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STABILITY AND ERROR ANALYSIS OF A NEW CLASS OF HIGHER-ORDER CONSISTENT SPLITTING SCHEMES FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. A new class of fully decoupled consistent splitting schemes for the Navier-Stokes equations are constructed and analyzed in this paper. The schemes are based on the Taylor expansion at $t^{n+\beta}$ with $\beta \geqslant 1$ being a free parameter. It is shown that by choosing $\beta = 3, 6, 9$ respectively for the second-, third- and fourth-order schemes, their numerical solutions are uniformed bounded in a strong norm, and admit optimal global-in-time convergence rates in both 2D and 3D. These results are the first stability and convergence results for any fully decoupled, higher than second-order schemes for the Navier-Stokes equations. Numerical results are provided to show that the third- and fourth-order schemes based on the usual backward differentiation formula (BDF) (i.e. $\beta = 1$) are not unconditionally stable while the new third- and fourth-order schemes with suitable β are unconditionally stable and lead to expected convergence rates.

1. INTRODUCTION

We consider in this paper the construction and error analysis of a new class of high order consistent splitting schemes for the incompressible Navier-Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{1.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.1b}$$

with suitable initial conditions in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ and no-slip boundary condition $\boldsymbol{u} = 0$ on $\partial\Omega$, and $\boldsymbol{f}$ is an external force.

The Navier-Stokes equations play an important role in many fields of science and engineering. Due to its importance in applications, there is an enormous amount of work devoted to the numerical approximation of the Navier-Stokes equations. These numerical methods can be roughly classified into two categories: coupled approach with a mixed formulation (cf. [2, 6, 7] and the references therein), and decoupled approach through a projection type method (including the pressure-correction and the velocity correction methods) [4, 8, 10–12, 19, 23–26, 30, 31], and the consistent splitting method [13, 18, 28, 32] (see also the gauge method [5, 22]). We refer to [9] for

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a review on the decoupled approach, and would like to point out that the projection type schemes suffer from a splitting error which prevents them from achieving full order accuracy in strong norms, while the consistent splitting schemes do not loose accuracy. However, it has been a long standing open question on how to construct unconditionally stable second- or higher-order decoupled scheme with a rigorous stability and error analysis.

In a recent work [16], we constructed a new second-order consistent splitting scheme, based on the Taylor expansions at time $t^{n+\beta}$, which, in the absence of nonlinear term, reads as follows:

(1.2)
$$\frac{(2\beta+1)\boldsymbol{u}^{n+1}-4\beta\boldsymbol{u}^n+(2\beta-1)\boldsymbol{u}^{n-1}}{2\delta t}-\nu\Delta(\beta\boldsymbol{u}^{n+1}-(\beta-1)\boldsymbol{u}^n)+\nabla((\beta+1)p^n-\beta p^{n-1}) = f^{n+\beta},$$

(1.3)
$$(\nabla p^{n+1},\nabla q)=(f^{n+1},\nabla q)-\nu(\nabla\times\nabla\times\boldsymbol{u}^{n+1},\nabla q),\quad \forall q\in H^1(\Omega),$$

where we use the identity $\Delta\boldsymbol{u}=\nabla\nabla\cdot\boldsymbol{u}-\nabla\times\nabla\times\boldsymbol{u}$ in (1.3). Note that by integration by parts, we can express the volume integral in the last equation as a boundary integral

$$(\nabla\times\nabla\times\boldsymbol{u}^{n+1},\nabla q)=\int_{\partial\Omega}\boldsymbol{n}\times\nabla\times\boldsymbol{u}^{n+1}\cdot\nabla q,$$

which makes it possible to implement with C^0 finite-element methods. We were able to prove that the above scheme with $\beta=5$ is unconditionally stable in a strong norm, which was the first such result for any fully decoupled second- or higher-order scheme for the time dependent Stokes problem. Then, by employing the generalized scalar auxiliary variable (GSAV) approach [15] to handle the nonlinear term, we also conducted a rigorous stability and error analysis for a corresponding second-order consistent splitting scheme for the Navier-Stokes equations.

While one can construct formally higher-order consistent splitting schemes based on the Taylor expansions at time $t^{n+\beta}$, it is an open question on how to prove its unconditional stability with a suitable β for third- and higher-order schemes. The main purpose of this paper is to provide an affirmative answer to this open question. More precisely, our main contributions include:

- We improve the results in [16] by showing that the second order consistent scheme based on the Taylor expansion at time t^{n+3}, instead of t^{n+5}, is unconditionally stable in $l^2(H^2)\cap l^\infty(H^1)$. Note that as β increases, so does the truncation error. Therefore, it is beneficial to use smaller β when possible.
- We show that the third-order (resp. fourth-order) consistent splitting schemes based on the Taylor expansion at time t^{n+6} (resp. t^{n+9}) is unconditionally stable in $l^2(H^2)\cap l^\infty(H^1)$, and also carry out a rigorous error analysis with global-in-time optimal error estimates both in 2D and 3D for the new second- to fourth-order consistent splitting schemes. Note that in [16] only local-in-time error estimate was established in 3D for a second-order consistent splitting scheme with $\beta=5$.

To the best of our knowledge, these schemes are the first higher than second-order fully decoupled schemes for the Navier-Stokes equations with a rigorous stability and error analysis.

We emphasize that the analysis in [16] for the second-order scheme cannot be easily extended to third- or higher-order schemes. A main difficulty is that stability in the higher-order cases cannot be derived with usual test functions. We recall that the stability of the usual higher-order BDF schemes for parabolic type equations relies on a result by Nevanlinna and Odeh [21] (see also [1] for the extension to the six-order BDF scheme) in which the existence of suitable multipliers that can lead to energy stability was established. Most recently in [17], we extended the Nevanlinna and Odeh Lemma to the generalized higher-order (up to order four) BDF schemes for parabolic type equations and carried out a rigorous error analysis. The technique used in [17] to identify suitable multipliers, as well as the Lemma on the Stokes commutator in [20], are the two essential tools in proving the unconditional stability of the new schemes proposed in this paper. However, unlike the parabolic type equations considered in [17], there is another essential difficulty to control the explicit treatment of the pressure in the consistent splitting schemes. In fact, the multipliers identified in [17] for parabolic type equations cannot be directly used here. A key and nontrivial step is to split the viscous term into suitable forms (see (3.16)) such that the explicit pressure terms can be controlled.

The rest of the paper is organized as follows. In the next section, we provide some preliminaries to be used in the sequel. In Section 3, we construct a new consistent splitting scheme for the time dependent Stokes equations and prove its unconditional stability in a strong norm. Then, in Section 4, we present the new high order consistent splitting scheme for the Navier-Stokes equations with explicit treatment for the nonlinear terms and present detailed error analysis. In the final section, we provide a numerical example to validate the accuracy of our scheme, and conclude with a few remarks.

2. Preliminaries

We first introduce some notations. Let W be a Banach space, we shall also use the standard notations $L^p(0,T;W)$ and $C([0,T];W)$. To simplify the notation, we often omit the spatial dependence for the exact solution u, i.e., $u(x,t)$ is often denoted by $u(t)$. We shall use bold faced letters to denote vectors and vector spaces, and use C to denote a generic positive constant independent of the discretization parameters. We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ the inner product and the norm in $L^2(\Omega)$, and $\|\cdot\|_1$, $\|\cdot\|_2$, the norm in $H^1(\Omega)$, $H^2(\Omega)$ respectively, and denote

$$\mathbb{V} = \left\{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \ : \ \nabla\cdot\mathbf{v} = 0\right\}.$$

Next, we define the trilinear form $b(\cdot,\cdot,\cdot)$ by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x}.$$

Using Hölder inequality and Sobolev inequality, we have [29]

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leqslant c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1^{1/2}\|\boldsymbol{v}\|_2^{1/2}\|\boldsymbol{w}\|, \quad d=2,3. \tag{2.1}$$

We also use frequently the following inequalities (see, for instance, Lemma 2.1 in [29]):

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leqslant \begin{cases} c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1; \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_0\|\boldsymbol{w}\|_1; \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_0; \\ c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_0; \\ c\|\boldsymbol{u}\|_0\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_1; \end{cases} \quad d \leqslant 3. \tag{2.2}$$

Note that the above inequalities, except the third one, are also valid when $d = 4$.

We will frequently use the following discrete versions of the Gronwall lemma.

Lemma 2.1 (Discrete Gronwall lemma)**.** (See, for instance, lemma 5.4 in [14].) Let a_n, b_n, c_n, and d_n be four nonnegative sequences satisfying

$$a_m + \tau\sum_{n=1}^{m} b_n \leqslant \tau\sum_{n=0}^{m-1} a_n d_n + \tau\sum_{n=0}^{m-1} c_n + C,\ m \geqslant 1,$$

where C and τ are two positive constants. Then

$$a_m + \tau\sum_{n=1}^{m} b_n \leqslant \exp\big(\tau\sum_{n=0}^{m-1} d_n\big)\big(\tau\sum_{n=0}^{m-1} c_n + \tilde{C}\big),\ m \geqslant 1,$$

where $\tilde{C}$ is a constant that depends on the initial data a_0, b_0, c_0, and the constant C.

In order to establish an unconditional stability result for (1.2)–(1.3), we need the following result about the Stokes pressure introduced in [20]. For any $\boldsymbol{u} \in H^2(\Omega, \mathbb{R}^N)$, the Stokes pressure $p_s = p_s(\boldsymbol{u})$ is defined as

$$\nabla p_s(\boldsymbol{u}) = (\Delta\mathcal{P} - \mathcal{P}\Delta)\boldsymbol{u}, \tag{2.3}$$

where $\mathcal{P}$ is the Leray-Helmholtz projection operator onto divergence-free fields with zero normal component, providing the Helmholtz decomposition $\boldsymbol{u} = \mathcal{P}\boldsymbol{u} + \nabla\phi$, where

$$\big(\mathcal{P}\boldsymbol{u}, \nabla q\big) = \big(\boldsymbol{u} - \nabla\phi, \nabla q\big) = 0, \quad \forall q \in H^1(\Omega). \tag{2.4}$$

Then it is proved in [20] that

Lemma 2.2. Let $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ be a connected bounded domain with C^3 boundary. Then for any $\varepsilon > 0$, there exists $C \geqslant 0$ such that for all vector fields $\boldsymbol{u} \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$,

$$\int_\Omega |(\Delta\mathcal{P} - \mathcal{P}\Delta)\boldsymbol{u}|^2 \leqslant \big(\frac{1}{2} + \varepsilon\big)\int_\Omega |\Delta\boldsymbol{u}|^2 + C\int_\Omega |\nabla\boldsymbol{u}|^2. \tag{2.5}$$

In order to make use of the energy techniques to conduct stability and error analysis, we need to find suitable multipliers with the help of Lemma 2.3 from Dahlquist's G-stability theory [3].

Lemma 2.3. Let $\alpha(\zeta) = \alpha_q\zeta^q + \cdots + \alpha_0$ and $\mu(\zeta) = \mu_q\zeta^q + \cdots + \mu_0$ be polynomials of degree at most q (and at least one of them of degree q) that have no common divisors. Let $(\cdot,\cdot)$ be an inner product with associated norm $|\cdot|$. If

$$\operatorname{Re}\frac{\alpha(\zeta)}{\mu(\zeta)} > 0 \quad \text{for } |\zeta| > 1, \tag{2.6}$$

then there exists a symmetric positive definite matrix $G = (g_{ij}) \in \mathbb{R}^{q\times q}$ and real $\delta_0, \dots, \delta_q$ such that for $\upsilon^0, \dots, \upsilon^q$ in the inner product space,

$$\Big(\sum_{i=0}^{q} \alpha_i \upsilon^i, \sum_{j=0}^{q} \mu_j \upsilon^j\Big) = \sum_{i,j=1}^{q} g_{ij}(\upsilon^i, \upsilon^j) - \sum_{i,j=1}^{q} g_{ij}(\upsilon^{i-1}, \upsilon^{j-1}) + \Big|\sum_{i=0}^{q} \delta_i \upsilon^i\Big|^2. \tag{2.7}$$

3. Higher-order consistent splitting scheme for the time dependent Stokes equations

We shall first present generalized BDF consistent splitting schemes based on the Taylor expansion at time $t^{n+\beta}$, and then show that the k-th ($k = 2, 3, 4$) order with suitable βs are unconditionally stable in the strong norm.

3.1. **The generalized BDF schemes.** We note that we constructed in [17] generalized BDF schemes based on the Taylor expansion at time $t^{n+\beta}$ for general parabolic type equations. Following [17], we can construct generalized BDF consistent splitting schemes as follows. Given an integer $k \geqslant 2$, denote $t^n = n\delta t$, it follows from the Taylor expansion at time $t^{n+\beta}$ that

$$\phi(t^{n+1-i}) = \sum_{m=0}^{k-1} [(1-i-\beta)\delta t]^m \frac{\phi^{(m)}(t^{n+\beta})}{m!} + \mathcal{O}(\delta t^k), \quad \forall i \geqslant 0. \tag{3.1}$$

Therefore we have

$$\frac{1}{\delta t}\sum_{q=0}^{k} a_{k,q}(\beta)\phi(t^{n+1-k+q}) = \partial_t \phi(t^{n+\beta}) + \mathcal{O}(\delta t^k), \tag{3.2}$$

with $a_{k,q}(\beta)$ can be obtained by solving the linear system:

$$\begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ \beta-1 & \beta & \dots & \dots & \beta+k-1 \\ (\beta-1)^2 & \beta^2 & \dots & \dots & (\beta+k-1)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\beta-1)^k & \beta^k & \dots & \dots & (\beta+k-1)^k \end{bmatrix} \begin{bmatrix} a_{k,k}(\beta) \\ a_{k,k-1}(\beta) \\ a_{k,k-2}(\beta) \\ \vdots \\ a_{k,0}(\beta) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \tag{3.3}$$

and

$$\sum_{q=0}^{k-1} b_{k,q}(\beta)\phi(t^{n+2-k+q}) = \phi(t^{n+\beta}) + \mathcal{O}(\delta t^k), \tag{3.4}$$

with $b_{k,q}(\beta)$ can be obtained by solving the linear system:

$$\begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ \beta-1 & \beta & \dots & \dots & \beta+k-2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\beta-1)^{k-1} & \beta^{k-1} & \dots & \dots & (\beta+k-2)^{k-1} \end{bmatrix} \begin{bmatrix} b_{k,k-1}(\beta) \\ b_{k,k-2}(\beta) \\ \vdots \\ b_{k,0}(\beta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \tag{3.5}$$

and finally

$$\sum_{q=0}^{k-1} c_{k,q}(\beta)\phi(t^{n+1-k+q}) = \phi(t^{n+\beta}) + \mathcal{O}(\delta t^k), \tag{3.6}$$

with $c_{k,q}(\beta)$ can be obtained by solving the linear system:

$$\begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ \beta & \beta+1 & \dots & \dots & \beta+k-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta^{k-1} & (\beta+1)^{k-1} & \dots & \dots & (\beta+k-1)^{k-1} \end{bmatrix} \begin{bmatrix} c_{k,k-1}(\beta) \\ c_{k,k-2}(\beta) \\ \vdots \\ c_{k,0}(\beta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.7}$$

Next, we would like to introduce the following notations to simplify the presentation below,

$$\begin{gathered} A_k^\beta(\phi^i) = \sum_{q=0}^{k} a_{k,q}(\beta)\phi^{i-k+q}, \quad B_k^\beta(\phi^i) = \sum_{q=0}^{k-1} b_{k,q}(\beta)\phi^{i-k+1+q}, \\ C_k^\beta(\phi^i) = \sum_{q=0}^{k-1} c_{k,q}(\beta)\phi^{i-k+1+q}. \end{gathered} \tag{3.8}$$

Now, with the above notations, the generalized k-th order BDF type schemes with explicit treatment of the pressure for the time dependent Stokes equation (in the absence of $\boldsymbol{f}$ and nonlinear term in (1.1)) are as follows:

$$\frac{A_k^\beta(\boldsymbol{u}^{n+1})}{\delta t} - \nu\Delta B_k^\beta(\boldsymbol{u}^{n+1}) + \nabla C_k^\beta(p^n) = 0, \tag{3.9a}$$

$$(\nabla p^{n+1}, \nabla q) = -\nu(\nabla \times \nabla \times \boldsymbol{u}^{n+1}, \nabla q), \quad \forall q \in H^1(\Omega). \tag{3.9b}$$

3.2. **Linear stability regions.** Before providing the stability proof for the new schemes (3.9), we would like to first investigate the linear stability regions of the new BDF type schemes. For the test equation $\phi_t = \lambda\phi$, by performing the Taylor expansions at $t^{n+\beta}$, a more general BDF type method can be written as

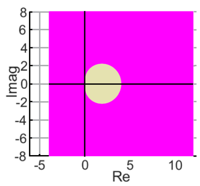
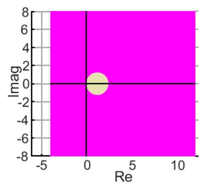
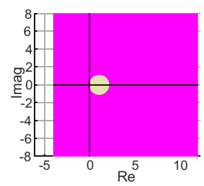
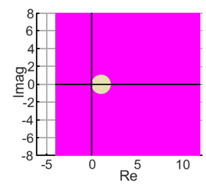
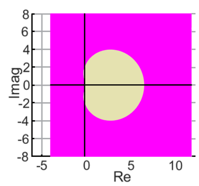
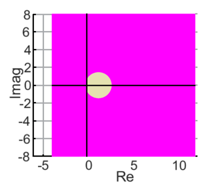
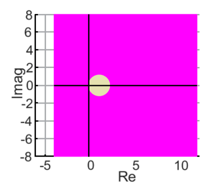
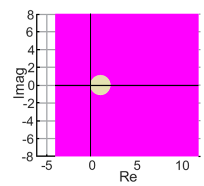
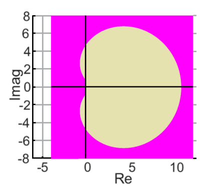
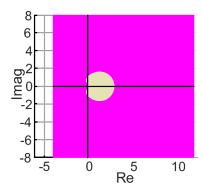
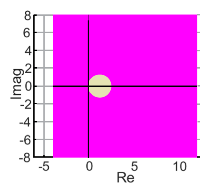
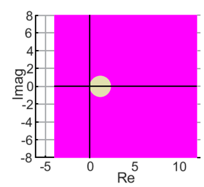
$$\frac{A_k^\beta(\phi^{n+1})}{\delta t} = \lambda B_k^\beta(\phi^{n+1}). \tag{3.10}$$

In order to study the stability region for $\beta \neq 1$, we set $\phi^n = \mu^n$ and $z = \lambda\delta t$ in (3.10) to obtain its characteristic polynomial

$$\sum_{q=0}^{k}(a_{k,q}(\beta) - b_{k,q-1}(\beta)z)\mu^q = 0, \tag{3.11}$$

where $a_{k,q}(\beta)$ and $b_{k,q}(\beta)$ are defined in (3.3) and (3.5) respectively and we further define $b_{k,-1} = 0$ in (3.11). Then the region of absolute stability is the set of all $z \in \mathbb{C}$ such that all roots μ of the characteristic equation (3.11) satisfy $|\mu| \leqslant 1$, and any root with $|\mu| = 1$ is simple. In Table 1, we plot the stability regions of the general BDF type method (3.10) for $\beta = 1, 3, 6, 9$. We observe that the stability regions increases as we increases β, at the expense of increased truncation error.

TABLE 1. The pink parts show the linear stability regions

	$\beta = 1$	$\beta = 3$	$\beta = 6$	$\beta = 9$
Second order				
Third order				
Fourth order				

3.3. **A uniform multiplier.** To conduct the stability and error analysis, we need to choose a suitable β for schemes of different orders. In the following, we choose $\beta = \beta_k$ as follows:

$$\beta_2 = 3,\ \beta_3 = 6,\ \beta_4 = 9. \tag{3.12}$$

These choices of β are sufficient for our purposes, but not necessarily the smallest possible. We recall that as β increases, so does the truncation error. So it is desirable to choose β as small as possible while maintaining stability. For the rest of the paper, we fix β_k as (3.12) and then the explicit expression of (3.9a) becomes: $k = 2,\ \beta = 3$:

$$\frac{7\boldsymbol{u}^{n+1} - 12\boldsymbol{u}^n + 5\boldsymbol{u}^{n-1}}{2\delta t} - \nu\Delta(3\boldsymbol{u}^{n+1} - 2\boldsymbol{u}^n) + \nabla(4p^n - 3p^{n-1}) = 0; \tag{3.13}$$

$k = 3,\ \beta = 6$:

$$\frac{146\boldsymbol{u}^{n+1} - 393\boldsymbol{u}^n + 354\boldsymbol{u}^{n-1} - 107\boldsymbol{u}^{n-2}}{6\delta t} - \nu\Delta(21\boldsymbol{u}^{n+1} - 35\boldsymbol{u}^n + 15\boldsymbol{u}^{n-1}) + \nabla(28p^n - 48p^{n-1} + 21p^{n-2}) = 0. \tag{3.14}$$

$k = 4,\ \beta = 9$:

$$\frac{2289\boldsymbol{u}^{n+1} - 8432\boldsymbol{u}^n + 11700\boldsymbol{u}^{n-1} - 7248\boldsymbol{u}^{n-2} + 1691\boldsymbol{u}^{n-3}}{12\delta t} - \nu\Delta(165\boldsymbol{u}^{n+1} - 440\boldsymbol{u}^n + 396\boldsymbol{u}^{n-1} - 120\boldsymbol{u}^{n-2}) + \nabla(220p^n - 594p^{n-1} + 540p^{n-2} - 165p^{n-3}) = 0. \tag{3.15}$$

A key step in the proof is to properly split $B_k^{\beta_k}(\boldsymbol{u}^{n+1})$ into three parts as follows:

$$B_k^{\beta_k}(\boldsymbol{u}^{n+1}) = \eta_k C_k^{\beta_k}(\boldsymbol{u}^{n+1}) + D_k^{\beta_k}(\boldsymbol{u}^{n+1}) + F_k^{\beta_k}(\boldsymbol{u}^{n+1}), \quad k = 2, 3, 4, \tag{3.16}$$

with η_k being a suitable positive number to be specified, and

(3.17a)

$$F_2^{\beta_2}(\boldsymbol{u}^{n+1}) := \sum_{q=0}^{1} f_{2,q}(\beta_2)\boldsymbol{u}^{n+q} = \frac{1}{100}\boldsymbol{u}^{n+1} + 0\boldsymbol{u}^n,$$

(3.17b)

$$F_3^{\beta_3}(\boldsymbol{u}^{n+1}) := \sum_{q=0}^{2} f_{3,q}(\beta_3)\boldsymbol{u}^{n-1+q} = \frac{1}{100}(27\boldsymbol{u}^{n+1} - 21\boldsymbol{u}^n) + 0\boldsymbol{u}^{n-1},$$

(3.17c)

$$F_4^{\beta_4}(\boldsymbol{u}^{n+1}) := \sum_{q=0}^{3} f_{4,q}(\beta_4)\boldsymbol{u}^{n-2+q} = \frac{2}{10^5}(215\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}) + 0\boldsymbol{u}^{n-2},$$

and

(3.18)

$$d_{k,q}(\beta_k) = b_{k,q}(\beta_k) - \eta_k c_{k,q}(\beta_k) - f_{k,q}(\beta_k), \quad D_k^{\beta_k}(\boldsymbol{u}^{n+1}) := \sum_{q=0}^{k-1} d_{k,q}(\beta_k)\boldsymbol{u}^{n-k+2+q}.$$

The reasons for the above splitting will become clear later. In the above, η_k should be chosen such that $\eta_k > \frac{\sqrt{2}}{2} \approx 0.7071$, the reason will be given in (3.35).

By choosing $F_k^{\beta_k}$ as in (3.17), we have the following inequalities, which are useful in the next section. The explicit telescoping terms given in appendix A imply there exists $U_k(\boldsymbol{u}^i, \dots, \boldsymbol{u}^{i+2-k}) \geqslant 0$, $k = 2, 3, 4$ such that

$$\begin{aligned}(3.19)\quad &\left(F_k^{\beta_k}(\boldsymbol{u}^{n+1}), C_k^{\beta_k}(\boldsymbol{u}^{n+1})\right) \\ &\geqslant \kappa_k\|\boldsymbol{u}^{n+1}\|^2 + U_k(\boldsymbol{u}^{n+1}, \dots, \boldsymbol{u}^{n+3-k}) - U_k(\boldsymbol{u}^n, \dots, \boldsymbol{u}^{n+2-k}),\end{aligned}$$

with

$$\kappa_2 = \frac{1}{100}, \quad \kappa_3 = \frac{3}{50}, \quad \kappa_4 = \frac{1}{10^4}. \tag{3.20}$$

In the following, we fix $\eta_k = 0.71$ and β_k as in (3.12) for $k = 2, 3, 4$. Then, we can establish two important lemmas which play key roles in the stability and error analysis. To this end, we introduce some polynomials with coefficients appearing in (3.8) and (3.18),

(3.21)

$$\tilde{A}_k^{\beta_k}(\zeta) = \sum_{q=0}^{k} a_{k,q}(\beta_k)\zeta^q, \quad \tilde{C}_k^{\beta_k}(\zeta) = \sum_{q=0}^{k-1} c_{k,q}(\beta_k)\zeta^q, \quad \tilde{D}_k^{\beta_k}(\zeta^q) = \sum_{q=0}^{k-1} d_{k,q}(\beta_k)\zeta^q.$$

Lemma 3.1. Given $\tilde{A}_k^{\beta_k}(\zeta), \tilde{C}_k^{\beta_k}(\zeta)$ defined in (3.21) and β_k as in (3.12), we have

$$\gcd\left(\tilde{A}_k^{\beta_k}(\zeta), \zeta\tilde{C}_k^{\beta_k}(\zeta)\right) = 1, \; k = 2, 3, 4, \tag{3.22}$$

i.e. they have no common divisor, and

$$\mathrm{Re}\, \frac{\tilde{A}_k^{\beta_k}(\zeta)}{\zeta\tilde{C}_k^{\beta_k}(\zeta)} > 0, \quad \text{for } |\zeta| > 1, \; k = 2, 3, 4. \tag{3.23}$$

The proof of the above lemma in a more general form was given in [17] (Theorem 1), which shows (3.22) and (3.23) are true for all $\beta_k > 1$.

Lemma 3.2. Given $\tilde{D}_k^{\beta_k}(\zeta), \tilde{C}_k^{\beta_k}(\zeta)$ defined in (3.21), β_k as in (3.12) and $\eta_k = 0.71$, we have

$$\gcd\big(\tilde{D}_k^{\beta_k}(\zeta), \tilde{C}_k^{\beta_k}(\zeta)\big) = 1,\ k = 2, 3, 4, \tag{3.24}$$

i.e. they have no common divisor, and

$$\mathrm{Re}\,\frac{\tilde{D}_k^{\beta_k}(\zeta)}{\tilde{C}_k^{\beta_k}(\zeta)} > 0, \quad \text{for } |\zeta| > 1,\ k = 2, 3, 4. \tag{3.25}$$

We shall defer the proof to Appendix B.

Several remarks are in order.

- One may choose other forms of $F_k^{\beta_k}$ in (3.17). As long as (3.19) with $\kappa_k > 0$ and Lemma 3.2 are still true.
- For larger values of η_k, a larger β may be required to prove Lemma 3.2, which in turn introduces a larger truncation error in the scheme. Therefore, we complete the proof by choosing $\eta_k = 0.71$-as small as possible while still satisfying $\eta_k > \frac{\sqrt{2}}{2}$.
- β_k can also be non-integer, for example, one can prove the above two lemmas by the same processes by choosing $\beta_2 = 2.9$ for the second order scheme.

3.4. **Unconditional stability.** With the help of Lemma 2.2–Lemma 3.2, we can prove the following results for the scheme (3.9).

Theorem 3.3. Suppose Ω satisfies the conditions in Lemma 2.2 and given $\boldsymbol{u}^i$, $i = 1, .., k-1$ such that $\|\nabla \boldsymbol{u}^i\|^2 + \delta t\|\Delta \boldsymbol{u}^i\|^2 \leqslant C\|\nabla \boldsymbol{u}^0\|^2$, $i = 1, .., k-1$. The scheme (3.9) with $\beta = \beta_k$ chosen as in (3.12) is unconditionally stable in the sense that, for all $n \geqslant 0$, we have

$$\|\nabla \boldsymbol{u}^{n+1}\|^2 + \delta t \sum_{i=k-1}^{n} \|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2 + \delta t \sum_{i=0}^{n} \|\Delta \boldsymbol{u}^{i+1}\|^2 + \delta t \sum_{i=0}^{n} \|\nabla p^{i+1}\|^2 \leqslant C,\ k = 2, 3, 4, \tag{3.26}$$

where C is a constant independent of the time step δt and n.

Proof. Taking the inner product of (3.9a) with $-\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})$, we deal with the three terms as follows. First, we split $B_k^{\beta_k}$ as in (3.16),

$$\begin{aligned} &\big(-\nu\Delta B_k^{\beta_k}(\boldsymbol{u}^{n+1}), -\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big) \\ &\quad = \nu\Big(\Delta(\eta_k C_k^{\beta_k}(\boldsymbol{u}^{n+1}) + D_k^{\beta_k}(\boldsymbol{u}^{n+1}) + F_k^{\beta_k}(\boldsymbol{u}^{n+1})), \Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\Big) \\ &\quad = \eta_k\nu\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\|^2 + \nu\big(\Delta D_k^{\beta_k}(\boldsymbol{u}^{n+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big) + \nu\big(\Delta F_k^{\beta_k}(\boldsymbol{u}^{n+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big). \end{aligned} \tag{3.27}$$

If we choose $\eta_k = 0.71$, it follows from Lemma 2.3 and Lemma 3.2 that there exists a symmetric positive definite matrix $H_k = (h_{ij}) \in \mathbb{R}^{(k-1)\times(k-1)}$ such that

$$\begin{aligned} &\big(\Delta D_k^{\beta_k}(\boldsymbol{u}^{n+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big) \\ &\qquad \geqslant \sum_{i,j=1}^{k-1} h_{ij}(\Delta \boldsymbol{u}^{n+2+i-k}, \Delta \boldsymbol{u}^{n+2+j-k}) - \sum_{i,j=1}^{k-1} h_{ij}(\Delta \boldsymbol{u}^{n+1+i-k}, \Delta \boldsymbol{u}^{n+1+j-k}), \end{aligned} \tag{3.28}$$

and (3.19) implies that

$$(3.29)\quad \big(\Delta F_k^{\beta_k}(\boldsymbol{u}^{n+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big) \geqslant \kappa_k \|\Delta \boldsymbol{u}^{n+1}\|^2 + U_k(\Delta \boldsymbol{u}^{n+1}, \dots, \Delta \boldsymbol{u}^{n+3-k}) - U_k(\Delta \boldsymbol{u}^{n}, \dots, \Delta \boldsymbol{u}^{n+2-k}).$$

For the pressure term,

$$(3.30)\quad \big(\nabla C_k^{\beta_k}(p^n), \Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big) \leqslant \frac{\gamma}{2\nu}\|\nabla C_k^{\beta_k}(p^n)\|^2 + \frac{\nu}{2\gamma}\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\|^2,$$

with γ can be any positive number. A key step is to deal with the first term in the above using Lemma 2.2. We recall from [20] that

$$(3.31)\quad (\nabla p_s(\boldsymbol{u}^n), \nabla q) = -(\nabla \times \nabla \times \boldsymbol{u}^n, \nabla q),$$

where $p_s(\boldsymbol{u}^n)$ is the Stokes pressure associated with $\boldsymbol{u}^n$ and it follows from (3.9b) that

$$(3.32)\quad \big(\nabla C_k^{\beta_k}(p^n), \nabla q\big) = -\nu\big(\nabla \times \nabla \times C_k^{\beta_k}(\boldsymbol{u}^n), \nabla q\big), \quad \forall q \in H^1(\Omega).$$

Taking $q = C_k^{\beta_k}(p^n)$ in (3.32) and in (3.31), we find from (3.31) with $\boldsymbol{u} = C_k^{\beta_k}(\boldsymbol{u}^n)$ that

$$(3.33)\quad \|\nabla C_k^{\beta_k}(p^n)\| \leqslant \nu \|\nabla p_s(C_k^{\beta_k}(\boldsymbol{u}^n))\|.$$

Now, we can use (3.33) and (2.5) to bound the first term as follows

$$(3.34)\quad \begin{aligned} \frac{\gamma}{2\nu}\|\nabla C_k^{\beta_k}(p^n)\|^2 &\leqslant \frac{\gamma\nu}{2}\|\nabla p_s(C_k^{\beta_k}(\boldsymbol{u}^n))\|^2 \\ &\leqslant \gamma\nu(\frac{1}{4} + \frac{\varepsilon}{2})\|\Delta C_k^{\beta_k}(\boldsymbol{u}^n)\|^2 + C\gamma\nu\|\nabla C_k^{\beta_k}(\boldsymbol{u}^n)\|^2, \end{aligned}$$

with $\varepsilon > 0$ which can be arbitrarily small. We observe from (3.27)–(3.34) that to ensure stability, we need

$$(3.35)\quad \eta_k \geqslant \min_{\gamma > 0}\big(\frac{1}{2\gamma} + \gamma(\frac{1}{4} + \frac{\varepsilon}{2})\big) \overset{\gamma = \sqrt{2}}{=} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\varepsilon}{2}.$$

As ε can be chosen arbitrarily small, we only need to choose $\eta_k > \frac{\sqrt{2}}{2} \approx 0.7071$ to ensure (3.35) and that is why we fix $\eta_k = 0.71$. It remains to deal with the last term:

$$(3.36)\quad \frac{1}{\delta t}\big(A_k^{\beta_k}(\boldsymbol{u}^{n+1}), -\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big).$$

Again, it follows from Lemma 2.3 and Lemma 3.1 that there exists symmetric positive definite matrix $G_k = (g_{ij}) \in \mathbb{R}^{k \times k}$ such that

$$(3.37)\quad \big(A_k^{\beta_k}(\boldsymbol{u}^{n+1}), -\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big) \geqslant \sum_{i,j=1}^{k} g_{ij}(\nabla \boldsymbol{u}^{n+1+i-k}, \nabla \boldsymbol{u}^{n+1+j-k}) - \sum_{i,j=1}^{k} g_{ij}(\nabla \boldsymbol{u}^{n+i-k}, \nabla \boldsymbol{u}^{n+j-k}).$$

With $\gamma = \sqrt{2}$ and $\eta_k = 0.71$, summing up $\delta t((3.27) + (3.30))$ and (3.37), using the estimates above, we find

(3.38)
$$\begin{aligned}
&\sum_{i,j=1}^{k} g_{ij}(\nabla \boldsymbol{u}^{n+1+i-k}, \nabla \boldsymbol{u}^{n+1+j-k}) - \sum_{i,j=1}^{k} g_{ij}(\nabla \boldsymbol{u}^{n+i-k}, \nabla \boldsymbol{u}^{n+j-k}) \\
&+ \nu\delta t \sum_{i,j=1}^{k-1} h_{ij}(\Delta \boldsymbol{u}^{n+2+i-k}, \Delta \boldsymbol{u}^{n+2+j-k}) \\
&- \nu\delta t \sum_{i,j=1}^{k-1} h_{ij}(\Delta \boldsymbol{u}^{n+1+i-k}, \Delta \boldsymbol{u}^{n+1+j-k}) + 0.71\nu\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\|^2 + \nu\kappa_k\delta t\|\Delta \boldsymbol{u}^{n+1}\|^2 \\
&+ \delta t\nu U_k(\Delta \boldsymbol{u}^{n+1}, \ldots, \Delta \boldsymbol{u}^{n+3-k}) - \delta t\nu U_k(\Delta \boldsymbol{u}^{n}, \ldots, \Delta \boldsymbol{u}^{n+2-k}) \\
\leqslant & \frac{\sqrt{2}\nu\delta t}{4}\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\|^2 + (\frac{\sqrt{2}}{4} + \frac{\sqrt{2}\varepsilon}{2})\nu\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{n})\|^2 + \sqrt{2}C\nu\delta t\|\nabla C_k^{\beta_k}(\boldsymbol{u}^{n})\|^2.
\end{aligned}$$

Now, we can choose ε small enough such that

$$0.71 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}\varepsilon}{2} = \rho > 0, \tag{3.39}$$

and take the sum of n from $k-1$ to $m \leqslant \frac{T}{\delta t} - 1$ on (3.38). Dropping some unnecessary terms, we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{k} g_{ij}(\nabla \boldsymbol{u}^{m+1+i-k}, \nabla \boldsymbol{u}^{m+1+j-k}) + \nu\delta t \sum_{i,j=1}^{k-1} h_{ij}(\Delta \boldsymbol{u}^{m+2+i-k}, \Delta \boldsymbol{u}^{n+2+j-k}) \\
&+ \rho\nu\delta t \sum_{n=k-1}^{m} \|\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\|^2 + \nu\kappa_k\delta t \sum_{n=k-1}^{m} \|\Delta \boldsymbol{u}^{n+1}\|^2 \\
\leqslant & C\nu\delta t \sum_{n=k-1}^{m} \|\nabla C_k^{\beta_k}(\boldsymbol{u}^{n})\|^2 + C_I \\
\leqslant & C\nu\delta t \sum_{n=0}^{m} \|\nabla \boldsymbol{u}^{n}\|^2 + C_{II},
\end{aligned}$$

where C_I is a constant depending on $\|\nabla \boldsymbol{u}^i\|^2$ and $\delta t\|\Delta \boldsymbol{u}^i\|^2$, $i = 0, 1, \ldots, k-1$. By the assumption on the initial k steps, we have that C_{II} only depends on $\boldsymbol{u}^0$. One the other hand, let λ_k^g and λ_k^h are the smallest eigenvalues of $G_k = (g_{ij})$ and $H_k = (h_{ij})$ respectively, then we have

$$\begin{aligned}
&\sum_{i,j=1}^{k} g_{ij}(\nabla \boldsymbol{u}^{m+1+i-k}, \nabla \boldsymbol{u}^{m+1+j-k}) + \nu\delta t \sum_{i,j=1}^{k-1} h_{ij}(\Delta \boldsymbol{u}^{m+2+i-k}, \Delta \boldsymbol{u}^{n+2+j-k}) \\
&\qquad \geqslant \lambda_k^g\|\nabla \boldsymbol{u}^{m+1}\|^2 + \lambda_k^h\nu\delta t\|\Delta \boldsymbol{u}^{m+1}\|^2.
\end{aligned}$$

Combining the above two inequalities, we have

$$\begin{aligned}
&\lambda_k^g\|\nabla \boldsymbol{u}^{m+1}\|^2 + \lambda_k^h\nu\delta t\|\Delta \boldsymbol{u}^{m+1}\|^2 + \rho\nu\delta t \sum_{n=k-1}^{m} \|\Delta C_k^{\beta_k}(\boldsymbol{u}^{n+1})\|^2 + \nu\kappa_k\delta t \sum_{n=k-1}^{m} \|\Delta \boldsymbol{u}^{n+1}\|^2 \\
&\leqslant C\nu\delta t \sum_{n=0}^{m} \|\nabla \boldsymbol{u}^{n}\|^2 + C_{II}.
\end{aligned}$$

We can then obtain the desired bound on the velocity by applying Lemma 2.1 to the above. Finally the bound on the pressure can be derived by taking $q = p^{n+1}$ in (1.3) and using Lemma 2.2. □

Remark 1. The above theorem provides the first unconditional stability results for any decoupled schemes of third- or higher-order for time-dependent Stokes equations. It also improves the previous result in [16] for the second-order scheme with $\beta = 5$ to $\beta = 3$.

4. The BDF-IMEX schemes and error analysis

In this section, we construct the k-th order BDF-IMEX schemes for the Navier-Stokes equations and carry out global-in-time error analysis up to fourth order scheme by induction.

4.1. **A general form of BDF-IMEX schemes.** Combining the new BDF type scheme with the consistent splitting schemes in [13], using the notations introduced in (3.8) and choosing β_k as (3.12), we construct the k-th ($k = 2, 3, 4$) order schemes for (1.1) as follows:

(4.1a)

$$\frac{A_k^{\beta_k}(\boldsymbol{u}^{n+1})}{\delta t} - \nu\Delta B_k^{\beta_k}(\boldsymbol{u}^{n+1}) + \nabla C_k^{\beta_k}(p^n) + C_k^{\beta_k}(\boldsymbol{u}^n)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^n) = \boldsymbol{f}^{n+\beta_k},$$

(4.1b)

$$(\nabla p^{n+1}, \nabla q) = (\boldsymbol{f}^{n+1} - \boldsymbol{u}^{n+1}\cdot\nabla\boldsymbol{u}^{n+1} - \nu\nabla\times\nabla\times\boldsymbol{u}^{n+1}, \nabla q), \quad \forall q \in H^1(\Omega).$$

4.2. **Error analysis.** To simplify the presentation, we take $\nu = 1$ in (4.1a) and denote

$$t^n = n\,\delta t, \quad \boldsymbol{e}^n = \boldsymbol{u}^n - \boldsymbol{u}(\cdot, t^n), \quad e_p^n = p^n - p(\cdot, t^n).$$

Theorem 4.1. Let $\Omega \subset \mathbb{R}^d$ satisfies the conditions in Lemma 2.2, $d = 2, 3$, $T > 0$, $\boldsymbol{u}_0 \in \mathbb{V} \cap \boldsymbol{H}_0^2$ and $\boldsymbol{u}$ be the solution of (1.1). Assuming that $\|\boldsymbol{f}(\cdot, t)\| \leqslant C_f$, $\forall t \in [0, T]$ and $\boldsymbol{u}^i$ are computed such that $\|\nabla\boldsymbol{e}^i\|^2 + \delta t\|\Delta\boldsymbol{e}^i\|^2 \leqslant C\delta t^{2k}\|\nabla\boldsymbol{u}^0\|^2$, $i = 0, .., k-1$. Let $\boldsymbol{u}^j$ ($j \geqslant k$) be the solution of (4.1) with $\beta = \beta_k$ chosen as in (3.12), and assume that the exact solutions are sufficiently smooth such that

(4.2)

$$\boldsymbol{u}\in L^2(0,T;H^2), \quad \frac{\partial^k\boldsymbol{u}}{\partial t^k}\in L^2(0,T;H^2), \quad \frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}\in L^2(0,T;L^2), \quad \frac{\partial^k p}{\partial t^k}\in L^2(0,T;H^1).$$

Then for $n + 1 \leqslant T/\delta t$ with δt sufficiently small, we have

$$\|\nabla\boldsymbol{e}^{n+1}\|^2 + \delta t\sum_{i=0}^{n+1}(\|\Delta\boldsymbol{e}^i\|^2 + \|\nabla e_p^i\|^2) \leqslant C\delta t^{2k}, \tag{4.3}$$

where the constants C are dependent on T, Ω and the exact solution $\boldsymbol{u}$, but are independent of δt.

Proof. Since our focus is on the error analysis for the semi-discrete scheme, we assume $\boldsymbol{f}^i = \boldsymbol{f}(t^i)$ $\forall i$, and $\boldsymbol{u}^i$, p^i, $i \leqslant k-1$ are computed with proper initialization procedure such that (4.3) holds for $n \leqslant k-1$.

Firstly, we denote

$$C_{H^1} := \max_{0\leqslant t\leqslant T} \|\nabla \boldsymbol{u}(\cdot,t)\| \text{ and } C_0 := C_{H^1}+1. \tag{4.4}$$

We need to prove a uniform bound of $\|\nabla \boldsymbol{u}^n\|$ by induction,

$$\|\nabla \boldsymbol{u}^i\| \leqslant C_0, \quad \forall i \leqslant T/\delta t. \tag{4.5}$$

In the following, we shall use C to denote a positive constant independent of δt, which can change from one step to another and we use $\varepsilon > 0$ to denote a constant which can be arbitrarily small.

Under the assumption, (4.5) certainly holds for $i = 0$. Now suppose we have

$$\|\nabla \boldsymbol{u}^i\| \leqslant C_0,\ \forall i \leqslant n, \tag{4.6}$$

we shall prove below

$$\|\nabla \boldsymbol{u}^{n+1}\| \leqslant C_0, \tag{4.7}$$

for the same constant C_0.

Step 1. **Bounds for $\delta t \sum_{q=0}^{i} \|\Delta \boldsymbol{u}^q\|^2$, $\forall i \leqslant n$.** Considering (4.1a) at step $i+1 \leqslant n$ and taking the inner product with $-\delta t \Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})$. For the first term on the left hand side, it follows from Lemma 2.3 and Lemma 3.1 that there exists a symmetric positive definite matrix $G_k = (g_{lj}) \in \mathbb{R}^{k\times k}$ such that

$$\begin{aligned}&\big(A_k^{\beta_k}(\boldsymbol{u}^{i+1}), -\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)\\ &\quad\geqslant \sum_{l,j=1}^{k} g_{lj}(\nabla \boldsymbol{u}^{i+1+l-k}, \nabla \boldsymbol{u}^{i+1+j-k}) - \sum_{l,j=1}^{k} g_{lj}(\nabla \boldsymbol{u}^{i+l-k}, \nabla \boldsymbol{u}^{i+j-k}).\end{aligned} \tag{4.8}$$

For the second term, we split $B_k^{\beta_k}(\boldsymbol{u}^{i+1})$ as (3.16) and choose $\eta_k = 0.71$, then we have

$$\begin{aligned}\delta t\big(-\Delta B_k^{\beta_k}(\boldsymbol{u}^{i+1}), -\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big) = 0.71\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2&\\ + \delta t\big(\Delta D_k^{\beta_k}(\boldsymbol{u}^{i+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)&\\ + \delta t\big(\Delta F_k^{\beta_k}(\boldsymbol{u}^{i+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big),&\end{aligned} \tag{4.9}$$

and for $\big(\Delta D_k^{\beta_k}(\boldsymbol{u}^{i+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)$, thanks to Lemma 2.3 and Lemma 3.2, there exists a symmetric positive definite matrix $H_k = (h_{lj}) \in \mathbb{R}^{(k-1)\times(k-1)}$ such that

$$\begin{aligned}&\big(\Delta D_k^{\beta_k}(\boldsymbol{u}^{i+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)\\ &\quad\geqslant \sum_{l,j=1}^{k-1} h_{lj}(\Delta \boldsymbol{u}^{i+2+l-k}, \Delta \boldsymbol{u}^{i+2+j-k}) - \sum_{l,j=1}^{k-1} h_{lj}(\Delta \boldsymbol{u}^{i+1+l-k}, \Delta \boldsymbol{u}^{i+1+j-k}),\end{aligned} \tag{4.10}$$

and for $\big(\Delta F_k^{\beta_k}(\boldsymbol{u}^{i+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)$, (3.19) implies

$$\begin{aligned}&\big(\Delta F_k^{\beta_k}(\boldsymbol{u}^{i+1}), \Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)\\ &\quad\geqslant \kappa_k\|\Delta \boldsymbol{u}^{i+1}\|^2 + U_k(\Delta \boldsymbol{u}^{i+1}, \ldots, \Delta \boldsymbol{u}^{i+3-k}) - U_k(\Delta \boldsymbol{u}^{i}, \ldots, \Delta \boldsymbol{u}^{i+2-k}).\end{aligned} \tag{4.11}$$

For the term with $C_k^{\beta_k}(\boldsymbol{u}^i)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i)$, making use of (2.1) and the Poincaré type inequality, we have

$$
\begin{aligned}
&\big(C_k^{\beta_k}(\boldsymbol{u}^i)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i),\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)\\
\leqslant&\Big|\big(C_k^{\beta_k}(\boldsymbol{u}^i)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i),\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)\Big|\\
\leqslant& c\|C_k^{\beta_k}(\boldsymbol{u}^i)\|_1\|C_k^{\beta_k}(\boldsymbol{u}^i)\|_1^{1/2}\|C_k^{\beta_k}(\boldsymbol{u}^i)\|_2^{1/2}\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|\\
\leqslant& C(\varepsilon)\|C_k^{\beta_k}(\boldsymbol{u}^i)\|_1^2\|C_k^{\beta_k}(\boldsymbol{u}^i)\|_1\|C_k^{\beta_k}(\boldsymbol{u}^i)\|_2+\varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2\\
\leqslant& C(\varepsilon)\|\nabla C_k^{\beta_k}(\boldsymbol{u}^i)\|^6+\varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2+\varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2,
\end{aligned}
\tag{4.12}
$$

where we used $\|C_k^{\beta_k}(\boldsymbol{u}^i)\|_2^2\leqslant C\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2$ in the last step.

For the term with $C_k^{\beta_k}(p^i)$, we have

$$
\Big|\big(\nabla C_k^{\beta_k}(p^i),-\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)\Big|\leqslant\|\nabla C_k^{\beta_k}(p^i)\|\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|.
\tag{4.13}
$$

To estimate $\|\nabla C_k^{\beta_k}(p^i)\|$, we follow a similar procedure as in [20]: first rewriting (4.1b) as

$$
\big(\nabla p^i,\nabla q\big)=\big(\boldsymbol{f}^i-\boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i,\nabla q\big)+\big(\nabla p_s(\boldsymbol{u}^i),\nabla q\big),\quad\forall i\leqslant n,
\tag{4.14}
$$

where $p_s(\boldsymbol{u}^i)$ is the Stokes pressure associated with $\boldsymbol{u}^i$ and hence

$$
\big(\nabla C_k^{\beta_k}(p^i),\nabla q\big)=\big(C_k^{\beta_k}(\boldsymbol{f}^i)-C_k^{\beta_k}(\boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i),\nabla q\big)+\big(\nabla p_s(C_k^{\beta_k}(\boldsymbol{u}^i)),\nabla q\big).
\tag{4.15}
$$

Now, taking $q=C_k^{\beta_k}(p^i)$, we have

$$
\|\nabla C_k^{\beta_k}(p^i)\|\leqslant\|C_k^{\beta_k}(\boldsymbol{f}^i)-C_k^{\beta_k}(\boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i)\|+\|\nabla p_s(C_k^{\beta_k}(\boldsymbol{u}^i))\|.
\tag{4.16}
$$

It follows from the Sobolev inequality and the elliptic regularity estimate that

(4.17)

$$
\begin{aligned}
\|C_k^{\beta_k}(\boldsymbol{f}^i)-C_k^{\beta_k}(\boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i)\|^2&\leqslant C\sum_{q=0}^{k-1}\|\boldsymbol{f}^{i-q}\|^2+C\sum_{q=0}^{k-1}\|\boldsymbol{u}^{i-q}\cdot\nabla\boldsymbol{u}^{i-q}\|^2\\
&\leqslant C\sum_{q=0}^{k-1}\|\boldsymbol{f}^{i-q}\|^2+C\sum_{q=0}^{k-1}\|\nabla\boldsymbol{u}^{i-q}\|^3\|\nabla\boldsymbol{u}^{i-q}\|_1\\
&\leqslant C\sum_{q=0}^{k-1}\|\boldsymbol{f}^{i-q}\|^2+C(\varepsilon)\sum_{q=0}^{k-1}\|\nabla\boldsymbol{u}^{i-q}\|^6+\varepsilon\sum_{q=0}^{k-1}\|\Delta\boldsymbol{u}^{i-q}\|^2,
\end{aligned}
$$

where we used the following inequality (cf. section 4 in [20]),

$$
\|\boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i\|^2\leqslant\|\boldsymbol{u}^i\|_{L^6}^2\|\nabla\boldsymbol{u}^i\|_{L^3}^2\leqslant C\|\nabla\boldsymbol{u}^i\|^3\|\nabla\boldsymbol{u}^i\|_1,\quad d=2,3.
$$

As a result, by making use of Lemma 2.2, we can estimate (4.13) as

(4.18)

$$
\begin{aligned}
&\big(\nabla C_k^{\beta_k}(p^i), -\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big)\\
\leqslant&\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|\big(\|C_k^{\beta_k}(\boldsymbol{f}^i) - C_k^{\beta_k}(\boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i)\| + \|\nabla p_s(C_k^{\beta_k}(\boldsymbol{u}^i))\|\big)\\
\leqslant&C(\varepsilon)\|C_k^{\beta_k}(\boldsymbol{f}^i) - C_k^{\beta_k}(\boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i)\|^2 + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2\\
&\qquad\qquad + \frac{\gamma}{2}\|\nabla p_s(C_k^{\beta_k}(\boldsymbol{u}^i))\|^2 + \frac{1}{2\gamma}\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2\\
\leqslant&C(\varepsilon)\sum_{q=0}^{k-1}\big(\|\boldsymbol{f}^{i-q}\|^2 + \|\nabla\boldsymbol{u}^{i-q}\|^6\big) + \varepsilon\Big(\sum_{q=0}^{k-1}\|\Delta\boldsymbol{u}^{i-q}\|^2 + \|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2\Big)\\
&+ \gamma\Big(\frac{1}{4}+\frac{\varepsilon}{2}\Big)\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 + C(\varepsilon)\|\nabla C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 + \frac{1}{2\gamma}\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2,
\end{aligned}
$$

with γ can be any positive number.

Finally, for the right hand side of (4.1a), we have

$$
\big(\boldsymbol{f}^{i+\beta_k}, -\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\big) \leqslant C(\varepsilon)\|\boldsymbol{f}^{i+\beta_k}\|^2 + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2. \tag{4.19}
$$

Combining (4.8) to (4.19) and choosing $\gamma = \sqrt{2}$ as before, we obtain

(4.20)

$$
\begin{aligned}
&\sum_{l,j=1}^{k} g_{lj}(\nabla\boldsymbol{u}^{i+1+l-k}, \nabla\boldsymbol{u}^{i+1+j-k}) - \sum_{l,j=1}^{k} g_{lj}(\nabla\boldsymbol{u}^{i+l-k}, \nabla\boldsymbol{u}^{i+j-k})\\
&\qquad\qquad + \delta t\sum_{l,j=1}^{k-1} h_{lj}(\Delta\boldsymbol{u}^{i+2+l-k}, \Delta\boldsymbol{u}^{i+2+j-k})\\
&- \delta t\sum_{l,j=1}^{k-1} h_{lj}(\Delta\boldsymbol{u}^{i+1+l-k}, \Delta\boldsymbol{u}^{i+1+j-k}) + 0.71\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2 + \kappa_k\delta t\|\Delta\boldsymbol{u}^{i+1}\|^2\\
&+ \delta t U_k(\Delta\boldsymbol{u}^{i+1},\ldots,\Delta\boldsymbol{u}^{i+3-k}) - \delta t U_k(\Delta\boldsymbol{u}^{i},\ldots,\Delta\boldsymbol{u}^{i+2-k})\\
\leqslant&C(\varepsilon)\delta t\Big(\sum_{q=0}^{k-1}\big(\|\boldsymbol{f}^{i-q}\|^2 + \|\nabla\boldsymbol{u}^{i-q}\|^6\big) + \|\boldsymbol{f}^{i+\beta_k}\|^2 + \|\nabla C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 + \|\nabla C_k^{\beta_k}(\boldsymbol{u}^i)\|^6\Big)\\
&+ \delta t\varepsilon\Big(\sum_{q=0}^{k-1}\|\Delta\boldsymbol{u}^{i-q}\|^2 + \|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 + \|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2\Big) + \Big(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}\varepsilon}{2}\Big)\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2\\
&+ \frac{\sqrt{2}}{4}\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+1})\|^2.
\end{aligned}
$$

Now, we can choose ε small enough such that there exists $\rho > 0$ such that

$$
0.71 - \Big(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}\varepsilon}{2} + 2\varepsilon\Big) \geqslant \rho > 0 \quad\text{and}\quad \kappa_k - k\varepsilon \geqslant \rho > 0. \tag{4.21}
$$

Then taking the sum on (4.20) for i from $k-1$ to $m-1$ with $m \leqslant n$ and dropping some unnecessary terms, we can obtain:

$$
\begin{aligned}
&\lambda_k^g \|\nabla \boldsymbol{u}^m\|^2 + \rho\delta t \sum_{i=k}^{m} (\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 + \|\Delta \boldsymbol{u}^i\|^2) \\
&\leqslant C\delta t \sum_{i=k-1}^{m-1} \left(\|\nabla \boldsymbol{u}^i\|^6 + \|\nabla C_k^{\beta_k}(\boldsymbol{u}^i)\|^6\right) + C\delta t \sum_{i=k-1}^{m-1} \|\nabla C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 \\
&\qquad + C\delta t \sum_{i=k-1}^{m-1} (\|\boldsymbol{f}^i\|^2 + \|\boldsymbol{f}^{i+\beta_k}\|^2) + M_0 \\
&\leqslant C\delta t \sum_{i=0}^{m-1} \|\nabla \boldsymbol{u}^i\|^6 + C\delta t \sum_{i=0}^{m-1} \|\nabla \boldsymbol{u}^i\|^2 + CTC_f^2 + M_0, \quad \forall m \leqslant n,
\end{aligned} \tag{4.22}
$$

where $\lambda_k^g > 0$ is the smallest eigenvalue of $G_k = (g_{lj})$, M_0 is a constant only depends on the data from initial $k-1$ steps and we used $\|\boldsymbol{f}(\cdot,t)\| \leqslant C_f$, $\forall t \in [0,T]$. Next, noting that $\|\nabla \boldsymbol{u}^i\| \leqslant C_0$, $\forall i \leqslant n$ under the induction assumption and $C_0 > 1$ from (4.4), we can obtain from (4.22):

$$
\|\nabla \boldsymbol{u}^m\|^2 + \delta t \sum_{i=k}^{m} (\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 + \|\Delta \boldsymbol{u}^i\|^2) \leqslant CT(C_0^6 + C_f^2) + M_0, \quad \forall m \leqslant n. \tag{4.23}
$$

Step 2. **Error estimate for $\|\nabla \boldsymbol{e}^{n+1}\|$.** From (1.1) and (4.1), we can write down the error equation for $\boldsymbol{u}^{i+1}$ and p^{i+1} as

$$
\begin{aligned}
&A_k^{\beta_k}(\boldsymbol{e}^{i+1}) - \delta t \Delta B_k^{\beta_k}(\boldsymbol{e}^{i+1}) + \delta t (C_k^{\beta_k}(\boldsymbol{u}^i) \cdot \nabla C_k^{\beta_k}(\boldsymbol{u}^i) - C_k^{\beta_k}[\boldsymbol{u}(t^i)] \cdot \nabla C_k^{\beta_k}[\boldsymbol{u}(t^i)]) \\
&\qquad + \delta t \nabla C_k^{\beta_k}(e_p^i) \\
&= \delta t P_k^i + \delta t Q_k^i + R_k^i + \delta t S_k^i,
\end{aligned} \tag{4.24}
$$

where P_k^i, Q_k^i, R_k^i S_k^i are given by

$$
\begin{aligned}
P_k^i &= \nabla p(t^{i+\beta_k}) - \nabla C_k^{\beta_k}(p(t^i)) \\
&= \frac{1}{(k-1)!} \sum_{q=0}^{k-1} c_{k,q}(\beta_k) \int_{t^{i+1+q-k}}^{t^{i+\beta_k}} (t^{i+1+q-k} - s)^{k-1} \nabla \frac{\partial^k p}{\partial t^k}(s) ds,
\end{aligned} \tag{4.25}
$$

with $c_{k,q}(\beta_k)$ defined in (3.6) and

$$
\begin{aligned}
Q_k^i &= -\Delta \boldsymbol{u}(t^{i+\beta_k}) + \Delta B_k^{\beta_k}(\boldsymbol{u}(t^{i+1})) \\
&= \frac{-1}{(k-1)!} \sum_{q=0}^{k-1} b_{k,q}(\beta_k) \int_{t^{i+2+q-k}}^{t^{i+\beta_k}} (t^{i+2+q-k} - s)^{k-1} \Delta \frac{\partial^k \boldsymbol{u}}{\partial t^k}(s) ds,
\end{aligned} \tag{4.26}
$$

$$
R_k^i = \delta t \boldsymbol{u}_t(t^{i+\beta_k}) - A_k^{\beta_k}(\boldsymbol{u}(t^{i+1})) = \frac{1}{k!} \sum_{q=0}^{k} a_{k,q}(\beta_k) \int_{t^{i+1+q-k}}^{t^{i+\beta_k}} (t^{i+1+q-k} - s)^k \frac{\partial^{k+1} \boldsymbol{u}}{\partial t^{k+1}}(s) ds, \tag{4.27}
$$

and

$$
\begin{aligned}
S_k^i &= \boldsymbol{u}(t^{i+\beta_k}) \cdot \nabla \boldsymbol{u}(t^{i+\beta_k}) - C_k^{\beta_k}[\boldsymbol{u}(t^i)] \cdot \nabla C_k^{\beta_k}[\boldsymbol{u}(t^i)] \\
&= \boldsymbol{u}(t^{i+\beta_k}) \cdot \nabla (\boldsymbol{u}(t^{i+\beta_k}) - C_k^{\beta_k}[\boldsymbol{u}(t^i)]) - (C_k^{\beta_k}[\boldsymbol{u}(t^i)] - \boldsymbol{u}(t^{i+\beta_k})) \cdot \nabla C_k^{\beta_k}[\boldsymbol{u}(t^i)].
\end{aligned} \tag{4.28}
$$

Next, we take the inner product of (4.24) with $-\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})$. For the first term on the left hand side, same as (4.8), we have

$$\begin{aligned}(4.29)\quad &\big(A_k^{\beta_k}(\boldsymbol{e}^{i+1}), -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\big)\\ &\geqslant \sum_{l,j=1}^{k} g_{lj}(\nabla \boldsymbol{e}^{i+1+l-k}, \nabla \boldsymbol{e}^{i+1+j-k}) - \sum_{l,j=1}^{k} g_{lj}(\nabla \boldsymbol{e}^{i+l-k}, \nabla \boldsymbol{e}^{i+j-k}).\end{aligned}$$

We handle the term with $B_k^{\beta_k}(\boldsymbol{e}^{i+1})$ similarly as in (4.9)–(4.11) to obtain

(4.30)
$$\begin{aligned}&\delta t\big(-\Delta B_k^{\beta_k}(\boldsymbol{e}^{i+1}), -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\big)\\ \geqslant &0.71\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2 + \kappa_k \delta t \|\Delta \boldsymbol{e}^{i+1}\|^2 + \delta t \sum_{l,j=1}^{k-1} h_{lj}(\Delta \boldsymbol{e}^{i+2+l-k}, \Delta \boldsymbol{e}^{i+2+j-k})\\ &- \delta t \sum_{l,j=1}^{k-1} h_{lj}(\Delta \boldsymbol{e}^{i+1+l-k}, \Delta \boldsymbol{e}^{i+1+j-k}) + \delta t U_k(\Delta \boldsymbol{e}^{i+1}, \dots, \Delta \boldsymbol{e}^{i+3-k})\\ &- \delta t U_k(\Delta \boldsymbol{e}^{i}, \dots, \Delta \boldsymbol{e}^{i+2-k}).\end{aligned}$$

For the third term on the left hand side of (4.24), we rewrite it as

$$\begin{aligned}&C_k^{\beta_k}(\boldsymbol{u}^i)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i) - C_k^{\beta_k}[\boldsymbol{u}(t^i)]\cdot\nabla C_k^{\beta_k}[\boldsymbol{u}(t^i)]\\ (4.31)\quad =&C_k^{\beta_k}(\boldsymbol{u}^i)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i) - C_k^{\beta_k}[\boldsymbol{u}(t^i)]\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i) + C_k^{\beta_k}[\boldsymbol{u}(t^i)]\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i)\\ &- C_k^{\beta_k}[\boldsymbol{u}(t^i)]\cdot\nabla C_k^{\beta_k}[\boldsymbol{u}(t^i)]\\ =&C_k^{\beta_k}(\boldsymbol{e}^i)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i) + C_k^{\beta_k}[\boldsymbol{u}(t^i)]\cdot\nabla C_k^{\beta_k}(\boldsymbol{e}^i).\end{aligned}$$

Therefore, it follows from (2.2) that

(4.32)
$$\begin{aligned}&\big(C_k^{\beta_k}(\boldsymbol{u}^i)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i) - C_k^{\beta_k}[\boldsymbol{u}(t^i)]\cdot\nabla C_k^{\beta_k}[\boldsymbol{u}(t^i)], -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\big)\\ =&\big(C_k^{\beta_k}(\boldsymbol{e}^i)\cdot\nabla C_k^{\beta_k}(\boldsymbol{u}^i), -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\big) + \big(C_k^{\beta_k}[\boldsymbol{u}(t^i)]\cdot\nabla C_k^{\beta_k}(\boldsymbol{e}^i), -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\big)\\ \leqslant &C\|\nabla C_k^{\beta_k}(\boldsymbol{e}^i)\|\|C_k^{\beta_k}(\boldsymbol{u}^i)\|_2\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\| + C\|C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|_2\|\nabla C_k^{\beta_k}(\boldsymbol{e}^i)\|\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|\\ \leqslant &C(\varepsilon)\|\nabla C_k^{\beta_k}(\boldsymbol{e}^i)\|^2\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 + C(\varepsilon)\|C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|_2^2\|\nabla C_k^{\beta_k}(\boldsymbol{e}^i)\|^2 + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2.\end{aligned}$$

For the term with $C_k^{\beta_k}(e_p^i)$, we have

$$(4.33)\qquad \big(\nabla C_k^{\beta_k}(e_p^i), -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\big) \leqslant \|\nabla C_k^{\beta_k}(e_p^i)\|\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|.$$

To estimate $\|\nabla C_k^{\beta_k}(e_p^i)\|$, same as in the last step, we make use of the Stokes pressure. First, from (4.1b), the error equation for e_p^i can be rewritten as

$$(4.34)\qquad (\nabla e_p^i, \nabla q) = \big(\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i, \nabla q\big) + (\nabla p_s(\boldsymbol{e}^i), \nabla q),$$

and hence,

(4.35)
$$(\nabla C_k^{\beta_k}(e_p^i), \nabla q) = \Big(C_k^{\beta_k}\big(\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i\big), \nabla q\Big) + (\nabla p_s(C_k^{\beta_k}(\boldsymbol{e}^i)), \nabla q),$$

where $p_s(C_k^{\beta_k}(\boldsymbol{e}^i))$ is the Stokes pressure associated with $C_k^{\beta_k}(\boldsymbol{e}^i)$. We let $q = C_k^{\beta_k}(e_p^i)$ in the above to obtain

$$\|\nabla C_k^{\beta_k}(e_p^i)\| \leqslant \|C_k^{\beta_k}(\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i)\| + \|\nabla p_s(C_k^{\beta_k}(\boldsymbol{e}^i))\|. \tag{4.36}$$

Similarly as in (4.31), we rewrite

$$\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i = -\boldsymbol{e}^i\cdot\nabla\boldsymbol{u}^i - \boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{e}^i, \tag{4.37}$$

then it follows from the Sobolev inequality and the Poincaré type inequality that

$$\|C_k^{\beta_k}(\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i)\|^2 \leqslant C\sum_{q=0}^{k-1}(\|\nabla\boldsymbol{e}^{i-q}\|^2\|\Delta\boldsymbol{u}^{i-q}\|^2 + \|\boldsymbol{u}(t^{i-q})\|_2^2\|\nabla\boldsymbol{e}^{i-q}\|^2). \tag{4.38}$$

Now, combining (4.33) to (4.38) and making use of Lemma 2.2 for the Stokes pressure, we can bound the term with $C_k^{\beta_k}(e_p^i)$ as

$$\begin{aligned}
&\big(\nabla C_k^{\beta_k}(e_p^i), -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\big)\\
\leqslant&\|C_k^{\beta_k}\big(\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i\big)\|\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\| + \|\nabla p_s(C_k^{\beta_k}(\boldsymbol{e}^i))\|\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|\\
\leqslant&C(\varepsilon)\|C_k^{\beta_k}(\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \boldsymbol{u}^i\cdot\nabla\boldsymbol{u}^i)\|^2 + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2 + \frac{\gamma}{2}\|\nabla p_s(C_k^{\beta_k}(\boldsymbol{e}^i))\|^2\\
&\qquad + \frac{1}{2\gamma}\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2\\
\leqslant&C(\varepsilon)\sum_{q=0}^{k-1}\|\nabla\boldsymbol{e}^{i-q}\|^2(\|\Delta\boldsymbol{u}^{i-q}\|^2 + \|\boldsymbol{u}(t^{i-q})\|_2^2) + (\varepsilon + \frac{1}{2\gamma})\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2\\
&\qquad + \gamma(\frac{1}{4} + \frac{\varepsilon}{2})\|\Delta C_k^{\beta_k}(\boldsymbol{e}^i)\|^2 + C(\varepsilon)\|\nabla C_k^{\beta_k}(\boldsymbol{e}^i)\|^2.
\end{aligned} \tag{4.39}$$

For the right hand side of (4.24), we derive from (4.25)–(4.27) that

$$\begin{aligned}
(P_k^i, -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})) &\leqslant C(\varepsilon)\|P_k^i\|^2 + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2\\
&\leqslant C(\varepsilon)\delta t^{2k-1}\int_{t^{i+1-k}}^{t^{i+\beta_k}}\Big\|\nabla\frac{\partial^k p}{\partial t^k}(s)\Big\|^2 ds + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2,
\end{aligned} \tag{4.40}$$

and similarly,

$$(Q_k^i, -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})) \leqslant C(\varepsilon)\delta t^{2k-1}\int_{t^{i+2-k}}^{t^{i+\beta_k}}\Big\|\Delta\frac{\partial^k\boldsymbol{u}}{\partial t^k}(s)\Big\|^2 ds + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2, \tag{4.41}$$

$$\begin{aligned}
(R_k^i, -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})) &\leqslant \frac{C(\varepsilon)}{\delta t}\|R_k^i\|^2 + \varepsilon\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2\\
&\leqslant C(\varepsilon)\delta t^{2k}\int_{t^{i+1-k}}^{t^{i+\beta_k}}\Big\|\frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}\Big\|^2 ds + \varepsilon\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2.
\end{aligned} \tag{4.42}$$

For the term with S_k^i, it follows from (2.2) and (4.28) that

(4.43)
$$\begin{aligned}
&(S_k^i, -\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1}))\\
\leqslant & C\|\boldsymbol{u}(t^{i+\beta_k})\|_2\|\nabla(\boldsymbol{u}(t^{i+\beta_k}) - C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|\\
&+ C\|C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|_2\|\nabla(\boldsymbol{u}(t^{i+\beta_k}) - C_k^{\beta_k}[\boldsymbol{u}(t^i)])\|\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|\\
\leqslant & C(\varepsilon)(\|\boldsymbol{u}(t^{i+\beta_k})\|_2^2 + \|C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|_2^2)\|\nabla(\boldsymbol{u}(t^{i+\beta_k}) - C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|^2 + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2\\
\leqslant & C(\varepsilon)\delta t^{2k-1}\int_{t^{i+1-k}}^{t^{i+\beta_k}} \left\|\nabla\frac{\partial^k \boldsymbol{u}}{\partial t^k}(s)\right\|^2 ds + \varepsilon\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2.
\end{aligned}$$

Now, we combine (4.29) to (4.43), choose $\gamma = \sqrt{2}$ and drop some unnecessary terms to obtain

(4.44)
$$\begin{aligned}
&\sum_{l,j=1}^{k} g_{lj}(\nabla\boldsymbol{e}^{i+1+l-k}, \nabla\boldsymbol{e}^{i+1+j-k}) - \sum_{l,j=1}^{k} g_{lj}(\nabla\boldsymbol{e}^{i+l-k}, \nabla\boldsymbol{e}^{i+j-k})\\
&\qquad\qquad + \delta t\sum_{l,j=1}^{k-1} h_{lj}(\Delta\boldsymbol{e}^{i+2+l-k}, \Delta\boldsymbol{e}^{i+2+j-k})\\
&- \delta t\sum_{l,j=1}^{k-1} h_{lj}(\Delta\boldsymbol{e}^{i+1+l-k}, \Delta\boldsymbol{e}^{i+1+j-k}) + 0.71\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2 + \kappa_k\delta t\|\Delta\boldsymbol{e}^{n+1}\|^2\\
&+ \delta t U_k(\Delta\boldsymbol{e}^{i+1}, \ldots, \Delta\boldsymbol{e}^{i+3-k}) - \delta t U_k(\Delta\boldsymbol{e}^{i}, \ldots, \Delta\boldsymbol{e}^{i+2-k})\\
\leqslant & C(\varepsilon)\delta t\|\nabla C_k^{\beta_k}(\boldsymbol{e}^i)\|^2(\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 + \|C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|_2^2 + 1) + (\varepsilon + \frac{\sqrt{2}}{4})\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{e}^{i+1})\|^2\\
&+ (\frac{\sqrt{2}}{4} + \frac{\sqrt{2}\varepsilon}{2})\delta t\|\Delta C_k^{\beta_k}(\boldsymbol{e}^i)\|^2 + C(\varepsilon)\delta t\sum_{q=0}^{k-1}\|\nabla\boldsymbol{e}^{i-q}\|^2(\|\Delta\boldsymbol{u}^{i-q}\|^2 + \|\boldsymbol{u}(t^{i-q})\|_2^2)\\
&+ C(\varepsilon)\delta t^{2k}\int_{t^{i+1-k}}^{t^{i+\beta_k}}\left(\left\|\nabla\frac{\partial^k p}{\partial t^k}(s)\right\|^2 + \left\|\Delta\frac{\partial^k \boldsymbol{u}}{\partial t^k}(s)\right\|^2 + \left\|\frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}\right\|^2 + \left\|\nabla\frac{\partial^k \boldsymbol{u}}{\partial t^k}(s)\right\|^2\right)ds.
\end{aligned}$$

Thanks to (4.21), we have

$$0.71 - \frac{\sqrt{2}}{2} - \varepsilon - \frac{\sqrt{2}\varepsilon}{2} > 0. \tag{4.45}$$

Taking the sum of (4.44) for i from $k-1$ to n. Under the assumption (4.2) on the exact solution and the initial steps $\boldsymbol{u}^i$, $\forall i \leqslant k-1$, we can obtain the following after dropping some unnecessary terms:

$$\begin{aligned}
&\lambda_k^g\|\nabla\boldsymbol{e}^{n+1}\|^2 + \kappa_k\delta t\sum_{i=k}^{n+1}\|\Delta\boldsymbol{e}^i\|^2\\
\leqslant & C\delta t\sum_{i=k-1}^{n}\|\nabla\boldsymbol{e}^i\|^2(\|\Delta\boldsymbol{u}^i\|^2 + \|\boldsymbol{u}(t^i)\|_2^2) + C\delta t\sum_{i=k-1}^{n}\|\nabla C_k^{\beta_k}(\boldsymbol{e}^i)\|^2(\|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2\\
&\qquad + \|C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|_2^2 + 1) + CT\delta t^{2k}\\
\leqslant & C\delta t\sum_{i=k-1}^{n}\|\nabla\boldsymbol{e}^i\|^2\Big(\|\Delta\boldsymbol{u}^i\|^2 + \|\boldsymbol{u}(t^i)\|_2^2 + \sum_{q=0}^{\min\{k-1,n-i\}}(\|\Delta C_k^{\beta_k}(\boldsymbol{u}^{i+q})\|^2\\
&\qquad + \|C_k^{\beta_k}[\boldsymbol{u}(t^{i+q})]\|_2^2 + 1)\Big) + CT\delta t^{2k}
\end{aligned} \tag{4.46}$$

where $\lambda_k^g > 0$ is the smallest eigenvalue of $G_k = (g_{lj})$. We can then derive from (4.23) and assumptions on the exact solution that there exists $C_1 > 1$, which is independent of C_0 and δt such that

$$\delta t \sum_{i=0}^{m} \|\Delta \boldsymbol{u}^i\|^2,\ \delta t \sum_{i=k-1}^{m} \|\Delta C_k^{\beta_k}(\boldsymbol{u}^i)\|^2 \leqslant C_1(C_0^6+1),\ \forall m \leqslant n; \tag{4.47a}$$

$$\|C_k^{\beta_k}[\boldsymbol{u}(t^i)]\|_2^2,\ \|\boldsymbol{u}(t^i)\|_2^2 \leqslant C_1,\ \forall t \leqslant T. \tag{4.47b}$$

Then noting that the assumption on the initial steps $\boldsymbol{u}^i$, $i = 0, \dots, k-1$ and applying the Gronwall Lemma 2.1 on (4.46), we obtain

$$\|\nabla \boldsymbol{e}^{n+1}\|^2 + \delta t \sum_{i=0}^{n+1} \|\Delta \boldsymbol{e}^i\|^2 \leqslant CT\delta t^{2k} \exp(CC_1(C_0^6+1)+T) =: C_{\boldsymbol{u}}^2 \delta t^{2k}, \tag{4.48}$$

where $C_{\boldsymbol{u}}$ is a constant independent of δt. Moreover, since

$$\|\nabla \boldsymbol{u}^{n+1}\| \leqslant \|\nabla \boldsymbol{u}(\cdot, t^{n+1})\| + \|\nabla \boldsymbol{e}^{n+1}\|, \tag{4.49}$$

it follows from the definition of C_0 in (4.4) that (4.7) is obviously true if we choose

$$\delta t \leqslant \min\{1, \frac{1}{C_{\boldsymbol{u}}}\}, \tag{4.50}$$

and hence the induction process is completed.

Step 3. **Error estimate for the pressure**. Let $q = e_p^i$ in (4.34), by using Lemma 2.2 and the Sobolev inequality, we have

$$\begin{aligned} \|\nabla e_p^i\|^2 &\leqslant 2\|\boldsymbol{u}(t^i)\cdot\nabla \boldsymbol{u}(t^i) - \boldsymbol{u}^i \cdot \nabla \boldsymbol{u}^i\|^2 + 2\|\nabla p_s(\boldsymbol{e}^i)\|^2 \\ &\leqslant C\|\nabla \boldsymbol{e}^i\|^2(\|\Delta \boldsymbol{u}^i\|^2 + \|\boldsymbol{u}(t^i)\|_2^2) + 2\|\Delta \boldsymbol{e}^i\|^2 + C\|\nabla \boldsymbol{e}^i\|^2. \end{aligned} \tag{4.51}$$

Now, take the sum on (4.51), then (4.47) and (4.48) together imply

$$\delta t \sum_{i=0}^{n+1} \|\nabla e_p^i\|^2 \leqslant C\delta t^{2k}. \tag{4.52}$$

Finally, we complete the proof by combining (4.48) and (4.52). □

5. Numerical validation and concluding remarks

We provide two numerical examples to show that (i) the higher-order consistent splitting schemes based on the usual BDF are not unconditionally stable but the new schemes with suitable β are, and (ii) the new schemes with suitable β achieve the expected convergence rates, followed by some concluding remarks.

5.1. **Numerical results.**

Example 1. In the first example, we first consider the stokes problem (in the absence of $\boldsymbol{f}$ and the nonlinear term in (1.1)) in $\Omega = (-1,1)\times(-1,1)$ with no-slip boundary condition, and the initial conditions are given as

$$u_1(x,y,0) = \sin(2\pi y)\sin^2(\pi x); \tag{5.1a}$$

$$u_2(x,y,0) = -\sin(2\pi x)\sin^2(\pi y). \tag{5.1b}$$

We set $\nu = 0.005$ and use the third- and fourth- order version of (3.9). We use the Legendre-Galerkin method [27] with $Nx = Ny = 128$ modes in space. In Figure 1, we plot the energy evolution obtained from the third- and fourth- order schemes.

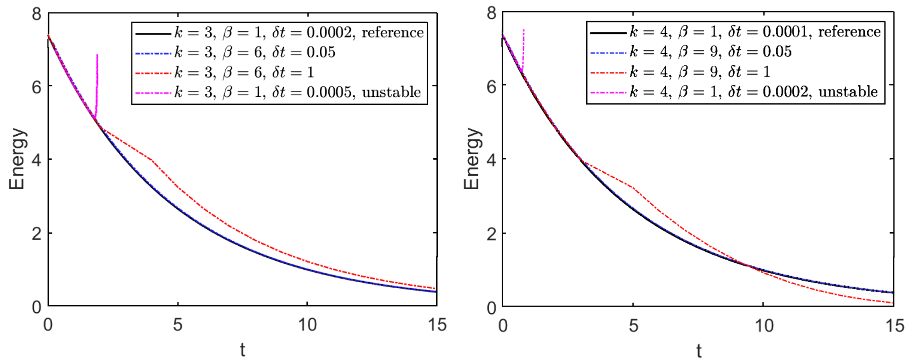


FIGURE 1. Energy evolution for the stokes problem. Left: third order scheme; Right: fourth order scheme

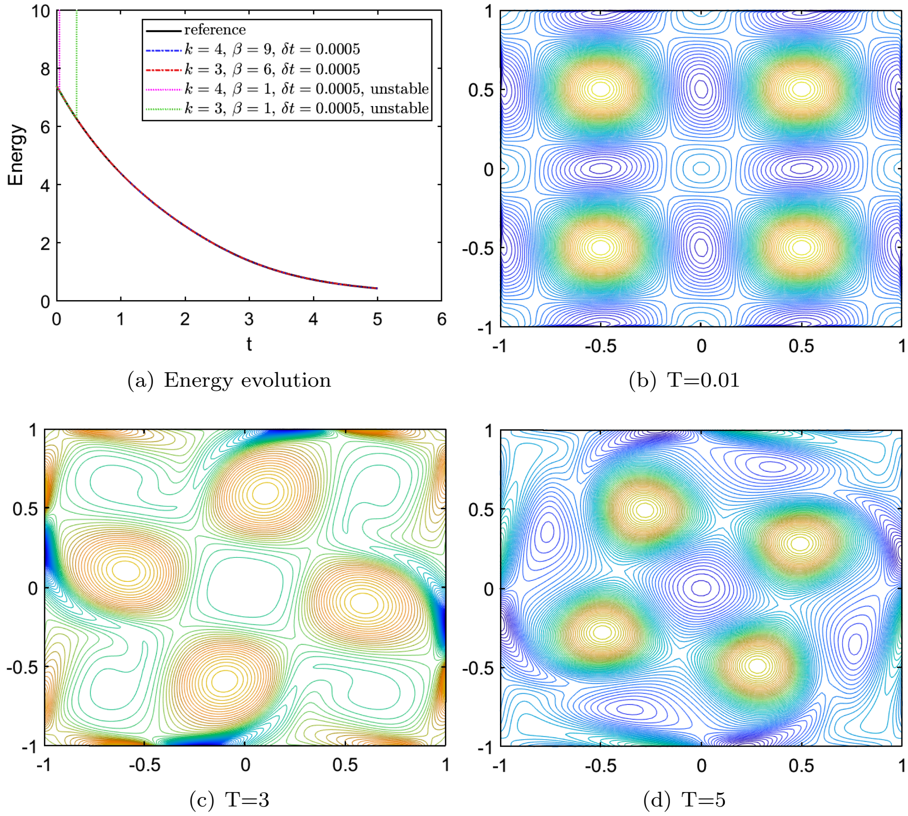
In both cases, we observe that the high-order schemes based on the usual BDF (with $\beta = 1$) are unstable even with an extremely small time step ($\delta t = 0.0005$ for the third order scheme and $\delta t = 0.0002$ for the fourth order scheme), while we can obtain correct solutions with large time step $\delta t = 0.05$ by choosing suitable β as specified in previous sections. We also observe that these schemes are still stable with $\delta t = 1$ although the solutions are no longer correct with such large time step.

Next, we consider the Navier-Stokes equation (1.1) with $\nu = 0.005$ and the initial conditions are still chosen as (5.1). We adopt the third- and fourth- order version of (4.1) and we use the Spectral-Galerkin method with $Nx = Ny = 128$ modes in space. In Figure 2, we plot the energy evolution obtained from the third- and fourth- order schemes, the reference solution is generated by the fourth-order scheme with $\beta = 9$, $Nx = Ny = 192$, $\delta t = 0.0002$. We observe from Figure that with the same time step $\delta t = 0.0005$, the usual BDF3 and BDF4 schemes (with $\beta = 1$) are unstable. On the other hand, we can obtain stable and correct solutions using the new third-order (resp. fourth-order) schemes with $\beta = 6$ (resp. $\beta = 9$). In Figure 2, we plot some snapshots of the vorticity contours at different times.

Example 2. In the second example, we validate the convergence order of the new schemes. Consider the Navier-Stokes equations (1.1) in $\Omega = (-1, 1) \times (-1, 1)$ with the exact solutions given by

$$\begin{aligned}
u_1(x, y, t) &= \sin(2\pi y)\sin^2(\pi x)\sin(t);\\
u_2(x, y, t) &= -\sin(2\pi x)\sin^2(\pi y)\sin(t);\\
p(x, y, t) &= \cos(\pi x)\sin(\pi y)\sin(t).
\end{aligned}$$

We set $\nu = 1$ in (1.1a), and use the Legendre-Galerkin method with $Nx = Ny = 32$ modes in space so that the spatial discretization error is negligible compared with the time discretization error. In Figure 3, we plot the convergence rate of the L^2 error for the velocity $error_{\boldsymbol{u}}$, the L^2 error for the pressure $error_p$ and the value of $\|\nabla \cdot \boldsymbol{u}\|$ at $T = 1$ by using the k-th ($k = 2, 3, 4$) order schemes (4.1) with $\beta_2 = 3, \beta_3 = 6, \beta_4 = 9$. We observe that the expected convergence rates are achieved in all test cases.



(a) Energy evolution (b) T=0.01

(c) T=3 (d) T=5

FIGURE 2. Energy evolution for the Navier-Stokes equations and snapshots of the vorticity contours at T=0.01, 3, 5

5.2. **Concluding remarks.** We considered in this paper the construction and analysis of semi-discrete higher-order consistent splitting schemes for the Navier-stokes equations. We constructed schemes based on the Taylor expansion at $t^{n+\beta}$ with $\beta \geqslant 1$ being a free parameter. Then, by using the multipliers identified in [17] and a delicate splitting of the viscous term, we showed that by choosing $\beta = 3, 6, 9$ respectively for the second-, third- and fourth-order schemes, these schemes are unconditionally stable in the absence of nonlinear terms. Then, we proved by induction optimal global-in-time convergence rates in both 2D and 3D for the nonlinear Navier-Stokes equations. There results are the first stability and convergence results for any fully decoupled, higher-than second-order schemes for the Navier-Stokes equations.

We provided numerical results to show that the third- and fourth-order schemes based on the usual BDF (i.e. $\beta = 1$) are not unconditionally stable while the new third- and fourth-order schemes with $\beta = \beta_k$ specified in (3.12) are unconditionally stable and lead to expected convergence rates.

Below are some problems related to this paper that deserve further investigation:

- We only carried out stability and error analysis for the second- to fourth-order consistent splitting schemes in this paper. It is still an open question

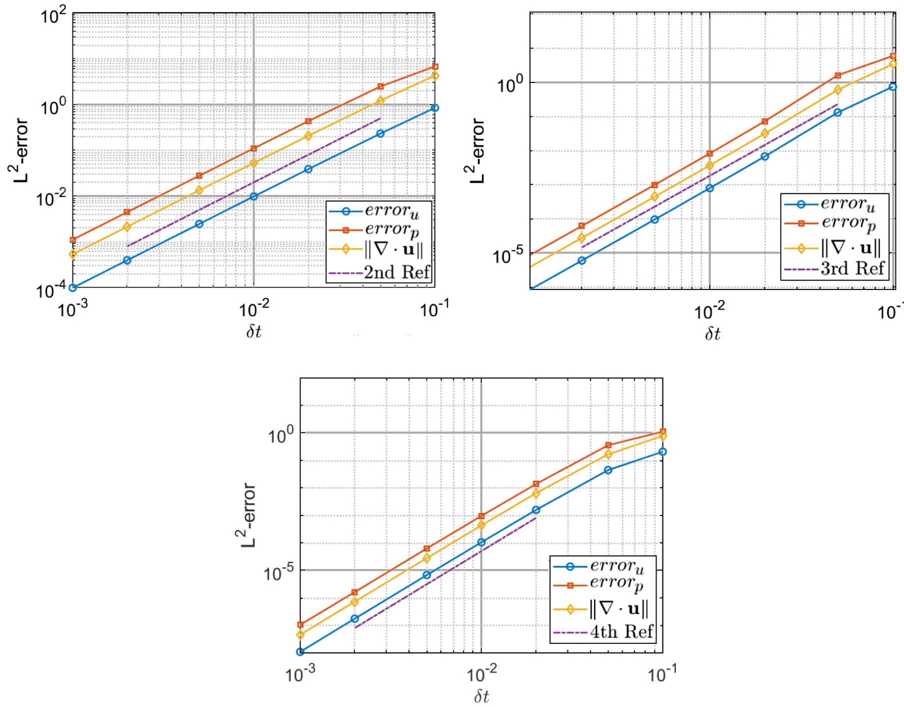


FIGURE 3. Convergence test for the general BDF type methods. Clockwise: second order, third order and fourth order schemes with $\beta_2 = 3$, $\beta_3 = 6, \beta_4 = 9$

whether these results can be extended to the fifth- and six-order consistent splitting schemes with suitable β.

- We only considered semi-discrete (in time) schemes in this paper. It is worthwhile to construct suitable space discretizations for these consistent splitting schemes and carry out corresponding stability and error analysis. Note that if one uses a spectral method or C^1 finite-element method, it is expected that the results established in this paper can be directly extended to the fully discrete cases. However, the case with a C^0 finite-element method would be much more delicate as we cannot directly test the scheme with Δv_h. We are currently working on a DG finite-element method to overcome this difficulty.
- A key element for the stability analysis is Lemma 2.2 which requires $\Omega \in C^3$. Our numerical results indicate that the proved stability and convergence rate are still valid in a square domain. However, it is not clear and beyond the scope of this paper whether the proof can be extended to polygonal domains.
- Since the Navier-Stokes equations are essential components of many coupled complex nonlinear systems, such as magneto-hydrodynamic equations,

Navier-Stokes-Cahn-Hilliard equations, etc, it would be interesting to extend the results in this paper for Navier-Stokes equations to coupled complex nonlinear systems involving Navier-Stokes equations.

Appendix A. Proof of (3.19)

Here, we provide the explicit telescoping forms for $\big(F_k^{\beta_k}(\boldsymbol{u}^{n+1}), C_k^{\beta_k}(\boldsymbol{u}^{n+1})\big)$ and hence prove (3.19):

(A.1)

$$k=2,\ \beta_2=3: \quad \big(F_2^3(\boldsymbol{u}^{n+1}), C_2^3(\boldsymbol{u}^{n+1})\big) = \big(\frac{1}{100}\boldsymbol{u}^{n+1}, 4\boldsymbol{u}^{n+1}-3\boldsymbol{u}^n\big)$$
$$= \frac{1}{100}\|\boldsymbol{u}^{n+1}\|^2 + \frac{3}{200}\|\boldsymbol{u}^{n+1}\|^2 - \frac{3}{200}\|\boldsymbol{u}^n\|^2 + \frac{3}{200}\|\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\|^2.$$

(A.2)

$k=3,\ \beta_2=6:$

$$\begin{aligned}
\big(F_3^6(\boldsymbol{u}^{n+1}), C_3^6(\boldsymbol{u}^{n+1})\big) &= \big(\frac{27}{100}\boldsymbol{u}^{n+1} - \frac{21}{100}\boldsymbol{u}^n, 28\boldsymbol{u}^{n+1} - 48\boldsymbol{u}^n + 21\boldsymbol{u}^{n-1}\big)\\
&= \big(\frac{27}{100}\boldsymbol{u}^{n+1} - \frac{21}{100}\boldsymbol{u}^n, \boldsymbol{u}^{n+1}\big) + \big(\frac{27}{100}\boldsymbol{u}^{n+1} - \frac{21}{100}\boldsymbol{u}^n, 27\boldsymbol{u}^{n+1} - 48\boldsymbol{u}^n + 21\boldsymbol{u}^{n-1}\big)\\
&= \frac{3}{50}\|\boldsymbol{u}^{n+1}\|^2 + \frac{21}{200}\|\boldsymbol{u}^{n+1}\|^2 - \frac{21}{200}\|\boldsymbol{u}^n\|^2 + \frac{21}{200}\|\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\|^2\\
&\quad + \frac{1}{200}\|27\boldsymbol{u}^{n+1} - 21\boldsymbol{u}^n\|^2 - \frac{1}{200}\|27\boldsymbol{u}^n - 21\boldsymbol{u}^{n-1}\|^2\\
&\quad + \frac{1}{200}\|27\boldsymbol{u}^{n+1} - 48\boldsymbol{u}^n + 21\boldsymbol{u}^{n-1}\|^2.
\end{aligned}$$

(A.3)

$k=4,\ \beta_2=9:$

$$\begin{aligned}
&\big(F_4^9(\boldsymbol{u}^{n+1}), C_4^9(\boldsymbol{u}^{n+1})\big)\\
=&\frac{2}{10^5}\big(215\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}, 220\boldsymbol{u}^{n+1} - 594\boldsymbol{u}^n + 540\boldsymbol{u}^{n-1} - 165\boldsymbol{u}^{n-2}\big)\\
=&\frac{2}{10^5}\big(215\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}, 5\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n\big)\\
&+ \frac{2}{10^5}\big(215\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}, 215\boldsymbol{u}^{n+1} - 590\boldsymbol{u}^n + 540\boldsymbol{u}^{n-1} - 165\boldsymbol{u}^{n-2}\big)\\
=&\frac{2}{10^5}\big(210\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}, 5\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n\big) + \frac{2}{10^5}\big(5\boldsymbol{u}^{n+1}, 5\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n\big)\\
&+ \frac{1}{10^5}\big(\|215\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}\|^2 - \|215\boldsymbol{u}^n - 375\boldsymbol{u}^{n-1} + 165\boldsymbol{u}^{n-2}\|^2\big)\\
&+ \frac{1}{10^5}\|215\boldsymbol{u}^{n+1} - 590\boldsymbol{u}^n + 540\boldsymbol{u}^{n-1} - 165\boldsymbol{u}^{n-2}\|^2,\\
=&\frac{2}{10^5}\big(210\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}, 5\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n\big) + \frac{1}{10^4}\|\boldsymbol{u}^{n+1}\|^2\\
&+ \frac{2}{10^4}\big(\|\boldsymbol{u}^{n+1}\|^2 - \|\boldsymbol{u}^n\|^2 + \|\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\|^2\big)\\
&+ \frac{1}{10^5}\big(\|215\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}\|^2 - \|215\boldsymbol{u}^n - 375\boldsymbol{u}^{n-1} + 165\boldsymbol{u}^{n-2}\|^2\big)\\
&+ \frac{1}{10^5}\|215\boldsymbol{u}^{n+1} - 590\boldsymbol{u}^n + 540\boldsymbol{u}^{n-1} - 165\boldsymbol{u}^{n-2}\|^2,
\end{aligned}$$

and finally, for the term $\big(210\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}, 5\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n\big)$, we have

(A.4)
$$\big(210\boldsymbol{u}^{n+1} - 375\boldsymbol{u}^n + 165\boldsymbol{u}^{n-1}, 5\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n\big)$$
$$= a\|\boldsymbol{u}^{n+1}\|^2 - a\|\boldsymbol{u}^n\|^2 + \|b\boldsymbol{u}^{n+1} + c\boldsymbol{u}^n\|^2 - \|b\boldsymbol{u}^n + c\boldsymbol{u}^{n-1}\|^2 + \|d\boldsymbol{u}^{n+1} + e\boldsymbol{u}^n + f\boldsymbol{u}^{n-1}\|^2,$$

with

(A.5)
$$e = -\sqrt{\frac{3375}{2}},\ f = \frac{-\sqrt{37.5} + \sqrt{1687.5}}{2},\ c = f,$$
$$d = \sqrt{37.5} + f,\ b = \frac{660 + 2ef}{2c},\ a = 1050 - b^2 - d^2 \approx 0.2188.$$

Appendix B. Proof of Lemma 3.2

Proof. The proof follows the basic process as in [1]. We consider the case $k = 2, 3, 4$ separately.

Case I. **k=2.** With $c_{2,q}$ obtained from (3.7) and $d_{2,q}$ defined in (3.18) and $\beta_2 = 3$, $\eta_k = 0.71$, the explicit form of $\tilde{C}_2^3(\zeta)$ and $\tilde{D}_2^3(\zeta)$ are given as

(2.1)
$$\tilde{C}_2^3(\zeta) = 4\zeta - 3, \quad \tilde{D}_2^3(\zeta) = \frac{3}{20}\zeta + \frac{13}{100},$$

which imply $\tilde{C}_2^3(\frac{3}{4}) = 0$ and $\tilde{D}_2^3(\frac{-13}{15}) = 0$. Hence $\tilde{C}_2^3(\zeta)$, $\tilde{D}_2^3(\zeta)$ have no common divisor and $\frac{\tilde{D}_2^3(\zeta)}{\tilde{C}_2^3(\zeta)}$ is holomorphic outside the unit disk. Moreover, we have

(2.2)
$$\lim_{|\zeta|\to\infty} \frac{\tilde{D}_2^3(\zeta)}{\tilde{C}_2^3(\zeta)} = \frac{3}{80} > 0.$$

Therefore, it follows from the maximum principle for harmonic functions, $\mathrm{Re}\,\frac{\tilde{D}_2^3(\zeta)}{\tilde{C}_2^3(\zeta)} > 0, \quad \forall|\zeta| > 1$ is equivalent to

(2.3)
$$\mathrm{Re}\,\frac{\tilde{D}_2^3(\zeta)}{\tilde{C}_2^3(\zeta)} \geqslant 0, \quad \forall\zeta \in \mathbb{S}^1,$$

with $\mathbb{S}^1$ is the unit circle in the complex plane and (2.3) is equivalent to

(2.4)
$$Re[\tilde{D}_2^3(e^{i\theta})\tilde{C}_2^3(e^{-i\theta})] \geqslant 0, \quad \theta \in [0, 2\pi).$$

Denote $y := \cos(\theta)$, then (2.4) is equivalent to

(2.5)
$$Re[\tilde{D}_2^3(e^{i\theta})\tilde{C}_2^3(e^{-i\theta})] = \frac{7}{100}y + \frac{21}{100} \geqslant 0, \quad \forall y \in [-1, 1],$$

which is obvious true and hence we proved Lemma 3.2 with $k = 2$.

Case II. **k=3.** With $k = 3$ and $\beta_3 = 6$, the explicit form of $\tilde{C}_3^6(\zeta)$ and $\tilde{D}_3^6(\zeta)$ are given as

(2.6)
$$\tilde{C}_3^6(\zeta) = 28\zeta^2 - 48\zeta + 21, \quad \tilde{D}_3^6(\zeta) = \frac{17}{20}\zeta^2 - \frac{71}{100}\zeta + \frac{9}{100},$$

and the zeros of $\tilde{C}_3^6(\zeta)$ are $\frac{12\pm\sqrt{3}i}{14}$, the zeros of $\tilde{D}_3^6(\zeta)$ are $\frac{71\pm\sqrt{1981}}{170}$, which imply $\tilde{C}_3^6(\zeta)$, $\tilde{D}_3^6(\zeta)$ have no common divisor and $|\frac{12\pm\sqrt{3}i}{14}| < 1$ implies $\frac{\tilde{D}_3^6(\zeta)}{\tilde{C}_3^6(\zeta)}$ is holomorphic outside the unit disk. Following the same process as the second order case, one can easily show $\mathrm{Re}\,\frac{\tilde{D}_3^6(\zeta)}{\tilde{C}_3^6(\zeta)} > 0, \quad \forall |\zeta| > 1$ is equivalent to

$$f_3(y) := \frac{2037}{50}y^2 - \frac{7991}{100}y + \frac{197}{5} \geqslant 0, \quad \forall y \in [-1,1], \tag{2.7}$$

which is true since

$$\min_{y\in[-1,1]} f_3(y) = f_3(\frac{7991}{8148}) \approx 0.214874 > 0. \tag{2.8}$$

Case III. **k=4.** With $k = 4$ and $\beta_4 = 9$, the explicit form of $\tilde{C}_4^9(\zeta)$ and $\tilde{D}_4^9(\zeta)$ are given as

$$\tilde{C}_4^9(\zeta) = 220\zeta^3 - 594\zeta^2 + 540\zeta - 165, \tag{2.9a}$$

$$\tilde{D}_4^9(\zeta) = \frac{1}{10^4}(87957\zeta^3 - 182525\zeta^2 + 125967\zeta - 28500), \tag{2.9b}$$

and the zeros of $\tilde{C}_4^9(\zeta)$ (with six decimal places) are

$$\zeta_{C1} = 0.858473,\ \zeta_{C2} = 0.920763 + 0.160745i,\ \zeta_{C3} = 0.920763 - 0.160745i, \tag{2.10}$$

and the zeros of $\tilde{D}_4^9(\zeta)$ (with six decimal places) are

$$\zeta_{D1} = 0.517951,\ \zeta_{D2} = 0.778605 + 0.139132i,\ \zeta_{D3} = 0.778605 - 0.139132i, \tag{2.11}$$

which imply $\tilde{C}_4^9(\zeta)$, $\tilde{D}_4^9(\zeta)$ have no common divisor and $|\zeta_{Ci}| < 1$, $i = 1, 2, 3$ implies $\frac{\tilde{D}_4^9(\zeta)}{\tilde{C}_4^9(\zeta)}$ is holomorphic outside the unit disk. Following the same process as the second order case, one can easily show $\mathrm{Re}\,\frac{\tilde{D}_4^9(\zeta)}{\tilde{C}_4^9(\zeta)} > 0, \quad \forall |\zeta| > 1$ is equivalent to

$$f_4(y) := \alpha_3 y^3 + \alpha_2 y^2 + \alpha_1 y + \alpha_0 \geqslant 0, \quad \forall y \in [-1,1], \tag{2.12}$$

with

$$\alpha_3 = -\frac{429 \times 9689}{500},\ \alpha_2 = \frac{9 \times 2716781}{1000},\ \alpha_1 = -\frac{241 \times 62141}{625},\ \alpha_0 = \frac{53 \times 3^{10}}{400}. \tag{2.13}$$

(2.12) is true since

$$\min_{y\in[-1,1]} f_4(y) = f_4(y^*) \approx 3.000376 \times 10^{-4} > 0, \tag{2.14}$$

with $y^* = \frac{-\alpha_2 + \sqrt{\alpha_2^2 - 3\alpha_1\alpha_3}}{3\alpha_3} \approx 0.959828$. The proof for all the cases is completed. □

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