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STABILITY AND ERROR ANALYSIS OF A SECOND-ORDER CONSISTENT SPLITTING SCHEME FOR THE NAVIER–STOKES EQUATIONS*

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Abstract. We present in this paper a new second-order consistent splitting scheme for the Navier–Stokes equations with no-slip boundary conditions based on (i) the Taylor expansions at time t^{n+5} which offer better stability than the usual expansion at time t^{n+1}, and (ii) the generalized scalar auxiliary variable (GSAV) approach which allows us to treat the nonlinear term explicitly while maintaining unconditional stability. We prove rigorously its unconditional stability in a strong norm for the time dependent Stokes equations by using a clever energy argument. Using this strong stability result combined with a weak stability offered by the GSAV approach, we are able to establish, for the first time, a global-in-time optimal error estimate in two dimensions and a local-in-time error estimate in three dimensions for a second-order consistent method for the Navier–Stokes equations.

Key words. Navier–Stokes, stability, error analysis, consistent splitting, generalized SAV approach

MSC codes. 65M12, 76D05, 65M15

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1. Introduction.

We consider in this paper the construction and error analysis of a second-order consistent splitting scheme for the following Navier–Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{1.1b}$$

with suitable initial conditions in a rectangular domain $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ and no-slip boundary condition $\boldsymbol{u}=0$ on $\partial\Omega$, and where $\boldsymbol{f}$ is an external force.

Due to its importance in applications and in analysis, there is an enormous amount of work devoted to the numerical approximation of the Navier–Stokes equations. These numerical methods can be roughly classified into two categories: coupled approach with a mixed formulation [1, 4, 5, 15, 22] and decoupled approach which includes projection type methods (including the pressure-correction and the velocity-correction methods) [2, 8, 9, 10, 11, 14, 20, 21, 24, 26, 30, 31], and the consistent splitting method [6, 13, 28, 32] (see also the gauge method [3, 19]). We refer the reader to [7] for a review on the decoupled approach, and we would like to point out that the projection type schemes suffer from a splitting error which prevents

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them from achieving full-order accuracy in strong norms, while the consistent splitting schemes do not lose accuracy.

Despite being used frequently in practice, the stability and error analysis of consistent splitting schemes is far from satisfactory. Guermond and Shen [6] (resp., Johnston and Liu [13]) established a stability result for the semidiscrete first-order consistent splitting scheme for the time dependent Stokes equations with no-slip boundary conditions (resp., with periodic-nonperiodic boundary conditions in a periodic channel). A series of work was carried out by Liu, Liu and Pego [16, 17] for the first-order version of the consistent splitting scheme for the nonlinear Navier–Stokes equations and established local-in-time stability and error estimates. Unfortunately, it appears that these techniques cannot be extended to the second-order case.

We recall that the second-order consistent splitting scheme for (1.1) introduced in [6] (see also [13]) reads as follows:

(1.2)
$$\frac{1}{2\delta t}(3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}) - \nu\Delta\boldsymbol{u}^{n+1} + \nabla(2p^n - p^{n-1}) = 2\boldsymbol{g}^n - \boldsymbol{g}^{n-1}, \quad \boldsymbol{u}|_{\partial\Omega} = 0,$$

$$(\nabla p^{n+1}, \nabla q) = (\boldsymbol{g}^{n+1} - \nu\nabla\times\nabla\times\boldsymbol{u}^{n+1}, \nabla q) \quad \forall q \in H^1(\Omega), \tag{1.3}$$

where $\boldsymbol{g} = \boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}$. However, it has been an open problem since its inception whether the above second-order consistent splitting scheme is unconditionally stable. The main difficulty is caused by the second-order extrapolation of the pressure term. This is reminiscent of the third-order pressure-correction scheme [25] in which the second-order explicit extrapolation of the pressure also prevents us from establishing its unconditionally stability. Therefore, new ideas are needed for dealing with the second-order consistent splitting scheme.

Note that the usual backward differentiation formula (BDF) and Adams–Bashforth extrapolation, including those used in (1.2), are based on the Taylor expansions at time t^{n+1}. But this leads to apparently insurmountable difficulty in dealing with the pressure extrapolation. In order to overcome this essential difficulty, it appears that we have to design a new scheme which possesses the same advantages as the scheme (1.2)–(1.3) while allowing us to establish unconditional stability. This leads us to consider a more general BDF formula where the Taylor expansions are performed at time t^{n+k} with k being an undetermined parameter. It turns out that as k increases, the stability region increases at the expense of increased truncation errors, and with a proper choice of k, the extra stability afforded by the new scheme will allow us to establish unconditional stability.

The main purpose of this paper is to construct a new second-order consistent splitting scheme for the Navier–Stokes equations with no-slip boundary conditions, to prove rigorously its unconditional stability, and to carry out an error analysis. Our main contributions include the following:

- We construct a new second-order consistent splitting scheme based on the Taylor expansion at time t^{n+5} which offers better stability than the usual scheme (1.2)–(1.3), and establish rigorously unconditional stability (in a strong norm $l^2(H^2)\cap l^\infty(H^1)$) by using a clever energy argument for the time dependent Stokes equations.
- For the Navier–Stokes equations, We employ the generalized scalar auxiliary variable (GSAV) approach which allows us to treat the nonlinear term explicitly while maintaining unconditional stability (in a weaker norm $l^2(H^1)\cap l^\infty(L^2)$).

- We establish a global-in-time optimal error estimate in two dimensions (2D) and a local-in-time error estimate in three dimensions (3D) for the Navier–Stokes equations.

To the best of our knowledge, this is the first rigorous stability and error analysis for any second-order consistent splitting scheme for the Navier–Stokes equations. The new idea (constructing a scheme with Taylor expansion at time t^{n+k}) introduced in this paper could be used to construct schemes with better stability in other situations, in particular, to explore the construction of higher-order unconditional stable consistent splitting schemes for the Navier–Stokes equations.

The rest of the paper is organized as follows. In the next section, we provide some preliminaries to be used in what follows. In section 3, we construct a new consistent splitting scheme for the time dependent Stokes equations and prove its unconditional stability in a strong norm. Then, in section 4, we couple the second-order consistent splitting scheme with a GSAV approach for the Navier–Stokes equations, and prove its unconditional stability in a weaker norm. In section 5, we present detailed error analysis. In the final section, we provide a numerical example to validate the accuracy of our scheme, and conclude with a few remarks.

2. Preliminaries. We first introduce some notation. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm in $L^2(\Omega)$, and denote

$$\mathbf{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \nabla \cdot \boldsymbol{v} = 0\}, \quad \mathbf{V} = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \nabla \cdot \boldsymbol{v} = 0\}.$$

Let V be a Banach space; we shall also use the standard notation $L^p(0,T;V)$ and $C([0,T];V)$. To simplify the notation, we often omit the spatial dependence for the exact solution u, i.e., $u(x,t)$ is often denoted by $u(t)$. We shall use boldface letters to denote vectors and vector spaces, and use C to denote a generic positive constant independent of the discretization parameters.

Next, we define the trilinear form $b(\cdot,\cdot,\cdot)$ by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x}.$$

In particular, we have

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \quad \forall \boldsymbol{u} \in \mathbf{H},\, \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega),$$

which implies

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0 \quad \forall \boldsymbol{u} \in \mathbf{H},\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{2.1}$$

Using the Hölder inequality and the Sobolev inequality, we have [29]

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le c\|\nabla\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{v}\|_2^{1/2}\|\nabla\boldsymbol{v}\|^{1/2}\|\boldsymbol{w}\|, \quad d=2; \tag{2.2}$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1^{1/2}\|\boldsymbol{v}\|_2^{1/2}\|\boldsymbol{w}\|, \quad d=3. \tag{2.3}$$

We also frequently use the following inequalities [29]:

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le \begin{cases} c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_0\|\boldsymbol{w}\|_1, \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_0, \\ c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_0, \\ c\|\boldsymbol{u}\|_0\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_1, \end{cases} \quad d \le 4. \tag{2.4}$$

We will frequently use the following two discrete versions of the Gronwall lemma.

LEMMA 1 (discrete Gronwall lemma 1 [27]). *Let y^k, h^k, g^k, f^k be four nonnegative sequences satisfying*

$$y^n + \delta t \sum_{k=0}^{n} h^k \le B + \delta t \sum_{k=0}^{n} (g^k y^k + f^k) \ \ \text{with} \ \ \delta t \sum_{k=0}^{T/\delta t} g^k \le M \quad \forall\, 0 \le n \le T/\delta t.$$

We assume $\delta t\, g^k < 1$ for all k and let $\sigma = \max_{0\le k\le T/\delta t}(1-\delta t g^k)^{-1}$. Then

$$y^n + \delta t \sum_{k=1}^{n} h^k \le \exp(\sigma M)\left(B + \delta t \sum_{k=0}^{n} f^k\right) \quad \forall\, n \le T/\delta t.$$

LEMMA 2 (discrete Gronwall lemma 2 [23]). *Let a_n, b_n, c_n, and d_n be four nonnegative sequences satisfying*

$$a_m + \tau \sum_{n=1}^{m} b_n \le \tau \sum_{n=0}^{m-1} a_n d_n + \tau \sum_{n=0}^{m-1} c_n + C,\, m \ge 1,$$

where C and τ are two positive constants. Then

$$a_m + \tau \sum_{n=1}^{m} b_n \le \exp\left(\tau \sum_{n=0}^{m-1} d_n\right)\left(\tau \sum_{n=0}^{m-1} c_n + C\right),\, m \ge 1.$$

We also recall the following lemma [18] which will be used to prove local error estimates in the three-dimensional case.

LEMMA 3. *Let $\Phi : (0,\infty) \to (0,\infty)$ be continuous and increasing, and let $M > 0$. Given T_* such that $0 < T_* < \int_M^\infty dz/\Phi(z)$, there exists $C_* > 0$ independent of $\delta t > 0$ with the following property. Suppose that quantities z_n, $w_n \ge 0$ satisfy*

$$z_n + \sum_{k=0}^{n-1} \delta t w_k \le y_n := M + \sum_{k=0}^{n-1} \delta t \Phi(z_k) \quad \forall\, n \le n_*,$$

with $n_* \delta t \le T_*$. Then $y_{n_*} \le C_*$.

In order to establish an unconditional stability result for (3.5)–(3.6), we need the following result about the Stokes pressure introduced in [16]. For any $\boldsymbol{u} \in H^2(\Omega, \mathbb{R}^N)$, the Stokes pressure $p_s = P_s(\boldsymbol{u})$ is defined as

$$\nabla p_s(\boldsymbol{u}) = (\Delta \mathcal{P} - \mathcal{P}\Delta)\boldsymbol{u}, \tag{2.5}$$

where $\mathcal{P}$ is the Leray–Helmholtz projection operator onto divergence-free fields with zero normal component, providing the Helmholtz decomposition $\boldsymbol{u} = \mathcal{P}\boldsymbol{u} + \nabla\phi$, where

$$\big(\mathcal{P}\boldsymbol{u}, \nabla q\big) = \big(\boldsymbol{u} - \nabla\phi, \nabla q\big) = 0 \quad \forall q \in H^1(\Omega). \tag{2.6}$$

Then it is proved in [16] that the following lemma holds.

LEMMA 4. *Let $\Omega \subset \mathbb{R}^N (N \ge 2)$ be a connected bounded domain with C^3 boundary. Then for any $\varepsilon > 0$, there exists $C \ge 0$ such that for all vector fields $\boldsymbol{u} \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$,*

$$\int_\Omega |(\Delta\mathcal{P} - \mathcal{P}\Delta)\boldsymbol{u}|^2 \le \left(\frac{1}{2} + \varepsilon\right)\int_\Omega |\Delta \boldsymbol{u}|^2 + C\int_\Omega |\nabla \boldsymbol{u}|^2. \tag{2.7}$$

3. A provable unconditionally stable second-order consistent splitting scheme for the time dependent Stokes problem. We shall first present a more general second-order BDF scheme based on the Taylor expansion at time t^{n+k} and show that its stability region increases with k, and then apply it with $k = 5$ to construct an unconditionally stable second-order consistent splitting scheme for the time dependent Stokes problem.

3.1. A more general second-order BDF scheme. Denote $t^n = n\delta t$; then it follows from the Taylor expansion at time t^{n+k} that

$$\frac{(2k+1)\phi(t^{n+1})-4k\phi(t^n)+(2k-1)\phi(t^{n-1})}{2\delta t} = \phi'(t^{n+k})+\frac{1-3k^2}{6}\phi'''(t^{n+k})\delta t^2+O(\delta t^3) \tag{3.1}$$

and

$$k\phi(t^{n+1}) - (k-1)\phi(t^n) = \phi(t^{n+k}) - \frac{k(k-1)}{2}\phi''(t^{n+k})\delta t^2 + O(\delta t^3). \tag{3.2}$$

Hence, for the test equation $\phi_t = \lambda\phi$, by performing the Taylor expansions at t^{n+k}, a more general second-order BDF method can be written as

$$\frac{(2k+1)\phi^{n+1} - 4k\phi^n + (2k-1)\phi^{n-1}}{2\delta t} = \lambda(k\phi^{n+1} - (k-1)\phi^n). \tag{3.3}$$

Note that with $k = 1$, it reduces to the usual second-order BDF scheme. It is well known that the usual second-order BDF method (with Taylor expansion at t^{n+1}) is A-stable. In order to study the stability region for $k \neq 1$, we set $\phi^n = \gamma^n$ and $z = \lambda\delta t$ in (3.3) to obtain its characteristic polynomial

$$(2k + 1 - 2kz)\gamma^2 + (2(k-1)z - 4k)\gamma + (2k - 1) = 0. \tag{3.4}$$

Then the region of absolute stability of method (3.3) is the set of all $z \in \mathbb{C}$ such that (3.4) holds for all $|\gamma| < 1$. In Figure 1, we plot the stability regions of the general BDF2 type method (3.3) for $k = 1, 3, 5$. We observe that the general BDF2 method is still A-stable for $k > 1$, and more importantly, the stability region increases as we increase k, at the expense of slightly increased truncation error.

3.2. A new second-order consistent splitting scheme for the time dependent Stokes problem. We now apply the general second-order BDF formula (3.3) to the time dependent Stokes problem. More precisely, for any positive integer k, a general second-order BDF type scheme with explicit treatment of the pressure for the time dependent Stokes equation is as follows:

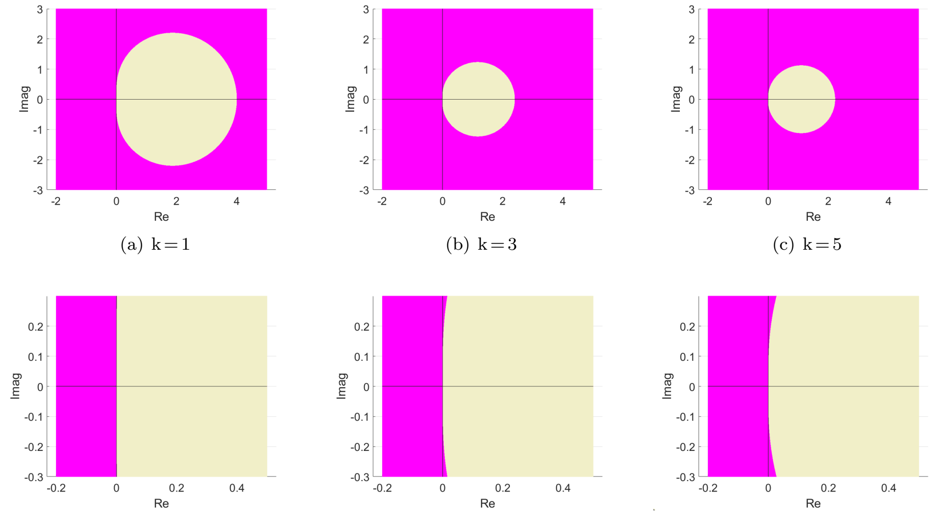
$$\frac{1}{2\delta t}((2k+1)\boldsymbol{u}^{n+1} - 4k\boldsymbol{u}^n + (2k-1)\boldsymbol{u}^{n-1}) - \nu\Delta(k\boldsymbol{u}^{n+1} - (k-1)\boldsymbol{u}^n) + \nabla((k+1)p^n - kp^{n-1}) = 0, \tag{3.5}$$

$$(\nabla p^{n+1}, \nabla q) = -\nu(\nabla\times\nabla\times\boldsymbol{u}^{n+1}, \nabla q) \quad \forall q \in H^1(\Omega), \tag{3.6}$$

One can easily check that the above is a second-order approximation to the time dependent Stokes equation at time t^{n+k} thanks to (3.1)–(3.2) and the fact that

$$(k+1)p(t^n) - kp(t^{n-1}) = p(t^{n+k}) - \frac{k(k+1)}{2}p''(t^{n+k})\delta t^2 + O(\delta t^3).$$

We observe that with $k = 1$, it reduces to the usual second-order consistent splitting scheme (1.2)–(1.3) (in the absence of $\boldsymbol{f}$ and a nonlinear term).



(a) k = 1 (b) k = 3 (c) k = 5

(d) k = 1, zoom in around the origin (e) k = 3, zoom in around the origin (f) k = 5, zoom in around the origin

FIG. 1. *The pink parts show the region of absolute stability of the general BDF2 type method with Taylor expansion at $n+k$, $k=1,3,5$. (Color figure available online.)*

THEOREM 5. *Assuming $\boldsymbol{u}^1$ is computed by using the first-order consistent splitting scheme* [6], *then the scheme* (3.5)–(3.6) *with $k=5$ is unconditionally stable in the sense that*

$$\|\nabla \boldsymbol{u}^{n+1}\|^2 + \delta t\sum_{i=0}^{n}\|\Delta \boldsymbol{u}^{i+1}\|^2 + \delta t\sum_{i=0}^{n}\|\nabla(6p^{i+1}-5p^i)\|^2 \le C, \tag{3.7}$$

where C is a constant independent of the time step δt and n.

Proof. Lemma 4 played an important role in the stability and error analysis for the first-order scheme [16, 17]. However, it appears not possible to extend the analysis to the usual second-order scheme with $k=1$ in (4.1a). More refined analysis below shows that this is due to the fact that the ratio between the coefficients of p^{n-1} and p^n, $\frac{k}{k+1}$, is too small, and by increasing k to 5, we will be able to establish the desired stability result.

Taking the inner product of (3.5) with $-\Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)$, we deal with the three terms as follows:

$$\begin{aligned}
&\big(-\nu\Delta(k\boldsymbol{u}^{n+1}-(k-1)\boldsymbol{u}^n), -\Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)\big)\\
&=\nu\left(\frac{k-1}{k}\Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)+\frac{1}{k}\Delta\boldsymbol{u}^{n+1}, \Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)\right)\\
&=\nu\left(\frac{k-1}{k}\Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n), \Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)\right)+\frac{\nu}{k}\big(\Delta\boldsymbol{u}^{n+1},\Delta\boldsymbol{u}^{n+1}\big)\\
&\quad+\frac{\nu}{k}\big(\Delta\boldsymbol{u}^{n+1},\Delta(k\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)\big)\\
&=\nu\frac{k-1}{k}\|\Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)\|^2+\frac{\nu}{k}\|\Delta\boldsymbol{u}^{n+1}\|^2\\
&\quad+\frac{\nu}{2}\big(\|\Delta\boldsymbol{u}^{n+1}\|^2-\|\Delta\boldsymbol{u}^n\|^2+\|\Delta\boldsymbol{u}^{n+1}-\Delta\boldsymbol{u}^n\|^2\big).
\end{aligned} \tag{3.8}$$

For the pressure term,

$$
(3.9)\qquad \begin{aligned}&\big(\nabla((k+1)p^n-kp^{n-1}),\Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)\big)\\ &\le \frac{1}{2\nu}\|\nabla((k+1)p^n-kp^{n-1})\|^2+\frac{\nu}{2}\|\Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)\|^2.\end{aligned}
$$

A key step is to deal with the first term in the above using Lemma 4. We recall from [16] that

$$
(3.10)\qquad (\nabla p_s(\boldsymbol{u}),\nabla q)=-(\nabla\times\nabla\times\boldsymbol{u},\nabla q),
$$

and it follows from (3.6) that

(3.11)

$$
\big(\nabla((k+1)p^n-kp^{n-1}),\nabla q\big)=-\nu\big(\nabla\times\nabla\times((k+1)\boldsymbol{u}^n-k\boldsymbol{u}^{n-1}),\nabla q\big)\quad \forall q\in H^1(\Omega).
$$

Taking $q=(k+1)p^n-kp^{n-1}$ in (3.11) and in (3.10), we find from (3.10) with $\boldsymbol{u}=(k+1)\boldsymbol{u}^n-k\boldsymbol{u}^{n-1}$ that

$$
(3.12)\qquad \|\nabla((k+1)p^n-kp^{n-1})\|\le\nu\|\nabla p_s((k+1)\boldsymbol{u}^n-k\boldsymbol{u}^{n-1})\|.
$$

Now, we can use (3.12) and (2.7) to bound the first term as follows:

(3.13)

$$
\begin{aligned}\frac{1}{2\nu}\|\nabla((k+1)p^n-kp^{n-1})\|^2&\le\frac{\nu}{2}\|\nabla p_s((k+1)\boldsymbol{u}^n-k\boldsymbol{u}^{n-1})\|^2\\ &\le\nu\left(\frac14+\varepsilon\right)\|\Delta((k+1)\boldsymbol{u}^n-k\boldsymbol{u}^{n-1})\|^2+C\nu\|\nabla((k+1)\boldsymbol{u}^n-k\boldsymbol{u}^{n-1})\|^2.\end{aligned}
$$

We observe from (3.8)–(3.13) that to ensure stability, we need

$$
(3.14)\qquad \frac{k-1}{k}-\frac12\ge\frac14+\varepsilon,
$$

which implies that $k>4$.

It remains to deal with the last term:

$$
(3.15)\qquad \frac{1}{2\delta t}\big((2k+1)\boldsymbol{u}^{n+1}-4k\boldsymbol{u}^n+(2k-1)\boldsymbol{u}^{n-1},-\Delta((k+1)\boldsymbol{u}^{n+1}-k\boldsymbol{u}^n)\big).
$$

A key step is to split the above in telescoping terms. To this end, we fix $k=5$. With a clever maneuvering, one can verify that the following equality holds:

(3.16)

$$
\begin{aligned}&\big(11\boldsymbol{u}^{n+1}-20\boldsymbol{u}^n+9\boldsymbol{u}^{n-1},-\Delta(6\boldsymbol{u}^{n+1}-5\boldsymbol{u}^n)\big)\\
&=\big(\nabla(11\boldsymbol{u}^{n+1}-20\boldsymbol{u}^n+9\boldsymbol{u}^{n-1}),\nabla(5\boldsymbol{u}^{n+1}-4\boldsymbol{u}^n)\big)\\
&\quad+\big(\nabla(11\boldsymbol{u}^{n+1}-20\boldsymbol{u}^n+9\boldsymbol{u}^{n-1}),\nabla(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\big)\\
&=\frac{1}{10}(\|\nabla\boldsymbol{u}^{n+1}\|^2-\|\nabla\boldsymbol{u}^n\|^2)+\left\|\frac{9\sqrt{10}}{5}\nabla\boldsymbol{u}^{n+1}-\frac{\sqrt{90}}{2}\nabla\boldsymbol{u}^n\right\|^2\\
&\quad-\left\|\frac{9\sqrt{10}}{5}\nabla\boldsymbol{u}^{n}-\frac{\sqrt{90}}{2}\nabla\boldsymbol{u}^{n-1}\right\|^2\\
&\quad+\left\|\frac{\sqrt{90}}{2}\nabla\boldsymbol{u}^{n+1}-\sqrt{90}\nabla\boldsymbol{u}^n+\frac{\sqrt{90}}{2}\nabla\boldsymbol{u}^{n-1}\right\|^2\\
&\quad+\frac{13}{2}\|\nabla(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\|^2-\frac92\|\nabla(\boldsymbol{u}^n-\boldsymbol{u}^{n-1})\|^2+\frac92\|\nabla(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1})\|^2.\end{aligned}
$$

Note that the above can be derived by assuming a desired form and then using the method of undetermined coefficients to find a set of suitable coefficients. Summing up $2\delta t((3.8)+(3.9)+(3.13))$ and (3.16), after dropping some unnecessary terms, we find

$$
\begin{aligned}
&\frac{1}{10}(\|\nabla \boldsymbol{u}^{n+1}\|^2 - \|\nabla \boldsymbol{u}^n\|^2) + \left\|\frac{9\sqrt{10}}{5}\nabla \boldsymbol{u}^{n+1} - \frac{\sqrt{90}}{2}\nabla \boldsymbol{u}^n\right\|^2 \\
&\quad - \left\|\frac{9\sqrt{10}}{5}\nabla \boldsymbol{u}^{n} - \frac{\sqrt{90}}{2}\nabla \boldsymbol{u}^{n-1}\right\|^2 \\
&\quad + \frac{13}{2}\left\|\nabla(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\right\|^2 - \frac{9}{2}\left\|\nabla(\boldsymbol{u}^{n}-\boldsymbol{u}^{n-1})\right\|^2 \\
&\quad + \frac{8\nu\delta t}{5}\left\|\Delta(6\boldsymbol{u}^{n+1}-5\boldsymbol{u}^n)\right\|^2 + \frac{2\nu\delta t}{5}\left\|\Delta \boldsymbol{u}^{n+1}\right\|^2 \\
&\le 2\nu\delta t\left(\frac{1}{4}+\varepsilon\right)\left\|\Delta(6\boldsymbol{u}^{n}-5\boldsymbol{u}^{n-1})\right\|^2 + 2C\nu\delta t\left\|\nabla(6\boldsymbol{u}^{n}-5\boldsymbol{u}^{n-1})\right\|^2 \\
&\quad + \nu\delta t\left\|\Delta(6\boldsymbol{u}^{n+1}-5\boldsymbol{u}^n)\right\|^2 .
\end{aligned}
\tag{3.17}
$$

Now, we can choose $\varepsilon = \frac{1}{20}$ and take the sum of n from 1 to $m \le \frac{T}{\delta t} - 1$ on (3.17) to obtain

$$
\begin{aligned}
\|\nabla \boldsymbol{u}^{m+1}\|^2 + 4\nu\delta t\sum_{n=1}^{m}\|\Delta \boldsymbol{u}^{n+1}\|^2 &\le C\nu\delta t\sum_{n=1}^{m}\|\nabla(6\boldsymbol{u}^n - 5\boldsymbol{u}^{n-1})\|^2 + C_I \\
&\le C\nu\delta t\sum_{n=0}^{m}\|\nabla \boldsymbol{u}^n\|^2 + C_{II},
\end{aligned}
$$

where C_I is a constant depending on $\|\nabla \boldsymbol{u}^0\|^2$, $\|\nabla \boldsymbol{u}^1\|^2$, and $\delta t\|\Delta \boldsymbol{u}^1\|^2$. Since $\boldsymbol{u}^1$ is computed by using the first-order consistent scheme, it is easy to see that $\|\nabla \boldsymbol{u}^1\|^2$ and $\delta t\|\Delta \boldsymbol{u}^1\|^2$ can be bounded by $\|\nabla \boldsymbol{u}^0\|^2$. Hence, C_{II} only depends on $\boldsymbol{u}^0$. We can then obtain the desired bound on the velocity by applying Lemma 2 to the above. Finally the bound on the pressure can be derived from (3.13). ∎

4. The BDF2-SAV schemes and stability results.

In this section, we construct second-order GSAV schemes for the Navier–Stokes equation with no-slip boundary conditions based on the general BDF2 type formulas, the GSAV approach for general dissipative systems [12], and the consistent splitting schemes [6], and establish a weak stability result.

4.1. A general form of BDF2-GSAV schemes.

In order to apply the GSAV approach, we introduce an SAV, $r(t) = E(\boldsymbol{u}(t)) + \bar{C}$, with $E(\boldsymbol{u}(t)) = \frac{1}{2}\|\boldsymbol{u}(t)\|^2$, $\bar{C} > 0$, and the condition on $\bar{C}$ is stated in Theorem 6 below. Then by combining the GSAV approach for general dissipative systems in [12] and the consistent splitting schemes in [6], using (3.1) and (3.2), we construct the general form of second-order schemes for (1.1) as

$$
\frac{(2k+1)\bar{\boldsymbol{u}}^{n+1} - 4k\bar{\boldsymbol{u}}^n + (2k-1)\bar{\boldsymbol{u}}^{n-1}}{2\delta t} - \nu\Delta(k\bar{\boldsymbol{u}}^{n+1} - (k-1)\bar{\boldsymbol{u}}^n) + \hat{\boldsymbol{u}}^n\cdot\nabla\hat{\boldsymbol{u}}^n + \nabla\hat{p}^n = \boldsymbol{f}^{n+k},
\tag{4.1a}
$$

$$
\left(\nabla p^{n+1}, \nabla q\right) = \left(\boldsymbol{f}^{n+1} - \bar{\boldsymbol{u}}^{n+1}\cdot\nabla\bar{\boldsymbol{u}}^{n+1} - \nu\nabla\times\nabla\times\bar{\boldsymbol{u}}^{n+1}, \nabla q\right),
\tag{4.1b}
$$

$$\frac{r^{n+1}-r^n}{\delta t}=\frac{r^{n+1}}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}}\big(-\nu\|\nabla\bar{\boldsymbol{u}}^{n+1}\|^2+(\boldsymbol{f}^{n+1},\bar{\boldsymbol{u}}^{n+1})\big), \tag{4.1c}$$

$$\xi^{n+1}=\frac{r^{n+1}}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}},\quad \eta^{n+1}=1-(1-\xi^{n+1})^2, \tag{4.1d}$$

$$\boldsymbol{u}^{n+1}=\eta^{n+1}\bar{\boldsymbol{u}}^{n+1}, \tag{4.1e}$$

where $\hat{\boldsymbol{u}}^n=(k+1)\boldsymbol{u}^n-k\boldsymbol{u}^{n-1}$ and $\hat{p}^n=(k+1)p^n-kp^{n-1}$.

4.2. A weak stability result.

THEOREM 6. *Let $\|\boldsymbol{f}(\cdot,t)\|\le C_f$ $\forall t\in[0,T]$, and let $\bar{\boldsymbol{u}}^{n+1}$, $\boldsymbol{u}^{n+1}$ be the solution of the scheme* (4.1). *For all $\delta t>0$, we choose $\bar{C}\ge\max\{2\delta t^2C_f^2,2C_f^2,1\}$. Then, given $r^n\ge 0$, we have $r^{n+1}\ge 0$, $\xi^{n+1}\ge 0$, and there exists a constant $M_T>0$ depending only on T such that*

$$\nu\delta t\sum_{j=0}^{n}\xi^{j+1}\|\nabla\bar{\boldsymbol{u}}^{j+1}\|^2,\ \|\boldsymbol{u}^{n+1}\|,\ r^{n+1}\le M_T\quad\forall n+1\le\frac{T}{\delta t}. \tag{4.2}$$

Proof. The main part of the proof essentially follows the proof of Theorem 1 in [32]. For the reader's convenience, we provide the proof here. By the assumptions on $\boldsymbol{f}$ and $\bar{C}$, we find

$$\begin{aligned}
&\left|\frac{\delta t(\boldsymbol{f}^{n+1},\bar{\boldsymbol{u}}^{n+1})}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}}\right|\le\frac{\delta t\|\boldsymbol{f}^{n+1}\|\,\|\bar{\boldsymbol{u}}^{n+1}\|}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}}\le\frac{\delta t\|\boldsymbol{f}^{n+1}\|\,\|\bar{\boldsymbol{u}}^{n+1}\|}{\frac12\|\bar{\boldsymbol{u}}^{n+1}\|^2+2\delta t^2\|\boldsymbol{f}^{n+1}\|^2}\le\frac12,\\
&\left|\frac{(\boldsymbol{f}^{n+1},\bar{\boldsymbol{u}}^{n+1})}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}}\right|\le\frac{\|\boldsymbol{f}^{n+1}\|\,\|\bar{\boldsymbol{u}}^{n+1}\|}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}}\le\frac{\|\boldsymbol{f}^{n+1}\|\,\|\bar{\boldsymbol{u}}^{n+1}\|}{\frac12\|\bar{\boldsymbol{u}}^{n+1}\|^2+2\|\boldsymbol{f}^{n+1}\|^2}\le\frac12.
\end{aligned} \tag{4.3}$$

Then, given $r^n>0$, it follows from (4.1c) and (4.3) that

$$r^{n+1}=r^n\left(1+\frac{\delta t\nu\|\nabla\bar{\boldsymbol{u}}^{n+1}\|^2}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}}-\frac{\delta t(\boldsymbol{f}^{n+1},\bar{\boldsymbol{u}}^{n+1})}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}}\right)^{-1}>0. \tag{4.4}$$

Next, taking the sum of (4.1c) for n from 0 to m, using again (4.3), we obtain

$$\begin{aligned}
r^{m+1}&=r^0-\nu\delta t\sum_{j=0}^{m}\xi^{j+1}\|\nabla\bar{\boldsymbol{u}}^{j+1}\|^2+\delta t\sum_{j=0}^{m}\xi^{j+1}(\boldsymbol{f}^{j+1},\bar{\boldsymbol{u}}^{j+1})\\
&\le r^0+\delta t\sum_{j=0}^{m}\frac{r^{j+1}}{E(\bar{\boldsymbol{u}}^{j+1})+\bar{C}}(\boldsymbol{f}^{j+1},\bar{\boldsymbol{u}}^{j+1})\\
&\le r^0+\delta t\sum_{j=0}^{m}\frac{\|\boldsymbol{f}^{j+1}\|\,\|\bar{u}^{j+1}\|}{E(\bar{\boldsymbol{u}}^{j+1})+\bar{C}}r^{j+1}\\
&\le r^0+\frac12 r^{m+1}+\frac{\delta t}{2}\sum_{j=0}^{m}r^j.
\end{aligned} \tag{4.5}$$

Applying the discrete Gronwall lemma 2 to the above, we obtain

$$r^{m+1}\le 2\exp(T)r^0:=C_Tr^0\quad\forall m+1\le T/\delta t, \tag{4.6}$$

which, along with $\bar{C}\ge 1$, implies that

$$|\xi^{n+1}|=\frac{r^{n+1}}{E(\bar{\boldsymbol{u}}^{n+1})+\bar{C}}\le\frac{2C_Tr^0}{\|\bar{\boldsymbol{u}}^{n+1}\|^2+2}. \tag{4.7}$$

Since $\eta^{n+1}=1-(1-\xi^{n+1})^2$, we have $\eta^{n+1}=\xi^{n+1}(2-\xi^{n+1})$. Then, we derive from inequality (4.7) that with $M_T:=2C_Tr^0(2+2C_Tr^0)>2C_Tr^0|2-\xi^{n+1}|>0$ we have

$$|\eta^{n+1}|=|\xi^{n+1}(2-\xi^{n+1})|\le\frac{M_T}{\|\bar{\boldsymbol{u}}^{n+1}\|^2+2},$$

which, along with $\boldsymbol{u}^{n+1}=\eta^{n+1}\bar{\boldsymbol{u}}^{n+1}$, implies

$$\|\boldsymbol{u}^{n+1}\|^2=(\eta^{n+1})^2\|\bar{\boldsymbol{u}}^{n+1}\|^2\le\left(\frac{M_T}{\|\bar{\boldsymbol{u}}^{n+1}\|^2+2}\right)^2\|\bar{\boldsymbol{u}}^{n+1}\|^2\le M_T^2,$$

where we use $(A+2)^2>A$ for any $A\ge0$ and $A=\|\bar{\boldsymbol{u}}^{n+1}\|^2$ in the case above. Finally, combining (4.3), (4.5), and (4.6), we have

$$\begin{aligned}\nu\delta t\sum_{j=0}^{m}\xi^{j+1}\|\nabla\bar{\boldsymbol{u}}^{j+1}\|^2&=r^0-r^{m+1}+\delta t\sum_{j=0}^{m}\xi^{j+1}(\boldsymbol{f}^{j+1},\bar{\boldsymbol{u}}^{j+1})\\&\le r^0+r^{m+1}+\frac{\delta t}{2}\sum_{j=0}^{m}r^{j+1}\\&\le r^0(1+C_T+TC_T).\end{aligned}\tag{4.8}$$

To simplify the notation, we can adjust the positive constant M_T such that $M_T\ge r^0(1+C_T+TC_T)$, and hence (4.2) is proved. ∎

Several remarks are in order:

- The above weak stability is established independent of the first step in (4.1a). However, combining this weak stability for the nonlinear Navier–Stokes equations and the strong stability established in Theorem 5 with $k=5$ for the linear Stokes problem will allow us to obtain an optimal error estimates.
- The main computational cost (4.1a) is to solve a sequence of Poisson type equations in (4.1a)–(4.1b), and it can be efficiently implemented as follows:
  - (i) Compute $\bar{\boldsymbol{u}}^{n+1}$ and p^{n+1} from (4.1a) and (4.1b);
  - (ii) With $\bar{\boldsymbol{u}}^{n+1}$ known, determine r^{n+1} explicitly from (4.1c);
  - (iii) Compute ξ^{n+1} and η_k^{n+1} from (4.1d);
  - (iv) Update $\boldsymbol{u}^{n+1}$ from (4.1e), goto the next step.

5. Error analysis. In this section, we carry out the error analysis for the second-order schemes (4.1) with $k=5$. To simplify the presentation, we take $\nu=1$ in (4.1), since the stability results were already proved in the previous sections for any $\nu>0$. We shall first carry out a detailed global-in-time error analysis in 2D, followed by pointing out necessary modifications needed to obtain a local-in-time error estimate in 3D.

5.1. Error analysis in 2D. With $k=5$ and $\nu=1$, we carry out below the error analysis for the second-order scheme (4.1). Now, (4.1a) becomes

$$\frac{11\bar{\boldsymbol{u}}^{n+1}-20\bar{\boldsymbol{u}}^n+9\bar{\boldsymbol{u}}^{n-1}}{2\delta t}-\Delta(5\bar{\boldsymbol{u}}^{n+1}-4\bar{\boldsymbol{u}}^n)+\hat{\boldsymbol{u}}^n\cdot\nabla\hat{\boldsymbol{u}}^n+\nabla\hat{p}^n=\boldsymbol{f}^{n+5},\tag{5.1}$$

where $\hat{\boldsymbol{u}}^n=6\boldsymbol{u}^n-5\boldsymbol{u}^{n-1}$ and $\hat{p}^n=6p^n-5p^{n-1}$. We denote

$$t^n=n\,\delta t,\quad s^n=r^n-r(t^n),\quad \bar{\boldsymbol{e}}^n=\bar{\boldsymbol{u}}^n-\boldsymbol{u}(\cdot,t^n),\quad \boldsymbol{e}^n=\boldsymbol{u}^n-\boldsymbol{u}(\cdot,t^n),\quad e_p^n=p^n-p(\cdot,t^n).$$

THEOREM 7. *Let $d=2$, $T>0$, $\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_0^2$, and $\boldsymbol{u}$ be the solution of* (1.1), *and suppose $\bar{\boldsymbol{u}}^{n+1}$, $\boldsymbol{u}^{n+1}$, and p^{n+1} are computed by the scheme* (4.1) *with $k=5$. We assume the conditions in Theorem* 6 *hold, and*

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;H^1), \quad \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^2(0,T;H^2), \quad \frac{\partial^3 \boldsymbol{u}}{\partial t^3} \in L^2(0,T;L^2), \quad \frac{\partial^2 p}{\partial t^2} \in L^2(0,T;H^1). \tag{5.2}$$

Then for $n+1 \le T/\delta t$ with $\delta t \le \frac{1}{1+2C_0^2}$, we have

$$\|\nabla \bar{\boldsymbol{e}}^{n+1}\|^2 + \|\nabla \boldsymbol{e}^{n+1}\|^2 + \delta t \sum_{i=0}^{n+1} (\|\Delta \bar{\boldsymbol{e}}^i\|^2 + \|\Delta \boldsymbol{e}^i\|^2 + \|\nabla e_p^i\|^2) \le C\delta t^4, \tag{5.3}$$

where the constants C_0 and C are dependent on T, Ω, and the exact solution $\boldsymbol{u}$, but are independent of δt.

Proof. As we focus on the error analysis for the semidiscrete scheme, we assume $\boldsymbol{f}^i = \boldsymbol{f}(t^i)$ for all i, and $\bar{\boldsymbol{u}}^1$, $\boldsymbol{u}^1$, p^1 are computed with proper initialization procedure such that (5.3) holds.

The main task is to prove by induction

$$|1-\xi^i| \le C_0\,\delta t \quad \forall i \le T/\delta t, \tag{5.4}$$

where the constant $C_0 > 1$ will be defined in the induction process below. In the following, we shall use C to denote a positive constant independent of $C_0, \delta t$, which can change from one step to another, and we use $\varepsilon > 0$ to denote a constant which can be arbitrarily small.

Under the assumption, (5.4) certainly holds for $i=0$. Now suppose we have

$$|1-\xi^i| \le C_0\,\delta t \;\; \forall i \le n; \tag{5.5}$$

we shall prove below

$$|1-\xi^{n+1}| \le C_0 \delta t \tag{5.6}$$

for the same constant C_0.

Step 1: Bounds for $\|\nabla \bar{\boldsymbol{u}}^i\|$ and $\|\nabla \boldsymbol{u}^i\|$ *for all $i \le n$***.** Under the assumption (5.5), if we choose δt small enough such that

$$\delta t \le \min\left\{\frac{1}{2C_0^2}, 1\right\}, \tag{5.7}$$

we have

$$\frac{1}{2} < 1 - \frac{1}{2C_0} < |\xi^i| < 1 + \frac{1}{2C_0} < 2 \quad \forall i \le n \tag{5.8}$$

and

$$\frac{1}{2} \le 1 - \frac{\delta t}{2} \le |\eta^i| \le 1 + \frac{\delta t}{2} \le 2 \quad \forall i \le n. \tag{5.9}$$

Then it follows from the above and (4.2) that

$$\delta t \sum_{i=0}^{n} \|\nabla \bar{\boldsymbol{u}}^i\|^2,\, \|\bar{\boldsymbol{u}}^i\| \le 2M_T \quad \forall i \le n. \tag{5.10}$$

Consider (5.1) at step $i+1 \le n$ and take the inner product with $-\Delta\tilde{\boldsymbol{u}}^{i+1}$ where $\tilde{\boldsymbol{u}}^{i+1} = 6\bar{\boldsymbol{u}}^{i+1} - 5\bar{\boldsymbol{u}}^i$. For the first term on the left-hand side, it follows from (3.16) that

$$
\begin{aligned}
&\big(11\bar{\boldsymbol{u}}^{i+1} - 20\bar{\boldsymbol{u}}^i + 9\bar{\boldsymbol{u}}^{i-1}, -\Delta\tilde{\boldsymbol{u}}^{i+1}\big) \\
&\quad = \frac{1}{10}(\|\nabla\bar{\boldsymbol{u}}^{i+1}\|^2 - \|\nabla\bar{\boldsymbol{u}}^i\|^2) + \left\|\frac{9\sqrt{10}}{5}\nabla\bar{\boldsymbol{u}}^{i+1} - \frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{u}}^i\right\|^2 \\
&\qquad - \left\|\frac{9\sqrt{10}}{5}\nabla\bar{\boldsymbol{u}}^{i} - \frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{u}}^{i-1}\right\|^2 \\
&\qquad + \left\|\frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{u}}^{i+1} - \sqrt{90}\nabla\bar{\boldsymbol{u}}^i + \frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{u}}^{i-1}\right\|^2 + \frac{13}{2}\|\nabla(\bar{\boldsymbol{u}}^{i+1} - \bar{\boldsymbol{u}}^i)\|^2 \\
&\qquad - \frac{9}{2}\|\nabla(\bar{\boldsymbol{u}}^{i} - \bar{\boldsymbol{u}}^{i-1})\|^2 + \frac{9}{2}\|\nabla(\bar{\boldsymbol{u}}^{i+1} - 2\bar{\boldsymbol{u}}^i + \bar{\boldsymbol{u}}^{i-1})\|^2.
\end{aligned} \tag{5.11}
$$

For the second term, taking $k=5$ in (3.8), we have

$$
\begin{aligned}
&\big(-\Delta(5\bar{\boldsymbol{u}}^{i+1} - 4\bar{\boldsymbol{u}}^i), -\Delta\tilde{\boldsymbol{u}}^{i+1}\big) \\
&= \frac{4}{5}\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2 + \frac{1}{5}\|\Delta\bar{\boldsymbol{u}}^{i+1}\|^2 + \frac{1}{2}\big(\|\Delta\bar{\boldsymbol{u}}^{i+1}\|^2 - \|\Delta\bar{\boldsymbol{u}}^{i}\|^2 + \|\Delta\bar{\boldsymbol{u}}^{i+1} - \Delta\bar{\boldsymbol{u}}^i\|^2\big).
\end{aligned} \tag{5.12}
$$

For the term with $\hat{\boldsymbol{u}}^i \cdot \nabla\hat{\boldsymbol{u}}^i$, noting that $\boldsymbol{u}^i = \eta^i\bar{\boldsymbol{u}}^i$ and $|\eta_i| < 2$, making use of (2.2), we have

$$
\begin{aligned}
\big(\hat{\boldsymbol{u}}^i \cdot \nabla\hat{\boldsymbol{u}}^i, \Delta\tilde{\boldsymbol{u}}^{i+1}\big) &\le 4\Big|\big(\tilde{\boldsymbol{u}}^i \cdot \nabla\tilde{\boldsymbol{u}}^i, \Delta\tilde{\boldsymbol{u}}^{i+1}\big)\Big| \\
&\le c\|\nabla\tilde{\boldsymbol{u}}^i\|\|\tilde{\boldsymbol{u}}^i\|^{1/2}\|\tilde{\boldsymbol{u}}^i\|_2^{1/2}\|\Delta\tilde{\boldsymbol{u}}^{i+1}\| \\
&\le C(\varepsilon)\|\nabla\tilde{\boldsymbol{u}}^i\|^2\|\tilde{\boldsymbol{u}}^i\|\|\tilde{\boldsymbol{u}}^i\|_2 + \varepsilon\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2 \\
&\le C(\varepsilon)M_T^2\|\nabla\tilde{\boldsymbol{u}}^i\|^4 + \varepsilon\|\Delta\tilde{\boldsymbol{u}}^i\|^2 + \varepsilon\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2,
\end{aligned} \tag{5.13}
$$

where we use the elliptic regularity estimate $\|\tilde{\boldsymbol{u}}^i\|_2^2 \le C\|\Delta\tilde{\boldsymbol{u}}^i\|^2$ in the last inequality above.

For the term with $\hat{p}^i$, we have

$$
\big(\nabla\hat{p}^i, -\Delta\tilde{\boldsymbol{u}}^{i+1}\big) \le \|\nabla\hat{p}^i\|\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|. \tag{5.14}
$$

In order to estimate $\|\nabla\hat{p}^i\|$, we follow the same process as in [16], first rewriting (4.1b) as

$$
\big(\nabla p^i, \nabla q\big) = \big(\boldsymbol{f}^i - \bar{\boldsymbol{u}}^i \cdot \nabla\bar{\boldsymbol{u}}^i, \nabla q\big) + \big(\nabla p_s(\bar{\boldsymbol{u}}^i), \nabla q\big) \quad \forall i \le n, \tag{5.15}
$$

where $p_s(\bar{\boldsymbol{u}}^i)$ is the Stokes pressure associated with $\bar{\boldsymbol{u}}^i$ and hence

$$
\big(\nabla\hat{p}^i, \nabla q\big) = \big(\hat{\boldsymbol{f}}^i - 6\bar{\boldsymbol{u}}^i \cdot \nabla\bar{\boldsymbol{u}}^i + 5\bar{\boldsymbol{u}}^{i-1} \cdot \nabla\bar{\boldsymbol{u}}^{i-1}, \nabla q\big) + \big(\nabla p_s(\tilde{\boldsymbol{u}}^i), \nabla q\big), \tag{5.16}
$$

where $\hat{\boldsymbol{f}}^i = 6\boldsymbol{f}^i - 5\boldsymbol{f}^{i-1}$. Now, taking $q = \hat{p}^i$, we have

$$
\|\nabla\hat{p}^i\| \le \|\hat{\boldsymbol{f}}^i - 6\bar{\boldsymbol{u}}^i \cdot \nabla\bar{\boldsymbol{u}}^i + 5\bar{\boldsymbol{u}}^{i-1} \cdot \nabla\bar{\boldsymbol{u}}^{i-1}\| + \|\nabla p_s(\tilde{\boldsymbol{u}}^i)\|. \tag{5.17}
$$

In the case $d=2$, it follows from the Sobolev inequality and the elliptic regularity estimate that

$$
\begin{aligned}
&\|\hat{\boldsymbol{f}}^{i} - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i + 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\|^2\\
&\le 3\|\hat{\boldsymbol{f}}^{i}\|^2 + 108\|\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i\|^2 + 75\|\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\|^2\\
&\le 3\|\hat{\boldsymbol{f}}^{i}\|^2 + C\|\bar{\boldsymbol{u}}^i\|\|\nabla\bar{\boldsymbol{u}}^i\|^2\|\nabla\bar{\boldsymbol{u}}^i\|_1 + C\|\bar{\boldsymbol{u}}^{i-1}\|\|\nabla\bar{\boldsymbol{u}}^{i-1}\|^2\|\nabla\bar{\boldsymbol{u}}^{i-1}\|_1\\
&\le 3\|\hat{\boldsymbol{f}}^{i}\|^2 + C(\varepsilon)M_T^2(\|\nabla\bar{\boldsymbol{u}}^i\|^4 + \|\nabla\bar{\boldsymbol{u}}^{i-1}\|^4) + \varepsilon(\|\Delta\bar{\boldsymbol{u}}^i\|^2 + \|\Delta\bar{\boldsymbol{u}}^{i-1}\|^2),
\end{aligned}
\tag{5.18}
$$

where we used the following inequality (cf. section 4 in [16]):

$$
\|\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n\|^2 \le \|\boldsymbol{u}^n\|_{L^4}^2\|\nabla\boldsymbol{u}^n\|_{L^4}^2 \le C\|\boldsymbol{u}^n\|\|\nabla\boldsymbol{u}^n\|^2\|\nabla\boldsymbol{u}^n\|_1, \quad d=2.
$$

As a result, by making use of Lemma 4, we can estimate (5.14) as

$$
\begin{aligned}
\big(\nabla\hat{p}^i, -\Delta\tilde{\boldsymbol{u}}^{i+1}\big) &\le \|\Delta\tilde{\boldsymbol{u}}^{i+1}\|\big(\|\hat{\boldsymbol{f}}^{i} - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i + 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\| + \|\nabla p_s(\tilde{\boldsymbol{u}}^i)\|\big)\\
&\le C(\varepsilon)\|\hat{\boldsymbol{f}}^{i} - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i + 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\|^2 + \varepsilon\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2\\
&\quad + \frac{1}{2}\|\nabla p_s(\tilde{\boldsymbol{u}}^i)\|^2 + \frac{1}{2}\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2\\
&\le C(\varepsilon)\big(\|\hat{\boldsymbol{f}}^{i}\|^2 + M_T^2(\|\nabla\bar{\boldsymbol{u}}^i\|^4 + \|\nabla\bar{\boldsymbol{u}}^{i-1}\|^4)\big)\\
&\quad + \varepsilon(\|\Delta\bar{\boldsymbol{u}}^i\|^2 + \|\Delta\bar{\boldsymbol{u}}^{i-1}\|^2 + \|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2)\\
&\quad + \left(\frac{1}{4} + \frac{\varepsilon}{2}\right)\|\Delta\tilde{\boldsymbol{u}}^i\|^2 + C(\varepsilon)\|\nabla\tilde{\boldsymbol{u}}^i\|^2 + \frac{1}{2}\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2.
\end{aligned}
\tag{5.19}
$$

Finally, for the right-hand side of (5.1), we have

$$
\big(\boldsymbol{f}^{i+5}, -\Delta\tilde{\boldsymbol{u}}^{i+1}\big) \le C(\varepsilon)\|\boldsymbol{f}^{i+5}\|^2 + \varepsilon\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2.
\tag{5.20}
$$

Now, combining (5.11)–(5.20) and dropping some unnecessary terms, we obtain

$$
\begin{aligned}
&\frac{1}{10}(\|\nabla\bar{\boldsymbol{u}}^{i+1}\|^2 - \|\nabla\bar{\boldsymbol{u}}^i\|^2) + \left\|\frac{9\sqrt{10}}{5}\nabla\bar{\boldsymbol{u}}^{i+1} - \frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{u}}^i\right\|^2 - \left\|\frac{9\sqrt{10}}{5}\nabla\bar{\boldsymbol{u}}^i - \frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{u}}^{i-1}\right\|^2\\
&\quad + \frac{13}{2}\|\nabla(\bar{\boldsymbol{u}}^{i+1} - \bar{\boldsymbol{u}}^i)\|^2 - \frac{9}{2}\|\nabla(\bar{\boldsymbol{u}}^i - \bar{\boldsymbol{u}}^{i-1})\|^2 + \frac{8\delta t}{5}\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2 + \frac{2\delta t}{5}\|\Delta\bar{\boldsymbol{u}}^{i+1}\|^2\\
&\quad + \delta t\big(\|\Delta\bar{\boldsymbol{u}}^{i+1}\|^2 - \|\Delta\bar{\boldsymbol{u}}^n\|^2\big)\\
&\le C(\varepsilon)M_T^2\delta t(\|\nabla\tilde{\boldsymbol{u}}^i\|^4 + \|\nabla\bar{\boldsymbol{u}}^i\|^4 + \|\nabla\bar{\boldsymbol{u}}^{i-1}\|^4) + \left(\frac{1}{2} + \varepsilon\right)\delta t\|\Delta\tilde{\boldsymbol{u}}^i\|^2 + \delta t\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2\\
&\quad + 2\varepsilon\delta t(\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2 + \|\Delta\tilde{\boldsymbol{u}}^i\|^2 + \|\Delta\bar{\boldsymbol{u}}^i\|^2 + \|\Delta\bar{\boldsymbol{u}}^{i-1}\|^2) + C(\varepsilon)\delta t\|\nabla\tilde{\boldsymbol{u}}^i\|^2\\
&\quad + C(\varepsilon)\delta t(\|\boldsymbol{f}^{i+5}\|^2 + \|\hat{\boldsymbol{f}}^{i}\|^2).
\end{aligned}
\tag{5.21}
$$

Now, we can choose ε small enough, such that

$$
\frac{8}{5} > \frac{1}{2} + \varepsilon + 1 + 4\varepsilon \quad \text{and} \quad \frac{2}{5} > 4\varepsilon.
\tag{5.22}
$$

For example, we can choose $\varepsilon = \frac{1}{100}$, then take the sum on (5.21) for i from 1 to $m-1$ with $m \leq n$ and drop some unnecessary terms:

$$
\begin{aligned}
&\|\nabla \bar{\boldsymbol{u}}^m\|^2 + \delta t \sum_{i=0}^{m} \|\Delta \bar{\boldsymbol{u}}^i\|^2 \\
&\quad \leq C M_T^2 \delta t \sum_{i=0}^{m-1} \|\nabla \bar{\boldsymbol{u}}^i\|^4 + C\delta t \sum_{i=0}^{m-1} \|\nabla \bar{\boldsymbol{u}}^i\|^2 \\
&\qquad + C\delta t \sum_{i=0}^{m-1} (\|\hat{\boldsymbol{f}}^i\|^2 + \|\boldsymbol{f}^{i+5}\|^2) + M_0 \\
&\quad \leq C M_T^2 \delta t \sum_{i=0}^{m-1} \|\nabla \bar{\boldsymbol{u}}^i\|^4 + C\delta t \sum_{i=0}^{m-1} \|\nabla \bar{\boldsymbol{u}}^i\|^2 + CTC_f^2 + M_0 \quad \forall m \leq n,
\end{aligned} \tag{5.23}
$$

where M_0 is a constant dependent only on the initial data and where we used $\|\boldsymbol{f}(\cdot,t)\| \leq C_f$ for all $t \in [0,T]$. Next, noting that $\delta t \sum_{i=0}^{m-1} \|\nabla \bar{\boldsymbol{u}}^i\|^2 \leq 2M_T$ from (5.10), we can make use of Lemma 2 on (5.23) and obtain

$$
\|\nabla \bar{\boldsymbol{u}}^m\|^2 + \delta t \sum_{i=0}^{m} \|\Delta \bar{\boldsymbol{u}}^i\|^2 \leq \exp(2CM_T^3)(2CM_T + CTC_f^2 + M_0) \quad \forall m \leq n. \tag{5.24}
$$

Since $\boldsymbol{u}^i = \eta^i \bar{\boldsymbol{u}}^i$ and $|\eta^i| > \frac{1}{2}$ for all $i \leq n$, we also have

$$
\delta t \sum_{i=0}^{m} \|\Delta \boldsymbol{u}^i\|^2 \leq 4 \exp\left(2CM_T^3\right)\left(2CM_T + CTC_f^2 + M_0\right) \quad \forall m \leq n. \tag{5.25}
$$

Step 2: Estimates for $\|\nabla \bar{e}^{n+1}\|$. From (5.1) and (4.1b), we can write the error equation for $\bar{\boldsymbol{u}}^{i+1}$ and p^{i+1} as

$$
\begin{aligned}
&11\bar{\boldsymbol{e}}^{i+1} - 20\bar{\boldsymbol{e}}^i + 9\bar{\boldsymbol{e}}^{i-1} - 2\delta t \Delta(5\bar{\boldsymbol{e}}^{i+1} - 4\bar{\boldsymbol{e}}^i) + 2\delta t(\hat{\boldsymbol{u}}^i \cdot \nabla \hat{\boldsymbol{u}}^i - \hat{\boldsymbol{u}}(t^i) \cdot \nabla \hat{\boldsymbol{u}}(t^i)) + 2\delta t \nabla \hat{e}_p^i \\
&\quad = 2\delta t P^i + 2\delta t Q^i + R^i + 2\delta t S^i,
\end{aligned} \tag{5.26}
$$

where $\hat{e}_p^i = 6e_p^i - 5e_p^{i-1}$, $\hat{\boldsymbol{u}}(t^i) = 6\boldsymbol{u}(t^i) - 5\boldsymbol{u}(t^{i-1})$, and P^i, Q^i, R^i S^i are given by

$$
\begin{aligned}
P^i &= \nabla p(t^{i+5}) - \nabla(6p(t^i) - 5p(t^{i-1})) \\
&= 6\int_{t^i}^{t^{i+5}} (t^i - s)\nabla \frac{\partial^2 p}{\partial t^2}(s)ds - 5\int_{t^{i-1}}^{t^{i+5}} (t^{i-1} - s)\nabla \frac{\partial^2 p}{\partial t^2}(s)ds,
\end{aligned} \tag{5.27}
$$

$$
\begin{aligned}
Q^i &= -\Delta \boldsymbol{u}(t^{i+5}) + \Delta(5\boldsymbol{u}(t^{i+1}) - 4\boldsymbol{u}(t^i)) \\
&= -5\int_{t^{i+1}}^{t^{i+5}} (t^{i+1} - s)\Delta \frac{\partial^2 \boldsymbol{u}}{\partial t^2}(s)ds + 4\int_{t^i}^{t^{i+5}} (t^i - s)\Delta \frac{\partial^2 \boldsymbol{u}}{\partial t^2}(s)ds,
\end{aligned} \tag{5.28}
$$

$$
\begin{aligned}
R^i &= -11\boldsymbol{u}(t^{i+1}) + 20\boldsymbol{u}(t^i) - 9\boldsymbol{u}(t^{i-1}) + 2\delta t \boldsymbol{u}_t(t^{i+5}) \\
&= \frac{11}{2}\int_{t^{i+1}}^{t^{i+5}} (t^{i+1} - s)^2 \frac{\partial^3 \boldsymbol{u}}{\partial t^3}(s)ds - 10\int_{t^i}^{t^{i+5}} (t^i - s)^2 \frac{\partial^3 \boldsymbol{u}}{\partial t^3}(s)ds \\
&\quad + \frac{9}{2}\int_{t^{i-1}}^{t^{i+5}} (t^{i-1} - s)^2 \frac{\partial^3 \boldsymbol{u}}{\partial t^3}(s)ds,
\end{aligned} \tag{5.29}
$$

and

$$S^i = \boldsymbol{u}(t^{i+5})\cdot\nabla\boldsymbol{u}(t^{i+5}) - \hat{\boldsymbol{u}}(t^i)\cdot\nabla\hat{\boldsymbol{u}}(t^i) \\ = \boldsymbol{u}(t^{i+5})\cdot\nabla(\boldsymbol{u}(t^{i+5}) - \hat{\boldsymbol{u}}(t^i)) - (\hat{\boldsymbol{u}}(t^i) - \boldsymbol{u}(t^{i+5}))\cdot\nabla\hat{\boldsymbol{u}}(t^i). \tag{5.30}$$

Take the inner product with $-\Delta\tilde{\boldsymbol{e}}^{i+1}$ on (5.26) with $\tilde{\boldsymbol{e}}^{i+1} = 6\bar{\boldsymbol{e}}^{i+1} - 5\bar{\boldsymbol{e}}^i$. For the first two terms on the left-hand side, same as (5.11) and (5.12), we have

$$\begin{aligned}&\big(11\bar{\boldsymbol{e}}^{i+1} - 20\bar{\boldsymbol{e}}^i + 9\bar{\boldsymbol{e}}^{i-1}, -\Delta\tilde{\boldsymbol{e}}^{i+1}\big)\\ &= \frac{1}{10}(\|\nabla\bar{\boldsymbol{e}}^{i+1}\|^2 - \|\nabla\bar{\boldsymbol{e}}^i\|^2) + \left\|\frac{9\sqrt{10}}{5}\nabla\bar{\boldsymbol{e}}^{i+1} - \frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{e}}^i\right\|^2\\ &\quad - \left\|\frac{9\sqrt{10}}{5}\nabla\bar{\boldsymbol{e}}^{i} - \frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{e}}^{i-1}\right\|^2\\ &\quad + \left\|\frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{e}}^{i+1} - \sqrt{90}\nabla\bar{\boldsymbol{e}}^i + \frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{e}}^{i-1}\right\|^2 + \frac{13}{2}\|\nabla(\bar{\boldsymbol{e}}^{i+1} - \bar{\boldsymbol{e}}^i)\|^2\\ &\quad - \frac{9}{2}\|\nabla(\bar{\boldsymbol{e}}^i - \bar{\boldsymbol{e}}^{i-1})\|^2 + \frac{9}{2}\|\nabla(\bar{\boldsymbol{e}}^{i+1} - 2\bar{\boldsymbol{e}}^i + \bar{\boldsymbol{e}}^{i-1})\|^2\end{aligned} \tag{5.31}$$

and

$$\begin{aligned}&\big(-2\delta t\Delta(5\bar{\boldsymbol{e}}^{i+1} - 4\bar{\boldsymbol{e}}^i), -\Delta\tilde{\boldsymbol{e}}^{i+1}\big)\\ &= \frac{8\delta t}{5}\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2 + \frac{2\delta t}{5}\|\Delta\bar{\boldsymbol{e}}^{i+1}\|^2 + \delta t\big(\|\Delta\bar{\boldsymbol{e}}^{i+1}\|^2 - \|\Delta\bar{\boldsymbol{e}}^i\|^2 + \|\Delta\bar{\boldsymbol{e}}^{i+1} - \Delta\bar{\boldsymbol{e}}^i\|^2\big).\end{aligned} \tag{5.32}$$

For the nonlinear term on the left-hand side of (5.26), we rewrite it as

$$\begin{aligned}\hat{\boldsymbol{u}}^i\cdot\nabla\hat{\boldsymbol{u}}^i - \hat{\boldsymbol{u}}(t^i)\cdot\nabla\hat{\boldsymbol{u}}(t^i) &= \hat{\boldsymbol{u}}^i\cdot\nabla\hat{\boldsymbol{u}}^i - \hat{\boldsymbol{u}}(t^i)\cdot\nabla\hat{\boldsymbol{u}}^i + \hat{\boldsymbol{u}}(t^i)\cdot\nabla\hat{\boldsymbol{u}}^i - \hat{\boldsymbol{u}}(t^i)\cdot\nabla\hat{\boldsymbol{u}}(t^i)\\ &= \hat{\boldsymbol{e}}^i\cdot\nabla\hat{\boldsymbol{u}}^i + \hat{\boldsymbol{u}}(t^i)\cdot\nabla\hat{\boldsymbol{e}}^i,\end{aligned} \tag{5.33}$$

where $\hat{\boldsymbol{e}}^i = 6\boldsymbol{e}^i - 5\boldsymbol{e}^{i-1}$. Therefore, it follows from (2.4) that

$$\begin{aligned}&\big(\hat{\boldsymbol{u}}^i\cdot\nabla\hat{\boldsymbol{u}}^i - \hat{\boldsymbol{u}}(t^i)\cdot\nabla\hat{\boldsymbol{u}}(t^i), -\Delta\tilde{\boldsymbol{e}}^{i+1}\big)\\ &= \big(\hat{\boldsymbol{e}}^i\cdot\nabla\hat{\boldsymbol{u}}^i, -\Delta\tilde{\boldsymbol{e}}^{i+1}\big) + \big(\hat{\boldsymbol{u}}(t^i)\cdot\nabla\hat{\boldsymbol{e}}^i, -\Delta\tilde{\boldsymbol{e}}^{i+1}\big)\\ &\le C\|\nabla\hat{\boldsymbol{e}}^i\|\|\hat{\boldsymbol{u}}^i\|_2\|\Delta\tilde{\boldsymbol{e}}^{i+1}\| + C\|\hat{\boldsymbol{u}}(t^i)\|_2\|\nabla\hat{\boldsymbol{e}}^i\|\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|\\ &\le C(\varepsilon)\|\nabla\hat{\boldsymbol{e}}^i\|^2\|\Delta\hat{\boldsymbol{u}}^i\|^2 + C(\varepsilon)\|\hat{\boldsymbol{u}}(t^i)\|_2^2\|\nabla\hat{\boldsymbol{e}}^i\|^2 + \varepsilon\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2.\end{aligned} \tag{5.34}$$

Noting that $\boldsymbol{u}^i = \eta^i\bar{\boldsymbol{u}}^i$, $|1-\eta^i| \le C_0^2\delta t^2$ for all $i \le n$, and $\|\nabla\bar{\boldsymbol{u}}^i\|$ is bounded above from (5.24), we can estimate $\hat{\boldsymbol{e}}^i$ as

$$\|\nabla\hat{\boldsymbol{e}}^i\|^2 = \|\nabla\hat{\boldsymbol{u}}^i - \nabla\tilde{\boldsymbol{u}}^i + \nabla\tilde{\boldsymbol{e}}^i\|^2 \le CC_0^4\delta t^4 + 2\|\nabla\tilde{\boldsymbol{e}}^i\|^2. \tag{5.35}$$

For the term with $\hat{e}_p^i$, we have

$$\big(\nabla\hat{e}_p^i, -\Delta\tilde{\boldsymbol{e}}^{i+1}\big) \le \|\nabla\hat{e}_p^i\|\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|. \tag{5.36}$$

To estimate $\|\nabla\hat{e}_p^i\|$, same as in the last step, we make use of the Stokes pressure. First, from (4.1b), the error equation for e_p^i can be rewritten as

$$(\nabla e_p^i, \nabla q) = (\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i, \nabla p) + (\nabla p_s(\bar{\boldsymbol{e}}^i), \nabla q), \tag{5.37}$$

and hence,

$$(\nabla \hat{e}_p^i, \nabla q) = (6\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i - 5\boldsymbol{u}(t^{i-1})\cdot\nabla\boldsymbol{u}(t^{i-1}) + 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}, \nabla p) + (\nabla p_s(\tilde{\boldsymbol{e}}^i), \nabla q), \tag{5.38}$$

where $p_s(\tilde{e}^i)$ is the Stokes pressure associated with $\tilde{e}^i$. We let $q = \hat{e}_p^i$ and obtain

$$\|\nabla \hat{e}_p^i\| \le \|6\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i\| + \|5\boldsymbol{u}(t^{i-1})\cdot\nabla\boldsymbol{u}(t^{i-1}) - 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\| + \|\nabla p_s(\tilde{\boldsymbol{e}}^i)\|. \tag{5.39}$$

As in (5.33), we first rewrite

$$6\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i = -6\bar{\boldsymbol{e}}^i\cdot\nabla\bar{\boldsymbol{u}}^i - 6\boldsymbol{u}(t^i)\cdot\nabla\bar{\boldsymbol{e}}^i; \tag{5.40}$$

then it follows from the Sobolev inequality and the elliptic regularity estimate that

$$\|6\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i\|^2 \le C\|\nabla\bar{\boldsymbol{e}}^i\|^2\|\Delta\bar{\boldsymbol{u}}^i\|^2 + C\|\boldsymbol{u}(t^i)\|_2^2\|\nabla\bar{\boldsymbol{e}}^i\|^2 \tag{5.41}$$

and, similarly,

$$\|5\boldsymbol{u}(t^{i-1})\cdot\nabla\boldsymbol{u}(t^{i-1}) - 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\|^2 \le C\|\nabla\bar{\boldsymbol{e}}^{i-1}\|^2\|\Delta\bar{\boldsymbol{u}}^{i-1}\|^2 + C\|\boldsymbol{u}(t^{i-1})\|_2^2\|\nabla\bar{\boldsymbol{e}}^{i-1}\|^2. \tag{5.42}$$

Now, combining (5.36)–(5.42) and making use of Lemma 4 for Stokes pressure, we can bound the term with $\hat{e}_p^i$ as

$$\begin{aligned}
&\big(\nabla\hat{e}_p^i, -\Delta\tilde{\boldsymbol{e}}^{i+1}\big)\\
&\le \|\Delta\tilde{\boldsymbol{e}}^{i+1}\|\big(\|6\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i\| + \|5\boldsymbol{u}(t^{i-1})\cdot\nabla\boldsymbol{u}(t^{i-1}) - 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\|\big)\\
&\quad + \|\Delta\tilde{\boldsymbol{e}}^{i+1}\|\|\nabla p_s(\tilde{\boldsymbol{e}}^i)\|\\
&\le C(\varepsilon)\big(\|6\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i\|^2 + \|5\boldsymbol{u}(t^{i-1})\cdot\nabla\boldsymbol{u}(t^{i-1}) - 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\|^2\big)\\
&\quad + \varepsilon\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2 + \frac{1}{2}\|\nabla p_s(\tilde{\boldsymbol{e}}^i)\|^2 + \frac{1}{2}\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2\\
&\le C(\varepsilon)\|\nabla\bar{\boldsymbol{e}}^i\|^2(\|\Delta\bar{\boldsymbol{u}}^i\|^2 + \|\boldsymbol{u}(t^i)\|_2^2) + C(\varepsilon)\|\nabla\bar{\boldsymbol{e}}^{i-1}\|^2(\|\Delta\bar{\boldsymbol{u}}^{i-1}\|^2 + \|\boldsymbol{u}(t^{i-1})\|_2^2)\\
&\quad + \left(\varepsilon + \frac{1}{2}\right)\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2 + \left(\frac{1}{4} + \frac{\varepsilon}{2}\right)\|\Delta\tilde{\boldsymbol{e}}^i\|^2 + C(\varepsilon)\|\nabla\tilde{\boldsymbol{e}}^i\|^2.
\end{aligned} \tag{5.43}$$

For the right-hand side of (5.26), (5.27)–(5.29) imply

$$(P^i, -\Delta\tilde{\boldsymbol{e}}^{i+1}) \le C(\varepsilon)\|P^i\|^2 + \varepsilon\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2 \le C(\varepsilon)\delta t^3\int_{t^{i-1}}^{t^{i+5}}\left\|\nabla\frac{\partial^2 p}{\partial t^2}(s)\right\|^2 ds + \varepsilon\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2, \tag{5.44}$$

and, similarly,

$$(Q^i, -\Delta\tilde{\boldsymbol{e}}^{i+1}) \le C(\varepsilon)\delta t^3\int_{t^i}^{t^{i+5}}\left\|\Delta\frac{\partial^2\boldsymbol{u}}{\partial t^2}(s)\right\|^2 ds + \varepsilon\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2, \tag{5.45}$$

$$(R^i, -\Delta\tilde{\boldsymbol{e}}^{i+1}) \le \frac{C(\varepsilon)}{\delta t}\|R^i\|^2 + \varepsilon\delta t\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2 \le C(\varepsilon)\delta t^4\int_{t^{i-1}}^{t^{i+5}}\left\|\frac{\partial^3\boldsymbol{u}}{\partial t^3}\right\|^2 ds + \varepsilon\delta t\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2. \tag{5.46}$$

For the term with S^i, it follows from (2.4) that

(5.47)
$$
\begin{aligned}
(S^i,-\Delta\tilde{\boldsymbol{e}}^{i+1}) \leq\; & C\|\boldsymbol{u}(t^{i+5})\|_2\|\nabla(\boldsymbol{u}(t^{i+5})-\hat{\boldsymbol{u}}(t^i))\|\|\Delta\tilde{\boldsymbol{e}}^{i+1}\| \\
& + C\|\hat{\boldsymbol{u}}(t^i)\|_2\|\nabla(\boldsymbol{u}(t^{i+5})-\hat{\boldsymbol{u}}(t^i))\|\|\Delta\tilde{\boldsymbol{e}}^{i+1}\| \\
\leq\; & C(\varepsilon)(\|\boldsymbol{u}(t^{i+5})\|_2^2+\|\hat{\boldsymbol{u}}(t^i)\|_2^2)\|\nabla(\boldsymbol{u}(t^{i+5})-\hat{\boldsymbol{u}}(t^i))\|^2+\varepsilon\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2 \\
\leq\; & C(\varepsilon)\delta t^3\int_{t^{i-1}}^{t^{i+5}}\|\nabla\frac{\partial^2\boldsymbol{u}}{\partial t^2}(s)\|^2ds+\varepsilon\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2.
\end{aligned}
$$

Now, combining (5.31)–(5.47) and dropping some unnecessary terms, we obtain

(5.48)
$$
\begin{aligned}
&\frac{1}{10}(\|\nabla\bar{\boldsymbol{e}}^{i+1}\|^2-\|\nabla\bar{\boldsymbol{e}}^i\|^2)+\left\|\frac{9\sqrt{10}}{5}\nabla\bar{\boldsymbol{e}}^{i+1}-\frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{e}}^i\right\|^2-\left\|\frac{9\sqrt{10}}{5}\nabla\bar{\boldsymbol{e}}^{i}-\frac{\sqrt{90}}{2}\nabla\bar{\boldsymbol{e}}^{i-1}\right\|^2 \\
&\quad+\frac{13}{2}\|\nabla(\bar{\boldsymbol{e}}^{i+1}-\bar{\boldsymbol{e}}^i)\|^2-\frac{9}{2}\|\nabla(\bar{\boldsymbol{e}}^{i}-\bar{\boldsymbol{e}}^{i-1})\|^2+\frac{8\delta t}{5}\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2+\frac{2\delta t}{5}\left\|\Delta\bar{\boldsymbol{e}}^{i+1}\right\|^2 \\
&\quad+\delta t\left(\|\Delta\bar{\boldsymbol{e}}^{i+1}\|^2-\|\Delta\bar{\boldsymbol{e}}^{n}\|^2\right) \\
&\leq C(\varepsilon)\delta t\|\nabla\tilde{\boldsymbol{e}}^i\|^2(\|\Delta\hat{\boldsymbol{u}}^i\|^2+\|\hat{\boldsymbol{u}}(t^i)\|_2^2+1)+(\varepsilon+1)\delta t\|\Delta\tilde{\boldsymbol{e}}^{i+1}\|^2+\left(\frac{1}{2}+\varepsilon\right)\delta t\|\Delta\tilde{\boldsymbol{e}}^i\|^2 \\
&\quad+C(\varepsilon)\delta t\|\nabla\bar{\boldsymbol{e}}^i\|^2(\|\Delta\bar{\boldsymbol{u}}^i\|^2+\|\boldsymbol{u}(t^i)\|_2^2)+C(\varepsilon)\delta t\|\nabla\bar{\boldsymbol{e}}^{i-1}\|^2(\|\Delta\bar{\boldsymbol{u}}^{i-1}\|^2+\|\boldsymbol{u}(t^{i-1})\|_2^2) \\
&\quad+C(\varepsilon)\delta t^4\int_{t^{i-1}}^{t^{i+5}}\left(\left\|\nabla\frac{\partial^2 p}{\partial t^2}(s)\right\|^2+\left\|\Delta\frac{\partial^2\boldsymbol{u}}{\partial t^2}(s)\right\|^2+\left\|\frac{\partial^3\boldsymbol{u}}{\partial t^3}\right\|^2+\left\|\nabla\frac{\partial^2\boldsymbol{u}}{\partial t^2}(s)\right\|^2\right)ds \\
&\quad+CC_0^4\delta t^5(\|\Delta\hat{\boldsymbol{u}}^i\|^2+\|\hat{\boldsymbol{u}}(t^i)\|_2^2).
\end{aligned}
$$

Next, we can choose ε small enough such that $\frac{8}{5}>\varepsilon+1+\frac{1}{2}+\varepsilon$ and take the sum on (5.48) for i from 1 to n. Under the assumption (5.2) on the exact solution, we can obtain the following after dropping some unnecessary terms:

(5.49)
$$
\begin{aligned}
\|\nabla\bar{\boldsymbol{e}}^{n+1}\|^2+\delta t\sum_{i=0}^{n+1}\|\Delta\bar{\boldsymbol{e}}^i\|^2 \leq\; & C\delta t\sum_{i=0}^{n}\|\nabla\bar{\boldsymbol{e}}^i\|^2(\|\Delta\bar{\boldsymbol{u}}^i\|^2+\|\boldsymbol{u}(t^i)\|_2^2) \\
& + CC_0^4\delta t^5\sum_{i=0}^{n}(\|\Delta\hat{\boldsymbol{u}}^i\|^2+\|\hat{\boldsymbol{u}}(t^i)\|_2^2) \\
& + C\delta t\sum_{i=0}^{n}\|\nabla\tilde{\boldsymbol{e}}^i\|^2(\|\Delta\hat{\boldsymbol{u}}^i\|^2+\|\hat{\boldsymbol{u}}(t^i)\|_2^2+1)+CT\delta t^4.
\end{aligned}
$$

Finally, note from (5.24), (5.25), and assumptions on the exact solution that

$$
\delta t\sum_{i=0}^{n}\|\Delta\bar{\boldsymbol{u}}^i\|^2,\ \delta t\sum_{i=0}^{n}\|\Delta\hat{\boldsymbol{u}}^i\|^2,\ \|\hat{\boldsymbol{u}}(t^i)\|_2^2,\ \|\boldsymbol{u}(t^i)\|_2^2\leq C_1 \tag{5.50}
$$

for some positive constant C_1, which is independent of C_0 and δt. Then applying the Gronwall lemma 2 on (5.49), we have

$$
\|\nabla\bar{\boldsymbol{e}}^{n+1}\|^2+\delta t\sum_{i=0}^{n+1}\|\Delta\bar{\boldsymbol{e}}^i\|^2\leq C\delta t^4\exp(2CC_1)(C_0^4C_1+T)\leq C_2\delta t^4(C_0^4+1), \tag{5.51}
$$

where we denote $C_2 := \max\{C\exp(2CC_1)C_1, C\exp(2CC_1)T\}$ to simplify the notation below. Under the condition (5.7) on δt, we can get a bound for $\|\nabla\bar{\boldsymbol{u}}^{n+1}\|^2$:

$$\|\nabla\bar{\boldsymbol{u}}^{n+1}\|^2 \le C_2\delta t^4(C_0^4+1)+C_1 \le 2C_2+C_1 := C_3. \tag{5.52}$$

Step 3: Estimate for $|1-\xi^{m+1}|$. It follows from (4.1c) that the equation for the errors can be written as

$$\begin{aligned} s^{i+1}-s^i = {} & \delta t\left(\|\nabla\boldsymbol{u}(t^{i+1})\|^2 - \frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1})+\bar{C}}\|\nabla(\bar{\boldsymbol{u}}^{i+1})\|^2\right) \\ & + \delta t\left(\frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1})+\bar{C}}(\boldsymbol{f}^{i+1},\bar{\boldsymbol{u}}^{i+1}) - (\boldsymbol{f}(t^{i+1}),\boldsymbol{u}(t^{i+1}))\right) + T^i, \end{aligned} \tag{5.53}$$

where T^i is given as

$$T^i = r(t^i) - r(t^{i+1}) + \delta t r_t(t^{i+1}) = \int_{t^i}^{t^{i+1}} (s-t^i) r_{tt}(s)ds. \tag{5.54}$$

Taking the sum of (5.53) for i from 0 to n, and noting that $s^0=0$, we have

$$\begin{aligned} s^{n+1} = {} & \delta t\sum_{i=0}^{n}\left(\|\nabla\boldsymbol{u}(t^{i+1})\|^2 - \frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1})+\bar{C}}\|\nabla(\bar{\boldsymbol{u}}^{i+1})\|^2\right) \\ & + \delta t\sum_{i=0}^{n}\left(\frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1})+\bar{C}}(\boldsymbol{f}^{i+1},\bar{\boldsymbol{u}}^{i+1}) - (\boldsymbol{f}(t^{i+1}),\boldsymbol{u}(t^{i+1}))\right) + \sum_{i=0}^{n} T^i. \end{aligned} \tag{5.55}$$

We can bound the terms on the right-hand side of (5.55) as follows. Recall that

$$r(t) = E(\boldsymbol{u}(t,\boldsymbol{x})) + \bar{C} = \frac{1}{2}\int_\Omega \boldsymbol{u}^2(t,\boldsymbol{x})d\boldsymbol{x} + \bar{C}.$$

By direct calculation, we have

$$r_{tt} = \int_\Omega \big(\boldsymbol{u}_t^2 + \boldsymbol{u}\boldsymbol{u}_{tt}\big)d\boldsymbol{x}, \tag{5.56}$$

then from (5.54) we have

$$|T^i| \le C\delta t\int_{t^n}^{t^{n+1}} |r_{tt}(s)|ds \le C\delta t\int_{t^n}^{t^{n+1}} \big(\|\boldsymbol{u}_t(s)\|^2 + \|\boldsymbol{u}(s)\|\|\boldsymbol{u}_{tt}(s)\|\big)ds. \tag{5.57}$$

Next,

$$\begin{aligned} & \left|\|\nabla\boldsymbol{u}(t^{i+1})\|^2 - \frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1})+\bar{C}}\|\nabla\bar{\boldsymbol{u}}^{i+1}\|^2\right| \\ & \le \|\nabla\boldsymbol{u}(t^{i+1})\|^2\left|1-\frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1})+\bar{C}}\right| + \frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1})+\bar{C}}\left|\|\nabla\boldsymbol{u}(t^{i+1})\|^2 - \|\nabla\bar{\boldsymbol{u}}^{i+1}\|^2\right| \\ & := W_1^i + W_2^i. \end{aligned} \tag{5.58}$$

For W_1^i, it follows from $E(\boldsymbol{u})+\bar{C}>1$ for all $\boldsymbol{u}$ and Theorem 6 that

$$\begin{aligned} W_1^i & \le C\left|1-\frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1})+\bar{C}}\right| \\ & \le C\left|\frac{r(t^{i+1})}{E[\boldsymbol{u}(t^{i+1})]+\bar{C}} - \frac{r^{i+1}}{E[u(t^{i+1})]+\bar{C}}\right| + C\left|\frac{r^{i+1}}{E[u(t^{i+1})]+\bar{C}} - \frac{r^{i+1}}{E(\bar{u}^{i+1})+\bar{C}}\right| \\ & \le C\Big(|E[\boldsymbol{u}(t^{i+1})] - E(\bar{\boldsymbol{u}}^{i+1})| + |s^{i+1}|\Big). \end{aligned} \tag{5.59}$$

For W_2^i, it follows from (5.52), $E(\boldsymbol{u}) + \bar{C} > 1$ for all $\boldsymbol{u}$, and Theorem 6 that

$$\begin{aligned} W_2^i &\le C\big|\|\nabla \bar{\boldsymbol{u}}^{i+1}\|^2 - \|\nabla \boldsymbol{u}(t^{i+1})\|^2\big| \\ &\le C\|\nabla \bar{\boldsymbol{u}}^{i+1} - \nabla \boldsymbol{u}(t^{i+1})\|(\|\nabla \bar{\boldsymbol{u}}^{i+1}\| + \|\nabla \boldsymbol{u}(t^{i+1})\|) \\ &\le CC_3\|\nabla \bar{\boldsymbol{e}}^{i+1}\|. \end{aligned} \tag{5.60}$$

On the other hand,

$$\begin{aligned} &\left|\frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1}) + \bar{C}}(\boldsymbol{f}^{i+1}, \bar{\boldsymbol{u}}^{i+1}) - (\boldsymbol{f}(t^{i+1}), \boldsymbol{u}(t^{i+1}))\right| \\ &\le |(\boldsymbol{f}(t^{i+1}), \boldsymbol{u}(t^{i+1}))| \left|1 - \frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1}) + \bar{C}}\right| \\ &\quad + \frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1}) + \bar{C}}\left|(\boldsymbol{f}^{i+1}, \bar{\boldsymbol{u}}^{i+1}) - (\boldsymbol{f}(t^{i+1}), \boldsymbol{u}(t^{i+1}))\right| \\ &:= W_3^i + W_4^i. \end{aligned} \tag{5.61}$$

For W_3^i, as for (5.59), we have

$$W_3^i \le C\Big(|E[\boldsymbol{u}(t^{i+1})] - E(\bar{\boldsymbol{u}}^{i+1})| + |s^{i+1}|\Big). \tag{5.62}$$

For W_4^i, since we assume $\boldsymbol{f}^i = \boldsymbol{f}(t^i)$ for all i, we have

$$W_4^i \le C\|\boldsymbol{f}^{i+1}\|\|\bar{\boldsymbol{u}}^{i+1} - \boldsymbol{u}(t^{i+1})\| \le C\|\bar{\boldsymbol{e}}^{i+1}\|. \tag{5.63}$$

On the other hand,

$$|E[\boldsymbol{u}(t^{i+1})] - E(\bar{\boldsymbol{u}}^{i+1})| \le \frac{1}{2}\big(\|\boldsymbol{u}(t^{i+1})\| + \|\bar{\boldsymbol{u}}^{i+1}\|\big)\|\boldsymbol{u}(t^{i+1}) - \bar{\boldsymbol{u}}^{i+1}\| \le C\|\bar{\boldsymbol{e}}^{i+1}\|. \tag{5.64}$$

Now, combining (5.51) and (5.55)–(5.64), we arrive at

$$\begin{aligned} |s^{n+1}| &\le \delta t \sum_{i=0}^{n} \big|\|\nabla \boldsymbol{u}(t^{i+1})\|^2 - \frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1}) + \bar{C}}\|\nabla(\bar{\boldsymbol{u}}^{i+1})\|^2\big| \\ &\quad + \delta t \sum_{i=0}^{n} \left|\frac{r^{i+1}}{E(\bar{\boldsymbol{u}}^{i+1}) + \bar{C}}(\boldsymbol{f}^{i+1}, \bar{\boldsymbol{u}}^{i+1}) - (\boldsymbol{f}(t^{i+1}), \boldsymbol{u}(t^{i+1}))\right| + \sum_{i=0}^{n} |T^i| \\ &\le C\delta t \sum_{i=0}^{n} |s^{i+1}| + CC_3\delta t \sum_{i=0}^{n} \|\nabla \bar{\boldsymbol{e}}^{i+1}\| + C\delta t \int_0^T (\|\boldsymbol{u}_t(s)\|^2 + \|\boldsymbol{u}(s)\|\|\boldsymbol{u}_{tt}(s)\|)ds \\ &\le C\delta t \sum_{i=0}^{n} |s^{i+1}| + CC_3\sqrt{C_2(C_0^4 + 1)}\delta t^2 + C\delta t. \end{aligned}$$

Applying the discrete Gronwall lemma 1 to the above inequality with $\delta t < \frac{1}{2C}$, we obtain

$$\begin{aligned} |s^{n+1}| &\le C\exp((1 - C\delta t)^{-1}T)\delta t\left(C_3\sqrt{C_2(C_0^4 + 1)}\delta t + 1\right) \\ &\le C_4\delta t\left(C_3\sqrt{C_2(C_0^4 + 1)}\delta t + 1\right), \end{aligned} \tag{5.65}$$

where C_4 is independent of C_0 and δt and can be defined as

$$C_4 := C \max\{\exp(2T), 2\}. \tag{5.66}$$

Then $\delta t < \frac{1}{2C}$ can be guaranteed by

$$\delta t < \frac{1}{C_4}. \tag{5.67}$$

Hence, noting (5.59), (5.64), (5.65), and (5.51), we have

$$\begin{aligned}
|1-\xi^{n+1}| &\leq C\big(|E[\boldsymbol{u}(t^{n+1})] - E(\bar{\boldsymbol{u}}^{n+1})| + |s^{n+1}|\big)\\
&\leq C(\|\bar{\boldsymbol{e}}^{n+1}\| + |s^{n+1}|)\\
&\leq C\delta t\left(\sqrt{C_2(C_0^4+1)}\delta t + C_4\left(C_3\sqrt{C_2(C_0^4+1)}\delta t + 1\right)\right)\\
&\leq C_5\delta t\left(\sqrt{1+C_0^4}\delta t + 1\right),
\end{aligned} \tag{5.68}$$

where the constant C_5 is independent of C_0 and δt. Without loss of generality, we assume $C_5 > \max\{C_4, 1\}$ to simplify the proof below.

As a result of (5.68), $|1-\xi^{n+1}| \leq C_0\delta t$ if we define C_0 such that

$$C_5\left(\sqrt{1+C_0^4}\delta t + 1\right) \leq C_0, \tag{5.69}$$

and the above can be satisfied if we choose $C_0 = 2C_5$ and $\delta t \leq \frac{1}{1+C_0^2}$:

$$C_5\left(\sqrt{1+C_0^4}\delta t + 1\right) \leq C_5[(1+C_0^2)\delta t + 1] \leq 2C_5 = C_0. \tag{5.70}$$

To summarize, under the condition

$$\delta t \leq \frac{1}{1+2C_0^2}, \tag{5.71}$$

we have $|1-\xi^{n+1}| \leq C_0\delta t$. Note that with $C_5 > \max\{C_4, 1\}$, (5.71) also implies (5.67). The induction process for (5.4) is complete.

Finally, besides (5.51), we want to show $\|\nabla \boldsymbol{e}^{n+1}\|^2$, $\delta t \sum_{i=0}^{n+1} \|\Delta \boldsymbol{e}^i\|^2 \leq C\delta t^4$.

We derive from (4.1e) and (5.52) that

$$\|\nabla(\boldsymbol{u}^{n+1} - \bar{\boldsymbol{u}}^{n+1})\|^2 \leq |\eta^{n+1}-1|^2\|\nabla\bar{\boldsymbol{u}}^{n+1}\|^2 \leq |\eta^{n+1}-1|^2 C_3. \tag{5.72}$$

On the other hand, we derive from (5.4) that

$$|\eta^{n+1} - 1| \leq C_0^2\delta t^2. \tag{5.73}$$

Then it follows from (5.51), (5.72), and (5.73) that

$$\begin{aligned}
\|\nabla \boldsymbol{e}^{n+1}\|^2 &\leq 2\|\nabla\bar{\boldsymbol{e}}^{n+1}\|^2 + 2\|\nabla(\boldsymbol{u}^{n+1} - \bar{\boldsymbol{u}}^{n+1})\|^2\\
&\leq 2C_2(C_0^4+1)\delta t^4 + 2C_3C_0^4\delta t^4
\end{aligned} \tag{5.74}$$

holds under the condition $\delta t < \frac{1}{1+2C_0^2}$. Similarly, we have

$$\|\Delta \boldsymbol{e}^{i+1}\|^2 \leq 2\|\Delta\bar{\boldsymbol{e}}^{i+1}\|^2 + 2\|\Delta(\boldsymbol{u}^{i+1} - \bar{\boldsymbol{u}}^{i+1})\|^2 \quad \forall i \leq n; \tag{5.75}$$

then (5.51) and (5.73) imply

$$\delta t \sum_{i=0}^{n+1} \|\Delta \boldsymbol{e}^i\|^2 \le C\delta t^4. \tag{5.76}$$

Error estimate for the pressure. Letting $q = e_p^i$ in (5.37), in the same manner as in (5.41), and making use of Lemma 4, we have

$$\begin{aligned} \|\nabla e_p^i\|^2 &\le 2\|\boldsymbol{u}(t^i)\cdot\nabla\boldsymbol{u}(t^i) - \bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i\|^2 + 2\|\nabla p_s(\bar{\boldsymbol{e}}^i)\|^2 \\ &\le C\|\nabla\bar{\boldsymbol{e}}^i\|^2(\|\Delta\bar{\boldsymbol{u}}^i\|^2 + \|\boldsymbol{u}(t^i)\|_2^2) + \|\Delta\bar{\boldsymbol{e}}^i\|^2 + C\|\nabla\bar{\boldsymbol{e}}^i\|^2. \end{aligned} \tag{5.77}$$

Now, take the sum on (5.77); then (5.50) and (5.51) together imply

$$\delta t \sum_{i=0}^{n+1} \|\nabla e_p^i\|^2 \le C\delta t^4. \tag{5.78}$$

The proof is complete. ∎

5.2. Error analysis in 3D. We observe that in the above proof for the two-dimensional case, only (5.13) and (5.18) are not valid in 3D. As a result, we can only get a local-in-time version for (5.24) and (5.25).

THEOREM 8. *Let $d=3$, $\boldsymbol{u}_0 \in \boldsymbol{V}\cap\boldsymbol{H}_0^2$, and $\boldsymbol{u}$ be the solution of* (1.1), *and suppose $\bar{\boldsymbol{u}}^{n+1}$, $\boldsymbol{u}^{n+1}$, and p^{n+1} are computed by the scheme* (4.1) *with $k=5$. We assume the conditions in Theorem* 6 *hold, and*

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;H^1), \quad \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^2(0,T;H^2), \quad \frac{\partial^3 \boldsymbol{u}}{\partial t^3} \in L^2(0,T;L^2), \quad \frac{\partial^2 p}{\partial t^2} \in L^2(0,T;H^1). \tag{5.79}$$

Then, there exists $T_* > 0$ such that for $0 < T < T_*$, $n+1 \le T/\delta t$ with $\delta t \le \frac{1}{1+2C_0^2}$, we have

$$\|\nabla\bar{\boldsymbol{e}}^{n+1}\|^2 + \|\nabla\boldsymbol{e}^{n+1}\|^2 + \delta t\sum_{i=0}^{n+1}(\|\Delta\bar{\boldsymbol{e}}^i\|^2 + \|\Delta\boldsymbol{e}^i\|^2 + \|\nabla e_p^i\|^2) \le C\delta t^4, \tag{5.80}$$

where the constants C_0 and C are dependent on T, Ω and the exact solution $\boldsymbol{u}$, but are independent of δt.

Proof. The proof follows essentially the same procedure as the proof for Theorem 7. However, since (2.2) is not valid when $d=3$, we have to deal with (5.13) and (5.18) in another way. To simplify the presentation, we shall only point out below how to deal with (5.13) and (5.18) to obtain a local version of (5.24) and (5.25) in **Step 1** in the proof of Theorem 7.

In **Step 1**, it follows from (2.3) that (5.13) becomes

$$\begin{aligned} (\hat{\boldsymbol{u}}^i\cdot\nabla\hat{\boldsymbol{u}}^i, \Delta\tilde{\boldsymbol{u}}^{i+1}) &\le 4\left|(\tilde{\boldsymbol{u}}^i\cdot\nabla\tilde{\boldsymbol{u}}^i, \Delta\tilde{\boldsymbol{u}}^{i+1})\right| \le c\|\nabla\tilde{\boldsymbol{u}}^i\|\|\nabla\tilde{\boldsymbol{u}}^i\|^{1/2}\|\tilde{\boldsymbol{u}}^i\|_2^{1/2}\|\Delta\tilde{\boldsymbol{u}}^{i+1}\| \\ &\le C(\varepsilon)\|\nabla\tilde{\boldsymbol{u}}^i\|^3\|\Delta\tilde{\boldsymbol{u}}^i\| + \varepsilon\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2 \\ &\le C(\varepsilon)\|\nabla\tilde{\boldsymbol{u}}^i\|^6 + \varepsilon\|\Delta\tilde{\boldsymbol{u}}^i\|^2 + \varepsilon\|\Delta\tilde{\boldsymbol{u}}^{i+1}\|^2, \end{aligned} \tag{5.81}$$

and in the case $d=3$, it follows from the Sobolev inequality and the elliptic regularity estimate that

$$\begin{aligned} &\|\hat{\boldsymbol{f}}^i - 6\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i + 5\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\|^2 \\ &\le 3\|\hat{\boldsymbol{f}}^i\|^2 + 108\|\bar{\boldsymbol{u}}^i\cdot\nabla\bar{\boldsymbol{u}}^i\|^2 + 75\|\bar{\boldsymbol{u}}^{i-1}\cdot\nabla\bar{\boldsymbol{u}}^{i-1}\|^2 \\ &\le 3\|\hat{\boldsymbol{f}}^i\|^2 + C\|\nabla\bar{\boldsymbol{u}}^i\|^3\|\nabla\bar{\boldsymbol{u}}^i\|_1 + C\|\nabla\bar{\boldsymbol{u}}^{i-1}\|^3\|\nabla\bar{\boldsymbol{u}}^{i-1}\|_1 \\ &\le 3\|\hat{\boldsymbol{f}}^i\|^2 + C(\varepsilon)(\|\nabla\bar{\boldsymbol{u}}^i\|^6 + \|\nabla\bar{\boldsymbol{u}}^{i-1}\|^6) + \varepsilon(\|\Delta\bar{\boldsymbol{u}}^i\|^2 + \|\Delta\bar{\boldsymbol{u}}^{i-1}\|^2). \end{aligned} \tag{5.82}$$

As a result, combining other estimates obtained in **Step 1** in the proof of Theorem 7, (5.23) becomes

(5.83)
$$\|\nabla\bar{\boldsymbol{u}}^m\|^2+\delta t\sum_{i=0}^{m}\|\Delta\bar{\boldsymbol{u}}^i\|^2\le C\delta t\sum_{i=0}^{m-1}\|\nabla\bar{\boldsymbol{u}}^i\|^6+C\delta t\sum_{i=0}^{m-1}\|\nabla\bar{\boldsymbol{u}}^i\|^2+CTC_f^2+M_0\ \forall m\le n,$$

where $M_0>0$ and C_f are the same as in Theorem 7. Now, if we define Φ as $\Phi(x)=x^3+x$ and let

$$0<T_*<\int_{CTC_f^2+M_0}^{\infty}dz/\Phi(z), \tag{5.84}$$

then Lemma 3 implies that there exists $C_*>0$ independent of δt and C_0 such that

$$\|\nabla\bar{\boldsymbol{u}}^m\|^2+\delta t\sum_{i=0}^{m}\|\Delta\bar{\boldsymbol{u}}^i\|^2\le C_*\quad\forall m\le n\le\frac{T_*}{\delta t}. \tag{5.85}$$

Since $\boldsymbol{u}^i=\eta^i\bar{\boldsymbol{u}}^i$ and $|\eta^i|>\frac{1}{2}$ for all $i\le n$, we also have

$$\delta t\sum_{i=0}^{m}\|\Delta\boldsymbol{u}^i\|^2\le 2C_*\quad\forall m\le n\le\frac{T_*}{\delta t}. \tag{5.86}$$

Now, with (5.85) and (5.86) holding true, we can then prove (5.80) by following the same procedures as in **Steps 2** and **3** in the proof of Theorem 7. ∎

6. Numerical validation and concluding remarks. We first provide a numerical example to verify the convergence rate of our numerical scheme, followed by some concluding remarks.

6.1. Numerical results. *Example* 1. In the first example, we validate the convergence order of the new schemes. Consider the Navier–Stokes equations (1.1) in $\Omega=(-1,1)\times(-1,1)$ with no-slip boundary condition. The exact solutions are given as

$$\begin{aligned}\boldsymbol{u}_1(x,y,t)&=\sin(2\pi y)\sin^2(\pi x)\sin(t);\\ \boldsymbol{u}_2(x,y,t)&=\sin(2\pi x)\sin^2(\pi y)\sin(t);\\ p(x,y,t)&=\cos(\pi x)\sin(\pi y)\sin(t).\end{aligned}$$

We set $\nu=1$ in (1.1a) and use the Spectral-Galerkin [27] method with 64×64 modes in space so that the spatial discretization error is negligible compared with the time discretization error. In Figure 2, we plot the convergence rate of the L^2 error for the velocity $error_{\boldsymbol{u}}$ and the L^2 error for the pressure $error_p$ at $T=1$ by using the second-order scheme (4.1) with $k=5$. We can observe the expected convergence rates.

In the second example, we aim to show the advantages of the new schemes with $k=5$ over the usual BDF2 (with $k=1$) and investigate the effect of the GSAV.

Example 2. Consider the Navier–Stokes equations (1.1) in $\Omega=(-1,1)\times(-1,1)$ with no-slip boundary condition. The initial condition is given as

$$\begin{aligned}u_1(x,y,0)&=\begin{cases}(1+x)(1-x)\tanh(\rho(y+0.5)),\ y\le 0,\\ (1+x)(1-x)\tanh(\rho(0.5-y)),\ y>0,\end{cases}\\ u_2(x,y,0)&=\delta\sin(\pi x).\end{aligned}$$

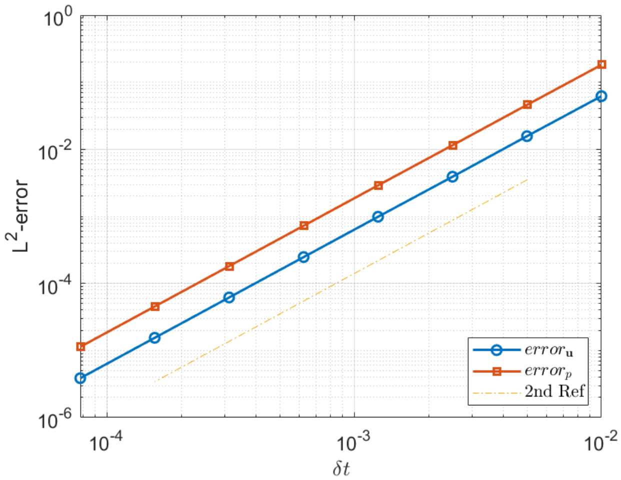
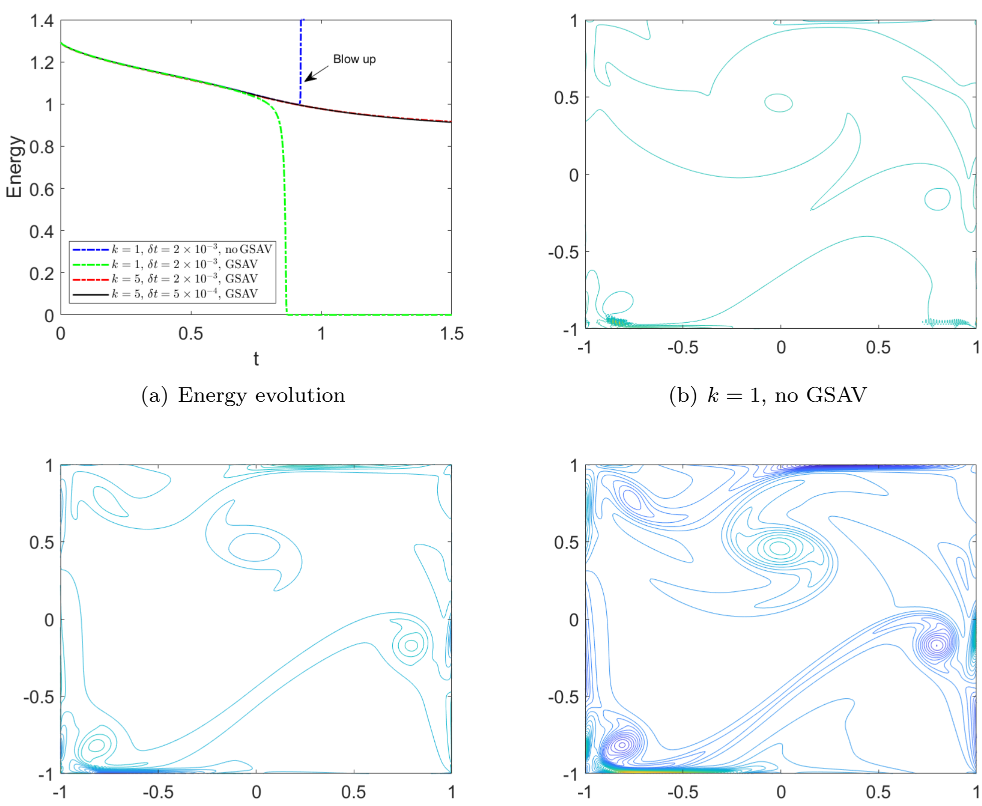


FIG. 2. *Convergence test of the second-order scheme for the Navier–Stokes equations.*



(a) Energy evolution

(b) $k = 1$, no GSAV

(c) $k = 1$, with GSAV

(d) $k = 5$, with GSAV

FIG. 3. *Energy evolution and snapshots at time* $T = 0.88$. *All three snapshots are generated with* $\delta t = 2 \times 10^{-3}$.

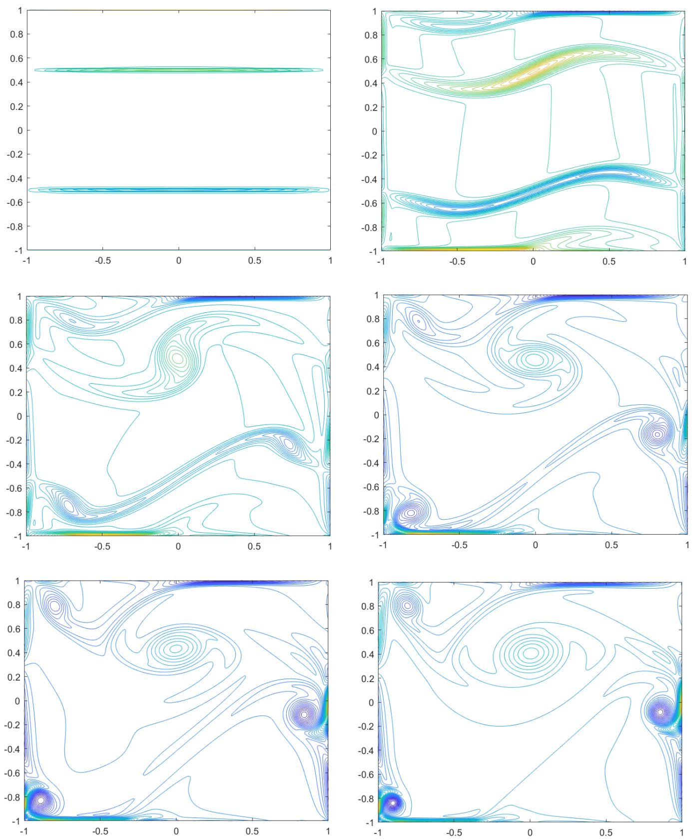


Fig. 4. *Snapshots of the vorticity contours at different times. Top:* $T = 0, 0.3, 0.6$. *Bottom:* $T = 0.9, 1.2, 1.5$. *The data are obtained by choosing* $k = 5$, $\delta t = 2 \times 10^{-3}$.

Note that this initial condition is not divergence free, but its divergence will quickly approach zero in a few steps. We fix $\boldsymbol{f} = 0$, $\nu = 0.002$ in (1.1), and other parameters in the initial condition are chosen as $\rho = 100$, $\delta = 0.5$. We use the spectral-Galerkin [27] method with 256×256 modes in space. In Figure 3, we plot the energy evolution of the system by choosing different k and δt and the snapshots of the vorticity contours at time $T = 0.88$, at which time the numerical solutions from the BDF2 scheme with $k = 1$ are no longer correct. From Figure 3, we observe that with a large time step $\delta t = 2 \times 10^{-3}$, the usual BDF2 scheme (with $k = 1$) without GSAV is unstable, and with the GSAV, the numerical solution is wrong although it does not blow up. On the other hand, within the same $\delta t = 2 \times 10^{-3}$, the new scheme (with $k = 5$) leads to a correct numerical solution which is the same as that obtained by using a smaller time step $\delta t = 5 \times 10^{-4}$. In Figure 4, we plot some snapshots of the vorticity contours, which are generated by choosing $k = 5$ and $\delta t = 2 \times 10^{-3}$.

6.2. Concluding remarks. We presented in this paper a new second-order consistent splitting scheme for the Navier–Stokes equations with no-slip boundary conditions based on (i) the Taylor expansions at time t^{n+5} which offer better stability than the usual expansion at time t^{n+1}, and (ii) the generalized scalar auxiliary variable (GSAV) approach which allows us to treat the nonlinear term explicitly while maintaining unconditional stability. Thanks to the extra stability afforded by the new scheme, we were able to establish its unconditional stability for the time dependent Stokes equations by using a clever energy argument. For the Navier–Stokes equations, we first established a weaker stability result using the GSAV approach. Then, by using the strong stability result for the linear equations and the weaker stability result for the nonlinear equations, we established error estimates in the strong norm which is global-in-time in 2D and a local-in-time in 3D for the Navier–Stokes equations.

To the best of our knowledge, this is the first rigorous stability and error analysis for any second-order consistent splitting scheme for the Navier–Stokes equations. The new idea (constructing scheme with Taylor expansion at time t^{n+k}) introduced in this paper opened up a new avenue in construct schemes with better stability for other problems. We are currently investigating the possibility of constructing higher-order unconditional stable consistent splitting schemes for the Navier–Stokes equations.

REFERENCES

[1] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Ser. Comput. Math. 15, Springer Science & Business Media, 2012.

[2] W. E and J.-G. Liu, *Projection method* I*: Convergence and numerical boundary layers*, SIAM J. Numer. Anal., 32 (1995), pp. 1017–1057, https://doi.org/10.1137/0732047.

[3] W. E and J. G. Liu, *Gauge method for viscous incompressible flows*, Commun. Math. Sci., 1 (2003), pp. 317–332.

[4] H. C. Elman, D. J. Silvester, and A. J. Wathen, *Finite Elements and Fast Iterative Solvers: With Applications in Incompressible Fluid Dynamics*, Oxford University Press, Oxford, 2014.

[5] V. Girault and P. A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Lecture Notes in Math. 749, Springer-Verlag, Berlin, New York, 1979.

[6] J. Guermond and J. Shen, *A new class of truly consistent splitting schemes for incompressible flows*, J. Comput. Phys., 192 (2003), pp. 262–276.

[7] J. L. Guermond, P. Minev, and J. Shen, *An overview of projection methods for incompressible flows*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 6011–6045.

[8] J.-L. Guermond, P. D. Minev, and A. J. Salgado, *Convergence Analysis of a Class of Massively Parallel Direction Splitting Algorithms for the Navier-Stokes Equations*, preprint, arXiv:1101.3587, 2011.

[9] J.-L. Guermond and A. J. Salgado, *Error analysis of a fractional time-stepping technique for incompressible flows with variable density*, SIAM J. Numer. Anal., 49 (2011), pp. 917–944, https://doi.org/10.1137/090768758.

[10] J. L. Guermond and J. Shen, *Velocity-correction projection methods for incompressible flows*, SIAM J. Numer. Anal., 41 (2003), pp. 112–134, https://doi.org/10.1137/S0036142901395400.

[11] J. L. Guermond and J. Shen, *On the error estimates for the rotational pressure-correction projection methods*, Math. Comp., 73 (2004), pp. 1719–1737.

[12] F. Huang and J. Shen, *A new class of implicit-explicit BDFk SAV schemes for general dissipative systems and their error analysis*, Comput. Methods Appl. Mech. Engrg., 392 (2022), 114718.

[13] H. Johnston and J. G. Liu, *Accurate, stable and efficient Navier-Stokes solvers based on explicit treatment of the pressure term*, J. Comput. Phys., 199 (2004), pp. 221–259.

[14] G. E. Karniadakis, M. Israeli, and S. A. Orszag, *High-order splitting methods for the incompressible Navier-Stokes equations*, J. Comput. Phys., 97 (1991), pp. 414–443.

[15] X. Li and H. Rui, *Superconvergence of characteristics marker and cell scheme for the Navier–Stokes equations on nonuniform grids*, SIAM J. Numer. Anal., 56 (2018), pp. 1313–1337, https://doi.org/10.1137/18M1175069.

[16] J.-G. Liu, J. Liu, and R. L. Pego, *Stability and convergence of efficient Navier-Stokes solvers via a commutator estimate*, Comm. Pure Appl. Math., 60 (2007), pp. 1443–1487.

[17] J.-G. Liu, J. Liu, and R. L. Pego, *Error estimates for finite-element Navier-Stokes solvers without standard inf-sup conditions*, Chinese Ann. Math. Ser. B, 30 (2009), pp. 743–768.

[18] J.-G. Liu and R. Pego, *Stable discretization of magnetohydrodynamics in bounded domains*, Commun. Math. Sci., 8 (2010), pp. 235–251.
[19] R. Nochetto and J.-H. Pyo, *Error estimates for semi-discrete gauge methods for the Navier-Stokes equations*, Math. Comp., 74 (2005), pp. 521–542.
[20] S. A. Orszag, M. Israeli, and M. O. Deville, *Boundary conditions for incompressible flows*, J. Sci. Comput., 1 (1986), pp. 75–111.
[21] A. Prohl, *Projection and Quasi-compressibility Methods for Solving the Incompressible Navier-Stokes Equations*, Springer, 1997.
[22] H. Rui and X. Li, *Stability and superconvergence of MAC scheme for Stokes equations on nonuniform grids*, SIAM J. Numer. Anal., 55 (2017), pp. 1135–1158, https://doi.org/10.1137/15M1050550.
[23] J. Shen, *Long time stability and convergence for fully discrete nonlinear Galerkin methods*, Appl. Anal., 38 (1990), pp. 201–229.
[24] J. Shen, *On error estimates of projection methods for Navier–Stokes equations: First-order schemes*, SIAM J. Numer. Anal., 29 (1992), pp. 57–77, https://doi.org/10.1137/0729004.
[25] J. Shen, *A remark on the projection-3 method*, Internat. J. Numer. Methods Fluids, 16 (1993), pp. 249–253.
[26] J. Shen, *Modeling and numerical approximation of two-phase incompressible flows by a phase-field approach*, in Multiscale Modeling and Analysis for Materials Simulation, World Scientific, Hackensack, NJ, 2012, pp. 147–195.
[27] J. Shen, T. Tang, and L.-L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer Ser. Comput. Math. 41, Springer Science & Business Media, 2011.
[28] J. Shen and X. Yang, *Error estimates for finite element approximations of consistent splitting schemes for incompressible flows*, Discrete Contin. Dyn. Syst. Ser. B, 8 (2007), pp. 663–676.
[29] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1983.
[30] L. J. Timmermans, P. D. Minev, and F. N. van De Vosse, *An approximate projection scheme for incompressible flow using spectral elements*, Internat. J. Numer. Methods Fluids, 22 (1996), pp. 673–688.
[31] C. Wang and J. G. Liu, *Convergence of gauge method for incompressible flow*, Math. Comp., 69 (2000), pp. 1385–1407.
[32] K. Wu, F. Huang, and J. Shen, *A new class of higher-order decoupled schemes for the incompressible Navier-Stokes equations and applications to rotating dynamics*, J. Comput. Phys., 458 (2022), 111097.