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STABILITY AND ERROR ANALYSIS OF A CLASS OF HIGH-ORDER IMEX SCHEMES FOR NAVIER–STOKES EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS*

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Abstract. We construct high-order semidiscrete-in-time, and fully discrete (with Fourier–Galerkin in space) schemes for the incompressible Navier–Stokes equations with periodic boundary conditions and carry out corresponding error analysis. The schemes are of implicit-explicit type based on a scalar auxiliary variable approach. It is shown that numerical solutions of these schemes are uniformly bounded without any restriction on time step size. These uniform bounds enable us to carry out a rigorous error analysis for the schemes up to fifth-order in a unified form and derive global error estimates in $l^\infty(0,T;H^1)\cap l^2(0,T;H^2)$ in the two-dimensional case as well as local error estimates in $l^\infty(0,T;H^1)\cap l^2(0,T;H^2)$ in the three-dimensional case. We also present numerical results confirming our theoretical convergence rates and demonstrating advantages of higher-order schemes for flows with complex structures in the double shear layer problem.

Key words. Navier–Stokes, stability, error analysis, high-order

AMS subject classifications. 65M15, 76D05, 65M70

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1. Introduction.

Numerical approximation of the Navier–Stokes equations has been a subject of intensive study for many decades and continues to attract considerable attention, as it plays a fundamental role in computational fluid dynamics. Most of the work is concerned with the Navier–Stokes equations with nonperiodic boundary conditions, as is the case with most applications. An enormous amount of work has been devoted to constructing efficient and stable numerical algorithms for solving the incompressible Navier–Stokes equations with nonperiodic boundary conditions; see [14, 43, 9, 15, 18, 32] and the references therein. In particular, the papers [3, 22, 12, 17, 20, 8], among others, are concerned with the error estimates for semidiscrete-in-time or fully discrete schemes.

We consider in this paper numerical approximation of the incompressible Navier–Stokes equations in primitive formulation:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = 0, \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{1.1b}$$

with a suitable initial condition $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ in a rectangular domain $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ with periodic boundary conditions. The unknowns are velocity $\boldsymbol{u}$ and the pressure p, which is assumed to have zero mean for uniqueness, and $\nu>0$ is the viscosity. To simplify the presentation, we have set the external force to be zero. But our schemes and analytical results can be naturally extended to the case with a nonzero external force.

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The incompressible Navier–Stokes equations with periodic boundary conditions retain the essential mathematical properties/difficulties of the system with nonperiodic boundary conditions but are amenable to very efficient numerical algorithms using the Fourier spectral method and are particularly useful in the study of homogeneous turbulence [31, 34, 29].

There exists also a significant number of works devoted to the numerical analysis for Navier–Stokes equations with periodic boundary conditions. For example, in [19], Hald proved the convergence of semidiscrete Fourier–Galerkin methods in two and three dimensions; in [11], E used semigroup theory to establish convergence and error estimates of the semidiscrete Fourier–Galerkin and Fourier-collocation methods in various energy norms and L^p-norms; in [45], Wang proved uniform bounds and convergence of long time statistics for a semidiscrete second-order implicit-explicit (IMEX) scheme for the two-dimensional Navier–Stokes equations with periodic boundary conditions in vorticity-stream function formulation (see also related work in [16, 44]); in [7], Cheng and Wang established uniform bounds for a semidiscrete higher-order (up to fourth-order) IMEX scheme for the two-dimensional Navier–Stokes equations with periodic boundary conditions in vorticity-stream function formulation; in [21], Heister, Olshanskii, and Rebholz proved uniform bounds for a fully discrete finite-element and second-order IMEX scheme for the two-dimensional Navier–Stokes equations with periodic boundary conditions in vorticity-velocity formulation. Note that the uniform bounds for semidiscrete IMEX schemes obtained in the above references are for two-dimensional cases only and require that the time step be sufficiently small.

It appears that, except some recently constructed schemes based on the scalar auxiliary variable (SAV) approach [27, 25], all other IMEX type schemes (i.e., the nonlinear term is treated explicitly) for Navier–Stokes equations require the time step to be sufficiently small to have a bounded numerical solution. Furthermore, to the best of our knowledge, there is no error analysis for any IMEX scheme for the three-dimensional Navier–Stokes equations, and no error estimate is available for any higher-order (≥ 3) IMEX scheme. The main difficulties in these cases are (i) lack of uniform bounds for the numerical solution and (ii) the explicit treatment of the nonlinear term.

The original SAV approach proposed in [39, 40] is a powerful approach to construct efficient time discretization schemes for gradient flows and has been applied to various problems (see, for instance, [36, 41] and the references therein). Numerous works have been devoted to the error analysis of SAV schemes, e.g., rigorous error analysis of the semidiscretized first-order original SAV schemes for L^2 and H^{-1} gradient flows with minimum assumptions in [38] (cf. [26] and [6] for the fully discretized SAV schemes with finite differences and finite elements), error analysis for a related semidiscretized gPAV scheme for the Cahn–Hilliard equation in [33], and error analysis for the SAV approach coupled with extrapolated and linearized Runge–Kutta methods in [2]. Recently, a new SAV approach which can be used for general dissipative systems was introduced in [23].

Inspired by the approach in [23], we construct in this paper semidiscrete and fully discrete with Fourier–Galerkin in space SAV IMEX schemes for the Navier–Stokes equations, and carry out a unified stability and error analysis. The main advantages, compared with other SAV approaches proposed in [27, 25] for Navier–Stokes equations is that our schemes are linear and decoupled and can be high-order. Moreover, in the two-dimensional case, we use a stronger energy dissipation law (2.6), which is true only for the two-dimensional Navier–Stokes equations with periodic boundary conditions, that leads to a uniform bound for the numerical solution in $l^\infty(0,T;H^1)$, as opposed to $l^\infty(0,T;L^2)$ in the three-dimensional case. Our main contributions include the following:

- Our semidiscrete and fully discrete schemes of arbitrary order in time are unconditionally stable without any restriction on time step size.
- Global error estimates in $l^\infty(0,T;H^1)\cap l^2(0,T;H^2)$ up to fifth-order in time are established for the two-dimensional case.
- Local error estimates in $l^\infty(0,T_*;H^1)\cap l^2(0,T_*;H^2)$ (with a $T_*\le T$) up to fifth-order in time are established for the three-dimensional case.

The rest of the paper is organized as follows. In the next section, we provide some preliminaries to be used in what follows. In section 3, we describe our semidiscrete and fully discrete with Fourier–Galerkin SAV schemes for the Navier–Stokes equations with periodic boundary condition, prove their unconditionally stability, and provide some numerical results to demonstrate the convergence rates and validate the robustness of our schemes. In section 4, we present a detailed error analysis for the kth-order schemes ($k=1,2,3,4,5$) in a unified form. Some concluding remarks are given in the last section.

2. Preliminaries. We first introduce some notation. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm in $L^2(\Omega)$ and denote

$$\boldsymbol{H}_p^k(\Omega)=\{u^j\ (j=0,1,\dots,k)\in L^2(\Omega): u^j\ (j=0,1,\dots,k-1)\quad \text{periodic}\}$$

with norm $\|\cdot\|_k$. For noninteger $s>0$, $H_p^s(\Omega)$ and the corresponding norm $\|\cdot\|_s$ are defined by space interpolation [1]. In particular, we set $H_p^0(\Omega)=L^2(\Omega)$. We denote $L_0^2(\Omega)=\{v\in L^2(\Omega):\int_\Omega v d\boldsymbol{x}=0\}$.

Letting V be a Banach space, we shall also use the standard notation $L^p(0,T;V)$ and $C([0,T];V)$. To simplify the notation, we often omit the spatial dependence for the exact solution u, i.e., $u(x,t)$ is often denoted by $u(t)$. We shall use boldface letters to denote vectors and vector spaces, and we use C to denote a generic positive constant independent of the discretization parameters.

We now define the following spaces, which are particularly used for Navier–Stokes equations:

$$\mathbf{H}=\{\boldsymbol{v}\in\boldsymbol{L}_0^2(\Omega):\nabla\cdot\boldsymbol{v}=0\},\quad \mathbf{V}=\{\boldsymbol{v}\in\boldsymbol{H}_p^1(\Omega):\nabla\cdot\boldsymbol{v}=0\}.$$

Letting $\boldsymbol{v}\in\boldsymbol{L}_0^2(\Omega)$, we define $w:=\Delta^{-1}\boldsymbol{v}$ as the solution of

$$\Delta\boldsymbol{w}=\boldsymbol{v}\quad \boldsymbol{x}\in\Omega;\quad \boldsymbol{w}\text{ periodic with zero mean.}$$

Note that in the periodic case, we can define the operators ∇, $\nabla\cdot$ and Δ^{-1} in the Fourier space by expanding functions and their derivatives in Fourier series, and one can easily show that these operators commute with each other.

We define a linear operator $\mathbf{A}$ in $\boldsymbol{L}_0^2(\Omega)$ by

$$\mathbf{A}\boldsymbol{v}:=\nabla\times\nabla\times\Delta^{-1}\boldsymbol{v}\quad\forall\boldsymbol{v}\in\boldsymbol{L}_0^2(\Omega). \tag{2.1}$$

Since

$$\|\Delta\boldsymbol{w}\|^2=\|\nabla\times\nabla\times\boldsymbol{w}\|^2+\|\nabla\nabla\cdot\boldsymbol{w}\|^2\quad\forall\boldsymbol{w}\in\boldsymbol{H}_p^2(\Omega),$$

we derive immediately from the above that

$$\|\mathbf{A}\boldsymbol{v}\|^2=\|\Delta\Delta^{-1}\boldsymbol{v}\|^2-\|\nabla\nabla\cdot\Delta^{-1}\boldsymbol{v}\|^2\le\|\boldsymbol{v}\|^2\ \ \forall\boldsymbol{v}\in\boldsymbol{L}_0^2(\Omega). \tag{2.2}$$

Next, we define the trilinear form $b(\cdot,\cdot,\cdot)$ and $b_{\mathbf{A}}(\cdot,\cdot,\cdot)$ by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x},\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_\Omega\mathbf{A}((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{w}d\boldsymbol{x}.$$

In particular, we have

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\ \forall\boldsymbol{u}\in\mathbf{H},\ \boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}_p^1(\Omega),$$

which implies

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})=0\ \forall\boldsymbol{u}\in\mathbf{H},\ \boldsymbol{v}\in\boldsymbol{H}_p^1(\Omega). \tag{2.3}$$

In the two-dimensional periodic case, we have also (cf. page 19, Lemma 3.1, in [42]):

$$b(\boldsymbol{u},\boldsymbol{u},\Delta\boldsymbol{u})=0\ \forall\boldsymbol{u}\in\boldsymbol{H}_p^2(\Omega). \tag{2.4}$$

Taking the inner product of (1.1) with $\boldsymbol{u}$, thanks to (2.3), we find that the solution of the Navier–Stokes equations (1.1) satisfies the energy dissipation law

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2=-\nu\|\nabla\boldsymbol{u}\|^2\qquad(d=2,3). \tag{2.5}$$

On the other hand, in the two-dimensional periodic case, taking the inner product of (1.1) with $-\Delta\boldsymbol{u}$, thanks to (2.4), we derive another energy dissipation law [42],

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2=-\nu\|\Delta\boldsymbol{u}\|^2\qquad(d=2). \tag{2.6}$$

Using (2.2), the Hölder inequality, and the Sobolev inequality, we have [42]

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\le c\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\|_2^{1/2}\|\mathbf{v}\|_1^{1/2}\|\mathbf{w}\|,\quad d=2; \tag{2.7}$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\le c\|\boldsymbol{u}\|_1\|\nabla\boldsymbol{v}\|_{1/2}\|\boldsymbol{w}\|,\quad d=3. \tag{2.8}$$

We also use frequently the following inequalities [42]:

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\le\begin{cases}c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1;\\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_0\|\boldsymbol{w}\|_1;\\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_0;\\ c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_0;\\ c\|\boldsymbol{u}\|_0\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_1;\end{cases}\qquad d\le 4. \tag{2.9}$$

Note that (2.4), (2.6), and (2.7) enable us to obtain global error estimates in the two-dimensional case.

3. The SAV schemes and stability results. In this section, we construct semidiscrete and fully discrete SAV schemes for the incompressible Navier–Stokes equations and establish stability results for both semidiscrete and fully discrete schemes. More precisely, we shall prove the uniform L^2 bound for the SAV scheme based on the dissipation law (2.5) in the three-dimensional case and prove a uniform H^1 bound for the SAV scheme based on the dissipation law (2.6) in the two-dimensional case.

3.1. The SAV schemes. Following the ideas in [23] for the general dissipative systems, we construct below unconditionally energy stable schemes for (1.1).

For Navier–Stokes equations with periodic boundary conditions, we can explicitly eliminate the pressure from (1.1). Indeed, taking the divergence on both sides of (1.1), we find

$$-\Delta p = \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}), \tag{3.1}$$

from which we derive

$$\begin{aligned} \nabla p &= \nabla \Delta^{-1} \Delta p = -\nabla \Delta^{-1} \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \\ &= -\nabla \nabla \cdot \Delta^{-1} (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -(\Delta + \nabla \times \nabla \times) \Delta^{-1} (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \\ &= -\boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla \times \nabla \times \Delta^{-1} (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\boldsymbol{u} \cdot \nabla \boldsymbol{u} - \mathbf{A}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}), \end{aligned} \tag{3.2}$$

where $\mathbf{A}$ is defined in (2.1). Hence, (1.1) is equivalent to (3.1) and

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} - \mathbf{A}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = 0. \tag{3.3}$$

In order to apply the SAV approach, we introduce a SAV, $r(t) = E(\boldsymbol{u}(t)) + 1$, and expand (3.3) as

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} - \mathbf{A}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = 0, \tag{3.4a}$$

$$\frac{dE}{dt} = \begin{cases} -\nu \frac{r(t)}{E(\boldsymbol{u}(t))+1} \|\Delta \boldsymbol{u}\|^2, & d = 2, \\ -\nu \frac{r(t)}{E(\boldsymbol{u}(t))+1} \|\nabla \boldsymbol{u}\|^2, & d = 3, \end{cases} \tag{3.4b}$$

where

$$E(\boldsymbol{u}) = \begin{cases} \frac{1}{2} \|\nabla \boldsymbol{u}\|^2, & d = 2, \\ \frac{1}{2} \|\boldsymbol{u}\|^2, & d = 3. \end{cases} \tag{3.5}$$

We construct below semidiscrete and fully discrete schemes for the expanded system (3.4).

3.1.1. Semidiscrete SAV schemes. We consider first the time discretization of (3.4) based on the IMEX BDF-k formulae in the following unified form.

Given r^n, $\boldsymbol{u}^j$ $(j = n, n-1, \dots, n-k+1)$, we compute $\bar{\boldsymbol{u}}^{n+1}$, r^{n+1}, p^{n+1}, ξ^{n+1}, and $\boldsymbol{u}^{n+1}$ consecutively by

$$\frac{\alpha_k \bar{\boldsymbol{u}}^{n+1} - A_k(\bar{\boldsymbol{u}}^n)}{\delta t} - \nu \Delta \bar{\boldsymbol{u}}^{n+1} - \mathbf{A}(B_k(\boldsymbol{u}^n) \cdot \nabla B_k(\boldsymbol{u}^n)) = 0, \tag{3.6a}$$

$$\frac{1}{\delta t}\left(r^{n+1} - r^n\right) = \begin{cases} -\nu \frac{r^{n+1}}{E(\bar{\boldsymbol{u}}^{n+1})+1} \|\Delta \bar{\boldsymbol{u}}^{n+1}\|^2, & d = 2, \\ -\nu \frac{r^{n+1}}{E(\bar{\boldsymbol{u}}^{n+1})+1} \|\nabla \bar{\boldsymbol{u}}^{n+1}\|^2, & d = 3; \end{cases} \tag{3.6b}$$

$$\xi^{n+1} = \frac{r^{n+1}}{E(\bar{\boldsymbol{u}}^{n+1}) + 1}; \tag{3.6c}$$

$$\boldsymbol{u}^{n+1} = \eta_k^{n+1} \bar{\boldsymbol{u}}^{n+1} \quad \text{with } \eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k. \tag{3.6d}$$

Whenever pressure is needed, it can be computed from

$$\Delta p^{n+1} = -\nabla \cdot (\boldsymbol{u}^{n+1} \cdot \nabla \boldsymbol{u}^{n+1}). \tag{3.7}$$

In the above, α_k, the operators A_k and B_k $(k = 1, 2, 3, 4, 5)$ are given by
first-order,

$$\alpha_1 = 1, \quad A_1(\bar{\boldsymbol{u}}^n) = \bar{\boldsymbol{u}}^n, \quad B_1(\boldsymbol{u}^n) = u^n; \tag{3.8}$$

second-order,

$$\alpha_2 = \frac{3}{2}, \quad A_2(\bar{\boldsymbol{u}}^n) = 2\bar{\boldsymbol{u}}^n - \frac{1}{2}\bar{\boldsymbol{u}}^{n-1}, \quad B_2(\boldsymbol{u}^n) = 2\boldsymbol{u}^n - \boldsymbol{u}^{n-1}; \tag{3.9}$$

third-order,

$$\alpha_3 = \frac{11}{6}, \quad A_3(\bar{\boldsymbol{u}}^n) = 3\bar{\boldsymbol{u}}^n - \frac{3}{2}\bar{\boldsymbol{u}}^{n-1} + \frac{1}{3}\bar{\boldsymbol{u}}^{n-2}, \quad B_3(\boldsymbol{u}^n) = 3\boldsymbol{u}^n - 3\boldsymbol{u}^{n-1} + \boldsymbol{u}^{n-2}; \tag{3.10}$$

fourth-order,

$$\begin{aligned} \alpha_4 = \frac{25}{12}, \ & A_4(\bar{\boldsymbol{u}}^n) = 4\bar{\boldsymbol{u}}^n - 3\bar{\boldsymbol{u}}^{n-1} + \frac{4}{3}\bar{\boldsymbol{u}}^{n-2} - \frac{1}{4}\bar{\boldsymbol{u}}^{n-3}, \\ & B_4(\boldsymbol{u}^n) = 4\boldsymbol{u}^n - 6\boldsymbol{u}^{n-1} + 4\boldsymbol{u}^{n-2} - \boldsymbol{u}^{n-3}; \end{aligned} \tag{3.11}$$

fifth-order,

$$\begin{aligned} \alpha_5 = \frac{137}{60}, \quad & A_5(\bar{\boldsymbol{u}}^n) = 5\bar{\boldsymbol{u}}^n - 5\bar{\boldsymbol{u}}^{n-1} + \frac{10}{3}\bar{\boldsymbol{u}}^{n-2} - \frac{5}{4}\bar{\boldsymbol{u}}^{n-3} + \frac{1}{5}\bar{\boldsymbol{u}}^{n-4}, \\ & B_5(\boldsymbol{u}^n) = 5\boldsymbol{u}^n - 10\boldsymbol{u}^{n-1} + 10\boldsymbol{u}^{n-2} - 5\boldsymbol{u}^{n-3} + \boldsymbol{u}^{n-4}. \end{aligned} \tag{3.12}$$

Several remarks are in order:

- We observe from (3.6b) that r^{n+1} is a first-order approximation to $E(u(\cdot, t_{n+1})) + 1$, which implies that ξ^{n+1} is a first-order approximation to 1.
- (3.6a) is a kth-order approximation to (3.3) with kth-order BDF for the linear terms and kth-order Adams–Bashforth extrapolation for the nonlinear terms. Hence, $\bar{\boldsymbol{u}}^{n+1}$ is a kth-order approximation to $\boldsymbol{u}(\cdot, t^{n+1})$, which, along with (3.6b) and (3.6a), implies that $\boldsymbol{u}^{n+1}$ and p^{n+1} are kth-order approximations for $\boldsymbol{u}(\cdot, t^{n+1})$ and $p(\cdot, t^{n+1})$.
- The main computational cost is to solve the Poisson type equation (3.6a).

3.1.2. Fully discrete schemes with Fourier spectral method in space. We now consider $\Omega = [0, L_x) \times [0, L_y) \times [0, L_z)$ with periodic boundary conditions. We partition the domain $\Omega = (0, L_x) \times (0, L_y) \times (0, L_z)$ uniformly with size $h_x = L_x/N_x, h_y = L_y/N_y, h_z = L_z/N_z$, and Nx, Ny, Nz are positive even integers. Then the Fourier approximation space can be defined as

$$\begin{aligned} S_N = \operatorname{span}\Bigg\{ e^{i\xi_j x} e^{i\eta_k y} e^{i\tau_l z} : -\frac{N_x}{2} \le j \le \frac{N_x}{2} - 1, -\frac{N_y}{2} \le k \\ \le \frac{N_y}{2} - 1, -\frac{N_z}{2} \le l \le \frac{N_z}{2} - 1 \Bigg\} \setminus \mathbb{R}, \end{aligned}$$

where $i = \sqrt{-1}$, $\xi_j = 2\pi j/L_x$, $\eta_k = 2\pi k/L_y$, and $\tau_l = 2\pi l/L_z$. Then, any function $u(x, y, z) \in L^2(\Omega)$ can be approximated by

$$u(x, y, z) \approx u_N(x, y, z) = \sum_{j=-\frac{N_x}{2}}^{\frac{N_x}{2}-1} \sum_{k=-\frac{N_y}{2}}^{\frac{N_y}{2}-1} \sum_{l=-\frac{N_z}{2}}^{\frac{N_z}{2}-1} \hat{u}_{j,k,l} e^{i\xi_j x} e^{i\eta_k y} e^{i\tau_l z},$$

with the Fourier coefficients defined as

$$\hat{u}_{j,k,l} = \frac{1}{|\Omega|}\int_\Omega u e^{-i(\xi_j x+\eta_k y+\tau_l z)} d\boldsymbol{x}.$$

In the following, we fix $N_x = N_y = N_z = N$ for simplicity.

Define the L^2-orthogonal projection operator $\Pi_N : L^2(\Omega) \to S_N$ by

$$(\Pi_N u - u, \Psi) = 0 \quad \forall\, \Psi \in S_N, \quad u \in L^2(\Omega);$$

then we have the following approximation results (cf. [24]).

LEMMA 3.1. *For any $0 \le k \le m$, there exists a constant C such that*

$$\|\Pi_N u - u\|_k \le C\|u\|_m N^{k-m} \;\forall\, u \in \boldsymbol{H}_p^m(\Omega). \tag{3.13}$$

We are now ready to describe our fully discrete schemes.

Given r^n and $\boldsymbol{u}_N^j \in S_N$ for $j = n, \ldots, n-k+1$, we compute $\bar{\boldsymbol{u}}_N^{n+1}$, r^{n+1}, p_N^{n+1}, ξ^{n+1}, and $\boldsymbol{u}_N^{n+1}$ consecutively by

$$\left(\frac{\alpha_k \bar{\boldsymbol{u}}_N^{n+1} - A_k(\bar{\boldsymbol{u}}_N^n)}{\delta t}, v_N\right) + \nu\big(\nabla\bar{\boldsymbol{u}}_N^{n+1}, \nabla v_N\big) - \big(\mathbf{A}(B_k(\boldsymbol{u}_N^n)\cdot\nabla B_k(\boldsymbol{u}_N^n)), v_N\big) = 0 \quad \forall v_N \in S_N, \tag{3.14a}$$

$$\frac{1}{\delta t}\big(r^{n+1} - r^n\big) = \begin{cases} -\nu\frac{r^{n+1}}{E(\bar{\boldsymbol{u}}_N^{n+1})+1}\|\Delta\bar{\boldsymbol{u}}_N^{n+1}\|^2, & d=2,\\ -\nu\frac{r^{n+1}}{E(\bar{\boldsymbol{u}}_N^{n+1})+1}\|\nabla\bar{\boldsymbol{u}}_N^{n+1}\|^2, & d=3; \end{cases} \tag{3.14b}$$

$$\xi^{n+1} = \frac{r^{n+1}}{E(\bar{\boldsymbol{u}}_N^{n+1})+1}; \tag{3.14c}$$

$$\boldsymbol{u}_N^{n+1} = \eta_k^{n+1}\bar{\boldsymbol{u}}_N^{n+1} \;\text{ with } \eta_k^{n+1} = 1-(1-\xi^{n+1})^k, \tag{3.14d}$$

where α_k and the operators A_k and B_k $(k = 1, 2, 3, 4, 5)$ are given in (3.8)–(3.12).

Note that Fourier approximation of Poisson type equations leads to diagonal matrix in the frequency space, so the above scheme can be efficiently implemented as follows:

(i) Compute $\bar{\boldsymbol{u}}_N^{n+1}$ from (3.14a), which is a Poisson type equation.
(ii) With $\bar{\boldsymbol{u}}_N^{n+1}$ known, determine r^{n+1} explicitly from (3.14b).
(iii) Compute ξ^{n+1}, η_k^{n+1}, and $\boldsymbol{u}_N^{n+1}$ from (3.14c) and (3.14d), goto the next step.

Finally, whenever pressure is needed, it can be computed from

$$\Delta p_N^{n+1} = -\Pi_N\nabla\cdot(\boldsymbol{u}_N^{n+1}\cdot\nabla\boldsymbol{u}_N^{n+1}). \tag{3.15}$$

3.2. Stability results. We have the following results concerning the stability of the above schemes.

THEOREM 3.2. *Let $\boldsymbol{u}_0 \in \boldsymbol{V}\cap\boldsymbol{H}_p^2$ if $d = 2$ and $\boldsymbol{u}_0 \in \boldsymbol{V}$ if $d = 3$. Let $\{r^k, \xi^k, \bar{\boldsymbol{u}}_N^k, \boldsymbol{u}_N^k\}$ be the solution of the fully discrete scheme* (3.14)*. Then, given $r^n \ge 0$, we have $r^{n+1} \ge 0$, $\xi^{n+1} \ge 0$, and for any k, the scheme* (3.14) *is unconditionally energy stable in the sense that*

$$r^{n+1} - r^n = \begin{cases} -\delta t\nu\xi^{n+1}\|\Delta\bar{\boldsymbol{u}}_N^{n+1}\|^2 \le 0, & d=2,\\ -\delta t\nu\xi^{n+1}\|\nabla\bar{\boldsymbol{u}}_N^{n+1}\|^2 \le 0, & d=3, \end{cases} \quad \forall n. \tag{3.16}$$

Furthermore, there exists $M_k > 0$ such that

$$\begin{aligned} &\|\nabla \boldsymbol{u}_N^{n+1}\|^2 \le M_k^2, \qquad d=2,\\ &\|\boldsymbol{u}_N^{n+1}\|^2 \le M_k^2, \qquad d=3, \end{aligned} \qquad \forall n. \tag{3.17}$$

The same results hold for the semidiscrete schemes (3.6) *with* $\bar{\boldsymbol{u}}_N^{n+1}$ *and* $\boldsymbol{u}_N^{n+1}$ *in* (3.16) *and* (3.17) *replaced by* $\bar{\boldsymbol{u}}^{n+1}$ *and* $\boldsymbol{u}^{n+1}$.

Proof. Since the proofs for the fully discrete scheme (3.14) and for the semidiscrete scheme (3.6) are essentially the same, we shall give only the proof for the fully discrete scheme (3.14) below.

Assume $r^n \ge 0$. Since $E(\bar{\boldsymbol{u}}_N^{n+1}) \ge 0$, it follows from (3.14b) that

$$r^{n+1} = \begin{cases} \dfrac{r^n}{1+\delta t\nu \frac{\|\Delta \bar{\boldsymbol{u}}_N^{n+1}\|^2}{E(\bar{\boldsymbol{u}}_N^{n+1})+1}} \ge 0, & d=2,\\ \dfrac{r^n}{1+\delta t\nu \frac{\|\nabla \bar{\boldsymbol{u}}_N^{n+1}\|^2}{E(\bar{\boldsymbol{u}}_N^{n+1})+1}} \ge 0, & d=3. \end{cases}$$

Then we derive from (3.14c) that $\xi^{n+1} \ge 0$ and obtain (3.16).

Denote $M := r^0 = E[\boldsymbol{u}(\cdot,0)]$; then (3.16) implies $r^n \le M\ \forall n$. It then follows from (3.14c) that

$$|\xi^{n+1}| = \frac{r^{n+1}}{E(\bar{\boldsymbol{u}}_N^{n+1})+1} \le \begin{cases} \frac{2M}{\|\nabla \bar{\boldsymbol{u}}_N^{n+1}\|^2+2}, & d=2,\\ \frac{2M}{\|\bar{\boldsymbol{u}}_N^{n+1}\|^2+2}, & d=3. \end{cases} \tag{3.18}$$

Since $\eta_k^{n+1} = 1-(1-\xi^{n+1})^k$, we have $\eta_k^{n+1} = \xi^{n+1}P_{k-1}(\xi^{n+1})$ with P_{k-1} being a polynomial of degree $k-1$. Then, we derive from (3.18) that there exists $M_k > 0$ such that

$$|\eta_k^{n+1}| = |\xi^{n+1}P_{k-1}(\xi^{n+1})| \le \begin{cases} \frac{M_k}{\|\nabla \bar{\boldsymbol{u}}_N^{n+1}\|^2+2}, & d=2,\\ \frac{M_k}{\|\bar{\boldsymbol{u}}_N^{n+1}\|^2+2}, & d=3, \end{cases}$$

which, along with $\boldsymbol{u}_N^{n+1} = \eta_k^{n+1}\bar{\boldsymbol{u}}_N^{n+1}$, implies

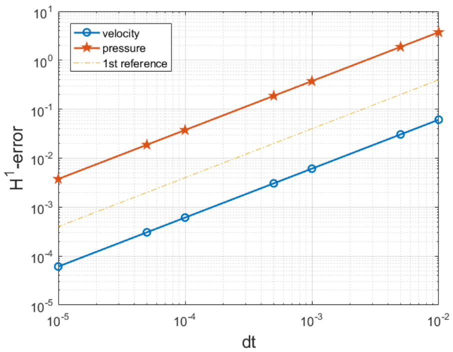
$$\begin{aligned} &\|\nabla \boldsymbol{u}_N^{n+1}\|^2 = (\eta_k^{n+1})^2\|\nabla \bar{\boldsymbol{u}}_N^{n+1}\|^2 \le \big(\tfrac{M_k}{\|\nabla \bar{\boldsymbol{u}}_N^{n+1}\|^2+2}\big)^2\|\nabla \bar{\boldsymbol{u}}_N^{n+1}\|^2 \le M_k^2, \qquad d=2,\\ &\|\boldsymbol{u}_N^{n+1}\|^2 = (\eta_k^{n+1})^2\|\bar{\boldsymbol{u}}_N^{n+1}\|^2 \le \big(\tfrac{M_k}{\|\bar{\boldsymbol{u}}_N^{n+1}\|^2+2}\big)^2\|\bar{\boldsymbol{u}}_N^{n+1}\|^2 \le M_k^2, \qquad d=3. \end{aligned}$$ ∎

3.3. Numerical examples. Before we start the error analysis, we provide numerical examples to demonstrate the convergence rates and compare the performance of the schemes with different orders on a classical benchmark problem.

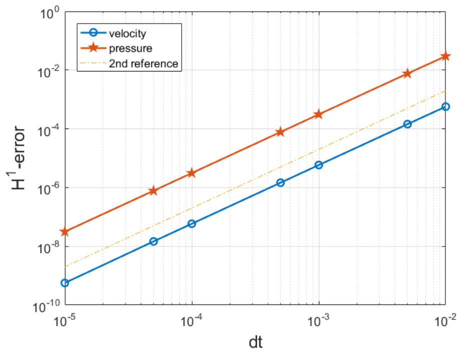
Example 1*: Convergence test.* Consider the Navier–Stokes equations (1.1) with an external forcing $\boldsymbol{f}$ in $\Omega = (0,2)\times(0,2)$ with periodic boundary condition such that the exact solution is given by

$$\begin{aligned} &u_1(x,y) = \pi\exp(\sin(\pi x))\exp(\sin(\pi y))\cos(\pi y)\sin^2(t);\\ &u_2(x,y) = -\pi\exp(\sin(\pi x))\exp(\sin(\pi y))\cos(\pi x)\sin^2(t);\\ &p(x,y) = \exp(\cos(\pi x)\sin(\pi y))\sin^2(t). \end{aligned}$$

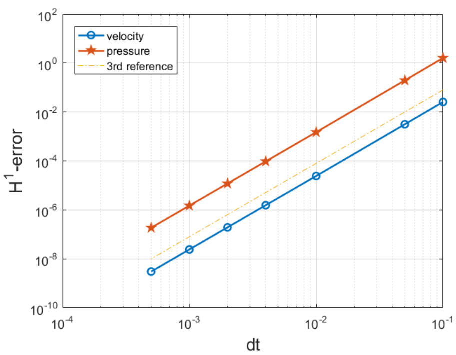
We set $\nu = 1$ in (1.1) and use the Fourier spectral method with 40×40 modes for space discretization so that the spatial discretization error is negligible with respect



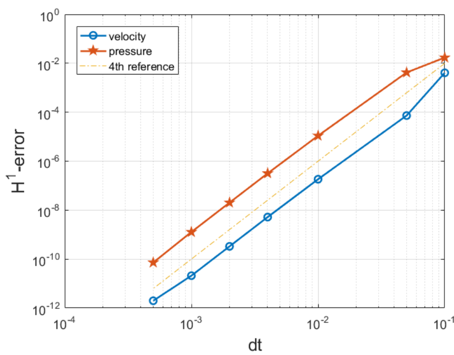
(a) BDF1 errors of velocity and pressure



(b) BDF2 errors of velocity and pressure



(c) BDF3 errors of velocity and pressure



(d) BDF4 errors of velocity and pressure

FIG. 1. *Convergence test for the Navier–Stokes equations using SAV/BDFk* ($k = 1, 2, 3, 4$).

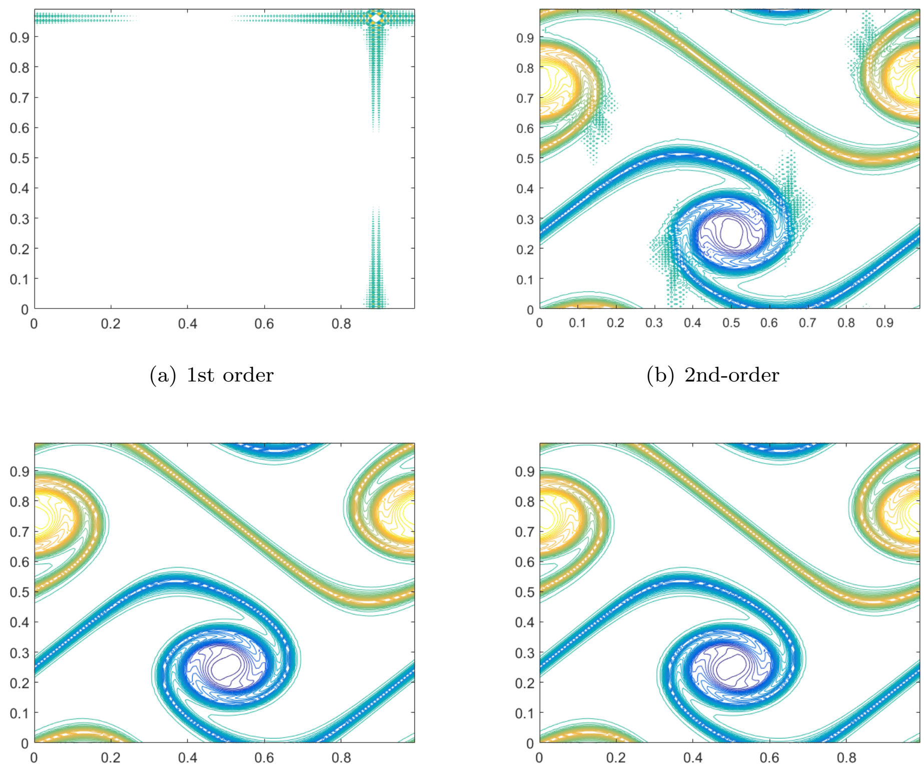
to the time discretization error. In Figure 1, we plot the convergence rate of the H^1 error for the velocity and the pressure at $T = 1$ by using first- to fourth-order schemes. We observe the expected convergence rates for both the velocity and the pressure.

Example 2: Double shear layer problem [4, 5, 10]. Consider the Navier–Stokes equations (1.1) in $\Omega = (0, 1) \times (0, 1)$ with periodic boundary conditions and the initial condition given by

$$u_1(x, y, 0) = \begin{cases} \tanh(\rho(y - 0.25)), & y \le 0.5, \\ \tanh(\rho(0.75 - y)), & y > 0.5, \end{cases}$$
$$u_2(x, y, 0) = \delta \sin(2\pi x),$$

where ρ determines the slope of the shear layer and δ represents the size of the perturbation. In our simulations, we fix $\delta = 0.05$.

We first test a *thick layer problem* by choosing $\rho = 30$ and $\nu = 0.0001$. We use the Fourier spectral method with 128×128 modes for the space discretization, and set $\delta t = 8 \times 10^{-4}$. In Figure 2, we show the vorticity contours at $T = 1.2$ obtained with first- to fourth-order schemes. We observe that a correct solution is obtained with the third- and fourth-order schemes while the first-order scheme gives totally wrong results and the second-order scheme leads to inaccurate results.



(a) 1st order

(b) 2nd-order

(c) 3rd-order

(d) 4th-order

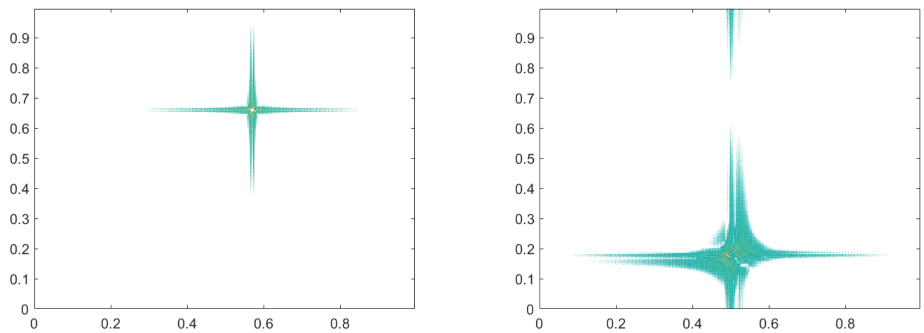
Fig. 2. *Thick layer problem: vorticity contours at* $T = 1.2$ *with* $\rho = 30$, $\nu = 0.0001$, *and* $\delta t = 8 \times 10^{-4}$.

Next, we test a *thin layer problem* by choosing $\rho = 100$ and $\nu = 0.00005$. We use first- to fourth-order schemes with 256×256 Fourier modes and $\delta t = 3 \times 10^{-4}$. In Figure 3, we plot the vorticity contours at $T = 1.2$. We observe that correct solutions are obtained with the third- and fourth-order schemes while first- and second-order schemes lead to wrong results.

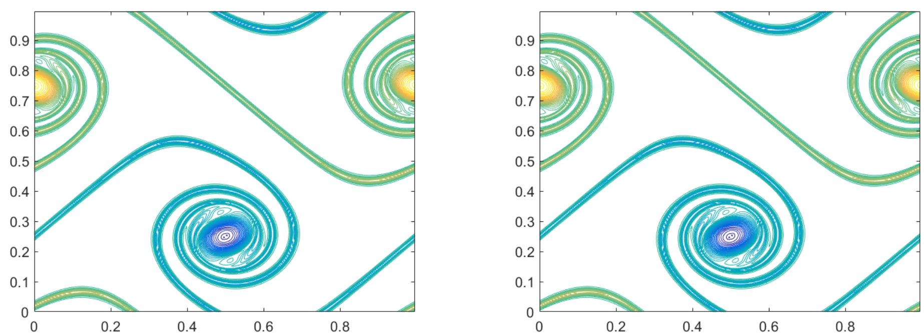
In order to examine the effect of the SAV approach, we plot in Figure 4 evolution of the SAV factor $\eta = 1-(1-\xi)^2$ and the vorticity contours at $T = 1.2$, computed with the second-order scheme with $\delta t = 2.5 \times 10^{-4}$. We observe that at around $t = 1.05$, the usual semi-implicit second-order scheme blows up (see Figure 4(a)) while the SAV factor dips slightly to allow the scheme continue to produce a correct simulation (see Figure 4(b), (c)).

Remark 1. These two tests indicate that for high Reynolds number flows with complex structures, higher-order schemes are preferred over lower-order schemes, as much smaller time steps have to be used to obtain correct solutions with lower-order schemes.

Note that if we use the usual semi-implicit schemes with the same time steps in the above tests, the first- and second-order schemes would blow up. So the SAV approach can effectively prevent the numerical solution from blowing up, although sufficient small time steps are needed to capture the correct solution. Thus, one is advised to adopt a suitable adaptive time stepping to take full advantage of the SAV schemes.



(a) 1st-order (b) 2nd-order



(c) 3rd-order (d) 4th-order

Fig. 3. *Thin layer problem: vorticity contours at* $T = 1.2$ *with* $\rho = 100$, $\nu = 0.00005$, *and* $\delta t = 3 \times 10^{-4}$.

4. Error analysis. In this section, we carry out a unified error analysis for the fully discrete schemes (3.14) with $1 \le k \le 5$ and state, as corollaries, similar results for the semidiscrete schemes (3.6).

We denote

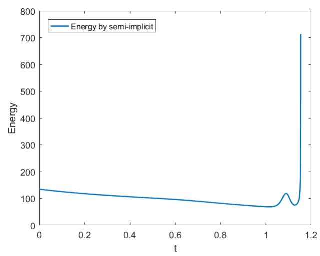
$$\begin{aligned} &t^n = n\,\delta t, \quad s^n = r^n - r(t^n),\\ &\bar{\boldsymbol{e}}_N^n = \bar{\boldsymbol{u}}_N^n - \Pi_N \boldsymbol{u}(\cdot, t^n), \quad \boldsymbol{e}_N^n = \boldsymbol{u}_N^n - \Pi_N \boldsymbol{u}(\cdot, t^n), \quad \boldsymbol{e}_\Pi^n = \Pi_N \boldsymbol{u}(\cdot, t^n) - \boldsymbol{u}(\cdot, t^n),\\ &\bar{\boldsymbol{e}}^n = \bar{\boldsymbol{u}}_N^n - \boldsymbol{u}(\cdot, t^n) = \bar{\boldsymbol{e}}_N^n + \boldsymbol{e}_\Pi^n, \quad \boldsymbol{e}^n = \boldsymbol{u}_N^n - \boldsymbol{u}(\cdot, t^n) = \boldsymbol{e}_N^n + \boldsymbol{e}_\Pi^n. \end{aligned}$$

To simplify the notation, we dropped the dependence on N for $\bar{\boldsymbol{e}}^n$ and $\boldsymbol{e}^n$ in the above, and will do so for some other quantities in what follows.

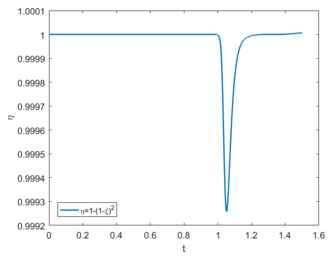
4.1. Several useful lemmas. We will frequently use the following two discrete versions of the Gronwall lemma.

Lemma 4.1 (discrete Gronwall lemma 1 [37]). *Let* y^k, h^k, g^k, f^k *be four nonnegative sequences satisfying*

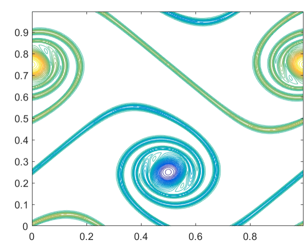
$$y^n + \delta t \sum_{k=0}^{n} h^k \le B + \delta t \sum_{k=0}^{n} (g^k y^k + f^k) \quad \textit{with} \quad \delta t \sum_{k=0}^{T/\delta t} g^k \le M, \; \forall\, 0 \le n \le T/\delta t.$$



(a) Evolution of Energy by a usual semi-implicit scheme



(b) Evolution of η



(c) Vorticity contours at $t = 1.2$

FIG. 4. *Thin layer problem: second-order scheme with* $\rho = 100$*,* $\nu = 0.00005$*, and* $\delta t = 2.5 \times 10^{-4}$*.*

We assume $\delta t\, g^k < 1\ \forall\ k$*, and let* $\sigma = \max_{0\le k\le T/\delta t}(1-\delta t g^k)^{-1}$*. Then*

$$y^n + \delta t \sum_{k=1}^{n} h^k \le \exp(\sigma M)\left(B + \delta t \sum_{k=0}^{n} f^k\right)\ \forall\, n \le T/\delta t.$$

LEMMA 4.2 (discrete Gronwall lemma 2 [35]). *Let* a_n*,* b_n*,* c_n*, and* d_n *be four nonnegative sequences satisfying*

$$a_m + \tau \sum_{n=1}^{m} b_n \le \tau \sum_{n=0}^{m-1} a_n d_n + \tau \sum_{n=0}^{m-1} c_n + C,\ m \ge 1,$$

where C *and* τ *are two positive constants. Then*

$$a_m + \tau \sum_{n=1}^{m} b_n \le \exp\left(\tau \sum_{n=0}^{m-1} d_n\right)\left(\tau \sum_{n=0}^{m-1} c_n + C\right),\ m \ge 1.$$

Based on Dahlquist's G-stability theory, Nevanlinna and Odeh [30] proved the following result, which plays an essential role in our error analysis.

LEMMA 4.3. *For* $1 \le k \le 5$*, there exist* $0 \le \tau_k < 1$*, a positive definite symmetric matrix* $G = (g_{ij}) \in \mathcal{R}^{k,k}$*, and real numbers* $\delta_0, \dots, \delta_k$ *such that*

$$\begin{aligned}(\alpha_k u^{n+1} - A_k(u^n), u^{n+1} - \tau_k u^n) = &\sum_{i,j=1}^{k} g_{ij}(u^{n+1+i-k}, u^{n+1+j-k}) \\ &- \sum_{i,j=1}^{k} g_{ij}(u^{n+i-k}, u^{n+j-k}) + \left\|\sum_{i=0}^{k} \delta_i u^{n+1+i-k}\right\|^2,\end{aligned}$$

where the smallest possible values of τ_k *are*

$$\tau_1 = \tau_2 = 0, \quad \tau_3 = 0.0836, \quad \tau_4 = 0.2878, \quad \tau_5 = 0.8160,$$

and α_k*,* A_k *are defined in* (3.10)–(3.12)*.*

We also recall the following lemma [28], which will be used to prove local error estimates in the three-dimensional case.

LEMMA 4.4. *Let* $\phi : (0,\infty) \to (0,\infty)$ *be continuous and increasing, and let* $M > 0$*. Given* T_* *such that* $0 < T_* < \int_M^\infty dz/\phi(z)$*, there exists* $C_* > 0$ *independent of* $\delta t > 0$ *with the following property. Suppose that quantities* z_n*,* $w_n \geq 0$ *satisfy*

$$z_n + \sum_{k=0}^{n-1} \delta t w_k \leq y_n := M + \sum_{k=0}^{n-1} \delta t \phi(z_k) \ \ \forall n \leq n_*$$

with $n_* \delta t \leq T_*$*. Then* $y_{n_*} \leq C_*$*.*

4.2. Error analysis for the velocity in two dimensions.

THEOREM 4.5. *Let* $d = 2$*,* $\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_p^m$ *with* $m \geq 3$ *and* $\boldsymbol{u}$ *be the solution of* (1.1)*. We assume that* $\bar{\boldsymbol{u}}_N^i$ *and* $\boldsymbol{u}_N^i$ $(i = 1, \dots, k-1)$ *are computed with a proper initialization procedure such that*

$$\begin{aligned} &\|\bar{\boldsymbol{u}}_N^i - \boldsymbol{u}(\cdot,t_i)\|_1,\ \|\boldsymbol{u}_N^i - \boldsymbol{u}(\cdot,t_i)\|_1 = O(\delta t^k + N^{1-m}), \\ &\|\bar{\boldsymbol{u}}_N^i - \boldsymbol{u}(\cdot,t_i)\|_2,\ \|\boldsymbol{u}_N^i - \boldsymbol{u}(\cdot,t_i)\|_2 = O(\delta t^k + N^{2-m}), \end{aligned} \quad i = 1, \dots, k-1. \tag{4.1}$$

Let $\bar{\boldsymbol{u}}_N^{n+1}$ *and* $\boldsymbol{u}_N^{n+1}$ *be computed with the kth-order scheme* (3.14) $(1 \leq k \leq 5)$*, and*

$$\eta_1^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad \eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k \ (k = 2,3,4,5).$$

Then for any $T > 0$*, and* $n+1 \leq T/\delta t$ *with* $\delta t \leq \frac{1}{1+2^{k+2}C_0^{k+1}}$ *and* $N \geq 2^{k+2}C_\Pi^{k+1} + 1$*, we have*

$$\|\bar{\boldsymbol{u}}_N^n - \boldsymbol{u}(\cdot,t^n)\|_1^2,\ \|\boldsymbol{u}_N^n - \boldsymbol{u}(\cdot,t^n)\|_1^2 \leq C\delta t^{2k} + CN^{2(1-m)}$$

and

$$\delta t \sum_{q=0}^{n} \|\bar{\boldsymbol{u}}_N^{q+1} - \boldsymbol{u}(\cdot,t^{q+1})\|_2^2,\ \delta t \sum_{q=0}^{n} \|\boldsymbol{u}_N^{q+1} - \boldsymbol{u}(\cdot,t^{q+1})\|_2^2 \leq C\delta t^{2k} + CN^{2(2-m)},$$

where the constants C_0*,* C_Π*, and* C *are dependent on* T*,* Ω*, the* $k \times k$ *matrix* $G = (g_{ij})$ *in Lemma* 4.3*, and the exact solution* $\boldsymbol{u}$ *but are independent of* δt *and* N*.*

Proof. It is shown in [42] that in the periodic case, $\boldsymbol{u}_0 \in \boldsymbol{H}_p^m$ implies that $\boldsymbol{u}(\cdot,t) \in \boldsymbol{H}_p^m \ \forall\, t \leq T$, and furthermore, it is shown in [13] that $\boldsymbol{u}$ has Gevrey class regularity. In particular, we have

$$\boldsymbol{u} \in C([0,T]; \boldsymbol{H}_p^m),\ m \geq 3,\ \frac{\partial^j \boldsymbol{u}}{\partial t^j} \in L^2(0,T; \boldsymbol{H}_p^2)\ 1 \leq j \leq k,\ \frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}} \in L^2(0,T; L^2). \tag{4.2}$$

To simplify the presentation, we assume $\bar{\boldsymbol{u}}_N^i = \boldsymbol{u}_N^i = \Pi_N \boldsymbol{u}(t_i)$ and $r^i = E[\boldsymbol{u}_N^i] + 1$ for $i = 1, \dots, k-1$ so that (4.1) is obviously satisfied.

The main task is to prove by induction,

$$|1-\xi^q| \le C_0\,\delta t + C_\Pi N^{2-m} \;\forall q \le T/\delta t, \tag{4.3}$$

where the constant C_0 and C_Π will be defined in the induction process below.

Under the assumption, (4.3) certainly holds for $q=0$. Now suppose we have

$$|1-\xi^q| \le C_0\,\delta t + C_\Pi N^{2-m} \;\forall q \le n; \tag{4.4}$$

we shall prove below

$$|1-\xi^{n+1}| \le C_0\delta t + C_\Pi N^{2-m}. \tag{4.5}$$

We shall first consider $k=2,3,4,5$ and point out the necessary modifications for the case $k=1$ later.

Step 1*: Bounds for* $\nabla\bar{\boldsymbol{u}}_N^q$, $\Delta\bar{\boldsymbol{u}}_N^q$, *and* $\Delta\boldsymbol{u}_N^q$, $\forall q \le n$. We first recall the inequality

$$(a+b)^k \le 2^{k-1}(a^k+b^k) \quad \forall a,b>0,\ k\ge 1. \tag{4.6}$$

Under the assumption (4.4), if we choose δt small enough and N large enough such that

$$\delta t \le \min\left\{\frac{1}{2^{k+2}C_0^k}, 1\right\}, \quad N \ge \max\{2^{k+2}C_\Pi^k, 1\}, \tag{4.7}$$

we have

(4.8)
$$1-\left(\frac{1}{2^{k+2}C_0^{k-1}}+\frac{N^{3-m}}{2^{k+2}C_\Pi^{k-1}}\right) \le |\xi^q| \le 1+\left(\frac{1}{2^{k+2}C_0^{k-1}}+\frac{N^{3-m}}{2^{k+2}C_\Pi^{k-1}}\right) \;\forall q \le n,$$

and

$$(1-\xi^q)^k \le \frac{\delta t^{k-1}}{4} + \frac{N^{k(2-m)+1}}{4} \;\forall q \le n,$$

and

$$\frac{1}{2} < 1-\left(\frac{\delta t^{k-1}}{4}+\frac{N^{k(2-m)+1}}{4}\right) \le |\eta_k^q| \le 1+\frac{\delta t^{k-1}}{4}+\frac{N^{k(2-m)+1}}{4} < 2 \;\forall q \le n.$$

Then it follows from the above and (3.17) that

$$\|\bar{\boldsymbol{u}}_N^q\|_1 \le 2M_k \;\forall q \le n. \tag{4.9}$$

Moveover, (3.16) and $m \ge 3$ imply that

$$\nu\delta t\sum_{q=1}^{n}\|\Delta\bar{\boldsymbol{u}}_N^q\|^2 \le \frac{2r^0}{|\xi^q|} \le 4r^0,\ C_0 \ge 1,\ C_\Pi \ge 1, \tag{4.10}$$

and

$$\nu\delta t\sum_{q=1}^{n}\|\Delta\boldsymbol{u}_N^q\|^2 \le 16r^0,\ C_0 \ge 1,\ C_\Pi \ge 1. \tag{4.11}$$

Step 2: Estimates for $\nabla \bar{\boldsymbol{e}}_N^{n+1}$ and $\Delta \bar{\boldsymbol{e}}_N^{n+1}$. By the assumptions on the exact solution $\boldsymbol{u}$ and (4.9), we can choose C large enough such that

$$\|\boldsymbol{u}(t)\|_{H^2}^2 \le C \ \forall t \le T, \ \|\bar{\boldsymbol{u}}_N^q\|_1 \le C \ \forall q \le n. \tag{4.12}$$

From (3.14a), we can write the error equation as

$$\big(\alpha_k \bar{\boldsymbol{e}}^{q+1} - A_k(\bar{\boldsymbol{e}}^q), v_N\big) + \delta t \nu \big(\nabla \bar{\boldsymbol{e}}^{q+1}, \nabla v_N\big) = \big(R_k^q, v_N\big) + \delta t \big(Q_k^q, v_N\big) \quad \forall v_N \in S_N, \tag{4.13}$$

where Q_k^q and R_k^q are given by

$$Q_k^q = -\mathbf{A}\big((B_k \boldsymbol{u}^q)\cdot\nabla)B_k(\boldsymbol{u}^q)\big) + \mathbf{A}\big(\boldsymbol{u}(t^{q+1})\cdot\nabla\boldsymbol{u}(t^{q+1})\big) \tag{4.14}$$

and

$$\begin{aligned} R_k^q &= -\alpha_k \boldsymbol{u}(t^{q+1}) + A_k(\boldsymbol{u}(t^q)) + \delta t \boldsymbol{u}_t(t^{q+1}) \\ &= \sum_{i=1}^k a_i \int_{t^{q+1-i}}^{t^{q+1}} (t^{q+1-i} - s)^k \frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}(s) ds, \end{aligned} \tag{4.15}$$

with a_i being some fixed and bounded constants determined by the truncation errors, for example, in the case $k=3$, we have

$$\begin{aligned} R_3^q = -3\int_{t^q}^{t^{q+1}} (t^q - s)^3 \frac{\partial^4 \boldsymbol{u}}{\partial t^4}(s) ds + \frac{3}{2}\int_{t^{q-1}}^{t^{n+1}} (t^{q-1} - s)^3 \frac{\partial^4 \boldsymbol{u}}{\partial t^4}(s) ds \\ - \frac{1}{3}\int_{t^{q-2}}^{t^{n+1}} (t^{q-2} - s)^3 \frac{\partial^4 \boldsymbol{u}}{\partial t^4}(s) ds. \end{aligned}$$

Letting $v_N = -\Delta \bar{\boldsymbol{e}}_N^{q+1} + \tau_k \Delta \bar{\boldsymbol{e}}_N^q$ in (4.13), it follows from Lemma 4.3 and (3.13) that

$$\begin{aligned} &\sum_{i,j=1}^k g_{ij}(\nabla \bar{\boldsymbol{e}}_N^{q+1+i-k}, \nabla \bar{\boldsymbol{e}}_N^{q+1+j-k}) - \sum_{i,j=1}^k g_{ij}(\nabla \bar{\boldsymbol{e}}_N^{q+i-k}, \nabla \bar{\boldsymbol{e}}_N^{q+j-k}) \\ &\quad + \left\|\sum_{i=0}^k \delta_i \nabla \bar{\boldsymbol{e}}_N^{q+1+i-k}\right\|^2 + \delta t \nu \|\Delta \bar{\boldsymbol{e}}_N^{q+1}\|^2 \\ &= \delta t \nu (\Delta \bar{\boldsymbol{e}}_N^{q+1}, \tau_k \Delta \bar{\boldsymbol{e}}_N^q) + (R_k^q, -\Delta \bar{\boldsymbol{e}}_N^{q+1} + \tau_k \Delta \bar{\boldsymbol{e}}_N^q) + \delta t (Q_k^n, -\Delta \bar{\boldsymbol{e}}_N^{q+1} + \tau_k \Delta \bar{\boldsymbol{e}}_N^q). \end{aligned} \tag{4.16}$$

Next, we bound the right-hand side of (4.16). It follows from (4.15) that

$$\|R_k^q\|^2 \le C \delta t^{2k+1} \int_{t^{q+1-k}}^{t^{q+1}} \left\|\frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}(s)\right\|^2 ds. \tag{4.17}$$

Therefore,

$$\begin{aligned} \left|\big(R_k^q, -\Delta \bar{\boldsymbol{e}}_N^{q+1} + \tau_k \Delta \bar{\boldsymbol{e}}_N^q\big)\right| &\le \frac{C(\varepsilon)}{\delta t}\|R_k^q\|^2 + \delta t \varepsilon \| - \Delta \bar{\boldsymbol{e}}_N^{q+1} + \tau_k \Delta \bar{\boldsymbol{e}}_N^q\|^2, \\ &\le \frac{C(\varepsilon)}{\delta t}\|R_k^q\|^2 + 2\delta t \varepsilon \|\Delta \bar{\boldsymbol{e}}_N^{q+1}\|^2 + 2\delta t \varepsilon \|\Delta \bar{\boldsymbol{e}}_N^q\|^2, \\ &\le 2\delta t \varepsilon \|\Delta \bar{\boldsymbol{e}}_N^{q+1}\|^2 + 2\delta t \varepsilon \|\Delta \bar{\boldsymbol{e}}_N^q\|^2 + C(\varepsilon)\delta t^{2k} \int_{t^{q+1-k}}^{t^{q+1}} \left\|\frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}(s)\right\|^2 ds. \end{aligned} \tag{4.18}$$

For the term with Q_k^q, we split it as

$$
\begin{aligned}
(Q_k^n, -\Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\boldsymbol{e}_N^q) = {} & \Big(\mathbf{A}\big([\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}^q)]\cdot\nabla\boldsymbol{u}(t^{q+1})\big), -\Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\boldsymbol{e}_N^q\Big) \\
& + \Big(\mathbf{A}\big(B_k(\boldsymbol{u}^q)\cdot\nabla[\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}(t^q))]\big), -\Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\Big) \\
& - \Big(\mathbf{A}\big(B_k(\boldsymbol{e}^q)\cdot\nabla B_k(\boldsymbol{e}^q)\big), -\Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\Big) \\
& - \Big(\mathbf{A}\big(B_k(\boldsymbol{u}(t^q))\cdot\nabla B_k(\boldsymbol{e}^q)\big), -\Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\Big).
\end{aligned}
\tag{4.19}
$$

We bound the terms on the right-hand side of (4.19) with the help of (2.7), (2.9), and (4.12):

$$
\begin{aligned}
&\Big(\mathbf{A}\big([\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}^q)]\cdot\nabla\boldsymbol{u}(t^{q+1})\big), -\Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\Big) \\
&\quad\le C\|\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}^q)\|_1\|\boldsymbol{u}(t^{q+1})\|_2\| - \Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\| \\
&\quad\le C(\varepsilon)\|\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}^q)\|_1^2\|\boldsymbol{u}(t^{q+1})\|_2^2 + \varepsilon\| - \Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\|^2 \\
&\quad\le C(\varepsilon)\|\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}(t^q))\|_1^2\|\boldsymbol{u}(t^{q+1})\|_2^2 + C(\varepsilon)\|B_k(\boldsymbol{e}^q)\|_1^2\|\boldsymbol{u}(t^{q+1})\|_2^2 \\
&\qquad + \varepsilon\| - \Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\|^2 \\
&\quad\le C(\varepsilon)\|\sum_{i=1}^k b_i\int_{t^{q+1-i}}^{t^{q+1}}(t^{q+1-i} - s)^{k-1}\frac{\partial^k\boldsymbol{u}}{\partial t^k}(s)ds\|_1^2 + C(\varepsilon)\|B_k(\boldsymbol{e}^q)\|_1^2 \\
&\qquad + 2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^{q+1}\|^2 + 2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^q\|^2 \\
&\quad\le C(\varepsilon)\delta t^{2k-1}\int_{t^{q+1-k}}^{t^{q+1}}\|\frac{\partial^k\boldsymbol{u}}{\partial t^k}(s)\|_1^2 ds + C(\varepsilon)\|B_k(\boldsymbol{e}^q)\|_1^2 + 2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^{q+1}\|^2 + 2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^q\|^2,
\end{aligned}
\tag{4.20}
$$

where b_i are some fixed and bounded constants determined by the truncation error. For example, in the case $k = 3$, we have

$$
\begin{aligned}
B_3(\boldsymbol{u}(t^q)) - \boldsymbol{u}(t^{q+1}) = {} & -\frac{3}{2}\int_{t^q}^{t^{q+1}}(t^q - s)^2\frac{\partial^3\boldsymbol{u}}{\partial t^3}(s)ds + \frac{3}{2}\int_{t^{q-1}}^{t^{q+1}}(t^{q-1} - s)^2\frac{\partial^3\boldsymbol{u}}{\partial t^3}ds \\
& - \frac{1}{2}\int_{t^{q-2}}^{t^{q+1}}(t^{q-2} - s)^2\frac{\partial^3\boldsymbol{u}}{\partial t^3}ds.
\end{aligned}
$$

For the other terms in the right-hand side of (4.19), we have

$$
\begin{aligned}
&\Big|\Big(\mathbf{A}\big(B_k(\boldsymbol{u}^q)\cdot\nabla[\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}(t^q))]\big), -\Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\Big)\Big| \\
&\qquad\le C\|B_k(\boldsymbol{u}^q)\|_1\|\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}(t^q))\|_2\| - \Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\| \\
&\qquad\le C(\varepsilon)\|B_k(\boldsymbol{u}^q)\|_1^2\|\boldsymbol{u}(t^{q+1}) - B_k(\boldsymbol{u}(t^q))\|_2^2 + \varepsilon\| - \Delta\bar{\boldsymbol{e}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{e}}_N^q\|^2 \\
&\qquad\le C(\varepsilon)\delta t^{2k-1}\int_{t^{q+1-k}}^{t^{q+1}}\left\|\frac{\partial^k\boldsymbol{u}}{\partial t^k}(s)\right\|_2^2 ds + 2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^{q+1}\|^2 + 2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^q\|^2.
\end{aligned}
\tag{4.21}
$$

Since $d=2$, we can use (2.7) to obtain

(4.22)
$$\begin{aligned}
&\Big|\Big(\mathbf{A}\big(B_k(\boldsymbol{e}^q)\cdot\nabla B_k(\boldsymbol{e}^q)\big), -\Delta\bar{\boldsymbol{e}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{e}}_N^q\Big)\Big|\\
&\quad\le C\|B_k(\bar{\boldsymbol{e}}^q)\|_1^{1/2}\|B_k(\bar{\boldsymbol{e}}^q)\|^{1/2}\|B_k(\bar{\boldsymbol{e}}^q)\|_2^{1/2}\|B_k(\bar{\boldsymbol{e}}^q)\|_1^{1/2}\|-\Delta\bar{\boldsymbol{e}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{e}}_N^q\|\\
&\quad\le C\|B_k(\boldsymbol{e}^q)\|_1\|B_k(\boldsymbol{e}^q)\|_2\|-\Delta\bar{\boldsymbol{e}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{e}}_N^q\| \ \text{(true in two and three dimensions)}\\
&\quad\le C(\varepsilon)\|B_k(\boldsymbol{e}^q)\|_1^2\|B_k(\boldsymbol{e}^q)\|_2^2+\varepsilon\|-\Delta\bar{\boldsymbol{e}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{e}}_N^q\|^2\\
&\quad\le C(\varepsilon)\|B_k(\boldsymbol{e}^q)\|_1^2\|B_k(\boldsymbol{e}^q)\|_2^2+2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^{q+1}\|^2+2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^q\|^2.
\end{aligned}$$

Thanks to (2.9), we have

$$\begin{aligned}
&\Big|\Big(\mathbf{A}\big(B_k(\boldsymbol{u}(t^q))\cdot\nabla B_k(\bar{\boldsymbol{e}}^q)\big), -\Delta\bar{\boldsymbol{e}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{e}}_N^q\Big)\Big|\\
&\quad\le C\|B_k(\boldsymbol{u}(t^q))\|_2\|B_k(\boldsymbol{e}^q)\|_1\|-\Delta\bar{\boldsymbol{e}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{e}}_N^q\|\\
&\quad\le C(\varepsilon)\|B_k(\boldsymbol{u}(t^q))\|_2^2\|B_k(\boldsymbol{e}^q)\|_1^2+\varepsilon\|-\Delta\bar{\boldsymbol{e}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{e}}_N^q\|^2\\
&\quad\le C(\varepsilon)\|B_k(\boldsymbol{e}^q)\|_1^2+2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^{q+1}\|^2+2\varepsilon\|\Delta\bar{\boldsymbol{e}}_N^q\|^2.
\end{aligned}\tag{4.23}$$

On the other hand, we derive from (4.6) and (4.4) that

$$|\eta_k^q-1|\le 2^kC_0^k\,\delta t^k+2^kC_\Pi^kN^{k(2-m)}\quad\forall q\le n.$$

Noting that $\boldsymbol{u}_N^q=\eta_k^q\bar{\boldsymbol{u}}_N^q$, we can estimate $\|B_k(\boldsymbol{e}^q)\|_1^2$ by

(4.24)
$$\begin{aligned}
\|B_k(\boldsymbol{e}^q)\|_1^2&=\|B_k(\boldsymbol{u}_N^q-\bar{\boldsymbol{u}}_N^q)+B_k(\bar{\boldsymbol{e}}_N^q)+B_k(\boldsymbol{e}_\Pi^q)\|_1^2\\
&\le CC_0^{2k}\delta t^{2k}+CC_\Pi^{2k}N^{2k(2-m)}+C\|B_k(\bar{\boldsymbol{e}}_N^q)\|_1^2+C\|\boldsymbol{u}(t^q)\|_m^2N^{2-2m}.
\end{aligned}$$

Combining (4.16)–(4.24) and dropping some unnecessary terms, we arrive at

(4.25)
$$\begin{aligned}
&\sum_{i,j=1}^k g_{ij}(\nabla\bar{\boldsymbol{e}}_N^{q+1+i-k},\nabla\bar{\boldsymbol{e}}_N^{q+1+j-k})-\sum_{i,j=1}^k g_{ij}(\nabla\bar{\boldsymbol{e}}_N^{q+i-k},\nabla\bar{\boldsymbol{e}}_N^{q+j-k})\\
&\qquad+\delta t\left(\frac{\nu}{2}-10\varepsilon\right)\|\Delta\bar{\boldsymbol{e}}_N^{q+1}\|^2\\
&\quad\le\delta t\left(\frac{\nu\tau_k^2}{2}+10\varepsilon\right)\|\Delta\bar{\boldsymbol{e}}_N^q\|^2+C(\varepsilon)\delta t\|B_k(\bar{\boldsymbol{e}}_N^q)\|_1^2+C(\varepsilon)\delta t\|B_k(\bar{\boldsymbol{e}}_N^q)\|_1^2\|B_k(\boldsymbol{e}_N^q)\|_2^2\\
&\qquad+C(\varepsilon)\delta t^{2k}\int_{t^{q+1-k}}^{t^{q+1}}\left(\left\|\frac{\partial^k\boldsymbol{u}}{\partial t^k}(s)\right\|_2^2+\left\|\frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}(s)\right\|^2\right)ds\\
&\qquad+C(\varepsilon)C_0^{2k}\delta t^{2k+1}(1+\|B_k(\boldsymbol{e}^q)\|_2^2)+\delta tC(\varepsilon)C_\Pi^{2k}N^{2k(2-m)}(1+\|B_k(\boldsymbol{e}^q)\|_2^2)\\
&\qquad+\delta tC(\varepsilon)\|\boldsymbol{u}(t^q)\|_m^2N^{2-2m}(1+\|B_k(\boldsymbol{e}^q)\|_2^2).
\end{aligned}$$

Since $\tau_k<1$, we can choose ε small enough such that

$$\frac{\nu}{2}-10\varepsilon>\frac{\nu\tau_k^2}{2}+10\varepsilon+\frac{\nu(1-\tau_k^2)}{4},\tag{4.26}$$

and then taking the sum of (4.25) on q from $k-1$ to n, noting that $G=(g_{ij})$ is a symmetric positive definite matrix with minimum eigenvalue λ_G, we obtain

(4.27)

$$\begin{aligned}&\lambda_G\|\nabla\bar{\boldsymbol{e}}_N^{n+1}\|^2+\frac{\delta t\nu(1-\tau_k^2)}{4}\sum_{q=0}^{n+1}\|\Delta\bar{\boldsymbol{e}}_N^q\|^2\\ &\quad\le\sum_{i,j=1}^{k}g_{ij}(\nabla\bar{\boldsymbol{e}}_N^{n+1+i-k},\nabla\bar{\boldsymbol{e}}_N^{n+1+j-k})+\frac{\delta t\nu(1-\tau_k^2)}{4}\sum_{q=0}^{n+1}\|\Delta\bar{\boldsymbol{e}}_N^q\|^2\\ &\quad\le C\delta t\sum_{q=0}^{n}\|\bar{\boldsymbol{e}}_N^q\|_1^2(\|B_k(\boldsymbol{e}^q)\|_2^2+1)\\ &\qquad+C\delta t^{2k}\left(\int_0^T\left(\left\|\frac{\partial^k\boldsymbol{u}}{\partial t^k}(s)\right\|_2^2+\left\|\frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}(s)\right\|^2\right)ds+C_0^{2k}\left(T+\delta t\sum_{q=0}^{n}\|B_k(\boldsymbol{e}^q)\|_2^2\right)\right)\\ &\qquad+C\big(C_\Pi^{2k}N^{2k(2-m)}+N^{2-2m}\big)\left(T+\delta t\sum_{q=0}^{n}\|B_k(\boldsymbol{e}^q)\|_2^2\right).\end{aligned}$$

Noting that (4.11) and (4.12) imply $\delta t\sum_{q=0}^{n}\|B_k(\boldsymbol{e}^q)\|_2^2<C_{H^2}$ for some constant C_{H^2} depends only on the exact solution $\boldsymbol{u}$. Applying the discrete Gronwall Lemma 4.2 to (4.27), we obtain

$$\begin{aligned}&\|\bar{\boldsymbol{e}}_N^{n+1}\|_1^2+\delta t\sum_{q=0}^{n+1}\|\bar{\boldsymbol{e}}_N^q\|_2^2\\ &\le C\exp\big(C_{H^2}+1)\big)\delta t^{2k}\int_0^T\left(\left\|\frac{\partial^k\boldsymbol{u}}{\partial t^k}(s)\right\|_2^2+\left\|\frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}(s)\right\|^2\right)ds\\ &+C\exp\big(C_{H^2}+1)\big)(\delta t^{2k}C_0^{2k}+C_\Pi^{2k}N^{2k(2-m)}+N^{2-2m})(T+C_{H^2})\\ &\le C_1(1+C_0^{2k})\delta t^{2k}+C_1(C_\Pi^{2k}N^{2k(2-m)}+N^{2-2m}),\end{aligned}\tag{4.28}$$

where C_1 is independent of δt, C_0, C_Π and can be defined as

(4.29)

$$C_1:=C\exp(C_{H^2}+1)\max\left(\int_0^T\left(\left\|\frac{\partial^k\boldsymbol{u}}{\partial t^k}(s)\right\|_2^2+\left\|\frac{\partial^{k+1}\boldsymbol{u}}{\partial t^{k+1}}(s)\right\|^2\right)ds,\,1,\,T+C_{H^2}\right).$$

Therefore, (4.28) implies

$$\|\bar{\boldsymbol{e}}_N^{n+1}\|_1^2,\ \delta t\sum_{q=0}^{n+1}\|\bar{\boldsymbol{e}}_N^q\|_2^2\le C_1(1+C_0^{2k})\delta t^{2k}+C_1(C_\Pi^{2k}N^{2k(2-m)}+N^{2-2m}).\tag{4.30}$$

Since $\bar{\boldsymbol{e}}^q=\bar{\boldsymbol{e}}_N^q+\bar{\boldsymbol{e}}_\Pi^q$, it follows from the triangle inequality that

$$\|\bar{\boldsymbol{e}}^{n+1}\|_1^2\le C_1(1+C_0^{2k})\delta t^{2k}+C_1(C_\Pi^{2k}N^{2k(2-m)}+N^{2-2m})+CN^{2(1-m)}\tag{4.31}$$

and

$$\delta t\sum_{q=0}^{n+1}\|\bar{\boldsymbol{e}}^q\|_2^2\le C_1(1+C_0^{2k})\delta t^{2k}+C_1(C_\Pi^{2k}N^{2k(2-m)}+N^{2-2m})+CN^{2(2-m)}.\tag{4.32}$$

Combining (4.12), (4.31), and (4.32), we find that, under the condition (4.7) and $m \geq 3$, we have

$$\begin{aligned}&\|\bar{\boldsymbol{u}}_N^{n+1}\|_1^2,\ \delta t\sum_{q=0}^{n+1}\|\bar{\boldsymbol{u}}_N^q\|_2^2\\ &\leq C_1\left(1+C_0^{2k}\frac{1}{2^{2k(k+2)}C_0^{2k^2}}\right)+C_1(C_\Pi^{2k}2^{-4k(k+1)}C_\Pi^{-4k^2}+1)+C\\ &\leq 4C_1+C:=\bar{C}.\end{aligned} \tag{4.33}$$

Step 3*: Estimate for* $|1-\xi^{n+1}|$. It follows from (3.14b) that the equation for $\{s^j\}$ can be written as

$$s^{q+1}-s^q=\delta t\nu\big(\|\Delta\boldsymbol{u}(t^{q+1})\|^2-\frac{r^{q+1}}{E(\bar{\boldsymbol{u}}_N^{q+1})+1}\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|^2\big)+T_q\ \forall q\leq n, \tag{4.34}$$

where T_q is the truncation error

$$T_q=r(t^q)-r(t^{q+1})+\delta t r_t(t^{q+1})=\int_{t^q}^{t^{q+1}}(s-t^q)r_{tt}(s)ds. \tag{4.35}$$

Taking the sum of (4.34) for q from 0 to n, and noting that $s^0=0$, we have

$$s^{n+1}=\delta t\nu\sum_{q=0}^{n}\left(\|\Delta\boldsymbol{u}(t^{q+1})\|^2-\frac{r^{q+1}}{E(\bar{\boldsymbol{u}}_N^{q+1})+1}\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|^2\right)+\sum_{q=0}^{n}T_q. \tag{4.36}$$

We bound the right-hand side of (4.36) as follows. By direct calculation, we have

$$r_{tt}=\int_\Omega((\nabla\boldsymbol{u})_t^2+\nabla\boldsymbol{u}(\nabla\boldsymbol{u})_{tt})dx; \tag{4.37}$$

then from (4.35), we have

$$|T_q|\leq C\delta t\int_{t^q}^{t^{q+1}}|r_{tt}|ds\leq C\delta t\int_{t^q}^{t^{q+1}}(\|\boldsymbol{u}_t\|_1^2+\|\boldsymbol{u}_{tt}\|_1^2)ds\ \forall q\leq n.$$

By the triangular inequality,

$$\begin{aligned}&\left|\|\Delta\boldsymbol{u}(t^{q+1})\|^2-\frac{r^{q+1}}{E(\bar{\boldsymbol{u}}_N^{q+1})+1}\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|^2\right|\\ &\leq\|\Delta\boldsymbol{u}(t^{q+1})\|^2\left|1-\frac{r^{q+1}}{E(\bar{\boldsymbol{u}}_N^{q+1})+1}\right|+\frac{r^{q+1}}{E(\bar{\boldsymbol{u}}_N^{q+1})+1}\big|\|\Delta\boldsymbol{u}(t^{q+1})\|^2-\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|^2\big|\\ &:=K_1^q+K_2^q.\end{aligned} \tag{4.38}$$

It follows from (4.12) and Theorem 3.2 that

$$\begin{aligned}K_1^q&\leq C\left|1-\frac{r^{q+1}}{E(\bar{\boldsymbol{u}}_N^{q+1})+1}\right|\\ &=C\left|\frac{r(t^{q+1})}{E[\boldsymbol{u}(t^{q+1})]+1}-\frac{r^{q+1}}{E[\boldsymbol{u}(t^{q+1})]+1}\right|+C\left|\frac{r^{q+1}}{E[\boldsymbol{u}(t^{q+1})]+1}-\frac{r^{q+1}}{E(\bar{\boldsymbol{u}}_N^{q+1})+1}\right|\\ &\leq C\big(|E[\boldsymbol{u}(t^{q+1})]-E(\bar{\boldsymbol{u}}_N^{q+1})|+|s^{q+1}|\big)\ \forall q\leq n,\end{aligned} \tag{4.39}$$

and it follows from (4.12) and Theorem 3.2 that

$$
\begin{aligned}
K_2^q &\le C\big|\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|^2 - \|\Delta\boldsymbol{u}(t^{q+1})\|^2\big| \\
&\le C\|\Delta\bar{\boldsymbol{u}}_N^{q+1} - \Delta\boldsymbol{u}(t^{q+1})\|(\|\Delta\bar{\boldsymbol{u}}^{q+1}\| + \|\Delta\boldsymbol{u}(t^{q+1})\|) \\
&\le C\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|\|\Delta\bar{\boldsymbol{e}}^{q+1}\| + C\|\Delta\bar{\boldsymbol{e}}^{q+1}\| \ \forall q \le n.
\end{aligned} \tag{4.40}
$$

We derive from the definition of $E(\boldsymbol{u})$ that

$$
\begin{aligned}
&|E(\boldsymbol{u}(t^{q+1})) - E(\bar{\boldsymbol{u}}_N^{q+1})| \\
&\quad\le \frac{1}{2}(\|\nabla\boldsymbol{u}(t^{q+1})\| + \|\nabla\bar{\boldsymbol{u}}_N^{q+1}\|)\|\nabla\boldsymbol{u}(t^{q+1}) - \nabla\bar{\boldsymbol{u}}_N^{q+1}\| \le C\|\nabla\bar{\boldsymbol{e}}^{q+1}\|.
\end{aligned} \tag{4.41}
$$

It follows from (4.32), (4.33), and the Cauchy–Schwarz inequality that

$$
\begin{aligned}
\delta t\sum_{q=0}^{n}\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|\|\Delta\bar{\boldsymbol{e}}^{q+1}\| &\le \big(\delta t\sum_{q=0}^{n}\|\Delta\bar{\boldsymbol{u}}^{q+1}\|^2\delta t\sum_{q=0}^{n}\|\Delta\bar{\boldsymbol{e}}^{q+1}\|^2\big)^{1/2} \\
&\le C\sqrt{C_1(1+C_0^{2k})\delta t^{2k} + C_1(C_\Pi^{2k}N^{2k(2-m)} + N^{2-2m}) + N^{2(2-m)}}.
\end{aligned} \tag{4.42}
$$

Now, we are ready to estimate s^{n+1}. Combining the estimates obtained above, (4.36) leads to

$$
\begin{aligned}
|s^{n+1}| &\le \delta t\nu\sum_{q=0}^{n}\left|\|\nabla\boldsymbol{u}(t^{q+1})\|^2 - \frac{r^{q+1}}{E(\bar{\boldsymbol{u}}^{q+1})+1}\|\nabla\bar{\boldsymbol{u}}_N^{q+1}\|^2\right| + \sum_{q=0}^{n}|T^q| \\
&\le C\delta t\sum_{q=0}^{n}|s^{q+1}| + C\delta t\sum_{q=0}^{n}\|\bar{\boldsymbol{e}}^{q+1}\|_2 + C\delta t\sum_{q=0}^{n}\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|\|\Delta\bar{\boldsymbol{e}}^{q+1}\| \\
&\quad + C\delta t\int_0^{t^{n+1}}(\|\boldsymbol{u}_t\|_1^2 + \|\boldsymbol{u}_{tt}\|_1^2)ds \\
&\le C\sqrt{C_1(1+C_0^{2k})\delta t^{2k} + C_1(C_\Pi^{2k}N^{2k(2-m)} + N^{2-2m}) + N^{2(2-m)}} \\
&\quad + C\delta t\sum_{q=0}^{n}|s^{q+1}| + C\delta t.
\end{aligned} \tag{4.43}
$$

Finally, applying Lemma 4.1 on (4.43) with $\delta t < \frac{1}{2C}$, we obtain the following estimate for s^{n+1}:

$$
\begin{aligned}
|s^{n+1}| &\le C\exp((1-\delta tC)^{-1}) \\
&\quad\times\left(\sqrt{C_1(1+C_0^{2k})\delta t^{2k} + C_1(C_\Pi^{2k}N^{2k(2-m)} + N^{2-2m}) + N^{2(2-m)}} + \delta t\right) \\
&\le C_2\left(\sqrt{C_1(1+C_0^{2k})\delta t^{2k} + C_1(C_\Pi^{2k}N^{2k(2-m)} + N^{2-2m}) + N^{2(2-m)}} + \delta t\right) \\
&\le C_2\delta t^k\sqrt{C_1(1+C_0^{2k})} + C_2\sqrt{C_1(C_\Pi^{2k}N^{2k(2-m)} + N^{2-2m}) + N^{2(2-m)}} + C_2\delta t,
\end{aligned} \tag{4.44}
$$

where $C_2 := C\exp(2)$ is independent of δt and C_0. Then $\delta t < \frac{1}{2C}$ can be guaranteed by

$$
\delta t < \frac{1}{C_2}. \tag{4.45}
$$

Thanks to (4.30), (4.39), (4.41), (4.44), and $m \geq 3$, we have

(4.46)
$$\begin{aligned}|1-\xi^{n+1}| &\leq C\big(|E[\boldsymbol{u}(t^{n+1})]-E(\bar{\boldsymbol{u}}^{n+1})|+|s^{n+1}|\big)\\ &\leq C(\|\nabla\bar{\boldsymbol{e}}^{n+1}\|+|s^{n+1}|)\\ &\leq C\sqrt{C_1(1+C_0^{2k})\delta t^{2k}+C_1(C_\Pi^{2k}N^{2k(2-m)}+N^{2-2m})+CN^{2(1-m)}}\\ &\quad+C_2\delta t^k\sqrt{C_1(1+C_0^{2k})}+C_2\sqrt{C_1(C_\Pi^{2k}N^{2k(2-m)}+N^{2-2m})+N^{2(2-m)}}+C_2\delta t\\ &\leq C_3\delta t\left(\sqrt{1+C_0^{2k}}\delta t^{k-1}+1\right)+C_3N^{2-m}\left(\sqrt{C_\Pi^{2k}N^{(4-2m)(k-1)}+N^{-2}+1}\right),\end{aligned}$$

where the constant C_3 is independent of C_0, C_Π , δt, and N. Without loss of generality, we assume $C_3 > \max\{C_1, C_2, 1\}$ to simplify the proof below.

For the cases $k = 2, 3, 4, 5$, we choose $C_0 = 2C_3$ and $\delta t \leq \frac{1}{1+C_0^k}$ to obtain

$$C_3\left(\sqrt{1+C_0^{2k}}\delta t^{k-1}+1\right)\leq C_3[(1+C_0^k)\delta t+1]\leq 2C_3=C_0, \tag{4.47}$$

and since $m \geq 3$, we can choose $C_\Pi = 3C_3$ and $N \geq C_\Pi^k + 1$ to obtain

$$C_3\left(\sqrt{C_\Pi^{2k}N^{(4-2m)(k-1)}+N^{-2}+1}\right)\leq C_3[C_\Pi^kN^{2-m}+2]\leq 3C_3=C_\Pi. \tag{4.48}$$

For the case $k = 1$, since $\eta_1^{n+1} = 1-(1-\xi^{n+1})^2$, we choose $C_0 = 2C_3$ and $\delta t \leq \frac{1}{1+C_0^2}$ so that

$$C_3\left(\sqrt{1+C_0^4}\delta t+1\right)\leq C_3[(1+C_0^2)\delta t+1]\leq 2C_3=C_0,$$

and since $m \geq 3$, we choose $C_\Pi = 3C_3$ and $N \geq C_\Pi^2 + 1$ to obtain

$$C_3\left(\sqrt{C_\Pi^4N^{(4-2m)}+N^{-2}+1}\right)\leq C_3[C_\Pi^2N^{2-m}+2]\leq 3C_3=C_\Pi. \tag{4.49}$$

To summarize, combining the above with (4.46), we derive from (4.46) that

$$|1-\xi^{n+1}|\leq C_0\delta t+C_\Pi N^{2-m}$$

under the conditions

$$\delta t\leq\frac{1}{1+2^{k+2}C_0^{k+1}},\quad N\geq 2^{k+2}C_\Pi^{k+1}+1,\quad 1\leq k\leq 5. \tag{4.50}$$

Note that the above implies (4.7), and with $C_3 > \max\{C_1, C_2, 1\}$, it also implies (4.45). The induction process for (4.3) is complete.

We derive from (3.14d) and (4.33) that

$$\|\boldsymbol{u}_N^{n+1}-\bar{\boldsymbol{u}}_N^{n+1}\|_1^2\leq|\eta_k^{n+1}-1|^2\|\bar{\boldsymbol{u}}_N^{n+1}\|_1^2\leq|\eta_k^{n+1}-1|^2C \tag{4.51}$$

and

(4.52)
$$\begin{aligned}\delta t\sum_{q=0}^n\|\boldsymbol{u}_N^{q+1}-\bar{\boldsymbol{u}}_N^{q+1}\|_2^2 &\leq \delta t\sum_{q=0}^n|\eta_k^{q+1}-1|^2\|\bar{\boldsymbol{u}}_N^{q+1}\|_2^2\\ &\leq\max_q|\eta_k^{q+1}-1|^2\delta t\sum_{q=0}^n\|\bar{\boldsymbol{u}}_N^{q+1}\|_2^2\\ &\leq\max_q|\eta_k^{q+1}-1|^2C.\end{aligned}$$

On the other hand, we derive from (4.3) that

$$|\eta_1^{q+1} - 1| \le 2^2 C_0^2 \delta t^2 + 2^2 C_\Pi^2 N^{2(2-m)} \qquad \forall q \le n, \quad k = 1, \tag{4.53a}$$

$$|\eta_k^{q+1} - 1| \le 2^k C_0^k \delta t^k + 2^k C_\Pi^k N^{k(2-m)} \qquad \forall q \le n, \quad k = 2, 3, 4, 5. \tag{4.53b}$$

Therefore, we derive from (4.31), (4.32), (4.51), (4.52), (4.53) and the triangle inequality that

$$\|\boldsymbol{e}^{n+1}\|_1^2 \le \|\bar{\boldsymbol{e}}^{n+1}\|_1^2 + \|\boldsymbol{u}_N^{n+1} - \bar{\boldsymbol{u}}_N^{n+1}\|_1^2$$

and

$$\|\boldsymbol{e}^{q+1}\|_2^2 \le \|\bar{\boldsymbol{e}}^{q+1}\|_2^2 + \|\boldsymbol{u}_N^{q+1} - \bar{\boldsymbol{u}}_N^{q+1}\|_2^2 \quad \forall q \le n,$$

under the condition (4.50) on δt and N. The proof is now complete since we already proved (4.31) and (4.32). ∎

Using exactly the same procedure as above without the spatial discretization, we can prove the following result for the semidiscrete schemes (3.6).

Corollary 1. *Let $d = 2$, $\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_p^2$, and $\boldsymbol{u}$ be the solution of* (1.1). *We assume that $\bar{\boldsymbol{u}}^i$ and $\boldsymbol{u}^i$ $(i = 1, \dots, k-1)$ are computed with a proper initialization procedure such that for $(i = 1, \dots, k-1)$,*

$$\begin{aligned} &\|\bar{\boldsymbol{u}}^i - \boldsymbol{u}(t_i)\|_1,\ \|\boldsymbol{u}^i - \boldsymbol{u}(t_i)\|_1 = O(\delta t^k); \\ &\|\bar{\boldsymbol{u}}^i - \boldsymbol{u}(t_i)\|_2,\ \|\boldsymbol{u}^i - \boldsymbol{u}(t_i)\|_2 = O(\delta t^k), \end{aligned} \qquad i = 1, 2, 3, 4, 5.$$

Let $\bar{\boldsymbol{u}}^{n+1}$ and $\boldsymbol{u}^{n+1}$ be computed with the kth-order scheme (3.6) $(1 \le k \le 5)$, *and*

$$\eta_1^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad \eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k \ (k = 2, 3, 4, 5).$$

Then for any $T > 0$, and $n + 1 \le T/\delta t$ and $\delta t \le \frac{1}{1 + 2^{k+2} C_0^{k+1}}$, we have

$$\|\bar{\boldsymbol{u}}^n - \boldsymbol{u}(\cdot, t^n)\|_1^2,\ \|\boldsymbol{u}^n - \boldsymbol{u}(\cdot, t^n)\|_1^2 \le C \delta t^{2k}$$

and

$$\delta t \sum_{q=0}^{n} \|\bar{\boldsymbol{u}}^{q+1} - \boldsymbol{u}(\cdot, t^{q+1})\|_2^2,\ \delta t \sum_{q=0}^{n} \|\boldsymbol{u}^{q+1} - \boldsymbol{u}(\cdot, t^{q+1})\|_2^2 \le C \delta t^{2k},$$

where the constants C_0 and C are dependent on T, Ω, the $k \times k$ matrix $G = (g_{ij})$ in Lemma 4.3, *and the exact solution $\boldsymbol{u}$ but are independent of δt.*

4.3. Error analysis for the velocity in three dimensions. In the three-dimensional case, it is no longer possible to obtain the global estimates (4.9), (4.10), and (4.11) as in the two-dimensional case. Instead, we shall derive local estimates in analogy to the local existence of strong solution for the three-dimensional Navier–Stokes equations.

Theorem 4.6. *Let $d = 3$, $\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_p^m$ with $m \ge 3$. We assume that* (1.1) *admits a unique strong solution $\boldsymbol{u}$ in $C([0,T]; \boldsymbol{H}_p^1) \cap L^2(0,T; \boldsymbol{H}_p^2)$. We assume* (4.1) *as in Theorem* 2 *and let $\bar{\boldsymbol{u}}_N^{n+1}$ and $\boldsymbol{u}_N^{n+1}$ be computed using the kth-order scheme* (3.14) $(1 \le k \le 5)$, *and*

$$\eta_1^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad \eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k \ (k = 2, 3, 4, 5).$$

Then, there exists $T_* > 0$ such that for $0 < T < T_*$, $n+1 \le T/\delta t$, and $\delta t \le \frac{1}{1+2^{k+2}C_0^{k+1}}$, $N \ge 2^{k+2}C_\Pi^{k+1}+1$, we have

$$\|\bar{\boldsymbol{u}}_N^n - \boldsymbol{u}(\cdot,t^n)\|_1^2,\ \|\boldsymbol{u}_N^n - \boldsymbol{u}(\cdot,t^n)\|_1^2 \le C\delta t^{2k} + CN^{2(1-m)} \tag{4.54}$$

and

$$\delta t \sum_{q=0}^{n} \|\bar{\boldsymbol{u}}_N^{q+1} - \boldsymbol{u}(\cdot,t^{q+1})\|_2^2,\ \delta t \sum_{q=0}^{n} \|\boldsymbol{u}_N^{q+1} - \boldsymbol{u}(\cdot,t^{q+1})\|_2^2 \le C\delta t^{2k} + CN^{2(2-m)}, \tag{4.55}$$

where the constants C_0, C_Π, C are dependent on T, Ω, the $k\times k$ matrix $G=(g_{ij})$ in Lemma 4.3, and the exact solution $\boldsymbol{u}$ but are independent of δt and N.

Proof. The proof follows essentially the same procedure as the proof for Theorem 4.5. However, since we only have the weak version of the stability in Theorem 3.2 and (2.7) is not valid when $d=3$, we can only get a local version of (4.9) and (4.10). To simplify the presentation, we shall only point out below the main differences with the proof for Theorem 4.5.

With $\boldsymbol{u}_0 \in \boldsymbol{H}_p^m$ and the existence of a unique strong solution $\boldsymbol{u}$ in $C([0,T];\boldsymbol{H}_p^1)\cap L^2(0,T;\boldsymbol{H}_p^2)$, regularity results in [42, 13] imply that (4.2) is also valid in the three-dimensional case.

In Step 1, we still assume (4.4) holds and choose δt and N satisfy (4.7). Letting $v_N = -\Delta\bar{\boldsymbol{u}}^{n+1} + \tau_k\Delta\bar{\boldsymbol{u}}^n$ in (3.14a), it follows from Lemma 4.3 that

$$\begin{aligned}
&\sum_{i,j=1}^{k} g_{ij}(\nabla\bar{\boldsymbol{u}}_N^{q+1+i-k}, \nabla\bar{\boldsymbol{u}}_N^{q+1+j-k}) - \sum_{i,j=1}^{k} g_{ij}(\nabla\bar{\boldsymbol{u}}_N^{q+i-k}, \nabla\bar{\boldsymbol{u}}_N^{q+j-k}) \\
&\quad + \left\|\sum_{i=0}^{k}\delta_i\nabla\bar{\boldsymbol{u}}_N^{q+1+i-k}\right\|^2 + \delta t\nu\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|^2 \\
&= \delta t\nu(\Delta\bar{\boldsymbol{u}}_N^{q+1}, \tau_k\Delta\bar{\boldsymbol{u}}_N^q) + \delta t(\mathbf{A}\big((B_k(\boldsymbol{u}_N^q)\cdot\nabla)B_k(\boldsymbol{u}_N^q)\big), -\Delta\bar{\boldsymbol{u}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{u}}_N^q).
\end{aligned} \tag{4.56}$$

We now bound the right-hand side of (4.56). Note that (4.7) implies

$$\frac{1}{2} < 1 - \left(\frac{\delta t^{k-1}}{4} + \frac{N^{k(2-m)+1}}{4}\right) \le |\eta_k^q| \le 1 + \frac{\delta t^{k-1}}{4} + \frac{N^{k(2-m)+1}}{4} < 2\ \forall q \le n.$$

First, we have

$$|\delta t\nu(\Delta\bar{\boldsymbol{u}}_N^{q+1}, \tau_k\Delta\bar{\boldsymbol{u}}_N^q)| \le \delta t\frac{\nu}{2}\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|^2 + \delta t\frac{\nu\tau_k}{2}\|\Delta\bar{\boldsymbol{u}}_N^q\|^2. \tag{4.57}$$

Next, it follows from (2.8) that

$$\begin{aligned}
&|(\mathbf{A}\big((B_k(\boldsymbol{u}_N^q)\cdot\nabla)B_k(\boldsymbol{u}_N^q)\big), -\Delta\bar{\boldsymbol{u}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{u}}_N^q)| \\
&\quad \le C\|B_k(\boldsymbol{u}_N^q)\|_1\|B_k(\nabla\boldsymbol{u}_N^q)\|_{1/2}\|-\Delta\bar{\boldsymbol{u}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{u}}_N^q\| \\
&\quad \le C\|B_k(\boldsymbol{u}_N^q)\|_1\|B_k(\boldsymbol{u}_N^q)\|_1^{1/2}\|B_k(\boldsymbol{u}_N^q)\|_2^{1/2}\|-\Delta\bar{\boldsymbol{u}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{u}}_N^q\| \\
&\quad \le C(\varepsilon)\|B_k(\boldsymbol{u}_N^q)\|_1^3\|B_k(\boldsymbol{u}_N^q)\|_2 + \varepsilon\|-\Delta\bar{\boldsymbol{u}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{u}}_N^q\|^2 \\
&\quad \le C(\varepsilon)\|B_k(\boldsymbol{u}_N^q)\|_1^6 + \varepsilon\|B_k(\boldsymbol{u}_N^q)\|_2^2 + 2\varepsilon\|\Delta\bar{\boldsymbol{u}}_N^{q+1}\|^2 + 2\varepsilon\|\Delta\bar{\boldsymbol{u}}_N^q\|^2.
\end{aligned} \tag{4.58}$$

Now, combining (4.56)–(4.58) and noting that $\boldsymbol{u}_N^q = \eta_k^q \bar{\boldsymbol{u}}_N^q$, we find after dropping some unnecessary terms that

$$
\begin{aligned}
&\sum_{i,j=1}^{k} g_{ij}(\nabla \bar{\boldsymbol{u}}_N^{q+1+i-k}, \nabla \bar{\boldsymbol{u}}_N^{q+1+j-k}) - \sum_{i,j=1}^{k} g_{ij}(\nabla \bar{\boldsymbol{u}}_N^{q+i-k}, \nabla \bar{\boldsymbol{u}}_N^{q+j-k}) \\
&\quad + \delta t \left(\frac{\nu}{2} - 2\varepsilon\right) \|\Delta \bar{\boldsymbol{u}}_N^{q+1}\|^2 \\
&\le \delta t \left(\frac{\nu \tau_k}{2} + 2\varepsilon\right) \|\Delta \bar{\boldsymbol{u}}_N^{q}\|^2 + \varepsilon \delta t \|B_k(\boldsymbol{u}_N^q)\|_2^2 + C(\varepsilon)\delta t \|B_k(\boldsymbol{u}_N^q)\|_1^6 \\
&\le \delta t \left(\frac{\nu \tau_k}{2} + 2\varepsilon\right) \|\Delta \bar{\boldsymbol{u}}_N^{q}\|^2 + 2^2\varepsilon \delta t \|B_k(\bar{\boldsymbol{u}}_N^q)\|_2^2 + 2^6 C(\varepsilon)\delta t \|B_k(\bar{\boldsymbol{u}}_N^q)\|_1^6.
\end{aligned}
\tag{4.59}
$$

Taking the sum of (4.59) for q from $k-1$ to $n-1$, noting that $G = (g_{ij})$ is a symmetric positive definite matrix with the minimum eigenvalue λ_G and $\tau_k < 1$, we can choose ε small enough such that

$$
\begin{aligned}
&\lambda_G \|\bar{\boldsymbol{u}}_N^n\|_1^2 + \frac{\delta t \nu (1-\tau_k)}{4} \sum_{q=0}^{n} \|\Delta \bar{\boldsymbol{u}}_N^q\|^2 \\
&\qquad \le \sum_{i,j=1}^{k} g_{ij}(\nabla \bar{\boldsymbol{u}}^{n+i-k}, \nabla \bar{\boldsymbol{u}}^{n+j-k}) + \frac{\delta t \nu (1-\tau_k)}{4} \sum_{q=0}^{n} \|\Delta \bar{\boldsymbol{u}}_N^q\|^2 \\
&\qquad \le C\delta t \sum_{q=0}^{n-1} \|\bar{\boldsymbol{u}}_N^q\|_1^6 + M_0,
\end{aligned}
$$

where $M_0 > 0$ is a constant only depending on $\bar{\boldsymbol{u}}_N^0, \dots, \bar{\boldsymbol{u}}_N^k$, g_{ij}. If we define ϕ as $\phi(x) = x^6$ and let

$$
0 < T_* < \int_{M_0}^{\infty} dz/\phi(z), \tag{4.60}
$$

then Lemma 4.4 implies that there exist $C_* > 0$ independent of δt such that

$$
\|\bar{\boldsymbol{u}}_N^n\|_1^2 + \delta t \sum_{q=0}^{n} \|\Delta \bar{\boldsymbol{u}}_N^q\|^2 \le C_* \quad \forall n < T_*/\delta t. \tag{4.61}
$$

With (4.61) holding true, we can then prove (4.54) and (4.55) by following the same procedures in Steps 2 and 3 in the proof of Theorem 4.5. ∎

Similarly, we can prove the following result for the semidiscrete scheme (3.6).

COROLLARY 2. *Let* $d = 3$, $\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_p^m$ *with* $m \ge 3$. *We assume that* (1.1) *admits a unique strong solution* $\boldsymbol{u}$ *in* $C([0,T]; \boldsymbol{H}_p^1) \cap L^2(0,T; \boldsymbol{H}_p^2)$. *We assume* (4.1) *as in Theorem* 2, *and let* $\bar{\boldsymbol{u}}^{n+1}$ *and* $\boldsymbol{u}^{n+1}$ *be computed using the kth-order schemes* (3.6), *and*

$$
\eta_1^{n+1} = 1 - (1-\xi^{n+1})^2, \quad \eta_k^{n+1} = 1 - (1-\xi^{n+1})^k \ (k = 2,3,4,5).
$$

Then, there exists $T_* > 0$ *such that for* $0 < T < T_*$, $n+1 \le T/\delta t$, *and* $\delta t \le \frac{1}{1+2^{k+2}C_0^{k+1}}$, $N \ge 2^{k+2}C_\Pi^{k+1} + 1$, *we have*

$$
\|\bar{\boldsymbol{u}}^n - \boldsymbol{u}(\cdot, t^n)\|_1^2,\ \|\boldsymbol{u}^n - \boldsymbol{u}(\cdot, t^n)\|_1^2 \le C\delta t^{2k}
$$

and

$$\delta t\sum_{q=0}^{n}\|\bar{\boldsymbol{u}}^{q+1}-\boldsymbol{u}(\cdot,t^{q+1})\|_2^2,\ \delta t\sum_{q=0}^{n}\|\boldsymbol{u}^{q+1}-\boldsymbol{u}(\cdot,t^{q+1})\|_2^2\le C\delta t^{2k},$$

where T_* is defined in (4.60)*, the constants C_0, C_Π, C are dependent on T_*, Ω, the $k\times k$ matrix $G=(g_{ij})$ in Lemma* 4.3*, and the exact solution $\boldsymbol{u}$ but are independent of δt.*

4.4. Error analysis for the pressure. With the established error estimates for the velocity $\boldsymbol{u}$, the error estimate for the pressure p can be derived directly from (3.7) or (3.15).

We denote

$$e_{pN}^n:=p_N^n-\Pi_N p(\cdot,t^n),\ e_{p\Pi}^n:=\Pi_N p(\cdot,t^n)-p(\cdot,t^n),\ \text{ and } e_p^n=e_{pN}^n+e_{p\Pi}^n.$$

THEOREM 4.7. *Under the same assumptions as in Theorems* 4.5 *and* 4.6*, we have*

$$\|p_N^{n+1}-p(\cdot,t^{n+1})\|^2\le\begin{cases} C\delta t^{2k}+CN^{2(1-m)}\ \forall n\le T/\delta t, & d=2,\\ C\delta t^{2k}+CN^{2(1-m)}\ \forall n\le T_*/\delta t, & d=3,\end{cases}\tag{4.62}$$

and

$$\delta t\sum_{q=0}^{n}\|\nabla(p_N^{n+1}-p(\cdot,t^{n+1}))\|^2\le\begin{cases} C\delta t^{2k}+CN^{2(2-m)}\ \forall n\le T/\delta t, & d=2,\\ C\delta t^{2k}+CN^{2(2-m)}\ \forall n\le T_*/\delta t, & d=3,\end{cases}\tag{4.63}$$

where p_N^{n+1} is computed from (3.15)*, T_* is defined in* (4.60)*, and C is a constant independent of δt and N.*

Proof. From (3.15), we can write the error equation for p_N^{n+1} as

$$\left(\nabla e_p^{q+1},\nabla v_N\right)=\left(\boldsymbol{u}_N^{q+1}\cdot\nabla\boldsymbol{u}_N^{q+1}-\boldsymbol{u}(t^{q+1})\cdot\nabla\boldsymbol{u}(t^{q+1}),\nabla v_N\right)\forall v_N\in S_N,\ \forall q+1\le n.\tag{4.64}$$

To prove (4.62), we set $v_N=\Delta^{-1}e_{pN}^{q+1}$ in (4.64) to obtain

$$\begin{aligned}\|e_{pN}^{q+1}\|^2=&\left(\boldsymbol{u}_N^{q+1}\cdot\nabla[\boldsymbol{u}_N^{q+1}-\boldsymbol{u}(t^{q+1})],\Delta^{-\frac12}e_{pN}^{q+1}\right)\\&-\left([\boldsymbol{u}(t^{q+1})-\boldsymbol{u}_N^{q+1}]\cdot\nabla\boldsymbol{u}(t^{q+1}),\Delta^{-\frac12}e_{pN}^{q+1}\right).\end{aligned}\tag{4.65}$$

We can bound the right-hand side of (4.65) by using (2.9), the stability result Theorem 3.2, and error analysis for the velocity, namely, we can obtain

$$\begin{aligned}\left|\left(\boldsymbol{u}_N^{q+1}\cdot\nabla[\boldsymbol{u}_N^{q+1}-\boldsymbol{u}(t^{q+1})],\Delta^{-\frac12}e_{pN}^{q+1}\right)\right|&\le C(\varepsilon)\|\boldsymbol{u}_N^{q+1}\|_1^2\|\boldsymbol{e}^{q+1}\|_1^2+\varepsilon\|\nabla e_{pN}^{q+1}\|^2\\&\le C(\varepsilon)(\delta t^{2k}+N^{2(1-m)})+\varepsilon\|e_{pN}^{q+1}\|^2\end{aligned}\tag{4.66}$$

and

$$\begin{aligned}\left|-\left([\boldsymbol{u}(t^{q+1})-\boldsymbol{u}_N^{q+1}]\cdot\nabla\boldsymbol{u}(t^{q+1}),\Delta^{-\frac12}e_{pN}^{q+1}\right)\right|&\le C(\varepsilon)\|\boldsymbol{u}(t^{q+1})\|_1^2\|\boldsymbol{e}^{q+1}\|_1^2+\varepsilon\|\nabla e_{pN}^{q+1}\|^2\\&\le C(\varepsilon)(\delta t^{2k}+N^{2(1-m)})+\varepsilon\|e_{pN}^{q+1}\|^2.\end{aligned}\tag{4.67}$$

Combining (4.65)–(4.67) with $\varepsilon = \frac{1}{4}$ we obtain

$$\|e_{pN}^{q+1}\|^2 \le C\delta t^{2k} + CN^{2(1-m)} \quad \forall q \le n. \tag{4.68}$$

To prove (4.63), we set $v_N = e_{pN}^{q+1}$ in (4.64) to obtain

$$\begin{aligned}\|\nabla e_{pN}^{q+1}\|^2 &= \Big(\boldsymbol{u}_N^{q+1}\cdot\nabla[\boldsymbol{u}_N^{q+1} - \boldsymbol{u}(t^{q+1})], \nabla e_{pN}^{q+1}\Big)\\ &\quad - \Big([\boldsymbol{u}(t^{q+1}) - \boldsymbol{u}_N^{q+1}]\cdot\nabla\boldsymbol{u}(t^{q+1}), \nabla e_{pN}^{q+1}\Big).\end{aligned} \tag{4.69}$$

Again, we can bound the right-hand side of (4.69) in a similar fashion as in (4.66)–(4.67), namely, we can obtain

$$\begin{aligned}\left|\Big(\boldsymbol{u}_N^{q+1}\cdot\nabla[\boldsymbol{u}_N^{q+1} - \boldsymbol{u}(t^{q+1})], \nabla e_{pN}^{q+1}\Big)\right| &\le C(\varepsilon)\|\boldsymbol{u}_N^{q+1}\|_1^2\|\boldsymbol{e}^{q+1}\|_2^2 + \varepsilon\|\nabla e_{pN}^{q+1}\|^2\\ &\le C(\varepsilon)\|\boldsymbol{e}^{q+1}\|_2^2 + \varepsilon\|\nabla e_{pN}^{q+1}\|^2\end{aligned} \tag{4.70}$$

and

$$\begin{aligned}\left|-\Big([\boldsymbol{u}(t^{q+1}) - \boldsymbol{u}_N^{q+1}]\cdot\nabla\boldsymbol{u}(t^{q+1}), \nabla e_{pN}^{q+1}\Big)\right| &\le C(\varepsilon)\|\boldsymbol{u}(t^{q+1})\|_2^2\|\boldsymbol{e}^{q+1}\|_1^2 + \varepsilon\|\nabla e_{pN}^{q+1}\|^2\\ &\le C(\varepsilon)(\delta t^{2k} + N^{2(1-m)}) + \varepsilon\|\nabla e_{pN}^{q+1}\|^2.\end{aligned} \tag{4.71}$$

Combining (4.69)–(4.71) with $\varepsilon = \frac{1}{4}$, we obtain

$$\|\nabla e_{pN}^{q+1}\|^2 \le C\|\boldsymbol{e}^{q+1}\|_2^2 + C\delta t^{2k} + CN^{2(1-m)} \quad \forall q \le n. \tag{4.72}$$

Taking the sum of (4.25) for q from 0 to n and multiplying δt on both sides, we arrive at

$$\delta t\sum_{q=0}^{n}\|\nabla e_{pN}^{q+1}\|^2 \le C\delta t\sum_{q=0}^{n}\|\boldsymbol{e}^{q+1}\|_2^2 + C\delta t^{2k} + CN^{2(1-m)}. \tag{4.73}$$

Now, with the estimates on $\|\boldsymbol{e}^n\|_2^2$ in Theorem 4.5 or Theorem 4.6, (4.73) leads to

$$\delta t\sum_{q=0}^{n}\|\nabla e_{pN}^{q+1}\|^2 \le C\delta t^{2k} + CN^{2(2-m)}. \tag{4.74}$$

Finally, we can obtain (4.62) and (4.63) from (4.68), (4.74), and

$$\|\nabla e_{p\Pi}^{q}\|^2 \le CN^{2(1-m)}.$$

□

Similarly, we can derive the following results for the semidiscrete scheme (3.6).

Corollary 3. *Under the same assumptions as in Corollaries* 1 *and* 2, *we have*

$$\|p^{n+1} - p(\cdot, t^{n+1})\|^2 \le \begin{cases} C\delta t^{2k}\ \forall n \le T/\delta t, & d = 2,\\ C\delta t^{2k}\ \forall n \le T_*/\delta t, & d = 3,\end{cases}$$

and

$$\delta t\sum_{q=0}^{n}\|\nabla(p^{q+1} - p(\cdot, t^{n+1}))\|^2 \le \begin{cases} C\delta t^{2k}\ \forall n \le T/\delta t, & d = 2,\\ C\delta t^{2k}\ \forall n \le T_*/\delta t, & d = 3,\end{cases}$$

where p^{n+1} *is computed from* (3.7), T_* *is defined in* (4.60), *and* C *is a constant independent of* δt.

5. Concluding remarks. We considered numerical approximations of the incompressible Navier–Stokes equations with periodic boundary conditions for which the pressure can be explicitly eliminated, allowing us to construct very efficient IMEX type schemes using Fourier–Galerkin approximation in space. Our high-order semidiscrete-in-time and fully discrete IMEX schemes are based on an SAV approach which enables us to derive uniform bounds for the numerical solution without any restriction on time step size. We also take advantage of an additional energy dissipation law (2.6), which is valid only for the two-dimensional Navier–Stokes equations with periodic boundary conditions, leading to a uniform bound in H^1-norm, instead of the usual L^2-norm. By using these uniform bounds and a delicate induction process, we derived global error estimates in $l^\infty(0,T;H^1)\cap l^2(0,T;H^2)$ in the two-dimensional case as well as local error estimates in $l^\infty(0,T;H^1)\cap l^2(0,T;H^2)$ in the three-dimensional case for our semidiscrete-in-time and fully discrete IMEX schemes up to fifth-order. We also validated our schemes with manufactured exact solutions and with the double shear layer problem. Our numerical results for the double shear layer problem indicate that the SAV approach can effectively prevent the numerical solution from blowing up and that higher-order schemes are preferable for flows with complex structures such as the double shear layer problem with thin layers.

To the best of our knowledge, our numerical schemes are the first unconditionally stable high-order IMEX type schemes for Navier–Stokes equations without any restriction on time step size, and our error estimates are the first for any IMEX type scheme for the Navier–Stokes equations in the three-dimensional case.

While the stability results can be extended to similar schemes for the Navier–Stokes equations with nonperiodic boundary conditions, it is nontrivial to carry out the corresponding error analysis, which will be left as a subject of future endeavors.

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