \tag{34}$$

where we apply the homogeneous Neumann boundary condition for ρ^{n+1}. Recall the mass conservation

$$\int_\Omega \rho^{n+1}\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\rho_+^{n+1} - \rho_-^{n+1}\big)\mathrm{d}\boldsymbol{x} = \int_\Omega \rho^n\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\rho_+^n - \rho_-^n\big)\mathrm{d}\boldsymbol{x}.$$

Subtracting the above from (34), we derive

$$\int_\Omega \rho_-^{n+1}\mathrm{d}\boldsymbol{x} \le \int_\Omega \rho_-^n\,\mathrm{d}\boldsymbol{x}, \quad \text{and} \quad \rho_-^{n+1} = \rho_-^n = 0,$$

due to $\rho^0 = \rho_0 \ge 0$ and $\rho_0 \not\equiv 0$. Therefore, $\rho^{n+1} \ge 0$.

To check that $\rho^{n+1} > 0$ in Ω when $\rho_0 > 0$ we proceed as follows. Since ρ_0 belongs to L^∞, then ρ^1 is smooth. Assume that c_0 is smooth. If there exists a x_0 in Ω such that $\rho^1(x_0) = 0$, then $\nabla\rho^1(x_0) = 0$ and $\Delta\rho^1(x_0) \ge 0$. Returning to equation (6), we get $\rho_0(x_0) \le 0$, which leads to a contradiction. We conclude by induction on n.

From equation (7) for c^{n+1}, we have in the parabolic-elliptic case $\tau = 0$,

$$c^{n+1} = \gamma(-\Delta + \alpha I)^{-1}\rho^{n+1}.$$

In the parabolic-parabolic case $\tau > 0$ the proof is similar as the above. Therefore, the positivity of c^{n+1} is consistent with that of ρ^{n+1}.

For the energy dissipation, we first rewrite (6) as

$$\frac{\rho^{n+1} - \rho^n}{\delta t} = \nabla\cdot\big(\rho^{n+1}\nabla\big(\log\rho^{n+1} - \chi c^n\big)\big). \tag{35}$$

Taking the inner product of (35) with $\delta t\big(\log\rho^{n+1} - \chi c^n\big)$, we get

$$\int_\Omega \big(\rho^{n+1} - \rho^n\big)\big(\log\rho^{n+1} - \chi c^n\big)\mathrm{d}\boldsymbol{x} = -\delta t\int_\Omega \rho^{n+1}|\nabla\big(\log\rho^{n+1} - \chi c^n\big)|^2\mathrm{d}\boldsymbol{x}. \tag{36}$$

Noting that $f(\rho^n) = \rho^n\log\rho^n - \rho^n$ is convex for $\rho^n > 0$, we can use $f'(\rho^n) = \log\rho^n$ to derive

$$\begin{aligned}\int_\Omega \big(\rho^{n+1} - \rho^n\big)f'(\rho^{n+1})\mathrm{d}\boldsymbol{x} &= \int_\Omega \left(f(\rho^{n+1}) - f(\rho^n) + \frac{1}{2}(\rho^{n+1} - \rho^n)^2 f''(\xi)\right)\mathrm{d}\boldsymbol{x} \\ &\ge \int_\Omega \big(f(\rho^{n+1}) - f(\rho^n)\big)\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{37}$$

Taking the inner product of (7) with $c^{n+1} - c^n$, we get

$$\begin{aligned}&\frac{\tau}{\delta t}\|c^{n+1} - c^n\|_{L^2}^2 + \frac{1}{2}\|\nabla c^{n+1}\|_{L^2}^2 - \frac{1}{2}\|\nabla c^n\|_{L^2}^2 + \frac{1}{2}\|\nabla c^{n+1} - \nabla c^n\|_{L^2}^2 \\ &\quad + \frac{\alpha}{2}\|c^{n+1}\|_{L^2}^2 - \frac{\alpha}{2}\|c^n\|_{L^2}^2 + \frac{\alpha}{2}\|c^{n+1} - c^n\|_{L^2}^2 - \gamma(\rho^{n+1}, c^{n+1} - c^n) = 0.\end{aligned} \tag{38}$$

We sum up (36)–(38) to obtain the discrete energy dissipation

$$\begin{aligned}&E_{\mathrm{tot}}(\rho^{n+1}, c^{n+1}) - E_{\mathrm{tot}}(\rho^n, c^n) \\ &\quad \le -\delta t\int_\Omega \rho^{n+1}|\nabla\big(\log\rho^{n+1} - \chi c^n\big)|^2\mathrm{d}\boldsymbol{x} - \frac{\chi}{2\gamma}\|\nabla c^{n+1} - \nabla c^n\|_{L^2}^2 \\ &\quad\quad - \frac{\chi}{\gamma}\left(\frac{\tau}{\delta t} + \frac{\alpha}{2}\right)\|c^{n+1} - c^n\|_{L^2}^2 \le 0.\end{aligned} \tag{39}$$

□

5. Bounds of the numerical solution

Following the analytical framework of Section 3, we derive L^q-bounds ($1 < q \le \infty$) for the numerical solution ρ^{n+1} of the semi-implicit Euler scheme (6) and (7) for any $n \in \{0, 1, 2, \dots, N_t - 1\}$. Additionally, we assume that the small mass assumption (8) is satisfied throughout this section and fix the time $T = \delta t N_t < \infty$.

Lemma 5.1. *There exists a positive constant K^0 such that $\delta t \sum_{n=0}^{N_t-1} \|\rho^{n+1}\|_{L^2}^2 \le K^0$.*

Proof. Applying the expansion (16) in Lemma 3.2, we have that

$$\begin{aligned}
&(1-(2+\tau)\chi\gamma C_{gn}M)\|\rho^{n+1}\|_{L^2}^2 \\
&\quad \le (1-(2+\tau)\chi\gamma C_{gn}M)C_{gn}M\left(\int_\Omega \frac{|\nabla\rho^{n+1}|^2}{\rho^{n+1}}\mathrm{d}\boldsymbol{x} + M\right) \\
&\quad \le C_{gn}M\int_\Omega \rho^{n+1}|\nabla\big(\log\rho^{n+1} - \chi c^{n+1}\big)|^2\mathrm{d}\boldsymbol{x} + \frac{\tau\chi}{\gamma}\left\|\frac{c^{n+1}-c^n}{\delta t}\right\|_{L^2}^2 + K,
\end{aligned} \tag{40}$$

where K is some positive constant depending on the parameters and the mass of cells M.

We note that the first term in the right hand side of (40) is not exactly the same as that in the energy dissipation (39) with respect to the discretization of c. To address this discrepancy, we proceed as follows.

$$\begin{aligned}
&\int_\Omega \rho^{n+1}|\nabla\big(\log\rho^{n+1} - \chi c^{n+1}\big)|^2\mathrm{d}\boldsymbol{x} \\
&\quad \le 2\int_\Omega \rho^{n+1}|\nabla\big(\log\rho^{n+1} - \chi c^{n}\big)|^2\mathrm{d}\boldsymbol{x} + 2\int_\Omega \chi\rho^{n+1}|\nabla c^{n+1} - \nabla c^n|^2\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{41}$$

Using Cauchy-Schwarz inequality and Gagliardo-Niremberg inequality yields that

$$\begin{aligned}
&2\int_\Omega \chi\rho^{n+1}|\nabla c^{n+1} - \nabla c^n|^2 \\
&\quad \le 2\chi\|\rho^{n+1}\|_{L^2}\|\nabla c^{n+1} - \nabla c^n\|_{L^4}^2 \\
&\quad \le 2\chi C_{gn}\|\rho^{n+1}\|_{L^2}\|\nabla c^{n+1} - \nabla c^n\|_{L^2}\|\Delta c^{n+1} - \Delta c^n\|_{L^2} \\
&\quad \le 2\chi C_{gn}\|\rho^{n+1}\|_{L^2}\|\nabla c^{n+1} - \nabla c^n\|_{L^2}\big(\|\Delta c^{n+1}\|_{L^2} + \|\Delta c^n\|_{L^2}\big).
\end{aligned} \tag{42}$$

Define the operator $A = -\Delta + \alpha I$. Squaring and integrating (7), we have in the parabolic-parabolic case

$$\|Ac^{n+1}\|_{L^2}^2 + \frac{\tau}{\delta t}\left\|A^{\frac12}c^{n+1}\right\|_{L^2}^2 \le \frac{\tau}{\delta t}\left\|A^{\frac12}c^{n}\right\|_{L^2}^2 + \gamma^2\|\rho^{n+1}\|_{L^2}^2. \tag{43}$$

Summing this inequality leads to $\frac{\tau}{\delta t}\left\|A^{\frac12}c^n\right\|_{L^2}^2 \le \gamma^2\sum_{k=0}^{n}\|\rho^k\|_{L^2}^2$ with $c^{-1} = 0$. Gathering this with (43), we obtain

$$\|Ac^{n+1}\|_{L^2}^2 \le \gamma^2\sum_{k=0}^{n+1}\|\rho^k\|_{L^2}^2.$$

Then, we have

$$\|-\Delta c^{n+1}\|_{L^2}^2 \le \|Ac^{n+1}\|_{L^2}^2 \le \gamma^2\sum_{k=0}^{n+1}\|\rho^k\|_{L^2}^2.$$

This estimate is also valid in the parabolic-elliptic case. Finally, we substitute this estimate into the inequality (42) and apply the ε-Young's inequality to get

$$\begin{aligned}
&2\int_\Omega \chi\rho^{n+1}|\nabla c^{n+1}-\nabla c^n|^2\\
&\le 4\chi C_{gn}\|\rho^{n+1}\|_{L^2}\|\nabla c^{n+1}-\nabla c^n\|_{L^2}\left(\sum_{k=0}^{n+1}\gamma^2\|\rho^k\|_{L^2}^2\right)^{\frac12}\\
&\le 4\chi\gamma C_{gn}\|\rho^{n+1}\|_{L^2}^2\|\nabla c^{n+1}-\nabla c^n\|_{L^2}+4\chi C_{gn}\|\rho^{n+1}\|_{L^2}\|\nabla c^{n+1}-\nabla c^n\|_{L^2}\left(\sum_{k=0}^{n}\gamma^2\|\rho^k\|_{L^2}^2\right)^{\frac12}\\
&\le \left(\varepsilon+4\chi\gamma C_{gn}\|\nabla c^{n+1}-\nabla c^n\|_{L^2}\right)\|\rho^{n+1}\|_{L^2}^2+\frac{1}{\varepsilon}\chi\gamma^2C_{gn}\|\nabla c^{n+1}-\nabla c^n\|_{L^2}^2\left(\sum_{k=0}^{n}\|\rho^k\|_{L^2}^2\right).
\end{aligned} \tag{44}$$

Since the energy E_{tot} is bounded from below by Lemma 3.1, the series $\sum_n\|\nabla c^{n+1}-\nabla c^n\|_{L^2}^2$ is convergent by applying the discrete energy dissipation (39). Therefore, there exists an integer N large enough and a ε small enough such that

$$\varepsilon+4\chi\gamma C_{gn}\|\nabla c^{n+1}-\nabla c^n\|_{L^2}\le\frac12(1-(2+\tau)\chi\gamma C_{gn}M). \tag{45}$$

Gathering (40) and (41), and (44) and (45), we obtain that for $n\ge N$

$$\begin{aligned}
\|\rho^{n+1}\|_{L^2}^2\le \widetilde{K}\|\nabla c^{n+1}-\nabla c^n\|_{L^2}^2\left(\sum_{k=0}^{n}\|\rho^k\|_{L^2}^2\right)+K\bigg(4\int_\Omega\rho^{n+1}|\nabla\big(\log\rho^{n+1}-\chi c^n\big)|^2\mathrm{d}\boldsymbol{x}\\
+\frac{2\tau\chi}{\gamma}\left\|\frac{c^{n+1}-c^n}{\delta t}\right\|_{L^2}^2+1\bigg).
\end{aligned} \tag{46}$$

Let $S_{n+1}=\delta t\sum_{k=0}^{n+1}\|\rho^k\|_{L^2}^2$, then (46) reads

$$\begin{aligned}
S_{n+1}-S_n\le\widetilde{K}\|\nabla(c^{n+1}-c^n)\|_{L^2}^2S_n+K\delta t\Big(4\int_\Omega\rho^{n+1}|\nabla\big(\log\rho^{n+1}-\chi c^n\big)|^2\mathrm{d}\boldsymbol{x}\\
+\frac{2\tau\chi}{\gamma\delta t}\|c^{n+1}-c^n\|_{L^2}^2+\delta t\Big).
\end{aligned} \tag{47}$$

Now we apply the discrete Gronwall lemma with two sequences

$$a_n=\widetilde{K}\|\nabla(c^{n+1}-c^n)\|_{L^2}^2,$$

and

$$b_n=K\delta t\Big(4\int_\Omega\rho^{n+1}|\nabla\big(\log\rho^{n+1}-\chi c^n\big)|^2\mathrm{d}\boldsymbol{x}+\frac{2\tau\chi}{\gamma\delta t}\|c^{n+1}-c^n\|_{L^2}^2+\delta t\Big).$$

The sequences a_n and b_n defined from (47) are summable due to the discrete dissipation energy (39). Moreover, $\delta t\sum_{n\le N}||\rho^n||_{L^2}^2$ is bounded. Therefore, there exists a positive constant K^0 such that

$$S_{N_t}=\delta t\sum_{n=0}^{N_t}\|\rho^n\|_{L^2}^2\le K^0.$$

□

Lemma 5.2. *For any $n \in \{0, 1, 2, \dots, N_t - 1\}$ and the initial data $\rho_0 \in L^2(\Omega)$, there exists a positive constant K^0 such that $\|\rho^{n+1}\|_{L^2} \le K^0$.*

Proof. Taking the scalar product of (6) with ρ^{n+1} leads to

$$\left(\frac{\rho^{n+1} - \rho^n}{\delta t}, \rho^{n+1}\right) - \left(\Delta \rho^{n+1}, \rho^{n+1}\right) = -\left(\chi \nabla \cdot \left(\rho^{n+1} \nabla c^n\right), \rho^{n+1}\right), \tag{48}$$

where

$$\left(\frac{\rho^{n+1} - \rho^n}{\delta t}, \rho^{n+1}\right) = \frac{1}{2\delta t}\left(\|\rho^{n+1}\|_{L^2}^2 - \|\rho^n\|_{L^2}^2 + \|\rho^{n+1} - \rho^n\|_{L^2}^2\right), \tag{49}$$

$$-\left(\Delta \rho^{n+1}, \rho^{n+1}\right) = \|\nabla \rho^{n+1}\|_{L^2}^2, \tag{50}$$

and

$$-\left(\chi \nabla \cdot \left(\rho^{n+1} \nabla c^n\right), \rho^{n+1}\right) = \chi\left(\rho^{n+1} \nabla c^n, \nabla \rho^{n+1}\right) = \frac{\chi}{2}\left(|\rho^{n+1}|^2, -\Delta c^n\right). \tag{51}$$

Using the non-negativity of c^n for any $n \in \{0, 1, 2, \dots, N_t - 1\}$, we derive from equation (7) that

$$\begin{aligned}\frac{\chi}{2}\left(|\rho^{n+1}|^2, -\Delta c^n\right) &= \frac{\chi}{2}\left(|\rho^{n+1}|^2, -\tau \frac{c^n - c^{n-1}}{\delta t} - \alpha c^n + \gamma \rho^n\right) \\ &\le \frac{\chi\gamma}{2}\left(|\rho^{n+1}|^2, \rho^n\right) - \frac{\tau\chi}{2}\left(|\rho^{n+1}|^2, \frac{c^n - c^{n-1}}{\delta t}\right).\end{aligned} \tag{52}$$

Applying Cauchy-Schwarz inequality, Gagliardo-Nirenberg inequality and the ε-Young's inequality in turn, we derive

$$\begin{aligned}\frac{\chi\gamma}{2}\left(|\rho^{n+1}|^2, \rho^n\right) &\le \frac{\chi\gamma}{2}\|\rho^n\|_{L^2}\|(\rho^{n+1})^2\|_{L^2} \\ &\le \frac{\chi\gamma C_{gn}}{2}\|\rho^n\|_{L^2}\|\rho^{n+1}\|_{L^2}(\|\rho^{n+1}\|_{L^2} + \|\nabla \rho^{n+1}\|_{L^2}) \\ &\le \frac{\chi\gamma C_{gn}}{2}\left(\frac{1}{4\varepsilon}\|\rho^n\|_{L^2}^2 + \|\rho^n\|_{L^2}\right)\|\rho^{n+1}\|_{L^2}^2 + \frac{\chi\gamma C_{gn}}{2}\varepsilon\|\nabla \rho^{n+1}\|_{L^2}^2,\end{aligned} \tag{53}$$

and similarly

$$\begin{aligned}-\frac{\tau\chi}{2}\left(|\rho^{n+1}|^2, \frac{c^n - c^{n-1}}{\delta t}\right) \le \frac{\tau\chi C_{gn}}{2}&\left(\frac{1}{4\varepsilon}\left\|\frac{c^n - c^{n-1}}{\delta t}\right\|_{L^2}^2 + \left\|\frac{c^n - c^{n-1}}{\delta t}\right\|_{L^2}\right)\|\rho^{n+1}\|_{L^2}^2 \\ &+ \frac{\tau\chi C_{gn}}{2}\varepsilon\|\nabla \rho^{n+1}\|_{L^2}^2.\end{aligned} \tag{54}$$

Substituting (49)–(54) into (48), we obtain

$$\begin{aligned}&\frac{1}{2\delta t}\left(\|\rho^{n+1}\|_{L^2}^2 - \|\rho^n\|_{L^2}^2 + \|\rho^{n+1} - \rho^n\|_{L^2}^2\right) + \|\nabla \rho^{n+1}\|_{L^2}^2 \\ &\le \frac{\chi C_{gn}}{2}\left(\frac{\gamma}{4\varepsilon}\|\rho^n\|_{L^2}^2 + \gamma\|\rho^n\|_{L^2} + \frac{\tau}{4\varepsilon}\left\|\frac{c^n - c^{n-1}}{\delta t}\right\|_{L^2}^2 + \tau\left\|\frac{c^n - c^{n-1}}{\delta t}\right\|_{L^2}\right)\|\rho^{n+1}\|_{L^2}^2 \\ &\quad + \frac{\chi C_{gn}}{2}\varepsilon(\gamma + \tau)\|\nabla \rho^{n+1}\|_{L^2}^2.\end{aligned} \tag{55}$$

As long as ε is small enough so that $\frac{\chi C_{gn}}{2}(\gamma + \tau)\varepsilon \le 1$, we have

$$\begin{aligned}\|\rho^{n+1}\|_{L^2}^2 \le \|\rho^n\|_{L^2}^2 + \chi C_{gn}\delta t&\left(\frac{\gamma}{\varepsilon}\|\rho^n\|_{L^2}^2 + \gamma\|\rho^n\|_{L^2} + \frac{\tau}{\varepsilon}\left\|\frac{c^n - c^{n-1}}{\delta t}\right\|_{L^2}^2\right. \\ &\left. + \tau\left\|\frac{c^n - c^{n-1}}{\delta t}\right\|_{L^2}\right)\|\rho^{n+1}\|_{L^2}^2.\end{aligned} \tag{56}$$

From Lemma 5.1, we have $\delta t \sum_{n=0}^{N_t} \|\rho^n\|_{L^2}^2 \le K^0$. Moreover, according to the energy bound (15) in Lemma 3.1 and the discrete energy dissipation formula (39), we know $\tau\delta t \sum_{n=1}^{N_t} \|\frac{c^n-c^{n-1}}{\delta t}\|_{L^2}^2 \le E_{\mathrm{tot}}(\rho_0,c_0) - E_{\mathrm{tot}}(\rho^{N_t},c^{N_t}) \le 2K_e^0$. Taking the sum of (56) on n and applying the discrete Gronwall lemma completes the proof. □

Lemma 5.3. *For any $n \in \{0,1,2,\dots,N_t-1\}$, $1 < q \le \infty$, and the initial data $\rho_0 \in L^q(\Omega)$, there exists a positive constant K^0 such that $\|\rho^{n+1}\|_{L^q} \le K^0$.*

Remark 5.1. We emphasize that for $q > 2$ the upper bound in Lemma 5.3 depends on T.

Proof. Taking the scalar product of (6) with $(\rho^{n+1})^{2q-1}$ leads to

$$\left(\frac{\rho^{n+1}-\rho^n}{\delta t}, (\rho^{n+1})^{2q-1}\right) - \big(\Delta\rho^{n+1}, (\rho^{n+1})^{2q-1}\big) = -\big(\chi\nabla\cdot\big(\rho^{n+1}\nabla c^n\big), (\rho^{n+1})^{2q-1}\big).$$

Using the convexity of the function $x \mapsto x^{2q}$ we have

$$\begin{aligned}\left(\frac{\rho^{n+1}-\rho^n}{\delta t}, (\rho^{n+1})^{2q-1}\right) &\ge \frac{1}{2q\delta t}(||\rho^{n+1}||_{L^{2q}}^{2q} - ||\rho^n||_{L^{2q}}^{2q}),\\ -\big(\Delta\rho^{n+1}, (\rho^{n+1})^{2q-1}\big) &= \frac{2q-1}{q^2}\|\nabla((\rho^{n+1})^q)\|_{L^2}^2,\end{aligned}$$

and

$$\begin{aligned}-\big(\chi\nabla\cdot\big(\rho^{n+1}\nabla c^n\big), (\rho^{n+1})^{2q-1}\big) &= \chi(2q-1)\big((\rho^{n+1})^{2q-1}\nabla c^n, \nabla\rho^{n+1}\big)\\ &= \frac{(2q-1)\chi}{2q}\big(|\rho^{n+1}|^{2q}, -\Delta c^n\big).\end{aligned}$$

According to the equation (7) for c^{n+1} and the non-negativity of c^{n+1} for any $n \in \{0,1,2,\dots,N_t-1\}$, we have

$$\begin{aligned}\big(|\rho^{n+1}|^{2q}, -\Delta c^n\big) &= \left(|\rho^{n+1}|^{2q}, -\tau\frac{c^n-c^{n-1}}{\delta t} - \alpha c^n + \gamma\rho^n\right)\\ &\le \gamma\big(|\rho^{n+1}|^{2q}, \rho^n\big) - \tau\left(|\rho^{n+1}|^{2q}, \frac{c^n-c^{n-1}}{\delta t}\right).\end{aligned}$$

Applying Cauchy-Schwarz inequality, Gagliardo-Nirenberg inequality and the ε-Young's inequality in turn, we derive

$$\begin{aligned}\gamma\big(|\rho^{n+1}|^{2q}, \rho^n\big) &\le \gamma\|\rho^n\|_{L^2}\|(\rho^{n+1})^{2q}\|_{L^2}\\ &\le \gamma C_{gn}\|\rho^n\|_{L^2}\|(\rho^{n+1})^q\|_{L^2}\big(\|\nabla((\rho^{n+1})^q)\|_{L^2} + \|(\rho^{n+1})^q\|_{L^2}\big)\\ &\le \gamma C_{gn}\left(\frac{1}{4\varepsilon}\|\rho^n\|_{L^2}^2 + \|\rho^n\|_{L^2}\right)\|(\rho^{n+1})^q\|_{L^2}^2 + \gamma C_{gn}\varepsilon\|\nabla((\rho^{n+1})^q)\|_{L^2}^2\\ &\le \gamma C_{gn}\left(\frac{1}{4\varepsilon}K^0 + (K^0)^{\frac{1}{2}}\right)\|(\rho^{n+1})^q\|_{L^2}^2 + \gamma C_{gn}\varepsilon\|\nabla((\rho^{n+1})^q)\|_{L^2}^2,\end{aligned}$$

where $\|\rho^n\|_{L^2}^2 \le K^0$ is known in Lemma 5.2. Similarly,

$$\begin{aligned}-\tau\left(|\rho^{n+1}|^{2q}, \frac{c^n-c^{n-1}}{\delta t}\right) \le \tau C_{gn}&\left(\frac{1}{4\varepsilon}\left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}^2 + \left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}\right)\|(\rho^{n+1})^q\|_{L^2}^2\\ &+ \tau C_{gn}\varepsilon\|\nabla((\rho^{n+1})^q)\|_{L^2}^2.\end{aligned}$$

Gathering the inequalities above, we obtain

$$\begin{aligned}
&\frac{1}{2q\delta t}\|\rho^{n+1}\|_{L^{2q}}^{2q}+\frac{2q-1}{q^2}\|\nabla((\rho^{n+1})^q)\|_{L^2}^2\\
&\quad\le \frac{1}{2q\delta t}\|\rho^{n}\|_{L^{2q}}^{2q}+\frac{(2q-1)\chi C_{gn}\varepsilon}{2q}(\gamma+\tau)\|\nabla((\rho^{n+1})^q)\|_{L^2}^2\\
&\qquad+\frac{(2q-1)\chi C_{gn}}{2q}\left(\frac{\gamma}{4\varepsilon}K^0+\gamma(K^0)^{\frac12}+\frac{\tau}{4\varepsilon}\left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}^2+\tau\left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}\right)\|(\rho^{n+1})^q\|_{L^2}^2.
\end{aligned}$$

If ε is small enough so that $\frac{\chi q C_{gn}}{2}(\gamma+\tau)\varepsilon\le 1$, we have

$$\begin{aligned}
\|\rho^{n+1}\|_{L^{2q}}^{2q}\le \|\rho^{n}\|_{L^{2q}}^{2q}+(2q-1)\chi C_{gn}\delta t\Bigg(&\frac{\gamma}{4\varepsilon}K^0+\gamma(K^0)^{\frac12}+\frac{\tau}{4\varepsilon}\left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}^2\\
&+\tau\left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}\Bigg)\|(\rho^{n+1})^q\|_{L^2}^2.
\end{aligned}\tag{57}$$

By the estimate $\tau\delta t\sum_{n=1}^{N_t}\|\frac{c^n-c^{n-1}}{\delta t}\|_{L^2}^2\le 2K_e^0$ in the proof of Lemma 5.2, we take the sum of (57) on n and apply the discrete Gronwall lemma to obtain

$$\|\rho^{n+1}\|_{L^{2q}}^{2q}\le\|\rho_0\|_{L^{2q}}^{2q}\exp\Big(qT\Big(K^0+(K^0)^{\frac12}\Big)+qK_e^0\Big)<\infty,\quad 1<q<\infty,$$

with $\frac{1}{\varepsilon}$ being absorbed by the positive constant K^0. Taking the $\frac{1}{q}$-th power of this inequality and letting $q\to\infty$, we derive the L^∞ estimate. □

6. Error estimates

We now rigorously derive the error estimate in L^p-norm $(1<p<\infty)$ for the semi-implicit Euler scheme (6) and (7). For consistency analysis, we assume that the exact solutions ρ and c are sufficiently smooth, i.e to ensure (61) and (62) hold. Moreover, the bounds of exact solutions in Section 3 and the bounds of numerical solutions in Section 5 require the small mass condition (8).

Firstly we define error functions as follows:

$$e_\rho^{n+1}=\rho(\cdot,t^{n+1})-\rho^{n+1},\quad e_c^{n+1}=c(\cdot,t^{n+1})-c^{n+1},\quad n=0,1,2,\dots,N_t-1.\tag{58}$$

Then, we replace approximate solutions (ρ^{n+1},c^{n+1}) in semi-discrete scheme (6) and (7) with exact solutions $(\rho(\cdot,t^{n+1}),c(\cdot,t^{n+1}))$ to get

$$\frac{\rho(t^{n+1})-\rho(t^n)}{\delta t}-\Delta\rho(t^{n+1})+\chi\nabla\cdot\big(\rho(t^{n+1})\nabla c(t^n)\big)=T_\rho(t^{n+1}),\tag{59}$$

$$\tau\frac{c(t^{n+1})-c(t^n)}{\delta t}-\Delta c(t^{n+1})+\alpha c(t^{n+1})-\gamma\rho(t^{n+1})=\tau T_c(t^{n+1}).\tag{60}$$

Assume that the solutions (ρ,c) of the KS equations (1) and (2) are smooth enough, by a standard Taylor expansion at t^{n+1}, we have the truncation error

$$T_\rho(t^{n+1})=-\delta t\left(\frac12\frac{\partial^2\rho(t^{n+1})}{\partial t^2}+\chi\nabla\cdot\left(\rho(t^{n+1})\nabla\frac{\partial c(t^{n+1})}{\partial t}\right)\right)+O(\delta t^2),\tag{61}$$

$$T_c(t^{n+1})=-\delta t\left(\frac12\frac{\partial^2 c(t^{n+1})}{\partial t^2}\right)+O(\delta t^2).\tag{62}$$

Subtracting the semi-discrete equation (6) from the consistency estimate (59), we have

$$\frac{e_\rho^{n+1}-e_\rho^n}{\delta t} - \Delta e_\rho^{n+1} + \chi\nabla\cdot\big(\rho(t^{n+1})\nabla c(t^n) - \rho^{n+1}\nabla c^n\big) = T_\rho(t^{n+1}).$$

Since that

$$\begin{aligned}\rho(t^{n+1})\nabla c(t^n) - \rho^{n+1}\nabla c^n \\ &= \rho(t^{n+1})\nabla c(t^n) - \rho(t^{n+1})\nabla c^n + \rho(t^{n+1})\nabla c^n - \rho^{n+1}\nabla c^n \\ &= \rho(t^{n+1})\nabla e_c^n + e_\rho^{n+1}\nabla c^n,\end{aligned}$$

we can rewrite the error equation for e_ρ^{n+1} as

$$\frac{e_\rho^{n+1}-e_\rho^n}{\delta t} - \Delta e_\rho^{n+1} + \chi\nabla\cdot\big(\rho(t^{n+1})\nabla e_c^n + e_\rho^{n+1}\nabla c^n\big) = T_\rho(t^{n+1}). \tag{63}$$

Subtract the semi-discrete equation (7) from the consistency estimate (60), we obtain the error equation for e_c^{n+1}

$$\tau\frac{e_c^{n+1}-e_c^n}{\delta t} - \Delta e_c^{n+1} + \alpha e_c^{n+1} - \gamma e_\rho^{n+1} = \tau T_c(t^{n+1}). \tag{64}$$

When analyzing the L^p-bound of e_ρ^{n+1}, the estimate of ∇e_c^{n+1} for any $n \in \{0,1,2,\dots,N_t-1\}$ is a key point. Hence, we give the following lemma in advance.

Lemma 6.1. *For any $n \in \{0,1,2,\dots,N_t-1\}$, in the parabolic-elliptic case $\tau = 0$, there exists a positive constant K such that*

$$\|\nabla e_c^{n+1}\|_{L^{2p}}^{2p} \le K\|e_\rho^{n+1}\|_{L^{2p}}^{2p}, \tag{65}$$

In the parabolic-parabolic case $\tau > 0$, there exists a positive constant K such that

$$\max_{1\le k\le n+1}\|\nabla e_c^k\|_{L^{2p}}^{2p} \le K\bigg(\max_{1\le k\le n+1}\|e_\rho^k\|_{L^{2p}}^{2p} + \delta t\bigg). \tag{66}$$

Proof. In the parabolic-elliptic case $\tau = 0$, the error equation (64) for e_c^{n+1} reads

$$-\Delta e_c^{n+1} + \alpha e_c^{n+1} = \gamma e_\rho^{n+1}.$$

Then, the inquality (65) follows from the standard elliptic estimate.

In the parabolic-parabolic case $\tau > 0$, the error equation (64) for e_c^{n+1} reads

$$\bigg(I + \frac{\delta t}{\tau}A\bigg)e_c^{n+1} = e_c^n + \delta t T_c(t^{n+1}) + \frac{\gamma\delta t}{\tau}e_\rho^{n+1},$$

where we set the operator $A = -\Delta + \alpha I$. With $e_c^0 = 0$, we have

$$\nabla e_c^{n+1} = \delta t\sum_{k=1}^{n+1}\nabla\bigg(I + \frac{\delta t}{\tau}A\bigg)^{k-n-2}\Big(T_c(t^k) + \frac{\gamma}{\tau}e_\rho^k\Big).$$

We recall the Sobolev embedding $H^s(\Omega) \subset L^{2p}(\Omega)$ for $s = 1 - \frac{1}{p}$ in [6] to get

$$\big\|\nabla e_c^{n+1}\big\|_{L^{2p}} \le C\big\|A^{\frac{s}{2}}\nabla e_c^{n+1}\big\|_{L^2} \le C\delta t\sum_{k=1}^{n+1}\bigg\|\nabla A^{\frac{s}{2}}\bigg(I + \frac{\delta t}{\tau}A\bigg)^{k-n-2}\bigg\|_{\mathcal{L}(L^2,L^2)}\Big\|T_c(t^k) + \frac{\gamma}{\tau}e_\rho^k\Big\|_{L^2}. \tag{67}$$

Applying Plancherel's theorem [12], we have that, for $k \geq 1$,

$$\left\| \nabla A^{\frac{s}{2}} \left(I + \frac{\delta t}{\tau} A \right)^{-k} \right\|_{\mathcal{L}(L^2, L^2)} = \sup_{|\xi|} \left| \frac{|\xi|(\alpha + |\xi|^2)^{\frac{s}{2}}}{\left(1 + \frac{\delta t}{\tau}(\alpha + |\xi|^2)\right)^k} \right|. \tag{68}$$

To bound the right hand side of (68), we consider two cases, namely, $|\xi|^2 \leq \alpha$ and $\alpha < |\xi|^2$. In the former case, the right hand side is bounded by $2^{\frac{s}{2}} \alpha^{\frac{1}{2} + \frac{s}{2}}$. In the later case, we observe that the maximum of the function $|\xi| \mapsto \frac{|\xi|^{1+s}}{(1 + \frac{\delta t}{\tau}|\xi|^2)^k}$ is achieved for $\frac{\delta t}{\tau}|\xi|^2(2k - 1 - s) = 1 + s$. Then, the right hand side of (68) is bounded by $\max\{2^{\frac{s}{2}} \alpha^{\frac{1}{2} + \frac{s}{2}}, (\frac{(1+s)\tau}{(2k-1-s)\delta t})^{\frac{1+s}{2}}\}$. Since

$$\sum_{k=1}^{n+1} (2k - 1 - s)^{-\frac{1+s}{2}} \leq C(n+1)^{\frac{1}{2} - \frac{s}{2}} \leq C N_t^{\frac{1}{2} - \frac{s}{2}} \leq C(\delta t)^{\frac{s}{2} - \frac{1}{2}} T^{\frac{1}{2} - \frac{s}{2}}, \tag{69}$$

we have

$$\delta t \sum_{k=0}^{n} \left\| \nabla A^{\frac{s}{2}} \left(I + \frac{\delta t}{\tau} A \right)^{k-n-1} \right\|_{\mathcal{L}(L^2, L^2)} \leq C. \tag{70}$$

Therefore, the conclusion follows the estimate (70), the consistency estimate $\|T_c(t^{n+1})\|_{L^2} \leq C\delta t$, and the inequality (67) for any $n \in \{0, 1, 2, \ldots, N_t - 1\}$. $\square$

With the above preparations, we are ready to prove the following error estimates.

Theorem 6.1. *Let $T < \infty$ be arbitrary. Assume that the initial data $\rho_0, c_0 \in L^q(\Omega)$ are positive for $(1 < q \leq \infty)$, and the mass $M = \int_\Omega \rho_0 \mathrm{d}\boldsymbol{x}$ satisfies the condition (8). Assume further that the exact solutions ρ and c of the KS model (1)–(4) are sufficiently smooth on $\Omega \times [0, T]$. Then there exists a positive constant K^0 such that the semi-discrete scheme (6) and (7) satisfies the error estimate*

$$\|e_\rho^{n+1}\|_{L^p} + \|e_c^{n+1}\|_{L^p} \leq K^0 \delta t, \quad n = 0, 1, 2, \ldots, N_t - 1, \quad 1 < p < \infty. \tag{71}$$

Proof. Taking the inner product of the error equation (63) with $(e_\rho^{n+1})^{2p-1}$ leads to

$$\begin{aligned} &\left(\frac{e_\rho^{n+1} - e_\rho^n}{\delta t}, (e_\rho^{n+1})^{2p-1} \right) - \left(\triangle e_\rho^{n+1}, (e_\rho^{n+1})^{2p-1} \right) \\ &\quad = -\chi\big(\nabla \cdot \big(\rho(t^{n+1}) \nabla e_c^n + e_\rho^{n+1} \nabla c^n\big), (e_\rho^{n+1})^{2p-1}\big) + \big(T_\rho(t^{n+1}), (e_\rho^{n+1})^{2p-1}\big). \end{aligned} \tag{72}$$

Next, we estimate the bound of each term of this equation. For the first term, applying the convexity of the function $x \mapsto x^{2p}$, we have

$$\left(\frac{e_\rho^{n+1} - e_\rho^n}{\delta t}, (e_\rho^{n+1})^{2p-1} \right) \geq \frac{1}{2p\delta t} (||e_\rho^{n+1}||_{L^{2p}}^{2p} - ||e_\rho^n||_{L^{2p}}^{2p}). \tag{73}$$

For the second term, we have

$$-\left(\triangle e_\rho^{n+1}, (e_\rho^{n+1})^{2p-1} \right) = \frac{2p-1}{p^2} \|\nabla (e_\rho^{n+1})^p\|_{L^2}^2. \tag{74}$$

For the last term, using Hölder inequality and Young's inequality, we obtain

$$\begin{aligned} \big(T_\rho(t^{n+1}), (e_\rho^{n+1})^{2p-1}\big) &\leq \|T_\rho(t^{n+1})\|_{L^{2p}} \|e_\rho^{n+1}\|_{L^{2p}}^{2p-1} \\ &\leq \frac{1}{2p} \|T_\rho(t^{n+1})\|_{L^{2p}}^{2p} + \frac{2p-1}{2p} \|e_\rho^{n+1}\|_{L^{2p}}^{2p}. \end{aligned} \tag{75}$$

For the third term, we decompose it into two parts $-\chi\big(\nabla\cdot(e_\rho^{n+1}\nabla c^n),\, (e_\rho^{n+1})^{2p-1}\big)$ and $-\chi\big(\nabla\cdot\big(\rho(t^{n+1})\nabla e_c^n\big),\, (e_\rho^{n+1})^{2p-1}\big)$ below. According to the divergence theorem, the equation (7) for c^{n+1}, and the non-negativity of c^{n+1} for any $n \in \{0,1,2,\ldots,N_t-1\}$, the first part is estimated as

$$
\begin{aligned}
&-\chi\big(\nabla\cdot\big(e_\rho^{n+1}\nabla c^n\big), (e_\rho^{n+1})^{2p-1}\big)\\
&\quad= \chi(2p-1)\big((e_\rho^{n+1})^{2p-1}\nabla c^n, \nabla e_\rho^{n+1}\big)\\
&\quad= \frac{(2p-1)\chi}{2p}\big(|e_\rho^{n+1}|^{2p}, -\Delta c^n\big)\\
&\quad= \frac{(2p-1)\chi}{2p}\left(|e_\rho^{n+1}|^{2p}, -\tau\frac{c^n-c^{n-1}}{\delta t} - \alpha c^n + \gamma\rho^n\right)\\
&\quad\le \frac{(2p-1)\chi}{2p}\big(|e_\rho^{n+1}|^{2p}, \gamma\rho^n\big) - \frac{(2p-1)\chi}{2p}\left(|e_\rho^{n+1}|^{2p}, \tau\frac{c^n-c^{n-1}}{\delta t}\right).
\end{aligned}
\tag{76}
$$

Applying Cauchy-Schwarz inequality, Gagliardo-Nirenberg inequality, and the ε-Young's inequality, we derive

$$
\begin{aligned}
\big(|e_\rho^{n+1}|^{2p}, \gamma\rho^n\big) &\le \gamma\|\rho^n\|_{L^2}\|(e_\rho^{n+1})^{2p}\|_{L^2}\\
&\le \gamma C_{gn}\|\rho^n\|_{L^2}\|(e_\rho^{n+1})^p\|_{L^2}\big(\|\nabla(e_\rho^{n+1})^p\|_{L^2} + \|(e_\rho^{n+1})^p\|_{L^2}\big)\\
&\le \gamma C_{gn}\left(\frac{1}{4\varepsilon}\|\rho^n\|_{L^2}^2 + \|\rho^n\|_{L^2}\right)\|(e_\rho^{n+1})^p\|_{L^2}^2 + \gamma C_{gn}\varepsilon\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2\\
&\le \gamma C_{gn}\left(\frac{1}{4\varepsilon}K^0 + (K^0)^{\frac12}\right)\|(e_\rho^{n+1})^p\|_{L^2}^2 + \gamma C_{gn}\varepsilon\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2,
\end{aligned}
\tag{77}
$$

where $\|\rho^n\|_{L^2}^2 \le K^0$ for any $n \in \{0,1,2,\ldots,N_t\}$ is known by Lemma 5.2. Similarly,

$$
\begin{aligned}
&-\tau\left(|e_\rho^{n+1}|^{2p}, \frac{c^n-c^{n-1}}{\delta t}\right)\\
&\quad\le \tau C_{gn}\left(\frac{1}{4\varepsilon}\left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}^2 + \left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}\right)\|(e_\rho^{n+1})^p\|_{L^2}^2 + \tau C_{gn}\varepsilon\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2.
\end{aligned}
\tag{78}
$$

Using the divergence theorem, Hölder inequality, Young's inequality, and the L^∞-bound of exact solution ρ in Lemma 3.4, *i.e.*, $\|\rho(t)\|_{L^\infty} \le K^0$ for all $t \in [0,T]$, the second part is estimated as

$$
\begin{aligned}
&-\chi\big(\nabla\cdot\big(\rho(t^{n+1})\nabla e_c^n\big), (e_\rho^{n+1})^{2p-1}\big)\\
&\quad= \chi\big(\rho(t^{n+1})\nabla e_c^n, \nabla(e_\rho^{n+1})^{2p-1}\big)\\
&\quad= \frac{(2p-1)\chi}{p}\big(\rho(t^{n+1})\nabla e_c^n, (e_\rho^{n+1})^{p-1}\nabla(e_\rho^{n+1})^p\big)\\
&\quad\le \frac{(2p-1)\chi}{p}\|\rho(t^{n+1})(\nabla e_c^n)(e_\rho^{n+1})^{p-1}\|_{L^2}\|\nabla(e_\rho^{n+1})^p\|_{L^2}\\
&\quad\le \frac{(2p-1)\chi}{p}\|\rho(t^{n+1})(\nabla e_c^n)\|_{L^{2p}}\|e_\rho^{n+1}\|_{L^{2p}}^{p-1}\|\nabla(e_\rho^{n+1})^p\|_{L^2}\\
&\quad\le \frac{(2p-1)\chi}{2p}\left(\frac{1}{\varepsilon}\|\rho(t^{n+1})(\nabla e_c^n)\|_{L^{2p}}^2\|e_\rho^{n+1}\|_{L^{2p}}^{2p-2} + \varepsilon\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2\right)\\
&\quad\le \frac{(2p-1)\chi}{2p\varepsilon}\left(\frac{1}{p}\|\rho(t^{n+1})(\nabla e_c^n)\|_{L^{2p}}^{2p} + \frac{2p-2}{2p}\|e_\rho^{n+1}\|_{L^{2p}}^{2p}\right) + \frac{(2p-1)\chi\varepsilon}{2p}\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2\\
&\quad\le \frac{(2p-1)\chi}{2p\varepsilon}\left(\frac{1}{p}\|\rho(t^{n+1})\|_{L^\infty}^{2p}\|\nabla e_c^n\|_{L^{2p}}^{2p} + \frac{2p-2}{2p}\|e_\rho^{n+1}\|_{L^{2p}}^{2p}\right) + \frac{(2p-1)\chi\varepsilon}{2p}\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2\\
&\quad\le \frac{(2p-1)\chi}{2p\varepsilon}\left(\frac{1}{p}(K^0)^{2p}\|\nabla e_c^n\|_{L^{2p}}^{2p} + \frac{2p-2}{2p}\|e_\rho^{n+1}\|_{L^{2p}}^{2p}\right) + \frac{(2p-1)\chi\varepsilon}{2p}\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2.
\end{aligned}
\tag{79}
$$

Substituting (73)–(79) into (72), we have

$$
\begin{aligned}
&\frac{1}{2p\delta t}\|e_\rho^{n+1}\|_{L^{2p}}^{2p}+\frac{2p-1}{p^2}\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2\\
&\quad\le \frac{1}{2p\delta t}\|e_\rho^{n}\|_{L^{2p}}^{2p}+\frac{1}{2p}\|T_\rho(t^{n+1})\|_{L^{2p}}^{2p}\\
&\qquad+\frac{(2p-1)\chi C_{gn}}{2p}\left(\frac{\gamma}{4\varepsilon}K^0+\gamma(K^0)^{\frac12}+\frac{\tau}{4\varepsilon}\left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}^2+\tau\left\|\frac{c^n-c^{n-1}}{\delta t}\right\|_{L^2}\right)\|e_\rho^{n+1}\|_{L^{2p}}^{2p}\\
&\qquad+\frac{(2p-1)\chi\varepsilon}{2p}(C_{gn}(\gamma+\tau)+1)\|\nabla(e_\rho^{n+1})^p\|_{L^2}^2\\
&\qquad+\frac{(2p-1)\chi}{2p\varepsilon}\left(\frac{(K^0)^{2p}}{p}\|\nabla e_c^n\|_{L^{2p}}^{2p}+\frac{2p-2}{2p}\|e_\rho^{n+1}\|_{L^{2p}}^{2p}\right).
\end{aligned}
\tag{80}
$$

We now chose sufficiently small ε such that $\frac{p\chi\varepsilon}{2}(C_{gn}(\gamma+\tau)+1)\le 1$. Applying the discrete energy dissipation (39) and the energy bound (15) in Lemma 3.1, we know that the $\frac{c^n-c^{n-1}}{\delta t}$ remains bounded in $L^2(\Omega)$. Then we infer from (80) that there exists a positive constant K^0 such that

$$
\|e_\rho^{n+1}\|_{L^{2p}}^{2p}\le\|e_\rho^{n}\|_{L^{2p}}^{2p}+\delta t\Big(\|T_\rho(t^{n+1})\|_{L^{2p}}^{2p}+K^0\|e_\rho^{n+1}\|_{L^{2p}}^{2p}+K^0\|\nabla e_c^n\|_{L^{2p}}^{2p}\Big),
\tag{81}
$$

with $\frac{1}{\varepsilon}$ being absorbed by the constant K^0. According to the estimation of $\|\nabla e_c^n\|_{L^{2p}}^{2p}$ in Lemma 6.1, we should discuss the parabolic-elliptic case and the parabolic-parabolic case, respectively.

In the parabolic-elliptic case, applying (65) leads to

$$
\|e_\rho^{n+1}\|_{L^{2p}}^{2p}\le\|e_\rho^{n}\|_{L^{2p}}^{2p}+\delta t\Big(\|T_\rho(t^{n+1})\|_{L^{2p}}^{2p}+K^0\|e_\rho^{n+1}\|_{L^{2p}}^{2p}+K^0\|e_\rho^n\|_{L^{2p}}^{2p}\Big).
\tag{82}
$$

Summing the above inequality (82) on n and applying of the discrete Gronwall Lemma, we obtain

$$
\|e_\rho^{n+1}\|_{L^{2p}}^{2p}\le\Big(\|e_\rho^{0}\|_{L^{2p}}^{2p}+T\|T_\rho(t^{n+1})\|_{L^{2p}}^{2p}\Big)\exp(K^0T).
\tag{83}
$$

Since $e_\rho^0=0$ and $\|T_\rho(t^{n+1})\|_{L^{2p}}\le K^0\delta t$, we have $\|e_\rho^{n+1}\|_{L^{2p}}\le K^0\delta t$.

In the parabolic-parabolic case, setting $\eta_n=\max\limits_{1\le k\le n}\|e_\rho^k\|_{L^{2p}}^{2p}$ and gathering (66) with (81) yield

$$
(1-K^0\delta t)\eta_{n+1}\le\eta_n+K^0(\delta t)^{2p}.
\tag{84}
$$

Then, the bound of $\|e_\rho^{n+1}\|_{L^{2p}}^{2p}$ follows the discrete Gronwall lemma.

To conclusion, whether the equation for c is elliptic or parabolic, we always have

$$
\|e_\rho^{n+1}\|_{L^{2p}}\le K^0\delta t,\quad n=0,1,2,\dots,N_t-1,\quad 1<p<\infty.
\tag{85}
$$

Next, we estimate L^p-bound for the error e_c^{n+1}. Taking the inner product of the error equation (64) with $(e_c^{n+1})^{2p-1}$ and applying Green formula, we obtain

$$
\begin{aligned}
&\frac{\tau}{\delta t}\|e_c^{n+1}\|_{L^{2p}}^{2p}+\frac{2p-1}{p^2}\|\nabla(e_c^{n+1})^p\|_{L^2}^2+\alpha\|e_c^{n+1}\|_{L^{2p}}^{2p}\\
&\qquad=\frac{\tau}{\delta t}\big(e_c^n,(e_c^{n+1})^{2p-1}\big)+\gamma\big(e_\rho^{n+1},(e_c^{n+1})^{2p-1}\big)+\tau\big(T_c(t^{n+1}),(e_c^{n+1})^{2p-1}\big).
\end{aligned}
\tag{86}
$$

Using Hölder inequality and Young's inequality, we have

$$
\gamma\big(e_\rho^{n+1},(e_c^{n+1})^{2p-1}\big)
$$

$$\leq \gamma \|e_\rho^{n+1}\|_{L^{2p}} \|e_c^{n+1}\|_{L^{2p}}^{2p-1} \tag{87}$$

$$\leq \gamma \left(\frac{1}{2p} \|e_\rho^{n+1}\|_{L^{2p}}^{2p} + \frac{2p-1}{2p} \|e_c^{n+1}\|_{L^{2p}}^{2p} \right), \tag{88}$$

$$\frac{\tau}{\delta t} \big(e_c^n, (e_c^{n+1})^{2p-1}\big) \leq \frac{\tau}{\delta t} \left(\frac{1}{2p} \|e_c^n\|_{L^{2p}}^{2p} + \frac{2p-1}{2p} \|e_c^{n+1}\|_{L^{2p}}^{2p} \right), \tag{89}$$

$$\tau \big(T_c(t^{n+1}), (e_c^{n+1})^{2p-1}\big) \leq \tau \left(\frac{1}{2p} \|T_c(t^{n+1})\|_{L^{2p}}^{2p} + \frac{2p-1}{2p} \|e_c^{n+1}\|_{L^{2p}}^{2p} \right). \tag{90}$$

If $\tau = 0$, then by (86) and (87)

$$\|e_c^{n+1}\|_{L^{2p}} \leq \frac{\gamma}{\alpha} \|e_\rho^{n+1}\|_{L^{2p}} \leq K^0 \delta t, \quad n = 0, 1, 2, \ldots, N_t - 1, \quad 1 < p < \infty. \tag{91}$$

If $\tau > 0$, then substituting (88)–(90) into (86) yields

$$\begin{aligned} \|e_c^{n+1}\|_{L^{2p}}^{2p} \leq \|e_c^n\|_{L^{2p}}^{2p} + \delta t \frac{\gamma}{\tau} \|e_\rho^{n+1}\|_{L^{2p}}^{2p} + \delta t \|T_c(t^{n+1})\|_{L^{2p}}^{2p} \\ + \delta t \frac{2p(\gamma + \tau - \alpha) - (\gamma + \tau)}{\tau} \|e_c^{n+1}\|_{L^{2p}}^{2p}. \end{aligned} \tag{92}$$

We take the sum of the above inequality (92) on n and use the discrete Gronwall lemma to obtain

$$\|e_c^{n+1}\|_{L^{2p}}^{2p} \leq \left(\|e_c^0\|_{L_{2p}}^{2p} + \frac{\gamma}{\tau} T \|e_\rho^{n+1}\|_{L^{2p}}^{2p} + T \|T_c(t^{n+1})\|_{L^{2p}}^{2p} \right) \exp\left(\frac{2p(\gamma + \tau - \alpha) - (\gamma + \tau)}{\tau} T \right). \tag{93}$$

Since that $e_c^0 = 0$, $\|e_\rho^{n+1}\|_{L^{2p}} \leq K^0 \delta t$ and $\|T_c(t^{n+1})\|_{L^{2p}} \leq K^0 \delta t$, we conclude

$$\|e_c^{n+1}\|_{L^{2p}} \leq K^0 \delta t, \quad n = 0, 1, 2, \ldots, N_t - 1, \quad 1 < p < \infty. \tag{94}$$

In summary, the proof is completed by estimates (85) and (94). □

7. Concluding remarks

In this paper, we study the numerical analysis of the semi-discrete scheme (6) and (7) for the KS model (1)–(4) in two dimensions, covering both the parabolic-elliptic case ($\tau = 0$) and the parabolic-parabolic case ($\tau > 0$). We rigorously prove that the scheme (6) and (7) preserves the essential properties of the continuous model, nalmely mass conservation, positivity of the cell density, and energy dissipation. Furthermore, we establish L^q-bounds ($1 < q \leq \infty$) for the numerical solution, and perform rigorous error analysis to obtain optimal error estimates in L^p-norm ($1 < p < \infty$) under the small mass assumption (8).

There are several immediate directions for future work: (i) constructing fully discrete schemes, based on the time discretization (6) and (7), that can preserve all essential properties of the KS equations at the fully discrete level; (ii) constructing second- and higher-order semi-discrete schemes that can satisfy essential properties of the KS equations; and (iii) extending the results of this paper to the three-dimensional case.

Funding

X. Huang's work is supported by China Scholarship Council grant 202306310119. O. Goubet's work is supported by Labex CEMPI ANR-11-LABX-0007-01. J. Shen's work is partially supported by NSFC grant 12371409.

Data availability statement

The research data associated with this article are included in the article.

References

[1] D. Acosta-Soba, F. Guillén-González and J.R. Rodriguez-Galván, An unconditionally energy stable and positive upwind DG scheme for the Keller-Segel model. *J. Sci. Comput.* **97** (2023) 1–27.

[2] G. Arumugam and J. Tyagi, Keller-Segel chemotaxis models: a review. *Acta Appl. Math.* **171** (2021) 1–82.

[3] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues. *Math. Models Meth. Appl. Sci.* **25** (2015) 1663–1763.

[4] P. Biler, Existence and nonexistence of solutions for a model of gravitational interaction of particles I. *Colloq. Math.* **68** (1995) 229–239.

[5] A. Blanchet, J.A. Carrillo, D. Kinderlehrer, M. Kowalczyk, P. Laurencot and S. Lisini, A hybrid variational principle for the Keller-Segel system in R^2. *ESAIM Math. Model. Numer. Anal.* **49** (2015) 1553–1576.

[6] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011).

[7] P. Chatzipantelidis and C. Pervolianakis, Error analysis of a backward Euler positive preserving stabilized scheme for a Chemotaxis system. Preprint arXiv:2210.04709 (2022).

[8] W. Chen, Q. Liu and J. Shen, Error estimates and blow-up analysis of a finite-element approximation for the parabolic-elliptic Keller-Segel system. *Int. J. Numer. Anal. Model.* **19** (2022) 275–298.

[9] J.I. Diaz and T. Nagai, Symmetrization in a parabolic-elliptic system related to chemotaxis. *Adv. Math. Sci. Appl.* **5** (1995) 659–680.

[10] E. Emmrich, Discrete Versions of Gronwall's Lemma and their Application to the Numerical Analysis of Parabolic Problems. Technische Universität Berlin: Fachbereich Mathematik (1999).

[11] Y. Epshteyn and A. Izmirlioglu, Fully discrete analysis of a discontinuous finite element method for the Keller-Segel chemotaxis model. *J. Sci. Comput.* **40** (2009) 211–256.

[12] L.C. Evans, Partial Differential Equations. American Mathematical Society (2022).

[13] X. Feng, X. Huang and K. Wang, Error estimate of unconditionally stable and decoupled linear positivity-preserving FEM for the chemotaxis-Stokes equations. *SIAM J. Numer. Anal.* **59** (2021) 3052–3076.

[14] H. Gajewski, K. Zacharias and K. Gröger, Global behaviour of a reaction-diffusion system modelling chemotaxis. *Math. Nachr.* **195** (1998) 77–114.

[15] J.V. Gutiérrez-Santacreu and J.R. Rodriguez-Galván, Analysis of a fully discrete approximation for the classical Keller-Segel model: lower and a priori bounds. *Comput. Math. Appl.* **85** (2021) 69–81.

[16] M. Herda, A. Trescases and A. Zurek, A finite volume scheme for the local sensing chemotaxis model. *SMAI J. Comput. Math.* **11** (2025) 637–676.

[17] T. Hillen and K.J. Painter, A user's guide to PDE models for chemotaxis. *J. Math. Biol.* **58** (2009) 183–217.

[18] D. Horstmann, Lyapunov functions and L^p-estimates for a class of reaction-diffusion systems. *Colloq. Math.* **87** (2001) 113–127.

[19] D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **105** (2003) 103–165.

[20] F. Huang and J. Shen, Bound/positivity preserving and energy stable SAV schemes for dissipative systems: applications to Keller-Segel and Poisson-Nernst-Planck equations. *SIAM J. Sci. Comput.* **43** (2021) A1832–A1857.

[21] X. Huang and J. Shen, Bound/positivity preserving SAV schemes for the Patlak-Keller-Segel-Navier-Stokes system. *J. Comput. Phys.* **480** (2023) 112034.

[22] X. Huang and J. Shen, Efficient numerical schemes for a two-species Keller-Segel model and investigation of its blowup phenomena in 3D. *Acta Appl. Math.* **190** (2024) 1–23.

[23] X. Huang, X. Xiao, J. Zhao and X. Feng, An efficient operator-splitting FEM-FCT algorithm for 3D chemotaxis models. *Eng. Comput.* **36** (2020) 1393–1404.

[24] T. Kato, Schrödinger operators with singular potentials. *Israel J. Math.* **13** (1972) 135–148.

[25] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.* **26** (1970) 399–415.

[26] E.F. Keller and L.A. Segel, Model for chemotaxis. *J. Theor. Biol.* **30** (1971) 225–234.

[27] P. De Leenheer, J. Gopalakrishnan and E. Zuhr, Nonnegativity of exact and numerical solutions of some chemotactic models. *Comput. Math. Appl.* **66** (2013) 356–375.

[28] J. Liu, L. Wang and Z. Zhou, Positivity-preserving and asymptotic preserving method for 2D Keller-Segal equations. *Math. Comput.* **87** (2018) 1165–1189.

[29] Y. Lu, C. Chen, X. Li and C. Liu, Finite difference approximation with ADI scheme for two-dimensional Keller-Segel equations. *Commun. Comput. Phys.* **35** (2024) 1352–1386.

[30] T. Nagai, Behavior of solutions to a parabolic-elliptic system modelling chemotaxis. *J. Korean Math. Soc.* **37** (2000) 721–733.

[31] T. Nagai, Global existence and blow-up of solutions to a chemotaxis system. *Nonlinear Anal. Theory Meth. Appl.* **47** (2001) 777–787.

[32] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. *Funkcialaj Ekvacioj* **40** (1997) 411–433.

[33] C.S. Patlak, Random walk with persistence and external bias. *Bull. Math. Biophys.* **15** (1953) 311–338.

[34] N. Saito, Error analysis of a conservative finite-element approximation for the Keller-Segel system of chemotaxis. *Commun. Pure Appl. Anal.* **11** (2011) 339–364.

[35] N. Saito and T. Suzuki, Notes on finite difference schemes to a parabolic-elliptic system modelling chemotaxis. *Appl. Math. Comput.* **171** (2005) 72–90.

[36] J. Shen and J. Xu, Unconditionally bound preserving and energy dissipative schemes for a class of Keller-Segel Equations. *SIAM J. Numer. Anal.* **58** (2020) 1674–1695.

[37] R. Strehl, A. Sokolov, D. Kuzmin and S. Turek, A flux-corrected finite element method for chemotaxis problems. *Comput. Meth. Appl. Math.* **10** (2010) 219–232.

[38] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York (2012).

[39] G. Zhou, An analysis on the finite volume schemes and the discrete Lyapunov inequalities for the chemotaxis system. *J. Sci. Comput.* **87** (2021) 1–47.

[40] G. Zhou and N. Saito, Finite volume methods for a Keller-Segel system: discrete energy, error estimates and numerical blow-up analysis. *Numer. Math.* **135** (2017) 265–311.