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Efficient Spectral-Galerkin Method and Analysis for Elliptic PDEs with Non-local Boundary Conditions

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Abstract We present an efficient Legendre–Galerkin method and its error analysis for a class of PDEs with non-local boundary conditions. We also present several numerical experiments, including the scattering problem from an open cavity, to demonstrate the accuracy and efficiency of the proposed method.

Keywords Non-local boundary conditions · Spectral-Galerkin method · Legendre polynomial · Error analysis

Mathematics Subject Classification 65N35 · 65N22 · 65F05 · 35J05 · 35J25

1 Introduction

PDEs with non-local boundary conditions appear in many scientific and engineering applications, cf. for instance [2,7,9,10] and the references therein. However, most of the numerical

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methods proposed for PDEs with local boundary conditions can not be directly applied, or the cost increases significantly when applied, to PDEs with non-local boundary conditions. Various numerical approaches have been developed for problems with non-local boundary conditions, e.g., finite difference methods [8,12,15,24–27,30–32] and finite element methods [4,16,28]. A summary of the progress on this topic can be found in [11]. Compared to finite difference method and finite element method, spectral methods are capable of providing superior accuracy with fewer unknowns if the solutions are sufficiently smooth [6,19,34,35], and can be especially attractive to deal with problems with non-local features. However, there are only a few efforts on using spectral methods for problems with non-local boundary conditions, e.g., Chebyshev spectral collocation method [17] and pseudospectral Legendre method [13], and particularly not much is available on how to efficiently solve the resulting linear systems and its error analysis.

The main purposes of this paper are (i) to develop an efficient Spectral-Galerkin method for PDEs with non-local boundary conditions, and (ii) to carry out a rigorous error analysis for the proposed method. We shall also present numerical results to validate the algorithm and its error estimates. The main idea for the efficient algorithm is to recognize the fact that linear systems from problems with non-local boundary conditions can be considered as low-rank perturbations of those from problems with local boundary conditions. Since the problems with local boundary conditions can be solved efficiently by the matrix diagonalization method (see for instance [33,35]), we can then solve the problems with non-local boundary conditions by using the well-known Sherman-Morrison-Woodbury formula. As for the error analysis, we first show that the problems with non-local boundary conditions under consideration are well-posed with suitable conditions on the kernel functions, and then use the coercivity of the bilinear form and polynomial approximation theory to derive optimal error estimates.

The paper is organized as follows. Section 2 is devoted to the one-dimensional elliptic equation with non-local boundary conditions: we prove the well-posedness of the problem, develop an efficient Spectral-Galerkin method, and carry out an error analysis. In Sect. 3, we extend the algorithm and analysis of Sect. 2 to the two dimensional case, in particular we develop an efficient algorithm, by using the Sherman-Morrison-Woodbury formula, which has the same computational complexity as the spectral algorithm for the same problem but with all local boundary conditions. Several extensions are discussed in Sect. 4, including in particular the case where the non-local operator is defined through Fourier transform. Numerical experiments are presented in Sect. 5 to verify the accuracy and efficiency of the method, and as an application, we used the proposed method to solve the difficult scattering problem from an open cavity. Some concluding remarks are given in the last section.

2 One Dimensional Case

To fix the idea, we consider the following second order elliptic equation with non-local boundary conditions:

$$\alpha u - u'' = f, \quad \text{in } \ I = (-1, 1), \tag{2.1}$$

$$u' + \int_I A^{\pm}(x) u(x) dx = 0, \quad \text{at } \ x = \pm 1, \tag{2.2}$$

where $\alpha > 0$. The weak formulation for problem (2.1)–(2.2) is: find $u \in H^1(I)$ such that for any $v \in H^1(I)$,

$$\alpha(u,v)+(u',v')+v(1)\int_I A^+(x)u(x)dx - v(-1)\int_I A^-(x)u(x)dx = (f,v), \tag{2.3}$$

where $(u,v)=\int_I uvdx$ is the inner product in $L^2(I)$, on which the norm is denoted by $\|\cdot\|_0$, and $H^1(I)$ is the usual Sobolev space with the norm $\|\cdot\|_1$.

2.1 Wellposedness

To study the wellposedness of the above weak formulation, we first recall the following inequality

$$\max_{x\in[-1,1]} |u(x)| \le c_0\|u\|_1. \tag{2.4}$$

Define a bilinear form on $H^1(I)$ by

$$a(u,v)=\alpha(u,v)+(u',v')+v(1)\int_I A^+(x)u(x)dx - v(-1)\int_I A^-(x)u(x)dx. \tag{2.5}$$

One can derive from (2.4) and Cauchy-Schwarz inequality that

$$\begin{aligned}\big|a(u,v)\big| &\le \alpha\|u\|_0\|v\|_0+\|u'\|_0\|v'\|_0+c_0\|v\|_1\|A^+\|_0\|u\|_0+c_0\|v\|_1\|A^-\|_0\|u\|_0\\ &\lesssim \|u\|_1\|v\|_1.\end{aligned}$$

Here and after, $A\lesssim B$ means that $A\le CB$ for some generic constant C.

On the other hand,

$$\begin{aligned}a(v,v) &\ge \alpha(v,v)+(v',v')-\left|v(1)\int_I A^+(x)v(x)dx - v(-1)\int_I A^-(x)v(x)dx\right|\\ &\ge \gamma\|v\|_1^2 - c_0\|A^+\|_0\|v\|_1^2 - c_0\|A^-\|_0\|v\|_1^2\\ &\ge \left(\gamma - c_0\|A^+\|_0 - c_0\|A^-\|_0\right)\|v\|_1^2,\end{aligned}$$

where $\gamma=\min(\alpha,1)$.

Hence, an application of the Lax-Milgram lemma to (2.3) leads to the following:

Theorem 2.1 *Assuming*

$$C_A := \min(\alpha,1) - c_0(\|A^+\|_0+\|A^-\|_0) > 0, \tag{2.6}$$

then (2.3) *admits a unique solution satisfying*

$$\|u\|_1 \lesssim \|f\|_0.$$

2.2 Spectral-Galerkin Approximation

Let $L_n(x)$ be the Legendre polynomial of degree n, and P_N be the space of polynomials of degree less than or equal to N. Let us denote

$$\begin{aligned}\varphi_k(x) &= \frac{1}{\sqrt{4k+6}}(L_k(x)-L_{k+2}(x)), \quad 0\le k\le N-2,\\ \varphi_{N-1}(x) &= \frac{1}{2}(L_0(x)+L_1(x)), \quad \varphi_N(x)=\frac{1}{2}(L_0(x)-L_1(x)).\end{aligned} \tag{2.7}$$

It is easy to see that $P_N=\text{span}\{\varphi_k(x): 0\le k\le N\}$.

The Legendre–Galerkin approximation of (2.1)–(2.2) is: Find $u_N \in P_N$ such that

$$\begin{aligned}\alpha(u_N, v_N) + (u_N', v_N') + v_N(1)\int_I A^+(x)u_N(x)dx\\ - v_N(-1)\int_I A^-(x)u_N(x)dx = (I_N f, v_N),\ \forall v_N \in P_N,\end{aligned} \tag{2.8}$$

where $I_N f$ is the interpolation polynomial of f with respect to the Legendre-Gauss-Lobatto points $\{x_n\}_{n=0}^N$. Denote $u_N(x) = \sum_{k=0}^N u_k\varphi_k(x)$, and take in (2.8) $v_N = \varphi_j$, $0 \le j \le N$, (2.8) is reduced to the following linear system:

$$(\alpha M + S + \tilde{B}^+ - \tilde{B}^-)\mathbf{u} = \mathbf{f}, \tag{2.9}$$

where

$$\begin{aligned}\mathbf{u} &= (u_0, u_1, \ldots, u_N)^T,\\ \mathbf{f} &= (f_0, f_1, \ldots, f_N)^T, \quad f_j = \int_I I_N f(x)\varphi_j(x)dx,\\ M &= (m_{jk})_{0\le j,k\le N}, \quad m_{jk} = \int_I \varphi_k(x)\varphi_j(x)dx,\\ S &= (s_{jk})_{0\le j,k\le N}, \quad s_{jk} = \int_I \varphi_k'(x)\varphi_j'(x)dx,\\ \tilde{B}^\pm &= (\tilde{b}^\pm_{jk})_{0\le j,k\le N}, \quad \tilde{b}^\pm_{jk} = \varphi_j(\pm 1)a_k^\pm, \quad a_k^\pm = \int_I A^\pm(x)\varphi_k(x)dx.\end{aligned}$$

Thanks to the orthogonal properties of Legendre polynomials, we can easily determine the values of the matrix entries in (2.9). Namely, M is a symmetric matrix of the form:

$$M = \begin{bmatrix} & & & & & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ & & & & & \frac{1}{3\sqrt{10}} & -\frac{1}{3\sqrt{10}}\\ & & \tilde{M} & & & 0 & 0\\ & & & & & \vdots & \vdots\\ & & & & & 0 & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{10}} & 0 & \cdots & 0 & \frac{2}{3} & \frac{1}{3}\\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{10}} & 0 & \cdots & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \tag{2.10}$$

where the non-zero elements of $\tilde{M}$ are

$$\tilde{m}_{jk} = \tilde{m}_{kj} = \begin{cases} \frac{1}{4k+6}\left(\frac{2}{2k+1} + \frac{2}{2k+5}\right), & j = k,\\ -\frac{1}{\sqrt{4k+6}}\frac{1}{\sqrt{4(k+2)+6}}\frac{2}{2k+5}, & j = k+2, \end{cases} \quad 0 \le j, k \le N-2.$$

S is a symmetric matrix of the form:

$$S = \begin{bmatrix} & & & 0 & 0\\ & \tilde{S} & & \vdots & \vdots\\ & & & 0 & 0\\ 0 & \cdots & 0 & \frac{1}{2} & -\frac{1}{2}\\ 0 & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \tag{2.11}$$

in which $\tilde{S}$ is a diagonal matrix with $\tilde{s}_{kk} = 1,\ 0 \le k \le N-2$. The matrices $\tilde{B}^+$ and $\tilde{B}^-$ are as follows:

$$\tilde{B}^+ = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ a_0^+ & \cdots & a_{N-2}^+ & a_{N-1}^+ & a_N^+ \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{B}^- = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ a_0^- & \cdots & a_{N-2}^- & a_{N-1}^- & a_N^- \end{bmatrix}.$$

Since $\tilde{S}$ is diagonal and $\tilde{M}$ is penta-diagonal (with only three non-zero diagonals), the linear system (2.9) can be easily solved in $O(N)$ operations.

Remark 2.1 It is clear that the approach presented above applies to the problem with more general boundary conditions of the kind

$$a_{\pm}u(\pm 1) + b_{\pm}u'(\pm 1) + c_{\pm}\int_I K^{\pm}(x)u(x)dx = d_{\pm},$$

with given constants $a_{\pm}, b_{\pm}, c_{\pm}, d_{\pm}$.

2.3 Error Estimates

For elliptic problems with local boundary conditions, the error behaviors of spectral approximations have been well studied (cf. for instance [6,35] and the references therein). However, not much is available for error analysis of spectral approximations to elliptic problems with non-local boundary conditions. Hence, we provide below a rigorous error analysis in the one-dimensional case.

We first introduce the non-uniformly Jacobi-weighted Sobolev space:

$$B^m_{\alpha,\beta}(I) := \{u : \partial_x^k u \in L^2_{\omega^{\alpha+k,\beta+k}}(I), \quad 0 \le k \le m\}, \quad m \in \mathbb{N},$$

$$(u, v)_{B^m_{\alpha,\beta}} = \sum_{k=0}^{m} (\partial_x^k u, \partial_x^k v)_{\omega^{\alpha+k,\beta+k}},$$

$$\|u\|_{B^m_{\alpha,\beta}} = (u, u)^{1/2}_{B^m_{\alpha,\beta}}, \quad |u|_{B^m_{\alpha,\beta}} = \|\partial_x^m u\|_{\omega^{\alpha+m,\beta+m}},$$

where $\omega^{a,b}(x) = (1-x)^a(1+x)^b, a, b > -1$, is the Jacobi weight function. Note that $\|u\|^2_{\omega^{a,b}} := \int_I u^2\omega^{a,b}dx$ while we still use $\|\cdot\|_k$ to denote the usual norm in H^k.

Let $\pi_N^1 : H^1(I) \to P_N$ be defined by

$$\left(\pi_N^1 u - u, v_N\right) + \left(\partial_x(\pi_N^1 u - u), \partial_x^1 v_N\right) = 0, \quad v_N \in P_N.$$

We recall below the error estimate for π_N^1 [35, Theorem 3.36 and Theorem 3.37].

Lemma 2.1 ([35]) If $u \in H^1(I)$ and $\partial_x u \in B^{m-1}_{0,0}(I)$ with $1 \le m$, then we have

$$\|\pi_N^1 u - u\|_\mu \lesssim N^{\mu-m}\|\partial_x^m u\|_{\omega^{m-1,m-1}}, \quad \mu = 0, 1.$$

Now we present the main theorem in this section.

Theorem 2.2 *Let u and u_N be the solutions of* (2.3) *and* (2.8)*, respectively. Then, under the condition* (2.6)*, we have for $m \geq 1$ and $k > 1$,*

$$\|u-u_N\|_1 \lesssim N^{1-m}\|\partial_x^m u\|_{\omega^{m-1,m-1}} + N^{-k}\|\partial_x^k f\|_{\omega^{k,k}}.$$

Proof Using (2.3) and (2.8) leads to the error equation

$$a(u-u_N, v_N) = (f - I_N f, v_N), \quad v_N \in P_N, \tag{2.12}$$

where

$$\begin{aligned} a(u-u_N, v_N) =& \alpha(u-u_N, v_N) + (u'-u_N', v_N') + v_N(1)\int_I A^+(x)(u-u_N(x))dx \\ & - v_N(-1)\int_I A^-(x)(u-u_N(x))dx. \end{aligned}$$

Denote $\hat{e}_N = \pi_N^1 u - u_N$ and $\tilde{e}_N = \pi_N^1 u - u$. Taking $v_N = \hat{e}_N \in P_N$ in the error equation (2.12), we obtain

$$\begin{aligned} \alpha\|\hat{e}_N\|_0^2 + \|\hat{e}_N'\|_0^2 =& \alpha\left(\tilde{e}_N, \hat{e}_N\right) + \left(\tilde{e}_N', \hat{e}_N'\right) - \hat{e}_N(1)\int_I A^+(x)(\hat{e}_N - \tilde{e}_N)dx \\ & + \hat{e}_N(-1)\int_I A^-(x)\left(\hat{e}_N - \tilde{e}_N\right)dx + \left(f - I_N f, \hat{e}_N\right). \end{aligned}$$

For $\alpha > 0$ we have

$$\alpha\|\hat{e}_N\|_0^2 + \|\hat{e}_N'\|_0^2 \geq \gamma\|\hat{e}_N\|_1^2, \quad \gamma := \min(\alpha, 1).$$

On the other hand, it follows from (2.4) that

$$\hat{e}_N(\pm 1)\int_I A^{\pm}(x)(\hat{e}_N - \tilde{e}_N)dx \leq c_0\|\hat{e}_N\|_1\|A^{\pm}\|_0(\|\hat{e}_N\|_1 + \|\tilde{e}_N\|_0).$$

We then derive from above

$$\begin{aligned} (\gamma - c_0\|A^+\|_0 - c_0\|A^-\|_0)\|\hat{e}_N\|_1^2 \leq & \alpha\|\tilde{e}_N\|_1\|\hat{e}_N\|_1 + \|\tilde{e}_N\|_1\|\hat{e}_N\|_1 \\ & + c_0\|A^+\|_0\|\hat{e}_N\|_1\|\tilde{e}_N\|_0 \\ & + c_0\|A^-\|_0\|\hat{e}_N\|_1\|\tilde{e}_N\|_0 + \|f - I_N f\|_0\|\hat{e}_N\|_1, \end{aligned}$$

which implies that, under the condition $\gamma - c_0\|A^+\|_0 - c_0\|A^-\|_0 > 0$, we have

$$\|\hat{e}_N\|_1 \lesssim \|\tilde{e}_N\|_1 + \|f - I_N f\|_0.$$

Therefore, we have

$$\|u - u_N\|_1 \leq \|\hat{e}_N\|_1 + \|\tilde{e}_N\|_1 \lesssim \|\tilde{e}_N\|_1 + \|f - I_N f\|_0.$$

We recall from Theorem 3.43 in [35] that

$$\|I_N f - f\|_0 \leq cN^{-m}\|\partial_x^m f\|_{\omega^{m,m}} \quad \forall f \in B_{0,0}^m(I). \tag{2.13}$$

The desired result follows from the above and Lemma 2.1. ◻

3 Two Dimensional Case

Consider the following second order elliptic equation with non-local boundary conditions in both x and y directions:

$$\alpha u - \Delta u = f, \quad \text{in } \Omega = (-1,1)^2, \tag{3.1}$$

$$\partial_x u(\pm 1, y) + \int_I K_1^{\pm}(\xi, y) u(\pm 1, \xi) d\xi = 0, \quad y \in I = (-1,1), \tag{3.2}$$

$$\partial_y u(x, \pm 1) + \int_I K_2^{\pm}(x, \xi) u(\xi, \pm 1) d\xi = 0, \quad x \in I = (-1,1). \tag{3.3}$$

The variational formulation for (3.1)–(3.3) is: find $u \in H^1(\Omega)$ such that ,

$$a(u, v) = (f, v), \quad \forall v \in H^1(\Omega), \tag{3.4}$$

where $(u, v) = \int_I \int_I uv dx dy$ and

$$\begin{aligned} a(u, v) := \alpha(u, v) + (\nabla u, \nabla v) &+ \Big(\int_I K_1^+(\xi, y) u(1, \xi) d\xi, v(1, y) \Big)_{y \in I} \\ &- \Big(\int_I K_1^-(\xi, y) u(-1, \xi) d\xi, v(-1, y) \Big)_{y \in I} \\ &+ \Big(\int_I K_2^+(x, \xi) u(\xi, 1) d\xi, v(x, 1) \Big)_{x \in I} \\ &- \Big(\int_I K_2^-(x, \xi) u(\xi, -1) d\xi, v(x, -1) \Big)_{x \in I}. \end{aligned}$$

3.1 Wellposedness

As in the one-dimensional case, $\|\cdot\|_\mu$ with $\mu \geq 0$ will denote the norm in $H^\mu(\Omega)$. Let us first show the following:

Lemma 3.1 *For any $u, v \in H^1(\Omega)$ and $K(x, \xi)$ such that $K(\cdot, \xi) \in L^2(I)$ for all $\xi \in I$ and $K(x, \cdot) \in L^2(I)$ for all $x \in I$, we have*

$$\Big| \int_I \int_I K(x, \xi) u(\xi, b) v(x, b) d\xi dx \Big| \leq c_0^2 \|K\|_0 \|u\|_1 \|v\|_1, \quad b = \pm 1,$$

$$\Big| \int_I \int_I K(\xi, y) u(a, \xi) v(a, y) d\xi dy \Big| \leq c_0^2 \|K\|_0 \|u\|_1 \|v\|_1, \quad a = \pm 1,$$

where c_0 is the constant in (2.4).

Proof We only need to show the first inequality with $b = 1$, the other cases can be shown by the same approach.

Denote $g(x) := \int_I K(x, \xi) u(\xi, 1) d\xi$. Then

$$\Big| \int_I \int_I K(x, \xi) u(\xi, 1) v(x, 1) d\xi dx \Big| \leq \Big(\int_I (g(x))^2 dx \Big)^{\frac{1}{2}} \Big(\int_I (v(x, 1))^2 dx \Big)^{\frac{1}{2}}.$$

Thanks to (2.4),

$$|v(x, 1)|^2 \leq c_0^2 \Big(\int_I (v(x, y))^2 dy + \int_I (v_y(x, y))^2 dy \Big).$$

Hence,

$$\int_I |v(x,1)|^2 dx \le c_0^2 \int_I \Big(\int_I (v(x,y))^2 dy + \int_I (v_y(x,y))^2 dy \Big) dx \le c_0^2 \|v\|_1^2.$$

On the other hand, we derive from Cauchy-Schwarz inequality and the above result that

$$|g(x)| \le \Big(\int_I (K(x,\xi))^2 d\xi \Big)^{\frac{1}{2}} \Big(\int_I (u(\xi,1))^2 d\xi \Big)^{\frac{1}{2}} \le \Big(\int_I (K(x,\xi))^2 d\xi \Big)^{\frac{1}{2}} \cdot c_0 \|u\|_1.$$

Hence

$$\int_I (g(x))^2 dx \le \int_I \int_I (K(x,\xi))^2 d\xi dx \cdot c_0^2 \|u\|_1^2 = c_0^2 \|K\|_0^2 \|u\|_1^2,$$

from which the conclusion follows. □

We can then derive from the above lemma that the bilinear form $a(\cdot,\cdot)$ is continuous. On the other hand, we derive from the above lemma that

$$\begin{aligned} a(u,u) &\ge \alpha \|u\|_0^2 + \|\nabla u\|_0^2 - c_0^2 (\|K_1^+\|_0 + \|K_1^-\|_0 + \|K_2^+\|_0 + \|K_2^-\|_0) \|u\|_1^2 \\ &\ge \big(\min(\alpha,1) - c_0^2 (\|K_1^+\|_0 + \|K_1^-\|_0 + \|K_2^+\|_0 + \|K_2^-\|_0) \big) \|u\|_1^2. \end{aligned}$$

Hence, $a(\cdot,\cdot)$ is coercive in $H^1(\Omega) \times H^1(\Omega)$ if

$$C_K := \min(\alpha,1) - c_0^2 (\|K_1^+\|_0 + \|K_1^-\|_0 + \|K_2^+\|_0 + \|K_2^-\|_0) > 0. \quad (3.5)$$

Therefore, applying the Lax-Milgram lemma to (3.4) leads to

Theorem 3.1 *Under the condition* (3.5)*, the problem* (3.4) *has a unique solution satisfying*

$$\|u\|_1 \lesssim \|f\|_0.$$

3.2 Spectral-Galerkin Approximation

Let $\{\varphi_k\}$ be the basis functions defined in (2.7), and denote

$$X_N = \text{span}\big\{ \varphi_k(x)\varphi_j(y) : 0 \le k, j \le N \big\}.$$

The Legendre–Spectral-Galerkin approximation to (3.4) is: Find $u_N \in X_N$ such that

$$a(u_N, v_N) = (I_N f, v_N) \quad \forall v_N \in X_N, \quad (3.6)$$

where $I_N : C(\Omega) \to X_N$ is the Legendre-Gauss-Lobatto interpolation operator. It is clear that the above problem admits a unique solution, as its continuous counter part (3.4).

Setting

$$
\begin{aligned}
u_N(x,y) &= \sum_{k,j=0}^{N} u_{kj}\varphi_k(x)\varphi_j(y), \quad U=(u_{kj})_{0\le k,j\le N},\\
F &= (f_{pq})_{0\le p,q\le N}, \quad f_{pq}=\big(I_N f, \varphi_p(x)\varphi_q(y)\big),\\
M_x &= \big((m_x)_{pk}\big)_{0\le p,k\le N}, \quad (m_x)_{pk}=\int_I \varphi_k(x)\varphi_p(x)dx,\\
M_y &= \big((m_y)_{qj}\big)_{0\le q,j\le N}, \quad (m_y)_{qj}=\int_I \varphi_j(y)\varphi_q(y)dy,\\
S_x &= \big((s_x)_{pk}\big)_{0\le p,k\le N}, \quad (s_x)_{pk}=\int_I \varphi_k'(x)\varphi_p'(x)dx,\\
S_y &= \big((s_y)_{qj}\big)_{0\le q,j\le N}, \quad (s_y)_{qj}=\int_I \varphi_j'(y)\varphi_q'(y)dy,\\
\tilde{B}_x^{\pm} &= \Big(\big(\tilde{b}_x^{\pm}\big)_{pk}\Big)_{0\le p,k\le N}, \quad \big(\tilde{b}_x^{\pm}\big)_{pk}=\int_I\int_I K_2^{\pm}(x,\xi)\varphi_k(\xi)d\xi\varphi_p(x)dx,\\
\tilde{B}_y^{\pm} &= \Big(\big(\tilde{b}_y^{\pm}\big)_{qj}\Big)_{0\le q,j\le N}, \quad \big(\tilde{b}_y^{\pm}\big)_{qj}=\int_I\int_I K_1^{\pm}(\xi,y)\varphi_j(\xi)d\xi\varphi_q(y)dy,\\
T^+ &= \big(t_{qj}^+\big)_{0\le q,j\le N}, \quad t_{(N-1)(N-1)}^+=1, \quad t_{qj}^+=0 \ \text{ elsewhere},\\
T^- &= \big(t_{qj}^-\big)_{0\le q,j\le N}, \quad t_{NN}^-=1, \quad t_{qj}^-=0 \ \text{ elsewhere},
\end{aligned}
$$

by taking $v_N=\varphi_p(x)\varphi_q(y)$ in (3.6) for all $0\le p,q\le N$, we find that (3.6) is reduced to the following matrix equation:

$$
\begin{aligned}
&\alpha M_x U M_y^T + S_x U M_y^T + M_x U S_y^T + T^+U(\tilde{B}_y^+)^T - T^-U(\tilde{B}_y^-)^T\\
&\quad + \tilde{B}_x^+ U T^+ - \tilde{B}_x^- U T^- = F.
\end{aligned} \tag{3.7}
$$

We note that $M_x=M_y=M$ given by (2.10), and $S_x=S_y=S$ given by (2.11).

Unlike for the problem with local boundary condition [33], the matrix system (3.7) can not be solved directly by the matrix diagonalization method. However, since the only difference between the problems with local and non-local boundary conditions is at the boundary, it is not hard to realize that the matrix in (3.7) is simply a low-rank perturbation of a corresponding matrix with local boundary conditions. Since the matrix system with local boundary conditions can be solved efficiently by using the matrix diagonalization method [33], we can use the Sherman-Morrison-Woodbury formula (cf. for instance [18])

$$
(A+\tilde{U}\tilde{V}^T)^{-1} = A^{-1} - A^{-1}\tilde{U}(I+\tilde{V}^T A^{-1}\tilde{U})^{-1}\tilde{V}^T A^{-1}, \tag{3.8}
$$

where A is a $n\times n$ matrix, U and V are $n\times k$ matrix, and I is the $k\times k$ identity matrix. We note that if $k\ll n$ and A can be inverted efficiently, the Sherman-Morrison-Woodbury formula provides an efficient algorithm to invert the perturbed matrix $A+\tilde{U}\tilde{V}^T$.

We proceed with the detailed approach below. First we denote

$$
A = \alpha M_y\otimes M_x + M_y\otimes S_x + S_y\otimes M_x. \tag{3.9}
$$

Then, (3.7) can be rewritten in the form

$$
\Big(A + \tilde{B}_y^+\otimes T^+ - \tilde{B}_y^-\otimes T^- + T^+\otimes\tilde{B}_x^+ - T^-\otimes\tilde{B}_x^-\Big)\mathbf{u} = \mathbf{f}, \tag{3.10}
$$

where $\mathbf{f}$ and $\mathbf{u}$ are the vectors of length $M := (N+1)^2$ formed by the columns of U and F, i.e.,

$$\mathbf{f} = \left(f_{00}, f_{10}, \dots, f_{N0}; f_{01}, f_{11}, \dots, f_{N1}; f_{0N}, f_{1N}, \dots, f_{NN}\right)^T.$$

According to the Sherman-Morrison-Woodbury formula (3.8), we need to find matrices $\tilde{U}$, $\tilde{V}$ of order $M \times K$ with $K \ll M$ such that

$$\tilde{U}\tilde{V}^T = \tilde{B}_y^+ \otimes T^+ - \tilde{B}_y^- \otimes T^- + T^+ \otimes \tilde{B}_x^+ - T^- \otimes \tilde{B}_x^-. \tag{3.11}$$

By a careful examination, we find a pair of $\tilde{U}, \tilde{V}$ of order $M \times K$ with $(M, K) = ((N+1)^2, 4(N+1))$ as follows.

$$\tilde{U} = \begin{bmatrix} (\tilde{B}_y^+)_0 & -(\tilde{B}_y^-)_0 & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ (\tilde{B}_y^+)_{N-2} & -(\tilde{B}_y^-)_{N-2} & \mathbf{0} & \mathbf{0} \\ (\tilde{B}_y^+)_{N-1} & -(\tilde{B}_y^-)_{N-1} & I_{N+1} & \mathbf{0} \\ (\tilde{B}_y^+)_N & -(\tilde{B}_y^-)_N & \mathbf{0} & I_{N+1} \end{bmatrix},$$

where I_{N+1} denotes the $(N+1)\times(N+1)$ identity matrix and $\mathbf{0}$ the zero matrix with the same dimension, while $\left(\tilde{B}_y^+\right)_j$ is the $(N+1)\times(N+1)$ matrix whose N-th row is the $(j+1)$-th row of the matrix $\tilde{B}_y^+$ and 0 otherwise, i.e.,

$$\left(\tilde{B}_y^+\right)_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \left(\tilde{b}_y^+\right)_{j0} & \left(\tilde{b}_y^+\right)_{j1} & \cdots & \left(\tilde{b}_y^+\right)_{j(N-1)} & \left(\tilde{b}_y^+\right)_{jN} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad 0 \le j \le N,$$

and $\left(\tilde{B}_y^-\right)_j$ is the $(N+1)\times(N+1)$ matrix whose $(N+1)$-th row is the $(j+1)$-th row of the matrix $\tilde{B}_y^-$ and 0 otherwise, i.e.,

$$\left(\tilde{B}_y^-\right)_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ (\tilde{b}_y^-)_{j0} & (\tilde{b}_y^-)_{j1} & \cdots & (\tilde{b}_y^-)_{j(N-1)} & (\tilde{b}_y^-)_{jN} \end{bmatrix}, \quad 0 \le j \le N.$$

$$\tilde{V}^T = \begin{bmatrix} E_{0(N-1)} & \cdots & E_{(N-2)(N-1)} & E_{(N-1)(N-1)} & E_{N(N-1)} \\ E_{0N} & \cdots & E_{(N-2)N} & E_{(N-1)N} & E_{NN} \\ \mathbf{0} & \cdots & \mathbf{0} & \tilde{B}_x^+ & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\tilde{B}_x^- \end{bmatrix},$$

where E_{ij} is the $(N+1)\times(N+1)$ matrix whose only non-zero entry is $E_{ij}(i+1, j+1) = 1$.

Thanks to (3.8), we can express the solution $\mathbf{u}$ of (3.10) by

$$\mathbf{u} = A^{-1}\mathbf{f} - A^{-1}\tilde{U}(I + \tilde{V}^T A^{-1}\tilde{U})^{-1}\tilde{V}^T A^{-1}\mathbf{f}. \tag{3.12}$$

We recall that the linear system $A\mathbf{v} = \mathbf{f}$ can be solved by using the matrix diagonalization method in a small multiple of N^3 operations (cf. [33] and Appendix). Our algorithm for computing (3.12) is:

1. Precompute the capacitance matrix $I + \tilde{V}^T A^{-1}\tilde{U}$ and its LU factorization. The dominating cost of this step is to compute $A^{-1}\tilde{U}$. A direct approach would require $O(N^4)$ operations, but the cost can be reduced to $O(N^3)$ by using a similar procedure as in the construction of the capacitance matrix for solving the biharmonic equation. We refer to [5,33] for more detail in this regard.
2. Compute $\mathbf{u}$ by (3.12). The cost of this step is $O(N^3)$ for each righthand side $\mathbf{f}$.

3.3 Error Estimates

The error analysis for the two-dimensional case can be carries out using a similar procedure as in the one-dimensional case.

Theorem 3.2 *Let u and u_N be the solutions of* (3.4) *and* (3.6), *respectively. Then, under the condition* (3.5), *we have*

$$\|u - u_N\|_1 \lesssim N^{1-m}\|u\|_m + N^{-k}\|f\|_k.$$

Proof We first recall that there exists an operator $\Pi_N^1 : H^1(\Omega) \to P_N \times P_N$ such that [3, Theorem 7.3]

$$\|u - \Pi_N^1 u\|_1 \lesssim N^{1-m}\|u\|_m, \ m \geq 1. \tag{3.13}$$

We also recall that [3, Theorem 13.4]

$$\|I_N f - f\|_\mu \lesssim N^{\mu-k}\|f\|_k, \quad 0 \leq \mu \leq 1, \ \ k > 1, \tag{3.14}$$

where $I_N : C(\Omega) \to P_N \times P_N$ is the interpolation operator based on the Legendre-Gauss-Lobatto points.

Using (3.4) and (3.6) leads to the error equation

$$a(u - u_N, v_N) = (f - I_N f, v_N), \quad v_N \in P_N \times P_N, \tag{3.15}$$

where

$$\begin{aligned} a(u - u_N, v_N) = {} & \alpha(u - u_N, v_N) + (\nabla(u - u_N), \nabla v_N) \\ & + \left(\int_I K_1^+(\xi, y)(u(1,\xi) - u_N(1,\xi))d\xi, v_N(1, y)\right)_{y\in I} \\ & - \left(\int_I K_1^-(\xi, y)(u(-1,\xi) - u_N(-1,\xi))d\xi, v_N(-1, y)\right)_{y\in I} \\ & + \left(\int_I K_2^+(x, \xi)(u(\xi,1) - u_N(\xi,1))d\xi, v_N(x, 1)\right)_{x\in I} \\ & - \left(\int_I K_2^-(x, \xi)(u(\xi,-1) - u_N(\xi,-1))d\xi, v_N(x, -1)\right)_{x\in I}. \end{aligned} \tag{3.16}$$

To estimate the error, we denote $\hat{e}_N = \Pi_N^1 u - u_N$ and $\tilde{e}_N = \Pi_N^1 u - u$, and take $v_N = \hat{e}_N \in P_N \times P_N$ in the error equation (3.15). We need to bound the last four terms involving integrals. Since the treatment for the four terms are essentially the same, we will only bound the last term. Thanks to Lemma 3.1

$$\begin{aligned} & \left|\left(\int_I K_2^-(x, \xi)(u(\xi,-1) - u_N(\xi,-1))d\xi, \hat{e}_N(x, -1)\right)_{x\in I}\right| \\ & \quad \leq c_0^2\|K_2^-\|_0\|u - u_N\|_1\|\hat{e}_N\|_1 \\ & \quad \leq c_0^2\|K_2^-\|_0(\|\hat{e}_N\|_1 + \|\tilde{e}_N\|_1)\|\hat{e}_N\|_1. \end{aligned} \tag{3.17}$$

We then derive from (3.16) with $v_N = \hat{e}_N$ and (3.17) that

$$\begin{aligned}\alpha\|\hat{e}_N\|_0^2 + \|\nabla\hat{e}_N\|_0^2 &\leq \alpha(\tilde{e}_N, \hat{e}_N) + (\nabla\tilde{e}_N, \nabla\hat{e}_N)\\ &\quad + c_0^2(\|K_1^+\|_0 + \|K_1^-\|_0 + \|K_2^+\|_0 + \|K_2^-\|_0)(\|\hat{e}_N\|_1 + \|\tilde{e}_N\|_1)\|\hat{e}_N\|_1.\end{aligned}$$

Therefore, under the condition (3.5), we have

$$C_K\|\hat{e}_N\|_1 \lesssim \|\tilde{e}_N\|_1 + \|f - I_N f\|_0.$$

Therefore, we have

$$\|u - u_N\|_1 \leq \|\hat{e}_N\|_1 + \|\tilde{e}_N\|_1 \lesssim \|\tilde{e}_N\|_1 + \|f - I_N f\|_0.$$

The desired result follows from the above, (3.13) and (3.14). □

4 Some Extensions

In this section we consider some immediate extensions to related problems that can be treated by a similar approach.

4.1 Problems with Local and Non-local Boundary Conditions

The problem we considered in the last section is with non-local boundary conditions at all four boundaries. The same approach can be easily extended to the case where we only have non-local boundary conditions at part of the boundaries.

Consider, as an example, the following second order elliptic equation with non-local boundary conditions only in the y-direction:

$$\alpha u - \Delta u = f, \quad \text{in } \Omega = (-1, 1)^2, \quad \alpha > 0, \tag{4.1}$$

$$u(\pm 1, y) = 0, \quad y \in I = (-1, 1), \tag{4.2}$$

$$\partial_y u(x, \pm 1) + \int_I K^{\pm}(x, \xi)u(\xi, \pm 1)d\xi = 0, \quad x \in I = (-1, 1). \tag{4.3}$$

Define the approximation space

$$\tilde{X}_N = \big\{u \in P_N \times P_N : u(\pm 1, y) = 0, \quad y \in (-1, 1)\big\},$$

we have $\tilde{X}_N = \text{span}\big\{\varphi_k(x)\varphi_j(y) : 0 \leq k \leq N-2; 0 \leq j \leq N\big\}$, where $\{\varphi_k\}$ are the basis functions defined in (2.7). Then, the Legendre–Galerkin method for (4.1)–(4.3) is to find $u_N \in \tilde{X}_N$ such that

$$\tilde{a}(u_N, v_N) = (I_N f, v_N) \quad \forall v_N \in \tilde{X}_N, \tag{4.4}$$

where

$$\begin{aligned}\tilde{a}(u, v) := \alpha(u, v) + (\nabla u, \nabla v) &+ \left(\int_I K^+(x, \xi)u(\xi, 1)d\xi, v(x, 1)\right)_{x\in I}\\ &- \left(\int_I K^-(x, \xi)u(\xi, -1)d\xi, v(x, -1)\right)_{x\in I}.\end{aligned}$$

Expanding the approximate solution as

$$u_N(x, y) = \sum_{k=0}^{N-2}\sum_{j=0}^{N} u_{kj}\varphi_k(x)\varphi_j(y),$$

and using similar notations as before, we find that (4.4) can be reduced to the matrix equation

$$\alpha M_x U M_y^T + S_x U M_y^T + M_x U S_y^T + \tilde{B}_x^+ U T^+ - \tilde{B}_x^- U T^- = F. \tag{4.5}$$

Hence, we can still apply the Sherman-Morrison-Woodbury formula (3.8) to solve the above equation efficiently with a capacitance matrix of the size $2(N-1)$.

4.2 Problems with Other Type of Non-local Boundary Conditions

In some applications, e.g., the scattering problem from an open cavity, the non-local boundary conditions can take different forms. Consider, for example, the problem (4.1) with (4.2) and the following non-local boundary conditions

$$\partial_y u(x, \pm 1) + \mathscr{T}^{\pm}(u(x, \pm 1)) = 0, \quad x \in I = (-1, 1), \tag{4.6}$$

where the nonlocal operator $\mathscr{T}^{\pm}$ is defined by

$$\mathscr{T}^{\pm}(u(x)) = \int_{\mathbb{R}} \mathrm{i}\beta^{\pm}(\xi)\hat{u}(\xi)e^{\mathrm{i}\xi x}d\xi, \tag{4.7}$$

with given functions $\beta^{\pm}(\xi)$ and $\hat{u}(\xi)$ being the Fourier transform of $u(x)$, i.e.,

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{-i\xi x} dx.$$

Once again, expanding the approximate solution as

$$u_N(x, y) = \sum_{k=0}^{N-2} \sum_{j=0}^{N} u_{kj} \varphi_k(x) \varphi_j(y),$$

we still find that the coefficient matrix $U = (u_{kj})$ satisfies the matrix equation (4.5) except that the matrices $\tilde{B}_x^{\pm}$ now are defined by

$$(\tilde{B}_x^{\pm})_{kj} = (\mathscr{T}^{\pm}\varphi_j, \varphi_k)_{L^2(I)}, \quad 0 \le k, j \le N-2. \tag{4.8}$$

Note that $\mathscr{T}^{\pm}u$ involve an integral over unbounded domain which is not easy to deal with. Different approaches have been proposed for computing (4.7), e.g. by Green's function method or the method of Fourier transform [1,20], or with Hadamard finite-part integral [14]. We describe below an elegant and accurate algorithm to compute (4.8).

The key is the following formula for the Fourier transform of Legendre polynomials (cf. Formula 18.17.19 in [29])

$$\hat{L}_n(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} L_n(x) e^{-\mathrm{i}\xi x} dx = \mathrm{i}^n \frac{\sqrt{2\pi}}{\xi} J_{n+\frac{1}{2}}(\xi), \tag{4.9}$$

where $\mathrm{i} = \sqrt{-1}$ and $J_\nu(\xi)$ is the Bessel function of order ν. Thus, we have $\hat{\varphi}_j(\xi) = \hat{L}_k(\xi) - \hat{L}_{k+2}(\xi)$, and by definition,

$$\begin{aligned}
(\tilde{B}_x^{\pm})_{kj} &= (\mathscr{T}^{\pm}\phi_j, \phi_k)_{L^2(I)} = \int_I \left(\int_{\mathscr{R}} \mathrm{i}\beta^{\pm}(\xi)\hat{\phi}_j(\xi)e^{\mathrm{i}\xi x} d\xi \right) \phi_k(x) dx \\
&= \int_{\mathscr{R}} \mathrm{i}\beta^{\pm}(\xi)\hat{\phi}_j(\xi) \left(\int_I \phi_k(x) e^{\mathrm{i}\xi x} dx \right) d\xi = \int_{\mathscr{R}} \mathrm{i}\beta^{\pm}(\xi)\hat{\phi}_j(\xi)\hat{\phi}_k(-\xi) d\xi.
\end{aligned} \tag{4.10}$$

We can compute the above integral accurately by using Hermite-Gauss quadrature, or by splitting the whole line $\mathbb{R} = (-\infty, -L) \cup [-L, L] \cup (L, \infty)$ (with a suitable $L > 0$) and using

Legendre-Gauss-Lobatto quadrature on $[-L, L]$ and Laguerre-Gauss-Radau quadrature on $(-\infty, -L)$ and (L, ∞).

4.3 Three-Dimensional Problems

The approach presented above can also be extended to three-dimensional problems with non-local boundary conditions. Consider, as an example, the following problem:

$$\alpha u - \Delta u = f, \quad \text{in } \Omega = (-1,1)^3, \quad \alpha > 0, \tag{4.11}$$

$$u(\pm 1, y, z) = 0, \quad y, z \in I = (-1,1), \tag{4.12}$$

$$u(x, \pm 1, z) = 0, \quad x, z \in I = (-1,1), \tag{4.13}$$

$$\partial_z u(x, y, \pm 1) + \int_I \int_I K^{\pm}(x, y; \tilde{x}, \tilde{y}) u(\tilde{x}, \tilde{y}, \pm 1) d\tilde{x} d\tilde{y} = 0, \ x, y \in I. \tag{4.14}$$

If the non-local boundary conditions in the above is replaced by local boundary conditions, an efficient Legendre-spectral Galerkin method based on matrix diagonalization is described in detail in [33]. To solve (4.11)–(4.14) with a Legendre–Galerkin method, we can first diagonalize in z-direction to reduce the approximate problem in 3-D to a sequence of approximate problems in 2-D with non-local boundary conditions that can be solved by using the approach presented above.

4.4 Parabolic Equations

The algorithm we presented above for elliptic problems are well-suited for parabolic problems. Consider, as an example, the following 2-D parabolic problem:

$$u_t - \Delta u = f, \quad \text{in } \Omega \times (0, T], \tag{4.15}$$

$$u(\pm 1, y, t) = 0, \quad y \in (-1,1), \quad t \in (0, T], \tag{4.16}$$

$$u_y(x, \pm 1, t) + \int_I K^{\pm}(x, \xi) u(\xi, \pm 1, t) d\xi = 0, \quad x \in (-1,1), \quad t \in (0, T], \tag{4.17}$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \bar{\Omega}, \tag{4.18}$$

where $\Omega = (-1,1)^2$. Using the same notations as in Sect. 4.1, the Legendre–Galerkin method for (4.15)–(4.18) leads to the following system of ODEs:

$$M_x U'(t) M_y^T + S_x U(t) M_y^T + M_x U(t) S_y^T + \tilde{B}_x^+ U(t) T^+ - \tilde{B}_x^- U(t) T^- = F(t).$$

Applying the second-order backward difference method in time leads to

$$\begin{aligned} M_x \frac{3U^{n+1} - 4U^n + U^{n-1}}{2\Delta t} M_y^T + S_x U^{n+1} M_y^T + M_x U^{n+1} S_y^T \\ + \tilde{B}_x^+ U^{n+1} T^+ - \tilde{B}_x^- U^{n+1} T^- = F^{n+1}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &3 M_x U^{n+1} M_y^T + 2\Delta t S_x U^{n+1} M_y^T + 2\Delta t M_x U^{n+1} S_y^T \\ &\quad + 2\Delta t \tilde{B}_x^+ U^{n+1} T^+ - 2\Delta t \tilde{B}_x^- U^{n+1} T^- \\ &\quad = M_x (4U^n - U^{n-1}) M_y^T + 2\Delta t F^{n+1}. \end{aligned} \tag{4.19}$$

Note that the above system is exactly the same as the system (4.5) so it can be efficiently solved using the Sherman-Morrison-Woodbury formula (3.8). Furthermore, the capacitance

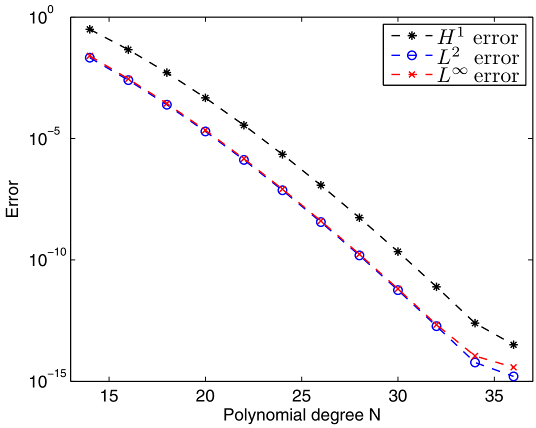

Fig. 1 Example 5.1: errors versus N in semi-log scale

matrix $I + \tilde{V}^T A^{-1} \tilde{U}$ will be the same at each time step so it only has to be precomputed once.

5 Numerical Validations and Applications

5.1 Numerical Validations

We present below several numerical experiments using the efficient Spectral-Galerkin method developed in previous sections. Here the integral terms on the boundary conditions are computed by using Legendre-Gauss-Lobatto quadrature. All computations are performed with MATLAB on a personal computer.

Example 5.1 Two dimensional problem with non-local boundary conditions in the y-direction:

$$
\begin{aligned}
&u-\Delta u = \sin(4\pi x)\Big(\big(1+16\pi^2\big)\big(\frac{1}{3\pi}\sin(3\pi y)+\frac{9}{8}y\big)+3\pi\sin(3\pi y)\Big), \quad \text{in } \Omega=(-1,1)^2,\\
&u(\pm 1, y) = 0, \quad y \in (-1,1),\\
&\partial_y u(x,\pm 1) + \int_I \Big(\mp\frac{1}{9}\Big)\sin(4\pi x)\sin(4\pi\xi)u(\xi,\pm 1)d\xi = 0, \quad x\in(-1,1).
\end{aligned}
$$

The exact solution is $u(x, y) = \sin(4\pi x)(\frac{1}{3\pi}\sin(3\pi y) + \frac{9}{8}y)$.

In Fig. 1, we plot the error $\|u - u_N\|$ in the H^1, L^2, L^∞ norms as a function of the polynomial degree N in semi-log scale. One can observe that the approximate solutions converge exponentially to the exact solution.

Example 5.2 Two dimensional problem with non-local boundary conditions in both x and y directions:

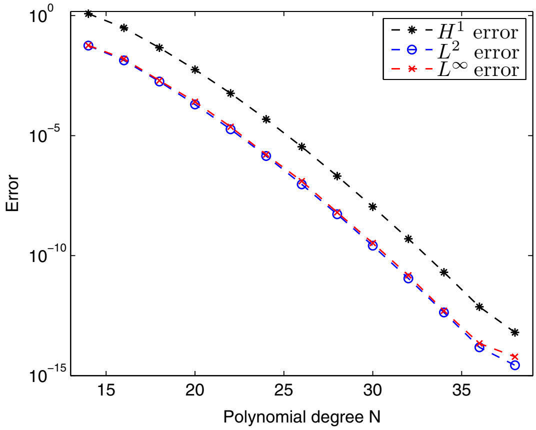

Fig. 2 Example 5.2: errors versus N in semi-log scale

$$
\begin{aligned}
&u - \Delta u = \left(1 + (a\pi)^2 + (b\pi)^2\right) \sin(a\pi x) \sin(b\pi y), \quad \text{in } \Omega = (-1, 1)^2,\\
&\partial_x u(\pm 1, y) + \int_I \mp c_1 \sin(a\pi \xi) \sin(b\pi y) u(\pm 1, \xi) d\xi = 0, \quad y \in (-1, 1),\\
&\partial_y u(x, \pm 1) + \int_I \mp c_2 \sin(a\pi x) \sin(b\pi \xi) u(\xi, \pm 1) d\xi = 0, \quad x \in (-1, 1),
\end{aligned}
$$

where $c_1 = \frac{2a^2\pi^2 \cot(a\pi)}{2a\pi - \sin(2a\pi)}$, $c_2 = \frac{2b^2\pi^2 \cot(b\pi)}{2b\pi - \sin(2b\pi)}$, and the exact solution is $u(x, y) = \sin(a\pi x) \sin(b\pi y)$, $a = b = 4.499$.

We plot in Fig. 2 the error $\|u - u_N\|$ in H^1, L^2, L^∞ norms. Numerical results show that the errors decay exponentially, and verify the error estimate established in Theorem 3.2.

Example 5.3 Two-dimensional parabolic equation with non-local boundary conditions.

$$
\begin{aligned}
&u_t - \Delta u = \cos(\pi t) \sin(4\pi x) \left(16\pi^2 \left(\frac{1}{3\pi} \sin(3\pi y) + \frac{9}{8} y\right) + 3\pi \sin(3\pi y)\right)\\
&\quad - \pi \sin(\pi t) \sin(4\pi x) \left(\frac{1}{3\pi} \sin(3\pi y) + \frac{9}{8} y\right), \quad \text{in } \Omega \times (0, T],\\
&\quad u(\pm 1, y, t) = 0, \quad y \in (-1, 1), \quad t \in (0, T],\\
&\quad u_y(x, \pm 1, t) + \int_I \left(\mp \frac{1}{9}\right) \sin(\pi x) \sin(\pi \xi) u(\xi, \pm 1, t) d\xi = 0, \ x \in I, \ t \in (0, T],\\
&\quad u(x, y, 0) = \sin(4\pi x) \left(\frac{1}{3\pi} \sin(3\pi y) + \frac{9}{8} y\right), \quad (x, y) \in \bar{\Omega}.
\end{aligned}
$$

The exact solution is $u(x, y, t) = \cos(\pi t) \sin(4\pi x)(\frac{1}{3\pi} \sin(3\pi y) + \frac{9}{8} y)$.

We first fix $\Delta t = 10^{-4}$ so that the time discretization error is negligible compared to the spatial error, and plot in Fig. 3 the errors vs N in semi-log scale. Since the solution is smooth, we observe as usual the exponential convergence until the errors are dominated by the time discretization. Then, we fix $N = 32$ so the spatial error is negligible compared with time error, and plot in Fig. 4 the errors in log-log scale. As expected we observe second-order accuracy in time.

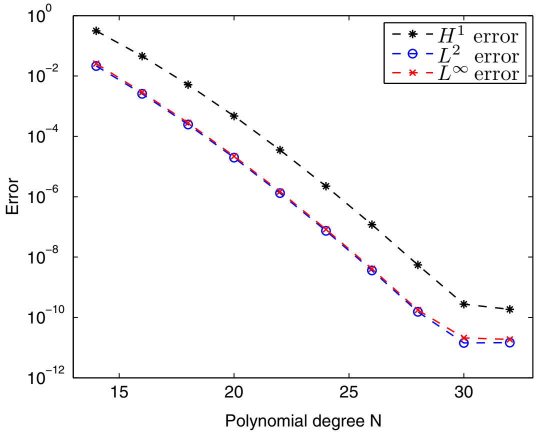

Fig. 3 Example 5.3: $\Delta t = 10^{-4}$, $T = 1$

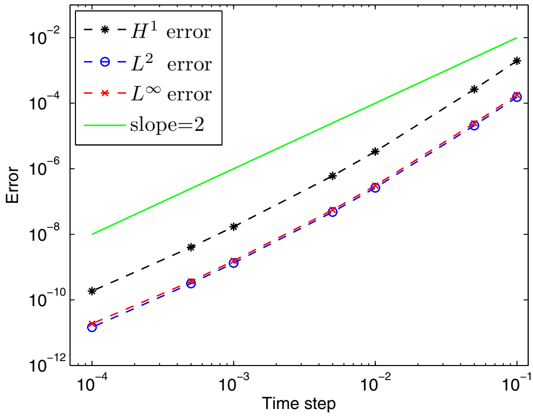

Fig. 4 Example 5.3: $N = 32$, $T = 1$

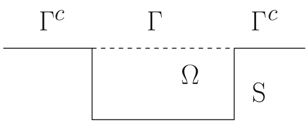

Fig. 5 Geometry of open cavity

5.2 Application to the Scattering Problem from an Open Cavity

The scattering problem from open cavities has important applications and is notorious difficult to compute, especially at high wave numbers (cf. [2,21,22] and the references therein). The geometry of the open cavity $\Omega = (a, b) \times (c, d)$, enclosed by the aperture Γ and the wall S with perfect conductivity, is shown in Fig. 5. Above the flat surface $\Gamma \cup \Gamma^c$, the medium is homogeneous with positive dielectric permittivity ε_0 and magnetic permeability μ_0; while inside the cavity, the medium has dielectric permittvity ε. Let ω be the angular frequency of the incident wave, the wave numbers above the ground and in the cavity are $\kappa_0 = \omega\sqrt{\varepsilon_0\mu_0}$, $\kappa = \omega\sqrt{\varepsilon\mu_0}$, respectively.

The scattered wave satisfies the following Helmholtz equation

$$\begin{aligned} &\Delta u + \kappa^2 u = 0, \quad \text{in } \Omega, \\ &u = 0, \qquad\qquad \text{on } S, \\ &\partial_n u = \mathscr{T} u + g, \quad \text{on } \Gamma, \end{aligned} \tag{5.1}$$

where the nonlocal (Dirichlet-to-Neumann) operator $\mathscr{T}$ is defined by the Fourier transform of u:

$$\mathscr{T}(u(x,d)) = \int_{\mathbb{R}} \mathrm{i}\beta(\xi)\hat{u}(\xi,d)e^{\mathrm{i}\xi x}d\xi, \tag{5.2}$$

where $\hat{u}(\xi, d)$ is the Fourier transform of $u(x, d)$, i.e.,

$$\hat{u}(\xi,d) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,d)e^{-\mathrm{i}\xi x}dx,$$

and $\beta(\xi)$ defined as

$$\beta(\xi) = \begin{cases} (\kappa_0^2 - \xi^2)^{1/2}, & \text{for } |\xi| < \kappa_0, \\ \mathrm{i}(\xi^2 - \kappa_0^2)^{1/2}, & \text{for } |\xi| > \kappa_0, \end{cases}$$

and $g(x) = -2\mathrm{i}\kappa_0 \cos\theta e^{\mathrm{i}\kappa_0 x \sin\theta}$ resulting from the incident field. Readers can refer to [2,23] for more detail.

Note that the non-local operator is of the type considered in Sect. 4.2, so we only have to compute the matrix

$$\begin{aligned} B_{kj} = (\mathscr{T}\phi_j, \phi_k)_{L^2(I)} &= \int_I \left(\int_{\mathbb{R}} \mathrm{i}\beta(\xi)\hat{\phi}_j(\xi)e^{\mathrm{i}\xi x}d\xi \right) \phi_k(x)dx \\ &= \int_{\mathbb{R}} \mathrm{i}\beta(\xi)\hat{\phi}_j(\xi)\hat{\phi}_k(-\xi)d\xi. \end{aligned} \tag{5.3}$$

It is clear from the odd-even property of the Legendre polynomials that

$$\hat{\phi}_n(-\xi) = (-1)^n \hat{\phi}_n(\xi). \tag{5.4}$$

Therefore, when $j+k$ is odd, we have $B_{kj} = 0$, and when $j+k$ is even,

$$B_{kj} = 2(-1)^k \mathrm{i} \int_{\mathbb{R}^+} \beta(\xi)\hat{\phi}_k(\xi)\hat{\phi}_j(\xi)d\xi. \tag{5.5}$$

We approximate the above integral as follows: First we split $\mathbb{R}_+ = [0, L] \cup (L, \infty)$ (with a suitable $L > 0$) and using Legendre-Gauss-Lobatto quadrature on $[0, L]$ and Laguerre-Gauss-Radau quadrature on (L, ∞).

We present below some numerical experiments by taking Ω as a box with coordinates: $[-0.5, 0.5] \times [-0.25, 0]$.

In order to compare our results to those in the literature [2,14], we take $\kappa_0 = \kappa = 32\pi$ and the angle of the incident wave $\theta = 0$. The magnitude of the scattered wave on Γ is plotted in Fig. 6. It matches well with existing results in [2,14]. To determine the order of convergence, we took the approximate solution with $N = 300$ as the reference solution, and plotted the error vs N in Fig. 7. We observe a second-order convergence. Note that we can not expect spectral accuracy due to the singularities at the corners of the cavity. We recall that only first-order convergence result was reported in [2] using the finite-difference method. It is well-known that for problems with corner singularities, spectral methods will double the convergence rate of FEM or FD methods with uniform meshes, thanks to the clustering of Legendre-Gauss points near the corners.

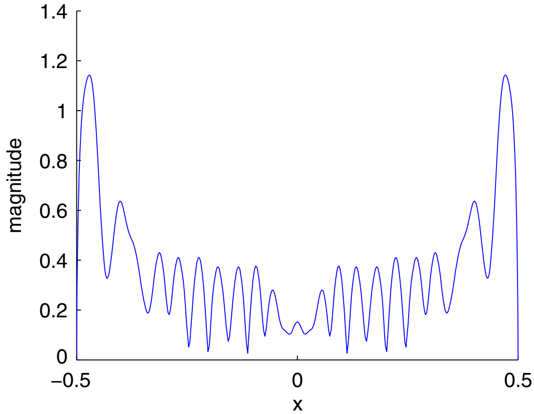

Fig. 6 Magnitude of scattered wave, $\kappa = 32\pi$

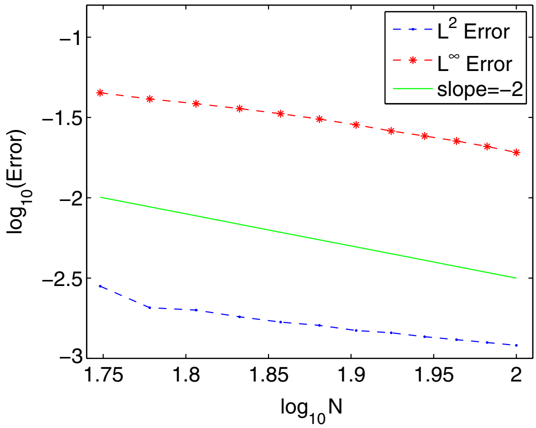

Fig. 7 log-log plot for cavity problem, compare with $u_{300}, \kappa = 32\pi$

6 Concluding Remarks

We developed an efficient Spectral-Galerkin method for elliptic equations with non-local boundary conditions. The method makes essential use of the Sherman-Morrison-Woodbury formula, which allows us to solve problems with non-local boundary conditions with the same computational complexity as required for problems with local boundary conditions. We also carried out a rigorous error analysis, and derived optimal error estimates for the proposed algorithms. Several numerical tests are provided to validate the algorithms and our error analysis. As an application, we used the proposed method to solve the scattering problem from an open cavity.

Appendix: Matrix Diagonalization Method

In this section we briefly recall the matrix diagonalization method in [33] for solving the linear system $A\mathbf{u} = \mathbf{f}$ where A is the matrix defined in (3.9). We can rewrite it as the following matrix equation:

$$\alpha M_x U M_y^T + S_x U M_y^T + M_x U S_y^T = F. \tag{6.1}$$

We diagonalize in x-direction and reduce the problem to $N+1$ one-dimension equations (in y-direction) following the steps below:

1. Consider the generalized eigenvalue problem:

$$M_x \bar{x} = \lambda S_x \bar{x}. \tag{6.2}$$

M_x and S_x are symmetric positive definite matrices. Let Λ be the diagonal matrix whose diagonal entries λ_p are the eigenvalues of (6.2), and let E be the matrix whose columns are the eigenvectors of (6.2). We have

$$M_x E = S_x E \Lambda, \quad E^{-1} = E^T. \tag{6.3}$$

2. Let $U = EV$, thanks to (6.3), the equation (6.1) becomes

$$\alpha S_x E \Lambda V M_y^T + S_x E V M_y^T + S_x E \Lambda V S_y^T = F.$$

Multiplying $E^T S_x^{-1}$ to both sides of the above equation yields

$$\alpha \Lambda V M_y^T + V M_y^T + \Lambda V S_y^T = E^T S_x^{-1} F := G. \tag{6.4}$$

3. Let $\mathbf{v}_p = (v_{p0}, v_{p1}, \ldots, v_{pN})^T$ and $\mathbf{g}_p = (g_{p0}, g_{p1}, \ldots, g_{pN})^T$, $0 \le p \le N$. Then the p-th row of the equation (6.4) can be written as

$$((\alpha\lambda_p + 1) M_y + \lambda_p S_y)\mathbf{v}_p = \mathbf{g}_p. \tag{6.5}$$

Since M_y and S_y are sparse, we can solve (6.5) in $O(N)$ operations for each p. Hence, the main cost of solving (6.1) is the two matrix-matrix multiplications which cost a small multiple of N^3 operations.

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