On Spectral Approximations Using Modified Legendre Rational Functions: Application to the Korteweg-de Vries Equation on the Half Line

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Dedicated to Ciprian Foias on the occasion of his retirement and to Roger Temam on the occasion of his 60th birthday.

ABSTRACT. A new set of modified Legendre rational functions which are mutually orthogonal in $L^2(0, +\infty)$ is introduced. Various projection and interpolation results using the modified Legendre rational functions are established. These results form the mathematical foundation of related spectral and pseudospectral methods for solving partial differential equations on the half line. A spectral scheme using the modified Legendre rational functions for the Korteweg-de Vries equation on the half line is investigated. The numerical solution of the scheme is shown to possess the essential conservation properties satisfied by the solution of the Korteweg-de Vries equation. The spectral convergence of the proposed scheme is established.

1. INTRODUCTION

How to accurately and efficiently solve partial differential equations in unbounded domains is a very important subject since many problems arising in science and engineering are set in unbounded domains, yet it is also a very difficult subject since the unboundedness of the domain introduce considerable theoretical and practical challenges which are not present in bounded domains.

While spectral approximations for partial differential equations (PDEs) in bounded domains have achieved great success and popularity in recent years (see e.g. [12, 7, 2]), spectral approximations for PDEs in unbounded domains have only received limited attention. Recently, a number of different spectral methods have been proposed for problems in unbounded domains: a first approach is to use spectral approximations associated with existing orthogonal systems such as the Laguerre or Hermite polynomials/functions, see, e.g., Maday, Pernaud-Thomas and Vandeven [26], Funaro and Kavian [11], Guo [15, 16], Guo and Shen [19], and Shen [27]; a second approach is to use a suitable mapping to reformulate the original problems in unbounded domains to singular/degenerate problems in bounded domains, and then use a suitable Jacobi approximation to treat the singular/degenerate problems [14, 17, 18]; another class of spectral methods is based on rational approximations, for example, Christov [8] and Boyd [4, 5] proposed some spectral methods on infinite intervals by using certain mutually orthogonal system of rational functions.

Recently, Guo, Shen, and Wang [20] proposed and analyzed a set of Legendre rational functions which are mutually orthogonal in $L^2_\chi(0,\infty)$ with a non-uniform weight function

$$\chi(x) = (x+1)^{-2}.$$

However, the non-uniform weight $\chi(x)$ may introduce serious difficulties in analysis and implementation for PDEs with global conservation properties. In particular, the numerical solutions based on a weighted formulation may not preserve these conservation properties satisfied. For example, the solutions of some important nonlinear differential equations, such as the system of conservation laws, the non-parabolic dissipative systems, the Schödinger equation, the Dirac equation, and the Korteweg-de Vries equation possess certain conservation properties which are essential in the theoretical analysis of these equations. Therefore, it is important that the numerical solutions satisfy as many conservation properties as possible. However, the numerical solutions based on a weighted formulations usually will not satisfy any of these conservation properties.

In this paper, we introduce a new set of modified Legendre rational functions which are mutually orthogonal with the uniform weight $\chi(x) \equiv 1$, and so the numerical solutions possess the essential conservation properties satisfied by the solutions of original problems. In the next section, we introduce the new set of orthogonal rational functions induced by the Legendre polynomials, and derive some of its basic properties. In Section 3, we study several orthogonal projection operators and derive optimal approximation results associated with them. Since we are interested in the approximation of Korteweg-de Vries equation that involves a third-order derivative operator, additional projection operators are needed and very delicate analyses using special recursively defined operators are introduced. In Section 4, we study the interpolation operators based on the Gauss and Gauss-Radau quadratures. In Section 5, we take the Korteweg-de Vries equation on the half line as an example to show how the modified Legendre rational spectral method would work for nonlinear problems with essential conservation properties. A modified Legendre rational spectral scheme is proposed and analyzed. We would like to emphasize that it seems impossible to derive a convergence result if the usual Legendre rational spectral method in [20] is used for approximating the Korteweg-de Vries equation.

2. Modified Legendre rational functions

Let us denote $\Lambda = \{x \mid 0 < x < \infty\}$. For $1 \le p \le \infty$, let

$$L^p(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda, \text{ and } \|v\|_{L^p} < \infty\},$$

where

$$\|v\|_{L^p} = \begin{cases} \left(\int_\Lambda |v(x)|^p \, dx\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\Lambda} |v(x)|, & p = \infty. \end{cases}$$

We denote in particular $\|v\|_\infty = \|v\|_{L^\infty(\Lambda)}$.

Let (u, v) and $\|v\|$ be respectively the inner product and the norm of the space $L^2(\Lambda)$, i.e.,

$$(u, v) = \int_\Lambda u(x)v(x)\, dx, \quad \|v\| = (v, v)^{1/2}.$$

For any non-negative integer m, we set

$$H^m(\Lambda) = \left\{ v \mid \partial_x^k v = \frac{d^k v}{dx^k} \in L^2(\Lambda),\ 0 \le k \le m \right\},$$

equipped with the inner product, the semi-norm, and the norm as follows:

$$(u, v)_{m,\chi} = \sum_{k=0}^{m} (\partial_x^k u, \partial_k^k v), \quad |v|_{m,\chi} = \|\partial_x^m v\|, \quad \|v\|_{m,\chi} = (v, v)_{m,\chi}^{1/2}.$$

For any real number $r > 0$, we define the space $H^r(\Lambda)$ with the norm $\|v\|_r$ by space interpolation as in Adams [1].

We denote by $L_\ell(x)$ the Legendre polynomial of degree ℓ, which is the eigenfunction of the singular Sturm-Liouville problem

$$\partial_x((1 - x^2)\partial_x v(x)) + \lambda v(x) = 0, \tag{2.1}$$

with the corresponding eigenvalues $\lambda_\ell = \ell(\ell + 1)$, $\ell = 0, 1, 2, \ldots$. They satisfy the recurrence relations

$$L_{\ell+1}(x) = \frac{2\ell + 1}{\ell + 1} x L_\ell(x) - \frac{\ell}{\ell + 1} L_{\ell-1}(x), \quad \ell \ge 1, \tag{2.2}$$

$$(2\ell + 1)L_\ell(x) = \partial_x L_{\ell+1}(x) - \partial_x L_{\ell-1}(x), \quad \ell \ge 1. \tag{2.3}$$

We also have the following identities:

$$
(2.4)\qquad \left|\begin{array}{ll} L_\ell(1) = 1, & L_\ell(-1) = (-1)^\ell, \\ \partial_x L_\ell(1) = \frac{1}{2}\ell(\ell+1), & \partial_x L_\ell(-1) = (-1)^{\ell+1}\frac{1}{2}\ell(\ell+1). \end{array}\right.
$$

We define the modified Legendre rational functions of degree ℓ by

$$
R_\ell(x) = \frac{\sqrt{2}}{x+1} L_\ell\left(\frac{x-1}{x+1}\right), \quad \ell = 0, 1, 2, \dots,
$$

By (2.1), $R_\ell(x)$ are the eigenfunctions of the singular Sturm-Liouville problem

$$
(2.5)\qquad (x+1)\partial_x(x(\partial_x((x+1)v(x))) + \lambda v(x) = 0, \quad x \in \Lambda,
$$

with the corresponding eigenvalues $\lambda_\ell = \ell(\ell+1)$, $\ell = 0, 1, 2, \dots$. Due to (2.2) and (2.3), they satisfy the recurrence relations

$$
(2.6)\qquad R_{\ell+1}(x) = \frac{2\ell+1}{\ell+1}\frac{x-1}{x+1}R_\ell(x) - \frac{\ell}{\ell+1}R_{\ell-1}(x), \quad \ell \geqslant 1,
$$

$$
(2.7)\qquad 2(2\ell+1)R_\ell(x) = (x+1)^2(\partial_x R_{\ell+1}(x) - \partial_x R_{\ell-1}(x)) + (x+1)(R_{\ell+1}(x) - R_{\ell-1}(x)).
$$

Furthermore,

$$
(2.8)\qquad \lim_{x\to\infty}(x+1)R_\ell(x) = \sqrt{2}, \quad \lim_{x\to\infty} x\partial_x((x+1)R_\ell(x)) = 0,
$$

By the orthogonality of the Legendre polynomials,

$$
(2.9)\qquad \int_\Lambda R_\ell(x)R_m(x)\,dx = \left(\ell + \frac{1}{2}\right)^{-1}\delta_{\ell,m},
$$

where $\delta_{\ell,m}$ is the Kronecker function. Thus the modified Legendre rational expansion of a function $v \in L^2(\Lambda)$ is

$$
v(x) = \sum_{\ell=0}^{\infty} \hat{v}_\ell R_\ell(x), \quad \text{with } \hat{v}_\ell = \left(\ell + \frac{1}{2}\right)\int_\Lambda v(x)R_\ell(x)\,dx.
$$

Let $R^1_\ell(x) = \partial_x((x+1)R_\ell(x))$ and $\omega_1(x) = x$. By virtue of (2.5), (2.8), and integration by parts,

$$
(2.10)\qquad \int_\Lambda R^1_\ell(x)R^1_m(x)\omega_1(x)\,dx = \ell(\ell+1)\left(\ell + \frac{1}{2}\right)^{-1}\delta_{\ell,m}.
$$

Hence, $\{R^1_\ell\}$ form a set of orthogonal rational functions in $L^2_{\omega_1}(\Lambda)$.

We end this section with an inverse inequality for the modified Legendre rational functions. Let N be any positive integer, and

$$\mathcal{R}_N = \text{span}\,\{R_0, R_1, \ldots, R_N\}.$$

Hereafter, c denotes a generic positive constant, independent of any function and N.

Theorem 2.1. *For any $\varphi \in \mathcal{R}_N$ and $r \geq 0$,*

$$|\varphi|_r \leq cN^{2r}\|(x+1)^{-r}\varphi\|.$$

Proof. Let $\tilde{\Lambda} = (-1,1)$. Then, $x = (1+y)/(1-y)$ maps $y \in \tilde{\Lambda}$ to $x \in \Lambda$. For any $\varphi \in \mathcal{R}_N$, we set $\psi(y) = \varphi((1+y)/(1-y))$. Obviously, we can write $\psi(y) = \frac{1}{2}(1-y)\psi_N(y)$ with some function $\psi_N \in \mathcal{P}_N$, where $\mathcal{P}_N$ is the set of all polynomials of degree at most N. By direct computation, we have

$$\|\partial_x\varphi\|^2 = \frac{1}{2}\int_{\tilde{\Lambda}} (\partial_y\psi(y))^2(1-y)^2\,dy, \tag{2.11}$$

and

$$\partial_y\psi(y) = \frac{1}{2}(1-y)\partial_y\psi_N(y) - \frac{1}{2}\psi_N(y). \tag{2.12}$$

Let $\chi^{(\alpha,\beta)}(y) = (1-y)^\alpha(1+y)^\beta$, α, $\beta > -1$. By Theorem 2.2 in Guo [17],

$$\|\partial_y\eta\|_{\chi^{(\alpha,\beta)}} \leq cN^2\|\eta\|_{\chi^{(\alpha,\beta)}}, \quad \forall\eta \in \mathcal{P}_N. \tag{2.13}$$

The above with (2.11) and (2.12) leads to

$$\begin{aligned}\|\partial_x\varphi\|^2 &\leq c\int_{\tilde{\Lambda}}(\partial_y\psi_N(y))^2(1-y)^4\,dy + c\int_{\tilde{\Lambda}}(\psi_N(y))^2(1-y)^2\,dy\\ &\leq cN^4\int_{\tilde{\Lambda}}(\psi_N(y))^2(1-y)^2\,dy = cN^4\int_{\tilde{\Lambda}}(\psi(y))^2\,dy\\ &= cN^4\|(x+1)^{-1}\varphi\|^2.\end{aligned}$$

Repeating the above procedure, we obtain the result for any non-negative integer r. Using a standard technique of space interpolation, we obtain the result for positive non-integer r. ◻

3. Projections using modified Legendre rational functions

In this section, we study several projection operators on the half line based on the modified Legendre rational functions.

The $L^2(\Lambda)$-orthogonal projection $P_N : L^2(\Lambda) \to \mathcal{R}_N$ is a mapping such that

$$(P_N v - v, \varphi) = 0, \quad \forall \varphi \in \mathcal{R}_N.$$

We introduce below a sequence of recursively defined Hilbert spaces, which play an essential role in the analysis of spectral methods based on the modified Legendre rational functions.

For any non-negative integer r, we define

$$H^r_{A_0}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda, \text{ and } \|v\|_{r,A_0} < \infty\},$$

with the norm

$$\|v\|_{r,A_0} = \Big(\sum_{k=0}^{r} \|(x+1)^{r/2+k}\partial_x^k v\|^2\Big)^{1/2}.$$

For any positive integers r, q such that $r \geq q \geq 1$, we define

$$H^r_{A_q}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda, \text{ and } \|v\|_{r,A_q} < \infty\},$$

with the norm

$$\|v\|_{r,A_q} = \|(x+1)\partial_x((x+1)v)\|_{r-1,A_{q-1}}.$$

For any real positive r such that $r \geq q$, the space $H^r_{A_q}(A)$ is defined by space interpolation. Let A be the Sturm-Liouville operator in (2.5), namely

$$Av(x) = -(x+1)\partial_x(x\partial_x((x+1)v(x))).$$

We can verify by induction that for any non-negative integer m,

$$A^m v(x) = \sum_{k=1}^{2m} (x+1)^{m+k} p_k(x)\partial_x^k v(x), \tag{3.1}$$

where $p_k(x)$ are some rational functions which are bounded uniformly on the whole interval Λ. Hence, A^m is a continuous mapping from $H^{2m}_{A_0}(\Lambda)$ to $L^2(\Lambda)$.

Theorem 3.1. *For any $v \in H^r_{A_0}(\Lambda)$ and $r \geq 0$,*

$$\|P_N v - v\| \leq cN^{-r}\|v\|_{r,A_0}.$$

Proof. We first assume that $r = 2m$. By virtue of (2.5), (2.9), and integration by parts,

$$
\begin{aligned}
\hat{v}_\ell &= \frac{2\ell+1}{2\ell(\ell+1)}\int_\Lambda v(x)A, R_\ell(x)\,dx \\
&= \frac{2\ell+1}{2\ell(\ell+1)}\int_\Lambda x\partial_x((x+1)v(x))\partial_x((x+1)R_\ell(x))\,dx \\
&= -\frac{2\ell+1}{2\ell(\ell+1)}\int_\Lambda (x+1)\partial_x(x\partial_x((x+1)v(x)))R_\ell(x)\,dx \\
&= \frac{2\ell+1}{2\ell(\ell+1)}\int_\Lambda Av(x)R_\ell(x)\,dx \\
&\qquad\vdots \\
&= \frac{2\ell+1}{2\ell^m(\ell+1)^m}\int_\Lambda A^m v(x)R_\ell(x)\,dx,
\end{aligned}
$$

Therefore, we derive from the above and the definition of $H^r_{A_0}(\Lambda)$ that

$$
\begin{aligned}
\|P_N v - v\|^2 &= \sum_{\ell=N+1}^{\infty}\hat{v}_\ell^2\|R_\ell\|^2 \le cN^{-4m}\sum_{\ell=N+1}^{\infty}\left(\frac{\int_\Lambda A^m v(x)R_\ell(x)\,dx}{\|R_\ell\|^2}\right)^2\|R_\ell\|^2 \\
&\le cN^{-4m}\|A^m v\|^2 \le cN^{-2r}||v||^2_{r,A_0}.
\end{aligned}
$$

Next, let $r = 2m+1$. By (2.5) and integration by parts, we have

$$
\begin{aligned}
\hat{v}_\ell &= \frac{2\ell+1}{2\ell^m(\ell+1)^m}\int_\Lambda A^m v(x)R_\ell(x)\,dx \\
&= -\frac{2\ell+1}{2\ell^{m+1}(\ell+1)^{m+1}}\int_\Lambda (x+1)A^m v(x)\partial_x(xR^1_\ell(x))\,dx \\
&= \frac{2\ell+1}{2\ell^{m+1}(\ell+1)^{m+1}}\int_\Lambda \partial_x((x+1)A^m v(x))R^1_\ell(x)\omega_1(x)\,dx.
\end{aligned}
$$

Thanks to (2.9), (2.10), and (3.1),

$$
\begin{aligned}
&\|P_N v - v\|^2 \\
&= \sum_{\ell=N+1}^{\infty}\frac{2\ell+1}{2(\ell(\ell+1))^{2m+2}}\left(\int_\Lambda \partial_x((x+1)A^m v(x))R^1_\ell(x)\omega_1(x)\,dx\right)^2 \\
&\le cN^{-2(2m+1)}\sum_{\ell=N+1}^{\infty}\left(\frac{\int_\Lambda \partial_x((x+1)A^m v(x))R^1_\ell(x)\omega_1(x)\,dx}{||R^1_\ell||^2_{\omega_1}}\right)^2||R^1_\ell||^2_{\omega_1} \\
&\le cN^{-2(2m+1)}||\partial_x((x+1)A^m v)||^2_{\omega_1} \le cN^{-2r}||v||^2_{r,A_0}.
\end{aligned}
$$

The proof is complete. ◻

We now consider the $H^m(\Lambda)$-orthogonal projection $P_N^m : H^m(\Lambda) \to \mathcal{R}_N$, which is defined by

$$(P_N^m v - v, \varphi)_m = 0, \quad \forall \varphi \in \mathcal{R}_N.$$

Theorem 3.2. *For any $v \in H^r_{A_m}(\Lambda)$ and $0 \le m \le r$,*

$$\|P_N^m v - v\|_m \le cN^{m-r}\|v\|_{r,A_m}.$$

Proof. We shall prove the result by induction on m. Clearly, Theorem 3.1 implies the desired result for $m = 0$. Now, we assume that the conclusion is true for P_N^k, $0 \le k \le m-1$. Given $v \in H^r_{A_m}(\Lambda)$, we introduce

$$u(x) = \int_0^x (z+1)\partial_z((z+1)v(z))\,dz. \tag{3.2}$$

Then $\partial_x u(x) = (x+1)\partial_x((x+1)v(x))$, and

$$v(x) = \frac{1}{x+1}\Big(\int_0^x (z+1)^{-1}\partial_z u(z)\,dz + v(0)\Big). \tag{3.3}$$

We also introduce

$$\varphi(x) = \frac{1}{x+1}\Big(\int_0^x (z+1)^{-1}P_{N-1}^{m-1}\partial_z u(z)\,dz + v(0)\Big). \tag{3.4}$$

By the definition of P_N^{m-1}, there exists a polynomial $q_{N-1} \in \mathcal{P}_{N-1}$ such that $P_{N-1}^{m-1}\partial_z u(z) = 1/(z+1)q_{N-1}((z-1)/(z+1))$. Therefore,

$$\begin{aligned}\varphi(x) &= \frac{1}{x+1}\left(\int_0^x (z+1)^{-2}q_{N-1}\left(\frac{z-1}{z+1}\right)dz + v(0)\right)\\ &= \frac{1}{2(x+1)}\left(\int_{-1}^{(x-1)/(x+1)} q_{N-1}(y)\,dy + 2v(0)\right),\end{aligned}$$

which implies that $\varphi \in \mathcal{R}_N$. By the Hardy inequality [22], (3.2)-(3.4), and the assumption of the induction, we have

$$\begin{aligned}\|\varphi - v\|^2 &\le \int_0^\infty \left(\frac{1}{x}\int_0^x (z+1)^{-1}(P_{N-1}^{m-1}\partial_z u(z) - \partial_z u(z))\,dz\right)^2 dx \\ &\le 4\int_0^\infty (x+1)^{-2}(P_{N-1}^{m-1}\partial_x u(x) - \partial_x u(x))^2\,dx \\ &\le 4\|P_{N-1}^{m-1}\partial_x u - \partial_x u\|^2 \le cN^{2m-2r}||\partial_x u||^2_{r-1,A_{m-1}} \\ &= cN^{2m-2r}||(x+1)\partial_x((x+1)v)||^2_{r-1,A_{m-1}} \\ &= cN^{2m-2r}\|v\|_{r,A_m}.\end{aligned} \tag{3.5}$$

On the other hand,

$$\partial_x \varphi(x) - \partial_x v(x) = \frac{1}{(x+1)^2}(P_{N-1}^{m-1}\partial_x u(x) - \partial_x u(x)) - \frac{1}{(x+1)^2}\int_0^x (z+1)^{-1}(P_{N-1}^{m-1}\partial_z u(z) - \partial_z u(z))\,dz. \tag{3.6}$$

By an induction argument, we can derive from (3.6) that for $1 \le k \le m$,

$$\partial_x^k \varphi(x) - \partial_x^k v(x) = F_k(x) + G_k(x),$$

where

$$F_k(x) = \sum_{j=0}^{k-1} d_j (x+1)^{-k+j-1}\partial_x^j (P_{N-1}^{m-1}\partial_x u(x) - \partial_x u(x)),$$
$$G_k(x) = (-1)^k k!(x+1)^{-k-1}\int_0^x (z+1)^{-1}(P_{N-1}^{m-1}\partial_z u(z) - \partial_z u(z))\,dz,$$

with d_j being some constants independent of N and u. Thus, by a similar argument as in (3.5), we can show that for $1 \le k \le m$,

$$\|F_k\| \le c\|P_{N-1}^{m-1}\partial_x u - \partial_x u\|_{k-1} \le cN^{2m-2r}||\partial_x u||_{r-1,A_{m-1}}^2 = cN^{2m-2r}||v||_{r,A_m}^2.$$

Similarly, we have

$$\|G_k\| \le cN^{2m-2r}||v||_{r,A_m}^2.$$

Hence,

$$\|P_N^m v - v\|_m \le \|\varphi - v\|_m \le cN^{m-r}\|v\|_{r,A_m}.$$

The proof is complete. □

When we apply the modified Legendre rational approximation to numerical solutions of differential equations with boundary conditions, we need to use orthogonal projections with built-in boundary conditions. To this end, let

$$H_0^m(\Lambda) = \{v \mid v \in H^m(\Lambda) \text{ and } \partial_x^k v(0) = 0, \text{ for } 0 \le k \le m-1\},$$

and $\mathcal{R}_N^{m,0} = \mathcal{R}_N \cap H_0^m(\Lambda)$. We denote in particular $\mathcal{R}_N^0 = \mathcal{R}_N^{1,0}$.

We define the orthogonal projection $P_N^{m,0} : H_0^m(\Lambda) \to \mathcal{R}_N^{m,0}$ by

$$(P_N^{m,0} v - v, \varphi)_m = 0, \quad \forall \varphi \in R_N^{m,0}.$$

Theorem 3.3. *For any* $v \in H^r_{A_m}(\Lambda) \cap H^m_0(\Lambda)$ *and* $0 \le m \le r$,

$$\|P^{m,0}_N v - v\|_m \le cN^{m-r}\|v\|_{r,A_m},$$

Proof. We define $u(x)$ as in (3.2), and set

$$\varphi(x) = \frac{1}{x+1}\int_0^x (z+1)^{-1} P^{m-1,0}_{N-1} \partial_z u(z)\, dz.$$

Clearly, $\varphi \in \mathcal{R}^{m,0}_N$. The desired result follows from the same argument as in the proof of Theorem 3.2. □

In order to analyze the modified Legendre rational approximation for the Korteweg-de Vries equation (see Section 5), we need another orthogonal projection. Let $\widetilde{H}^m_0(\Lambda) = H^m(\Lambda) \cap H^1_0(\Lambda)$. We define the orthogonal projection $\widetilde{P}^{m,0}_N : \widetilde{H}^m_0(\Lambda) \to \mathcal{R}^0_N$ by

$$(\widetilde{P}^{m,0}_N v - v, \varphi)_m = 0, \quad \forall \varphi \in \mathcal{R}^0_N.$$

Theorem 3.4. *For any* $v \in H^r_{A_m}(\Lambda) \cap \widetilde{H}^m_0(\Lambda)$ *and* $0 \le m \le r$,

$$\|\widetilde{P}^{m,0}_N v - v\|_m \le cN^{m-r}\|v\|_{r,A_m}.$$

Proof. Again we define $u(x)$ as in (3.2), and

$$\varphi(x) = \frac{1}{x+1}\int_0^x (z+1)^{-1} \widetilde{P}^{m-1,0}_{N-1} \partial_z u(z)\, dz \in \mathcal{R}^0_N.$$

Then, the result can be established in the same manner as in the proof of Theorem 3.2. □

We will need another special projection operator in the analysis of modified Legendre rational interpolations. To this end, we set

$$H^1_{\hat{A}_0}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda, \text{ and } \|v\|_{r,\hat{A}_0} < \infty\},$$

with

$$\|v\|_{1,\hat{A}_0} = (\|(x+1)^{3/2}\partial_x v\|^2 + \|v\|^2)^{1/2},$$

and

$$H^r_B(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda, \text{ and } \|v\|_{r,B} < \infty\},$$

with

$$\|v\|_{r,B} = \|(x+1)\partial_x((x+1)^2 v)\|_{r-1,A_0}.$$

Now, we define the $H^1_{\hat{A}_0}$-orthogonal projection $\hat{P}^1_N : H^1_{\hat{A}_0}(\Lambda) \to \mathcal{R}_N$ by

$$\int_\Lambda \partial_x(\hat{P}^1_N v(x) - v(x))\partial_x \varphi(x)(x+1)^3\, dx + \int_\Lambda (\hat{P}^1_N v(x) - v(x))\varphi(x) dx = 0, \qquad \forall \varphi \in R_N.$$

Theorem 3.5. *For any $v \in H^r_B(\Lambda)$ and $r \geq 1$,*

$$\|\hat{P}^1_N u - v\|_{1,\hat{A}_0} \leq cN^{1-r}\|v\|_{r,B}.$$

Proof. Let us denote

$$u(x) = \int_0^x (z+1)^{3/2}\partial_z((z+1)^2 v(z))\, dz.$$

Then, we have

$$\partial_x u(x) = (x+1)^{3/2}\partial_x((x+1)^2 v(x)),$$

and

$$v(x) = (x+1)^{-2}\Big(\int_0^x (z+1)^{-3/2}\partial_z u(z)\, dz + v(0)\Big).$$

Setting

$$\varphi(x) = (x+1)^{-2}\Big(\int_0^x (z+1)^{-1} P_{N-2}((z+1)^{-1/2}\partial_z u(z))\, dz + v(0)\Big),$$

then, by the definition of P_{N-2}, there exists a polynomial $q_{N-2} \in \mathcal{P}_{N-2}$ such that

$$\varphi(x) = \frac{1}{2(x+1)^2}\Big(\int_{-1}^{(x-1)/(x+1)} q_{N-2}(z)\, dz + 2v(0)\Big).$$

Using the identity $1/(x+1) = -\frac{1}{2}((x-1)/(x+1) - 1)$, we find that $\varphi \in \mathcal{R}_N$. Furthermore, by an argument as in the proof of Theorem 3.2 and the Hardy inequality, we have

$$\|\varphi - v\|^2 \leq \int_0^\infty \frac{1}{x^4}\Big(\int_0^x (z+1)^{-1}\big(P_{N-2}((z+1)^{-1/2}\partial_z u(z)) - (z+1)^{-1/2}\partial_z u(z)\big)\, dz\Big)^2 dx \quad \leq$$

$$\le \frac{4}{3}\int_0^\infty (x+1)^{-4}(P_{N-2}((x+1)^{-1/2}\partial_x u(x)) - (x+1)^{-1/2}\partial_x u(x))^2\,dx$$
$$\le \frac{4}{3}\|P_{N-1}((x+1)^{-1/2}\partial_x u) - (x+1)^{-1/2}\partial_x u\|^2$$
$$\le cN^{2-2r}||(x+1)^{-1/2}\partial_x u||^2_{r-1,A_0}$$
$$= cN^{2-2r}\|(x+1)\partial_x((x+1)^2 v)\|_{r-1,A_0} \le cN^{2-2r}\|v\|_{r,B}.$$

Next, we write

$$\partial_x\varphi(x) - \partial_x v(x) = F(x) + G(x),$$

with

$$F(x) = \frac{1}{(x+1)^3}(P_{N-2}((x+1)^{-1/2}\partial_x u(x)) - (x+1)^{-1/2}\partial_x u(x)),$$
$$G(x) = -\frac{2}{(x+1)^3}\int_0^x (z+1)^{-1}(P_{N-2}((z+1)^{-1/2}\partial_z u(z)) - (z+1)^{-1/2}\partial_z u(z))\,dz.$$

It can be shown that

$$\|(x+1)^{3/2}F\|^2 \le \|P_{N-2}((x+1)^{-1/2}\partial_x u) - (x+1)^{-1/2}\partial_x u\|^2 \le cN^{2-2r}||(x+1)^{-1/2}\partial_x u||^2_{r-1,A_0} \le cN^{2-2r}||v||^2_{r,B}.$$

Similarly,

$$\|(x+1)^{3/2}G\|^2 \le c\int_0^\infty \frac{1}{x^3}\left(\int_0^x (z+1)^{-1}(P_{N-2}((z+1)^{-1/2}\partial_z u(z)) - (z+1)^{-1/2}\partial_z u(z))\,dz)^2\right)dx$$
$$\le cN^{2-2r}||v||^2_{r,B}.$$

Therefore,

$$||\hat{P}_N^1 v - v||^2_{1,\hat{A}_0} \le ||\varphi - v||^2_{1,\hat{A}_0} \le \|(x+1)^{3/2}(F+G)\|^2 + \|\varphi - v\|^2 \le cN^{2-2r}||v||^2_{r,B}.$$ ∎

Finally, when we use the modified Legendre rational spectral method for nonlinear problems, we may need to estimate the upper-bounds of various orthogonal projections. In particular, the following result will be used in Section 5.

Theorem 3.6. *Let* $0 \le \mu < m - \frac{1}{2}$*, with a positive integer* m*. Then, for any* $v \in H^m_{A_m}(\Lambda)$,

$$\|P^m_N v\|_{\mu,\infty} \le c\|v\|_{m,A_m}.$$

For any $v \in H^m_{A_m}(\Lambda) \cap H^m_0(\Lambda)$,

$$\|P^{m,0}_N v\|_{\mu,\infty} \le c\|v\|_{m,A_m}.$$

Proof. Let $d = m - \mu > \frac{1}{2}$. By embedding theory and Theorem 3.3,

$$\begin{aligned}\|P^m_N v\|_{\mu,\infty} &\le \|v\|_{\mu,\infty} + \|P^m_N v - v\|_{\mu,\infty}\\ &\le \|v\|_{\mu+d} + c\|P^m_N v - v\|_{\mu+d}\\ &\le \|v\|_{\mu+d} + c\|v\|_{m,A_m} \le c\|v\|_{m,A_m}.\end{aligned}$$

The second result can be proved similarly. □

4. Interpolations using modified Legendre rational functions

We first consider the modified Legendre-Gauss rational interpolation. Let $\{\zeta_{N,j}\}_{j=0,\dots,N}$ be the $N+1$ distinct roots of $R_{N+1}(x)$. Indeed, we have

$$\zeta_{N,j} = (1+\sigma_{N,j})(1-\sigma_{N,j})^{-1}, \tag{4.1}$$

where $\sigma_{N,j}$ are the roots of $L_{N+1}(x)$. We denote

$$\omega_{N,j} = \frac{1}{2}\rho_{N,j}(\zeta_{N,j}+1)^2, \quad 0 \le j \le N, \tag{4.2}$$

where $\rho_{N,j}$ are the weights of the Legendre-Gauss quadrature,

$$\rho_{N,j} = \frac{2}{(1-\sigma^2_{N,j})(\partial_x L_{N+1}(\sigma_{N,j}))^2}, \quad 0 \le j \le N,$$

By virtue of (15.3.10) in Szegö [28], we have

$$\rho_{N,j} \sim \frac{2\pi}{N+1}(1-\sigma^2_{N,j})^{1/2}. \tag{4.3}$$

Thanks to the above and (4.2), we have

$$\omega_{N,j} \sim \frac{2\pi}{N+1}\zeta^{1/2}_{N,j}(\zeta_{N,j}+1). \tag{4.4}$$

We now introduce the discrete inner product and the discrete norm associated with $\{\zeta_{N,j}\}_{j=0,\dots,N}$.

$$(u,v)_N = \sum_{j=0}^{N} u(\zeta_{N,j})v(\zeta_{N,j})\omega_{N,j}, \quad \|v\|_N = (v,v)_N^{1/2}.$$

Let $y = (x-1)/(x+1)$, $\tilde{\Lambda} = (-1,1)$, and $\mathcal{P}_N$ be the set of all polynomials of degree at most N. For any $\varphi \in \mathcal{R}_N$ and $\psi \in \mathcal{R}_{N+1}$, we can write

$$\varphi(x) = \frac{1}{x+1} q_N\left(\frac{x-1}{x+1}\right), \quad \psi(x) = \frac{1}{x+1} q_{N+1}\left(\frac{x-1}{x+1}\right),$$

with $q_N \in \mathcal{P}_N$ and $q_{N+1} \in \mathcal{P}_{N+1}$. By the property of the Legendre-Gauss quadrature and (4.2), we have

$$(4.5) \quad \begin{aligned}(\varphi,\psi) &= \int_\Lambda \frac{1}{(x+1)^2} q_N\left(\frac{x-1}{x+1}\right) q_{N+1}\left(\frac{x-1}{x+1}\right) dx \\ &= \frac{1}{2}\int_{\tilde{\Lambda}} q_N(y)q_{N+1}(y)\,dy = \frac{1}{2}\sum_{j=0}^{N} q_N(\sigma_{N,j})q_{N+1}(\sigma_{N,j})\rho_j \\ &= \sum_{j=0}^{N} \varphi(\zeta_{N,j})\psi(\zeta_{N,j})\omega_{N,j} = (\varphi,\psi)_N, \quad \forall \varphi \in \mathcal{R}_N,\ \psi \in \mathcal{R}_{N+1}.\end{aligned}$$

In particular, we have

$$(4.6) \qquad \|\varphi\|_N = \|\varphi\|, \quad \forall \varphi \in \mathcal{R}_N.$$

For any $v \in C(\Lambda)$, the modified Legendre-Gauss rational interpolation operator $I_N v \in \mathcal{R}_N$ is defined by

$$I_N v(\zeta_{N,j}) = v(\zeta_{N,j}), \quad 0 \le j \le N,$$

or equivalently

$$(I_N v - v, \varphi)_N = 0, \quad \forall \varphi \in \mathcal{R}_N.$$

The following theorem is related to the stability of the interpolation.

Theorem 4.1. *For any* $v \in H^1_{\hat{A}_0}(\Lambda)$,

$$\|I_N v\| \le c(\|v\| + N^{-1}\|(x+1)^{3/2}\partial_x v\|).$$

Proof. By (4.4) and (4.6),

$$\|I_N v\|^2 = \|I_N v\|_N^2 = \sum_{j=0}^{N} v^2(\zeta_{N,j})\omega_{N,j} \le cN^{-1}\sum_{j=0}^{N} v^2(\zeta_{N,j})\zeta_{N,j}^{1/2}(\zeta_{N,j}+1).$$

Let $x = (1+y)/(1-y)$, $y = \cos\vartheta$, and $\hat{v}(\vartheta) = v((1+\cos\vartheta)/(1-\cos\vartheta))$. Then

$$\|I_N v\|^2 \le CN^{-1}\sum_{j=0}^{N} \hat{v}^2(\vartheta_{N,j})(1+\cos\vartheta_{N,j})^{1/2}(1-\cos\vartheta_{N,j})^{-3/2}.$$

According to (4.1) and Theorem 8.9.1 in Szegö [28],

$$\vartheta_{N,j} = \frac{1}{N+1}(j\pi + O(1)), \quad 0 \le j \le N, \tag{4.7}$$

where $O(1)$ is bounded uniformly for all $0 \le j \le N$. Now let $a_0 = O(1)/(N+1)$ and $a_1 = (N\pi + O(1))/(N+1)$. Then, we have $\vartheta_{N,j} \in K_j \subset [a_0, a_1]$, where the size of K_j is of the order $1/(N+1)$. Consequently,

$$\|I_N v\|^2 \le cN^{-1}\sum_{j=0}^{N}\sup_{\vartheta\in K_j}|\hat{v}(\vartheta)\lambda(\vartheta)|^2, \quad \text{with } \lambda(\vartheta) = \sqrt{\frac{\cos\vartheta/2}{\sin^3\vartheta/2}}.$$

We recall the following inequality (see (13.7) in Bernardi and Maday [2]):

$$\|f\|_{L^\infty(a,b)} \le c\left(\frac{1}{\sqrt{b-a}}\|f\|_{L^2(a,b)} + \sqrt{b-a}\,\|\partial_x f\|_{L^2(a,b)}\right), \tag{4.8}$$

for all $f \in H^1(a,b)$. Using the above inequality on each of the interval K_j, we find that

$$\begin{aligned}\|I_N v\|^2 &\le \sum_{j=0}^{N}(\|\hat{v}\lambda\|_{L^2(K_j)}^2 + N^{-2}\|\partial_\vartheta(\hat{v}\lambda)\|_{L^2(K_j)}^2)\\ &\le c(\|\hat{v}\lambda\|_{L^2(0,\pi)}^2 + N^{-2}\|\partial_\vartheta(\hat{v}\lambda)\|_{L^2(a_0,a_1)}^2)\\ &\le c(\|\hat{v}\lambda\|_{L^2(0,\pi)}^2 + N^{-2}\|\lambda\partial_\vartheta\hat{v}\|_{L^2(0,\pi)}^2 + N^{-2}\|\hat{v}\partial_\vartheta\lambda\|_{L^2(0,\pi)}).\end{aligned}$$

Using the identity

$$\partial_\vartheta\lambda(\vartheta) = -\frac{1}{4}\lambda(\vartheta)\left(\cos^{-1}\frac{\vartheta}{2}\sin\frac{\vartheta}{2} + 3\cos\frac{\vartheta}{2}\sin^{-1}\frac{\vartheta}{2}\right),$$

we derive that

$$|\partial_\vartheta \lambda(\vartheta)| \le cN|\lambda(\vartheta)|.$$

Moreover,

$$\begin{aligned}\frac{d\vartheta}{dx} &= \frac{d\vartheta}{dy}\frac{dy}{dx} = \left(\frac{dy}{d\vartheta}\right)^{-1}\frac{dy}{dx} = \frac{-2}{(x+1)^2}\sin^{-1}\vartheta \\ &= -\frac{1}{2}(1-y)^2\sin^{-1}\vartheta = -2\sin^4\frac{\vartheta}{2}\sin^{-1}\vartheta = -\lambda^{-2}(\vartheta).\end{aligned}$$

Therefore,

$$\begin{aligned}\|I_N v\|^2 &\le c(\|\hat{v}\lambda\|^2_{L^2(0,\pi)} + N^{-2}\|\lambda\partial_\vartheta \hat{v}\|_{L^2(0,\pi)}) \\ &\le c(\|v\|^2 + N^{-2}\|(x+1)^{3/2}\partial_x v\|^2).\end{aligned}$$

□

Theorem 4.2. *For any* $v \in H^r_B(\Lambda)$ *with* $r \ge 1$ *and* $0 \le \nu \le 1$,

$$\|I_N v - v\|_\nu \le cN^{2\nu+1-r}\|v\|_{r,B}.$$

Proof. Since $I_N(\hat{P}^1_N v)$ coincides with $\hat{P}^1_N v$, we derive from Theorems 3.5 and 4.1 that

$$\begin{aligned}\|I_N v - \hat{P}^1_N v\| &\le c(\|\hat{P}^1_N v - v\| + N^{-1}\|(x+1)^{3/2}\partial_x(\hat{P}^1_N v - v)\|) \\ &\le c\|\hat{P}^1_N v - v\|_{1,\hat{A}_0} \le cN^{1-r}\|v\|_{r,B}.\end{aligned}$$

Using Theorem 3.5 again yields

$$\|I_N v - v\| \le \|\hat{P}^1_N v - v\| + \|I_N v - \hat{P}^1_N v\| \le cN^{1-r}\|v\|_{r,B}.$$

Furthermore, by virtue of Theorems 2.1 and 3.5,

$$\begin{aligned}|I_N v - v|_1 &\le |\hat{P}^1_N v - v|_1 + |I_N(v - \hat{P}^1_N v)|_1 \\ &\le c\|\hat{P}^1_N v - v\|_{1,\hat{A}_0} + cN^2\|(x+1)^{-1}I_N(v - \hat{P}^1_N v)\| \\ &\le cN^{1-r}\|v\|_{r,B} + cN^2\|I_N v - \hat{P}^1_N v\| \le cN^{3-r}\|v\|_{r,B}.\end{aligned}$$

The desired result for $0 < \nu < 1$ follows from space interpolation. □

We now turn to the modified Gauss-Radau rational interpolation. We denote by $\{\hat{\zeta}_{N,j}\}_{j=0,\dots,N}$ the $N+1$ distinct roots of $R_N(x) + R_{N+1}(x)$. Indeed, we have

$$\hat{\zeta}_{N,j} = (1+\hat{\sigma}_{N,j})(1-\hat{\sigma}_{N,j})^{-1}, \tag{4.9}$$

where $\hat{\sigma}_{N,j}$ are the roots of $L_N(x) + L_{N+1}(x)$ in descending order. In particular, $\hat{\zeta}_{N,N} = 0$. We denote

$$\hat{\omega}_{N,j} = \frac{1}{2}\hat{\rho}_{N,j}(\hat{\zeta}_{N,j} + 1)^2, \quad 0 \le j \le N, \tag{4.10}$$

where $\hat{\rho}_{N,j}$ are the weights of the Legendre-Gauss quadrature,

$$\begin{aligned} \hat{\rho}_{N,j} &= \frac{1}{(N+1)^2}\frac{1-\hat{\sigma}_{N,j}}{(L_N(\hat{\sigma}_{N,j}))^2}, \quad 0 \le j \le N-1, \\ \hat{\rho}_{N,N} &= \frac{2}{(N+1)^2}. \end{aligned}$$

Thanks to (15.3.10) in Szegö [28], we have

$$\hat{\rho}_{N,j} \sim \frac{2\pi}{N+1}(1-\hat{\sigma}^2_{N,j})^{1/2}, \quad 0 \le j \le N-1, \tag{4.11}$$

which implies that

$$\hat{\omega}_{N,j} \sim \frac{4\pi}{N+1}\hat{\zeta}^{1/2}_{N,j}(\hat{\zeta}_{N,j} + 1), \quad 0 \le j \le N-1. \tag{4.12}$$

We now introduce the discrete product and the discrete norm associated to $\{\hat{\zeta}_{N,j}\}_{j=0,\dots,N}$.

$$(u,v)_{N,\sim} = \sum_{j=0}^{N} u(\hat{\zeta}_{\mu j})v(\hat{\zeta}_{\mu j})\hat{\omega}_{N,j}, \quad \|v\|_{N,\sim} = (v,v)^{1/2}_{N,\sim}.$$

Similar to (4.5), we can prove that

$$(\varphi,\psi) = (\varphi,\psi)_{N,\sim}, \quad \|\varphi\| = \|\varphi\|_{N,\sim}, \quad \forall \varphi,\, \psi \in \mathcal{R}_N. \tag{4.13}$$

For any $v \in C(\bar{\Lambda})$, the modified Legendre-Gauss-Radau rational interpolation operator $\hat{I}_N v \in \mathcal{R}_N$ is defined by

$$\hat{I}_N v(\hat{\zeta}_{N,j}) = v(\hat{\zeta}_{N,j}), \quad 0 \le j \le N,$$

or equivalently

$$(\hat{I}_N v - v,\, \varphi)_{N,\sim} = 0, \quad \forall \varphi \in \mathcal{R}_N.$$

Theorem 4.3. *For any $v \in H^1_{\hat{A}_0}(\Lambda)$,*

$$\|\hat{I}_N v\| \le c(\|v\| + N^{-1}\|(x+1)^{3/2}\partial_x v\|).$$

Proof. Thanks to (4.11)-(4.13), we have

$$\|\hat{I}_N v\|^2 = \|\hat{I}_N v\|^2_{N,\sim} = \sum_{j=0}^{N} v^2(\hat{\zeta}_{N,j})\hat{\omega}_{N,j} \le CN^{-1}\sum_{j=0}^{N-1} v^2(\hat{\zeta}_{N,j})\hat{\zeta}_{N,j}^{1/2}(1+\hat{\zeta}_{N,j}) + (N+1)^{-2}v^2(0). \tag{4.14}$$

By the trace theorem, we have $|v(0)| \le c\|v\|_1$. Let y, ϑ, and $\hat{v}(\vartheta)$ be the same as in the proof of Theorem 4.1. Then

$$\|\hat{I}_N v\|^2 \le CN^{-1}\sum_{j=0}^{N-1} \hat{v}^2(\hat{\vartheta}_{N,j})(1+\cos\hat{\vartheta}_{N,j})^{1/2}(1-\cos\hat{\vartheta}_{N,j})^{-3/2} + CN^{-2}\|v\|_1^{-2}.$$

According to (4.9), Theorem 8.9.1 in Szegö [28], and the relation between $\sigma_{N,j}$ and $\hat{\sigma}_{N,j}$, we also have

$$\hat{\vartheta}_{N,j} = \frac{1}{N}(j\pi + O(1)), \quad 0 \le j \le N-1.$$

Then the result follows from an argument as in the late part of the proof of Theorem 4.1. □

Theorem 4.4. *For any $v \in H_B^r(\Lambda)$ with $r \ge 1$ and $0 \le \nu \le 1$,*

$$\|\hat{I}_N v - v\|_\nu \le cN^{2\nu+1-r}\|v\|_{r,B}.$$

Proof. This result can be proved by using the same argument as for Theorem 4.2, with I_N replaced by $\hat{I}_N$. □

5. APPROXIMATION OF THE KORTEWEG-DE VRIES EQUATION USING THE MODIFIED LEGENDRE RATIONAL FUNCTIONS

The main advantage of using the modified Legendre rational functions is that they are orthogonal in $L^2(\Lambda)$. If we consider elliptic equations or parabolic equations, we may use, for example, the usual Legendre rational approximation by Guo, Shen, and Wang [20]. However, for some important nonlinear differential equations with essential conservation properties, the non-uniform weight for the usual Legendre rational functions may destroy these conservation properties. Hence, the usual Legendre rational functions are not suitable for these type of equations. In this section, we take the Korteweg-de Vries equation as an example to show how to deal with such problems by using the modified Legendre rational functions.

There exists a large body of literature concerning the Cauchy problem of Korteweg-de Vries equation, see, e.g., Zabursky and Kruskal [31], Lax [25], and Bullough and Caudrey [6], see also Temam [29] for an analysis of the problem using parabolic regularization, and Eden, Foias, Nicolaenko and Temam [10] for a discussion on the long-term behavior of the problem.

In most existing numerical work on Korteweg-de Vries equation, finite difference or finite element methods were used, for example, Kuo and Wu [24], and Kuo and Sanz Serna [23] first proved the convergences of some semi-discrete and fully discrete schemes on the whole line, and Bona, Dougalis and Karakashian [3] proved some higher-order convergence results for an implicit Runge-Kutta Galerkin finite element scheme for the Korteweg-de Vries equation with periodic boundary conditions. However, there are very few results for the initial boundary value problem of the Korteweg-de Vries equation on the half line, which is the subject of this section. The Korteweg-de Vries equation on the half line is physically relevant in situations such as water waves in a narrow and shallow stream coming from a large water reservoir or waves originated by a wave maker.

The Korteweg-de Vries equation on the half line is as follows:

$$\begin{aligned}
&\partial_t U(x,t) + U(x,t)\partial_x U(x,t) + \partial_x^3 U(x,t) = f(x,t), && \\
& && x\in\Lambda,\ 0<t\le T,\\
&U(0,t) = g(t), && 0\le t\le T,\\
&\lim_{x\to\infty} U(x,t) = \lim_{x\to\infty}\partial_x U(x,t) = 0, && 0\le t\le T,\\
&U(x,0) = U_0(x), && x\in\Lambda.
\end{aligned} \tag{5.1}$$

Chu, Xiang and Baransky [9], and Guo and Weideman [21] discovered the possibility of producing solitary waves with suitable boundary values at $x = 0$. Guo [13] also proved the convergence of the finite difference scheme for (5.1) used in Guo and Weideman [21]. We now present a modified Legendre rational approximation for this problem.

Without loss of generality, we take $g(t) \equiv 0$. Then the weak form of (5.1) is to find $U(x,t) \in \widetilde{H}_0^2(\Lambda)$ for all $0 \le t \le T$, such that

$$\begin{aligned}
&(\partial_t U(t), v) - \tfrac{1}{2}(U^2(t), \partial_x v) - (\partial_x^2 U(t), \partial_x v) = (f(t), v),\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in \widetilde{H}_0^2(\Lambda),\\
&U(0) = U_0.
\end{aligned} \tag{5.2}$$

The well-posedness of (5.2) can be established as in Ton [30]. Moreover, by the skew symmetry of the operators ∂_x and ∂_x^3, the solution of (5.2) possesses certain conservation properties. For instance, if $f \equiv 0$, then,

$$\|U(t)\| = \|U_0\|, \quad \forall t \in (0, T]. \tag{5.3}$$

Now let $u_N(x,t)$ be the numerical solution of (5.2), defined as follows. For any $t \in (0,T]$, find $u_N(x,t) \in \mathcal{R}_N^0$ such that

$$(5.4) \qquad \begin{aligned} &(\partial_t u_N(t), \varphi) - \tfrac{1}{2}(u_N^2(t), \partial_x \varphi) - (\partial_x^2 u_N(t), \partial_x \varphi) \\ &\qquad\qquad = (f(t), \varphi), \quad \forall \varphi \in \mathcal{R}_N^0, \\ &u_N(0) = P_N U_0. \end{aligned}$$

Take $\varphi = u_N(t)$ in (5.4), we find that for $f \equiv 0$,

$$(5.5) \qquad \|u_N(t)\| = \|u_N(0)\|, \quad \forall t \in (0,T].$$

Therefore, the modified Legendre rational approximation are well suited for numerical approximation of the Korteweg-de Vries equation. Indeed, we have the following result concerning the convergence and error estimate for (5.2).

Theorem 5.1. *If for $r \geq 3$, $U \in L^\infty(0,T;W^{1,\infty}(\Lambda)) \cap H^1(0,T;H^r_{A_3}(\Lambda))$ and $U_0 \in H^r_{A_3}(\Lambda)$, then we have*

$$\|U - u_N\| \leq C_3(U) N^{3-r}, \quad \forall t \in (0,T],$$

where $C_3(U)$ is a positive constant depending only on the norms of U and U_0 in the spaces mentioned above.

Proof. Let $U_N = \tilde{P}_N^{3,0} U$. We derive from (5.2) that

$$(5.6) \qquad \begin{aligned} &(\partial_t U_N(t), \varphi) - \tfrac{1}{2}(U_N^2(t), \partial_x \varphi) - (\partial_x^2 U_N(t), \partial_x \varphi) \\ &\qquad + \sum_{j=1}^{3} G_j(t,\varphi) = (f(t), \varphi), \quad \forall \varphi \in \mathcal{R}_N^0, \\ &U_N(0) = \tilde{P}_N^{3,0} U_0, \end{aligned}$$

where

$$\begin{aligned} G_1(t,\varphi) &= (\partial_t U(t) - \partial_t U_N(t),\, \varphi), \\ G_2(t,\varphi) &= -\tfrac{1}{2}(U^2(t) - U_N^2(t),\, \partial_x \varphi), \\ G_3(t,\varphi) &= -(\partial_x^2 U(t) - \partial_x^2 U_N(t),\, \partial_x \varphi). \end{aligned}$$

Let $\tilde{U}_N = u_N - U_N$. Subtracting (5.6) from (5.4) yields

$$
(5.7)\qquad
\begin{aligned}
&(\partial_t \widetilde{U}_N(t),\varphi) - \tfrac{1}{2}(\widetilde{U}_N^2(t),\partial_x\varphi) - (\partial_x^2\widetilde{U}_N(t),\partial_x\varphi)\\
&\qquad\qquad\qquad\qquad = \sum_{j=1}^{4} G_j(t,\varphi), \quad \forall \varphi\in \mathcal{R}_N^0,\\
&\widetilde{U}_N(0) = P_N U_0 - \widetilde{P}_N^{3,0} U_0,
\end{aligned}
$$

where $G_4(t,\varphi) = (U_N(t)\widetilde{U}_N(t),\partial_x\varphi)$.

Take $\varphi = \widetilde{U}_N$ in (5.7). It can be shown that

$$
(\widetilde{U}_N^2(t),\partial_x\widetilde{U}_N(t)) = 0, \quad (\partial_x^2\widetilde{U}_N(t),\partial_x\widetilde{U}_N(t)) = -(\partial_x U_N(0))^2.
$$

Therefore,

$$
(5.8)\qquad \frac{d}{dt}\|\widetilde{U}_N(t)\|^2 \le 2\sum_{j=1}^{4} |G_j(t,\widetilde{U}_N(t))|.
$$

Now, we estimate the terms at the right side of (5.8). Firstly, by Theorem 3.4,

$$
|G_1(t,\widetilde{U}_N(t))| \le \|\widetilde{U}_N(t)\|^2 + cN^{6-2r}||\partial_t U(t)||_{r,A_3}^2.
$$

Next, by Theorems 3.4 and 3.6,

$$
\begin{aligned}
|G_2(t,\widetilde{U}_N(t))| &= |(U(t)\partial_x U(t) - U_N(t)\partial_x U_N(t),\widetilde{U}_N(t))|\\
&\le \|\widetilde{U}_N(t)\|^2 + \|(U(t)-U_N(t))\partial_x U(t)\|^2\\
&\qquad\qquad + \|U_N(t)(\partial_x U(t) - \partial_x U_N(t))\|^2\\
&\le \|\widetilde{U}_N(t)\|^2 + c||U(t)||_{1,\infty}^2\|U(t)-U_N(t)\|^2\\
&\qquad\qquad + c||U_N(t)||_{\infty}^2\,|U(t)-U_N(t)|_1^2\\
&\le \|\widetilde{U}_N(t)\|^2 + cN^{6-2r}(||U(t)||_{1,\infty}^2 + ||U(t)||_{3,A_3}^2)||U(t)||_{r,A_3}^2.
\end{aligned}
$$

Using Theorem 3.4 again yields

$$
\begin{aligned}
|G_3(t,\widetilde{U}_N(t))| &= |(\partial_x^3 U(t) - \partial_x^3 U_N(t),\widetilde{U}_N(t))|\\
&\le \|\widetilde{U}_N(t)\|^2 + cN^{6-2r}||U(t)||_{r,A_3}^2.
\end{aligned}
$$

By integration by parts and Theorem 3.6,

$$
\begin{aligned}
|G_4(t,\widetilde{U}_N(t))| &= \tfrac{1}{2}|(\partial_x U_N(t),\widetilde{U}_N^2(t))| \le \|U_N(t)\|_{1,\infty}\|\widetilde{U}_N^2(t)\|^2\\
&\le c\|U(t)\|_{3,A_3}\|\widetilde{U}_N(t)\|^2.
\end{aligned}
$$

In addition, Theorems 3.1 and 3.4 lead to

$$\|U_N(0)\|^2 \le \|U_0 - P_N U_0\|^2 + \|U_0 - \widetilde{P}_N^{3,0} U_0\|^2 \le cN^{6-2r}||U_0||^2_{r,A_3}.$$

Substituting the previous estimates into (5.8) and integrating the result with respect to t, we obtain that

$$\|\widetilde{U}_N(t)\|^2 \le C_1(U) \int_0^t \|\widetilde{U}_N(s)\|^2\,ds + C_2(U)N^{6-2r},$$

where $C_1(U)$ is a positive constant depending only on $\|U\|_{L^\infty(0,T;H^3_{A_3}(\Lambda)\cap W^{1,\infty}(\Lambda))}$, and $C_2(U)$ is a positive constant depending only on $\|U\|_{H^1(0,T;H^r_{A_3}(\Lambda))}$ and $\|U_0\|_{H^r_{A_3}(\Lambda)}$. The desired result then follows from the Gronwall inequality and Theorem 3.4. □

6. CONCLUDING REMARKS

We introduced a new set of modified Legendre rational functions which are mutually orthogonal in $L^2(0,+\infty)$, and studied various projection operators and interpolation operators associated with them. These resulted form the mathematical foundation of approximations, by using the modified Legendre rational functions for partial differential equations on the half line. These new rational functions are particularly suitable for approximations of PDEs with essential global conservation properties. As an example, we proposed a spectral scheme using the modified Legendre rational functions for the Korteweg-de Vries equation on the half line. We showed that the numerical solution of the scheme possesses the essential conservation properties satisfied by the exact solution of the Korteweg-de Vries equation, and consequently, we were able to prove that the scheme convergences with spectral accuracy. It is noted that we were not able to obtain this type of results using the usual Legendre rational functions.

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