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# GENERALIZED LAGUERRE APPROXIMATION AND ITS APPLICATIONS TO EXTERIOR PROBLEMS *1)

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**Abstract**

Approximations using the generalized Laguerre polynomials are investigated in this paper. Error estimates for various orthogonal projections are established. These estimates generalize and improve previously published results on the Laguerre approximations. As an example of applications, a mixed Laguerre-Fourier spectral method for the Helmholtz equation in an exterior domain is analyzed and implemented. The proposed method enjoys optimal error estimates, and with suitable basis functions, leads to a sparse and symmetric linear system.

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*Key words*: Generalized Laguerre polynomials, Exterior problems, Mixed Laguerre-Fourier spectral method.

## 1. Introduction

Many practical problems in science and engineering require solving partial differential equations in exterior domains. Considerable progress has been made recently in using spectral methods for solving partial differential equations in unbounded domains. The first approach is based on the classical orthogonal systems in the unbounded domains, namely, the Hermite (cf. [7, 12, 10]) and Laguerre (cf. [16, 6, 17, 14, 18, 19, 20]) polynomials/functions. The second approach is to map the original problem in a unbounded domain to a singular problem in a bounded domain (cf. [8, 11, 13]). The third approach is based on rational approximations (cf. [3, 2, 5, 15, 9]). However, none of the methods mentioned above has yet been analyzed for multidimensional exterior problems.

In this paper, we investigate the spectral approximation using generalized Laguerre polynomials which form a mutually orthogonal system in the weighted Sobolev space $L^2_{\omega_\alpha}(0,\infty)$ with $\omega_\alpha(\rho)=\rho^\alpha\exp(-\rho)$. The orthogonal projection in $L^2_{\omega_\alpha}(0,\infty)$ has been analyzed in [6]. Other projection and interpolation operators for the special case $\alpha=0$ have been studied in [16, 17, 14, 20]. However, the usual weighted Sobolev spaces used in these papers are not the most appropriate. Here, we study the generalized Laguerre approximations in non-uniformly weighted spaces, i.e., with different weights for derivatives of different orders, and we obtain

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optimal results for several projection operators for all $\alpha > -1$. These new results enable us to study numerical approximations of a large class of problems in unbounded domains.

As an example of applications, we consider the Helmholtz equation in the two dimensional exterior domain $\Omega = \{(\rho, \theta) : \rho > 1,\ \theta \in [0, 2\pi)\}$. We propose a mixed Laguerre-Fourier spectral method using Laguerre polynomials for the radial direction and Fourier series for the azimuthal direction. Thanks to the new results on generalized Laguerre approximations, we are able to prove optimal error estimates for the mixed Laguerre-Fourier method applied to the transformed equation. Furthermore, by choosing a set of suitable basis functions, we are also able to construct an efficient numerical algorithm in which the linear system is symmetric and sparse, and hence can be efficiently solved.

The paper is organized as follows. In the next section, we present several basic approximation results using generalized Laguerre polynomials. Then, we study the mixed Laguerre-Fourier approximation outside a disk in Section 3. We construct the mixed Laguerre-Fourier spectral scheme for a model problem, and prove its convergence in Section 4. In Section 5, we present implementation details and an illustrative numerical result. Some concluding remarks are presented in the final section.

## 2. Generalized Laguerre Approximation

### 2.1 Notations and preliminaries

Let us first introduce some notations. Let $\Lambda = \{\rho \mid 0 < \rho < \infty\}$ and $\chi(\rho)$ be a certain weight function in the usual sense. We define

$$L^2_\chi(\Lambda) = \{\ v \mid v \ \text{ is measurable on } \Lambda \text{ and } \ \|v\|_{L^2_{\chi,\Lambda}} < \infty\ \}$$

with the following inner product and norm,

$$(u, v)_{\chi,\Lambda} = \int_\Lambda u(\rho)v(\rho)\chi(\rho)d\rho, \qquad \|v\|_{\chi,\Lambda} = (v, v)^{\frac{1}{2}}_{\chi,\Lambda}.$$

For simplicity, we denote by $\partial^k_\rho v$ the $k$-th derivative of $v(\rho)$ with respect to $\rho$. For any non-negative integer $m$, we define the weighted Sobolev space

$$H^m_\chi(\Lambda) = \left\{v \mid \partial^k_\rho v \in L^2_\chi(\Lambda),\ 0 \le k \le m\right\}$$

equipped with the following inner product, semi-norm and norm

$$(u, v)_{m,\chi,\Lambda} = \sum_{0\le k\le m} (\partial^k_\rho u, \partial^k_\rho v)_{\chi,\Lambda}, \quad |v|_{m,\chi,\Lambda} = \|\partial^m_\rho v\|_{\chi,\Lambda}, \quad \|v\|_{m,\chi,\Lambda} = (v, v)^{\frac{1}{2}}_{m,\chi,\Lambda}.$$

For any real $r > 0$, the space $H^r_\chi(\Lambda)$ and its norm $\|v\|_{r,\chi,\Lambda}$ are defined by space interpolation as in Adams [1]. In particular, we denote

$${}_0H^1_\chi(\Lambda) = \{v \mid v \in H^1_\chi(\Lambda) \text{ and } v(0) = 0\}.$$

Let $\omega_\alpha(\rho) = \rho^\alpha \mathrm{e}^{-\rho}$. We denote in particular $\omega(\rho) = \omega_0(\rho) = e^{-\rho}$. The generalized Laguerre polynomials of degree $l$ are defined by

$$\mathcal{L}^{(\alpha)}_l(\rho) = \frac{1}{l!}\rho^{-\alpha}\mathrm{e}^\rho \partial^l_\rho(\rho^{l+\alpha}\mathrm{e}^{-\rho}), \qquad l = 0, 1, 2, \cdots,\ \alpha > -1.$$

They are eigenfunctions of the Sturm-Liouville problem

$$\partial_\rho(\omega_{\alpha+1}(\rho)\partial_\rho v(\rho)) + \lambda\omega_\alpha(\rho)v(\rho) = 0, \qquad 0 < \rho < \infty, \tag{2.1}$$

with corresponding eigenvalues $\lambda_l = l$, and satisfy the recurrence relations

$$\mathcal{L}_l^{(\alpha)}(\rho) = \mathcal{L}_l^{(\alpha+1)}(\rho) - \mathcal{L}_{l-1}^{(\alpha+1)}(\rho) = \partial_\rho \mathcal{L}_l^{(\alpha)}(\rho) - \partial_\rho \mathcal{L}_{l+1}^{(\alpha)}(\rho), \tag{2.2}$$

$$\partial_\rho \mathcal{L}_l^{(\alpha)}(\rho) = -\mathcal{L}_{l-1}^{(\alpha+1)}(\rho) = \frac{1}{\rho}(l\mathcal{L}_l^{(\alpha)}(\rho) - (l+\alpha)\mathcal{L}_{l-1}^{(\alpha)}(\rho)). \tag{2.3}$$

The set of generalized Laguerre polynomials forms an orthogonal system in $L^2_{\omega_\alpha}(\Lambda)$, namely,

$$(\mathcal{L}_l^{(\alpha)}, \mathcal{L}_m^{(\alpha)})_{\omega_\alpha,\Lambda} = \begin{cases} \gamma_l^{(\alpha)}, & \text{for } l = m \\ 0, & \text{for } l \neq m \end{cases}, \tag{2.4}$$

where

$$\gamma_l^{(\alpha)} = \frac{\Gamma(l+\alpha+1)}{l!}. \tag{2.5}$$

Hence, for any $v \in L^2_{\omega_\alpha}(\Lambda)$, we can write

$$v(\rho) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha)} \mathcal{L}_l^{(\alpha)}(\rho) \text{ with } \hat{v}_l^{(\alpha)} = \frac{1}{\gamma_l^{(\alpha)}}(v, \mathcal{L}_l^{(\alpha)})_{\omega_\alpha,\Lambda}. \tag{2.6}$$

In order to describe our approximation results, for any integer $r \geq 0$, we define the non-uniformly weighted spaces $A^r_\alpha(\Lambda)$ as follows:

$$A^r_\alpha(\Lambda) = \left\{ v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{A^r_\alpha,\Lambda} < \infty \right\}$$

equipped with the following semi-norm and norm

$$|v|_{A^r_\alpha,\Lambda} = \|\partial_\rho^r v\|_{\omega_{\alpha+r},\Lambda}, \qquad \|v\|_{A^r_\alpha,\Lambda} = \left(\sum_{k=0}^{r} |v|^2_{A^k_\alpha,\Lambda}\right)^{\frac{1}{2}}$$

For any $r > 0$, we define the space $A^r_\alpha(\Lambda)$ and its norm by space interpolation.

Let $N$ be any positive integer and $\mathcal{P}_N(\Lambda)$ be the set of all algebraic polynomials of degree at most $N$. We define the orthogonal projection $P_{N,\alpha} : L^2_{\omega_\alpha}(\Lambda) \to \mathcal{P}_N(\Lambda)$ by

$$(P_{N,\alpha}v - v, \phi)_{\omega_\alpha,\Lambda} = 0, \qquad \forall \phi \in \mathcal{P}_N(\Lambda).$$

In the sequel, we denote by $c$ a generic positive constant independent of any function and $N$.

The following simple, but important, result generalizes and improves previously published results on the Laguerre approximations.

**Theorem 2.1.** *Let $r$ be an integer and $0 \leq s \leq r$. Then,*

$$\|P_{N,\alpha}v - v\|_{A^s_\alpha,\Lambda} \leq cN^{\frac{s-r}{2}} |v|_{A^r_\alpha,\Lambda}, \ \forall v \in A^r_\alpha(\Lambda).$$

*Proof.* We first consider the integer case. Since

$$P_{N,\alpha}v(\rho) - v(\rho) = -\sum_{l=N+1}^{\infty} \hat{v}_l^{(\alpha)} \mathcal{L}_l^{(\alpha)}(\rho),$$

we derive from (2.3) that for $N \geq r-1$,

$$\partial_\rho^s(P_{N,\alpha}v(\rho) - v(\rho)) = -\sum_{l=N+1}^{\infty} (-1)^s \hat{v}_l^{(\alpha)} \mathcal{L}_{l-s}^{(\alpha+s)}(\rho).$$

Thus by (2.4),

$$\|\partial_\rho^s(P_{N,\alpha}v - v)\|^2_{\omega_{\alpha+s},\Lambda} = \sum_{l=N+1}^{\infty} (\hat{v}_l^{(\alpha)})^2 \gamma_{l-s}^{\alpha+s}. \tag{2.7}$$

By the same argument,

$$\|\partial_\rho^r v\|^2_{\omega_{\alpha+r},\Lambda} = \sum_{l=r}^{\infty} (\hat{v}_l^{(\alpha)})^2 \gamma_{l-r}^{(\alpha+r)}. \tag{2.8}$$

A direct calculation gives

$$\frac{\gamma_{l-s}^{(\alpha+s)}}{\gamma_{l-r}^{(\alpha+r)}} = \frac{(l-r)!}{(l-s)!} \leq cN^{s-r}. \tag{2.9}$$

The combination of (2.7)-(2.9) leads to

$$\|\partial_\rho^s(P_{N,\alpha}v - v)\|_{\omega_{\alpha+s},\Lambda} \leq cN^{\frac{s-r}{2}} \|\partial_\rho^r v\|_{\omega_{\alpha+r},\Lambda}.$$

Finally, the result for the non-integer $s$ is proved by space interpolation.

**Remark 2.1.** Funaro [6] obtained the same result as Theorem 2.1 for integer $r \geq 0$ and $s = 0$. Maday, Pernaud-Thomas and Vandeven [16] derived another upper bound for $\|P_{N,\alpha}v - v\|_{\omega_\alpha,\Lambda}$ with $\alpha = 0$. In fact, they defined the space

$$H^r_{\omega_0,\beta}(\Lambda) = \left\{ v \in H^r_{\omega_0}(\Lambda) \mid \rho^{\frac{\beta}{2}} v \in H^r_{\omega_0}(\Lambda) \right\}$$

equipped with the norm $\|v\|_{r,\omega_0,\beta,\Lambda} = \|v(1+\rho)^{\frac{\beta}{2}}\|_{r,\omega_0,\Lambda}$, and proved that for any real $r \geq 0$,

$$\|P_{N,0}v - v\|_{\omega_0,\Lambda} \leq cN^{-\frac{r}{2}} \|v\|_{r,\omega_0,\beta,\Lambda},$$

where $\beta$ is the largest integer for which $\beta < r+1$. Since $\|v\|_{r,\omega_0,\beta,\Lambda}$ is not a semi-norm and the weights for all derivatives of $v$ are the same, i.e., $(1+\rho)^{\frac{\beta}{2}} e^{-\rho}$, its application is cumbersome and may not lead to optimal error estimates for certain functions, e.g., those behaving like $O(\frac{1}{\rho^\gamma})$ as $\rho \to \infty$. However, the result in Theorem 2.1 is sharper and allow us to obtain optimal estimates for a large class of problems, in particular, for the exterior problems considered in Sections 3 and 4 of this paper.

**Remark 2.2.** Mastroianni and Monegato [17] also studied the generalized Laguerre approximation. They defined the space

$$B^r_{\alpha,\Lambda} = \left\{ v \in L^2_{\omega_\alpha}(\Lambda) \mid \|v\|_{B^r_{\alpha,\Lambda}} < \infty \right\}$$

with the norm

$$\|v\|_{B^r_{\alpha,\Lambda}} = \left(\sum_{l=0}^{\infty} (l+1)^r (\hat{v}_l^{(\alpha)})^2\right)^{\frac{1}{2}},$$

and proved that for any $0 \leq s \leq r$,

$$\|P_{N,\alpha}v - v\|_{B^s_{\alpha,\Lambda}} \leq cN^{\frac{s-r}{2}} \|v\|_{B^r_{\alpha,\Lambda}}. \tag{2.10}$$

By Lemma 2.3 of [17], for any integer $r \geq 0$, the norm $\|v\|_{B^r_{\alpha,\Lambda}}$ is equivalent to the norm $\|v\|_{A^r_{\alpha,\Lambda}}$. So Theorem 2.1 improves the result (2.10) in the sense that the approximation error only depends on the semi-norm $\|\partial_\rho^r v\|_{\omega_{\alpha+r}}$.

### 2.2 Other projection operators

To carry out numerical analyses of the Laguerre spectral method for PDEs in unbounded domains, we need to consider other projection operators related to the PDEs under consideration. Let us denote

$$H^1_{\omega_\alpha,\omega_\beta}(\Lambda) = \left\{ v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{1,\omega_\alpha,\omega_\beta,\Lambda} < \infty \right\},$$

equipped with the norm

$$\|v\|_{1,\omega_\alpha,\omega_\beta,\Lambda} = \left( \|\partial_\rho v\|^2_{\omega_\alpha,\Lambda} + \|v\|^2_{\omega_\beta,\Lambda} \right)^{\frac{1}{2}}.$$

In particular, we set

$${}_0H^1_{\omega_\alpha,\omega_\beta}(\Lambda) = \left\{ v \in H^1_{\omega_\alpha,\omega_\beta}(\Lambda) \mid v(0) = 0 \right\}.$$

We define the orthogonal projection $P^1_{N,\alpha,\beta} : H^1_{\omega_\alpha,\omega_\beta}(\Lambda) \to \mathcal{P}_N(\Lambda)$ by

$$\left(\partial_\rho(P^1_{N,\alpha,\beta}v - v), \partial_\rho\phi\right)_{\omega_\alpha,\Lambda} + \left(P^1_{N,\alpha,\beta}v - v, \phi\right)_{\omega_\beta,\Lambda} = 0, \qquad \forall \phi \in \mathcal{P}_N(\Lambda). \tag{2.11}$$

We set ${}_0\mathcal{P}_N(\Lambda) = \{v \in \mathcal{P}_N(\Lambda) \mid v(0) = 0\}$ and define the orthogonal projection ${}_0P^1_{N,\alpha}(\Lambda)$: ${}_0H^1_{\omega_\alpha}(\Lambda) \to {}_0\mathcal{P}_N(\Lambda)$ by

$$\left(\partial_\rho({}_0P^1_{N,\alpha}v - v), \partial_\rho\phi\right)_{\omega_\alpha,\Lambda} = 0, \qquad \forall \phi \in {}_0\mathcal{P}_N(\Lambda). \tag{2.12}$$

In order to derive approximation results for these projections, we need several embedding inequalities.

**Lemma 2.1.** *Let $-1 < \beta \le \alpha \le \beta + 2$. We assume that there exists $\rho_0$ such that $v(\rho_0) = 0$, $\rho_0 > 0$ for $\beta \le 1$ and $\rho_0 > 2\sqrt{\beta(\beta-1)}$ for $\beta > 1$. Then, if $\partial_\rho v \in L^2_{\omega_\alpha}(\Lambda)$, we have*

$$\|v\|_{\omega_\beta,\Lambda} \le c\|\partial_\rho v\|_{\omega_\alpha,\Lambda}.$$

*Proof.* Let $\Lambda_1 = (\rho_0, \infty)$, $\Lambda_2 = (0, \rho_0)$ and

$$\|v\|_{\omega_\beta,\Lambda_j} = \left( \int_{\Lambda_j} \omega_\beta(\rho) v^2(\rho) d\rho \right)^{\frac{1}{2}}, \qquad j = 1, 2.$$

For any $\rho \in \Lambda_1$,

$$\begin{aligned} \omega_\beta(\rho)v^2(\rho) &= \int_{\rho_0}^{\rho} \partial_\xi(\omega_\beta(\xi)v^2(\xi))d\xi \\ &= 2\int_{\rho_0}^{\rho} \omega_\beta(\xi)v(\xi)\partial_\xi v(\xi)d\xi + \beta\int_{\rho_0}^{\rho} \omega_{\beta-1}(\xi)v^2(\xi)d\xi - \int_{\rho_0}^{\rho} \omega_\beta(\xi)v^2(\xi)d\xi. \end{aligned}$$

Letting $\rho \to \infty$ and using the Cauchy-Schwarz inequality, we obtain

$$\|v\|^2_{\omega_\beta,\Lambda_1} \le \frac{1}{2}\|v\|^2_{\omega_\beta,\Lambda_1} + 2\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1} + \beta\|v\|^2_{\omega_{\beta-1},\Lambda_1}.$$

Thus, for any $\beta$,

$$\|v\|^2_{\omega_\beta,\Lambda_1} \le 4\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1} + 2\beta\|v\|^2_{\omega_{\beta-1},\Lambda_1}. \tag{2.13}$$

If $\beta \le 0$, (2.13) implies that for $\beta \le \alpha$,

$$\|v\|^2_{\omega_\beta,\Lambda_1} \le 4\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1} \le 4\rho_0^{\beta-\alpha}\|\partial_\rho v\|^2_{\omega_\alpha,\Lambda_1}. \tag{2.14}$$

Otherwise, an integration by parts yields

$$2\beta\|v\|^2_{\omega_{\beta-1},\Lambda_1} = 4\beta\int_{\Lambda_1}\rho^{\beta-1}\mathrm{e}^{-\rho}v(\rho)\partial_\rho v(\rho)d\rho + 2\beta(\beta-1)\|v\|^2_{\omega_{\beta-2},\Lambda_1}. \tag{2.15}$$

Moreover, by the Cauchy-Schwarz inequality,

$$4\beta\int_{\Lambda_1}\rho^{\beta-1}\mathrm{e}^{-\rho}v(\rho)\partial_\rho v(\rho)d\rho \le 4\beta\|\partial_\rho v\|^2_{\omega_{\beta-1},\Lambda_1} + \beta\|v\|^2_{\omega_{\beta-1},\Lambda_1}.$$

Therefore, for $0 < \beta \le 1$,

$$2\beta\|v\|^2_{\omega_{\beta-1},\Lambda_1} \le 8\beta\|\partial_\rho v\|^2_{\omega_{\beta-1},\Lambda_1}.$$

The above inequality together with (2.13) implies that for $0 < \beta \le 1$ and $\beta \le \alpha$, we have

$$\begin{aligned}\|v\|^2_{\omega_\beta,\Lambda_1} &\le 4\|\partial_\rho v\|^2_{\omega_\alpha,\Lambda_1} + 8\beta\|\partial_\rho v\|^2_{\omega_{\beta-1},\Lambda_1} \le 4(1+2\beta\rho_0^{-1})\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1}\\ &\le 4(1+2\beta\rho_0^{-1})\rho_0^{\beta-\alpha}\|\partial_\rho v\|^2_{\omega_\alpha,\Lambda_1}.\end{aligned} \tag{2.16}$$

For $\beta > 1$, we have

$$4\beta\int_{\Lambda_1}\rho^{\beta-1}\mathrm{e}^{-\rho}v(\rho)\partial_\rho v(\rho)d\rho \le \frac{2\beta}{\beta-1}\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1} + 2\beta(\beta-1)\|v\|^2_{\omega_{\beta-2},\Lambda_1}.$$

This inequality together with (2.15) leads to

$$2\beta\|v\|^2_{\omega_{\beta-1},\Lambda_1} \le \frac{2\beta}{\beta-1}\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1} + 4\beta(\beta-1)\|v\|^2_{\omega_{\beta-2},\Lambda_1}.$$

We infer from the above and (2.13) that for $\beta > 1$,

$$\begin{aligned}\|v\|^2_{\omega_\beta,\Lambda_1} &\le \frac{2(3\beta-2)}{\beta-1}\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1} + 4\beta(\beta-1)\|v\|^2_{\omega_{\beta-2},\Lambda_1}\\ &\le \frac{2(3\beta-2)}{\beta-1}\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1} + 4\beta(\beta-1)\rho_0^{-2}\|v\|^2_{\omega_\beta,\Lambda_1}.\end{aligned}$$

If $\rho_0 > 2\sqrt{\beta(\beta-1)}$, then for $1 < \beta \le \alpha$,

$$\begin{aligned}\|v\|^2_{\omega_\beta,\Lambda_1} &\le \frac{2\rho_0^2(3\beta-2)}{(\rho_0^2-4\beta(\beta-1))(\beta-1)}\|\partial_\rho v\|^2_{\omega_\beta,\Lambda_1}\\ &\le \frac{2\rho_0^{2+\beta-\alpha}(3\beta-2)}{(\rho_0^2-4\beta(\beta-1))(\beta-1)}\|\partial_\rho v\|^2_{\omega_\alpha,\Lambda_1}.\end{aligned} \tag{2.17}$$

Next, for any $\rho \in \Lambda_2$,

$$\begin{aligned}\rho^{\beta+1}v^2(\rho) &= -\int_\rho^{\rho_0}\partial_\xi(\xi^{\beta+1}v^2(\xi))d\xi\\ &= -2\int_\rho^{\rho_0}\xi^{\beta+1}v(\xi)\partial_\xi v(\xi)d\xi - (\beta+1)\int_\rho^{\rho_0}\xi^\beta v^2(\xi)d\xi.\end{aligned}$$

Letting $\rho \to 0$ and using the Cauchy-Schwarz inequality, we find that for $\beta > -1$,

$$(\beta+1)\int_0^{\rho_0}\rho^\beta v^2(\rho)d\rho \le \frac{2}{\beta+1}\int_0^{\rho_0}\rho^{\beta+2}(\partial_\rho v(\rho))^2d\rho + \frac{\beta+1}{2}\int_0^{\rho_0}\rho^\beta v^2(\rho)d\rho.$$

Therefore

$$||v||^2_{\omega_\beta,\Lambda_2} \le \int_0^{\rho_0} \rho^\beta v^2(\rho)d\rho \le \frac{4}{(\beta+1)^2}\int_0^{\rho_0} \rho^{\beta+2}(\partial_\rho v(\rho))^2 d\rho \le \frac{4e^{\rho_0}}{(\beta+1)^2}||\partial_\rho v||^2_{\omega_{\beta+2},\Lambda_2}.$$

Accordingly, for $\alpha \le \beta+2$ and $\beta > -1$,

$$||v||^2_{\omega_\beta,\Lambda_2} \le \frac{4e^{\rho_0}\rho_0^{2+\beta-\alpha}}{(\beta+1)^2}||\partial_\rho v||^2_{\omega_\alpha,\Lambda_2}. \tag{2.18}$$

The combination of (2.14), (2.16), (2.17) and (2.18) leads to the desired result.

**Lemma 2.2.**
*(i) For any $v \in {}_0H^1_{\omega_\alpha}(\Lambda)$ and $\alpha < 1$,*

$$||v||^2_{\omega_\alpha,\Lambda} \le c_\alpha |v|^2_{1,\omega_\alpha,\Lambda}$$

*where $c_\alpha = 4$ for $\alpha \le 0$, and $c_\alpha = \dfrac{2(2-\alpha)}{1-\alpha}$ for $0 < \alpha < 1$;*
*(ii) For any $v \in {}_0H^1_{\omega_1}(\Lambda) \cap L^2_{\omega_{-1}}(\Lambda)$,*

$$||v||^2_{\omega_1,\Lambda} \le 2(\sqrt{2}+1)(|v|^2_{1,\omega_1,\Lambda} + ||v||^2_{\omega_{-1},\Lambda});$$

*(iii) For any $v \in H^1_{\omega_\alpha}(\Lambda) \cap L^2_{\omega_{\alpha-2}}(\Lambda)$ and $\alpha > 1$,*

$$||v||^2_{\omega_\alpha,\Lambda} \le \frac{2(3\alpha-2)}{\alpha-1}|v|^2_{1,\omega_\alpha,\Lambda} + 4\alpha(\alpha-1)||v||^2_{\omega_{\alpha-2},\Lambda}.$$

*(iv) For any $v \in A^1_{\omega_\alpha}(\Lambda)$,*

$$||\omega^{\frac{1}{2}}_{\alpha+1} v||^2_{L^\infty(\Lambda)} \le \max(\alpha+1,2)||v||^2_{A^1_{\omega_\alpha}}, \qquad ||v||^2_{\omega_{\alpha+1},\Lambda} \le 2\max(\alpha+1,2)||v||^2_{A^1_{\omega_\alpha}}.$$

*Proof.* Following the same argument as in the derivation of (2.13), we deduce that if $v(0)=0$ or $\alpha > 0$, then

$$||v||^2_{\omega_\alpha,\Lambda} \le 4||\partial_\rho v||^2_{\omega_\alpha,\Lambda} + 2\alpha||v||^2_{\omega_{\alpha-1},\Lambda}, \tag{2.19}$$

The result (i) for $\alpha \le 0$ follows (2.19) immediately. On the other hand, similar to (2.15), we have

$$2\alpha||v||^2_{\omega_{\alpha-1},\Lambda} = 4\alpha\int_\Lambda \rho^{\alpha-1}\mathrm{e}^{-\rho}v(\rho)\partial_\rho v(\rho)d\rho + 2\alpha(\alpha-1)||v||^2_{\omega_{\alpha-2},\Lambda}. \tag{2.20}$$

For $0 < \alpha < 1$, we derive by using the Cauchy-Schwarz inequality that

$$4\alpha\int_\Lambda \rho^{\alpha-1}\mathrm{e}^{-\rho}v(\rho)\partial_\rho v(\rho)d\rho \le \frac{2\alpha}{1-\alpha}|v|^2_{1,\omega_\alpha,\Lambda} + 2\alpha(1-\alpha)||v||^2_{\omega_{\alpha-2},\Lambda}.$$

Substituting the above and (2.20) into (2.19), we obtain the result (i) for $0 < \alpha < 1$.

For $\alpha > 1$, we have

$$4\alpha\int_\Lambda \rho^{\alpha-1}\mathrm{e}^{-\rho}v(\rho)\partial_\rho v(\rho)d\rho \le \frac{2\alpha}{\alpha-1}|v|^2_{1,\omega_\alpha,\Lambda} + 2\alpha(\alpha-1)||v||^2_{\omega_{\alpha-2},\Lambda}.$$

Substituting the above and (2.20) into (2.19), we obtain the result (iii).

Now, if $v(0)=0$ and $\alpha = 1$, an integration by parts leads to

$$2||v||^2_{\omega_0,\Lambda} = 4\int_\Lambda \mathrm{e}^{-\rho}v(\rho)\partial_\rho v(\rho)d\rho \le 2(\sqrt{2}-1)||\partial_\rho v||^2_{\omega_1,\Lambda} + 2(\sqrt{2}+1)||v||^2_{\omega_{-1},\Lambda}.$$

The above with (2.19) implies the result (ii).

Finally, we derive from

$$\rho^{\alpha+1}\mathrm{e}^{-\rho}v^2(\rho)=\int_0^\rho \partial_\xi\left(\xi^{\alpha+1}\mathrm{e}^{-\xi}v^2(\xi)\right)d\xi$$

that

$$\begin{aligned}\omega_{\alpha+1}(\rho)v^2(\rho)&+\int_0^\rho \omega_{\alpha+1}(\xi)v^2(\xi)d\xi\\ &=2\int_0^\rho \omega_{\alpha+1}(\xi)v(\xi)\partial_\xi v(\xi)d\xi+(\alpha+1)\int_0^\rho \omega_\alpha(\xi)v^2(\xi)d\xi\\ &\le \frac{1}{2}\int_0^\rho \omega_{\alpha+1}(\xi)v^2(\xi)d\xi+2\|\partial_\rho v\|^2_{\omega_{\alpha+1},\Lambda}+(\alpha+1)\|v\|^2_{\omega_\alpha,\Lambda}\end{aligned}$$

from which the result (iv) follows.

The following embedding inequality is also useful.

**Lemma 2.3.** *If $\partial_\rho v\in L^2_{\omega_{\alpha+2}}(\Lambda)$ and $v^2(\rho)\rho^{\alpha+1}\to 0$ as $\rho\to 0$, then for $\alpha\ne -1$,*

$$\|v\|^2_{\omega_\alpha,\Lambda}\le \frac{4}{(\alpha+1)^2}\|\partial_\rho v\|^2_{\omega_{\alpha+2},\Lambda}.$$

*Proof.* By integration by parts and the Cauchy-Schwartz inequality,

$$\begin{aligned}\|v\|^2_{\omega_\alpha,\Lambda}&=-\frac{2}{\alpha+1}\int_\Lambda \rho^{\alpha+1}\mathrm{e}^{-\rho}v(\rho)\partial_\rho v(\rho)d\rho+\frac{1}{\alpha+1}\int_0^\infty \rho^{\alpha+1}\mathrm{e}^{-\rho}v^2(\rho)d\rho\\ &\le \frac{2}{|\alpha+1|}\|v\|_{\omega_\alpha,\Lambda}\|\partial_\rho v\|_{\omega_{\alpha+2},\Lambda},\end{aligned}$$

which implies the desired result.

We now turn to the error estimates for various orthogonal approximations.

**Theorem 2.2.** *Let $-1<\beta\le\alpha\le\beta+2$ and integer $r\ge 1$. If $v\in H^1_{\omega_\alpha,\omega_\beta}(\Lambda)$ and $\partial_\rho v\in A^{r-1}_\alpha(\Lambda)$, then*

$$\|P^1_{N,\alpha,\beta}v-v\|_{1,\omega_\alpha,\omega_\beta,\Lambda}\le cN^{\frac{1-r}{2}}|\partial_\rho v|_{A^{r-1}_\alpha,\Lambda}.$$

*Proof.* By the definition (2.11) and the projection theorem, we have

$$\|P^1_{N,\alpha,\beta}v-v\|_{1,\omega_\alpha,\omega_\beta,\Lambda}\le \|\phi-v\|_{1,\omega_\alpha,\omega_\beta,\Lambda},\qquad \forall\phi\in\mathcal{P}_N(\Lambda).$$

We now take

$$\phi(\rho)=\int_0^\rho P_{N-1,\alpha}\partial_\xi v(\xi)d\xi+\lambda$$

where $\lambda$ is chosen in such a way that $\phi(\rho_0)=v(\rho_0)$, and $\rho_0$ is the same as in Lemma 2.1. Then, by Lemma 2.1 and Theorem 2.1 with $s=0$, we assert that for any integer $r\ge 1$,

$$\begin{aligned}\|\phi-v\|_{1,\omega_\alpha,\omega_\beta,\Lambda}&\le c|\phi-v|_{1,\omega_\alpha,\Lambda}=c\|P_{N-1,\alpha}\partial_\rho v-\partial_\rho v\|_{\omega_\alpha,\Lambda}\\ &\le cN^{\frac{1-r}{2}}\|\partial^r_\rho v\|_{\omega_{\alpha+r-1},\Lambda}=cN^{\frac{1-r}{2}}|\partial_\rho v|_{A^{r-1}_\alpha,\Lambda}.\end{aligned}$$

**Theorem 2.3.** *If $v\in L^2_{\omega_\alpha}(\Lambda)$, $\partial_\rho v\in A^{r-1}_\alpha(\Lambda)$ and $v(0)=0$, then for integer $r\ge 1$,*

$$\|\partial_\rho({}_0P^1_{N,\alpha}v-v)\|_{\omega_\alpha,\Lambda}\le cN^{\frac{1-r}{2}}|\partial_\rho v|_{A^{r-1}_\alpha,\Lambda}.$$

*If, in addition, $|\alpha|<1$, then*

$$\|{}_0P^1_{N,\alpha}v-v\|_{1,\omega_\alpha,\Lambda}\le cN^{\frac{1-r}{2}}|\partial_\rho v|_{A^{r-1}_\alpha,\Lambda}.$$

*Proof.* By the definition (2.12), for any $\phi \in {}_0\mathcal{P}_N(\Lambda)$,

$$\begin{aligned}\|\partial_\rho({}_0P^1_{N,\alpha}v - v)\|^2_{\omega_\alpha,\Lambda} &= (\partial_\rho({}_0P^1_{N\alpha}v - v), \partial_\rho(\phi - v))_{\omega_\alpha,\Lambda} \\ &\leq \|\partial_\rho({}_0P^1_{N,\alpha}v - v)\|_{\omega_\alpha,\Lambda}\|\partial_\rho(\phi - v)\|_{\omega_\alpha,\Lambda}.\end{aligned}$$

Taking $\phi(\rho) = \int_0^\rho P_{N-1,\alpha}\partial_\xi v(\xi)d\xi \in {}_0\mathcal{P}_N(\Lambda)$ in the above and using an argument similar to the proof of the last theorem lead to the first desired result.

If in addition $|\alpha| < 1$, then the second result follows from Lemma 2.2.

## 3. Mixed Laguerre-Fourier Approximation for Exterior Domains

In this section, we investigate the Laguerre-Fourier approximation for exterior problems. To this end, we need several results related to the Laplace operator in the polar coordinates. Let us consider first an auxiliary projection related to the generalized Laguerre approximation with $\alpha = 2$ and $\beta = 0$.

Let $\omega(\rho) = \omega_0(\rho) = e^{-\rho}$ and $\eta(\rho) = (\rho + 1)^2 e^{-\rho}$. We define the orthogonal projection ${}_0\Pi^1_N : {}_0H^1_\eta(\Lambda) \to {}_0\mathcal{P}_N(\Lambda)$ by

$$\left({}_0\Pi^1_N v - v, \phi\right)_{1,\eta,\Lambda} = 0, \qquad \forall \phi \in {}_0\mathcal{P}_N(\Lambda).$$

For simplicity, we denote $A^r_0(\Lambda)$ by $A^r(\Lambda)$ in the sequel.

**Lemma 3.1.** *For any $v \in H^1_{\omega_2,\omega}(\Lambda) \cap A^r(\Lambda)$ with $v(0) = 0$ and integer $r \geq 2$,*

$$\|{}_0\Pi^1_N v - v\|_{1,\eta,\Lambda} \leq cN^{1-\frac{r}{2}}|v|_{A^r,\Lambda}.$$

*Proof.* By the projection theorem,

$$\|{}_0\Pi^1_N v - v\|_{1,\eta,\Lambda} \leq \|\phi - v\|_{1,\eta,\Lambda}, \qquad \forall \phi \in {}_0\mathcal{P}_N(\Lambda).$$

Let

$$\phi(\rho) = \int_0^\rho P^1_{N-1,2,0}(\partial_\xi v(\xi))d\xi.$$

Clearly $\phi \in {}_0\mathcal{P}_N(\Lambda)$. Thus, it suffices to estimate $\|\phi - v\|_{1,\eta,\Lambda}$. In other words, we only need to estimate $\|\partial_\rho(\phi - v)\|_{\omega_k,\Lambda}$ and $\|\phi - v\|_{\omega_k,\Lambda}$ for $k = 0, 2$. In fact, a direct calculation reveals that

$$\|\partial_\rho(\phi - v)\|_{\omega_k,\Lambda} = \|P^1_{N-1,2,0}\partial_\rho v - \partial_\rho v\|_{\omega_k,\Lambda}, \qquad k = 0, 2. \tag{3.1}$$

Thanks to Lemma 2.2 with $\alpha = 2$ and Theorem 2.2 with $\alpha = 2$ and $\beta = 0$, we have

$$\begin{aligned}\|\partial_\rho(\phi - v)\|^2_{\omega_2,\Lambda} &+ \|\partial_\rho(\phi - v)\|^2_{\omega,\Lambda} \leq c(\|\partial^2_\rho(\phi - v)\|^2_{\omega_2,\Lambda} + \|\partial_\rho(\phi - v)\|^2_{\omega,\Lambda}) \\ &= c\|\partial_\rho(\phi - v)\|^2_{1,\omega_2,\omega,\Lambda} = c\|P^1_{N-1,2,0}\partial_\rho v - \partial_\rho v\|_{1,\omega_2,\omega,\Lambda} \\ &\leq cN^{2-r}\|\partial^r_\rho v\|^2_{\omega_r,\Lambda} = cN^{2-r}|v|^2_{A^r,\Lambda}.\end{aligned} \tag{3.2}$$

Next, thanks to Lemma 2.2 with $\alpha = 0$, we get

$$\|\phi - v\|^2_{\omega,\Lambda} \leq 4\|\partial_\rho(\phi - v)\|^2_{\omega,\Lambda} \leq cN^{2-r}|v|^2_{A^r,\Lambda}. \tag{3.3}$$

Finally, using (3.2), (3.3) and Lemma 2.2 with $\alpha = 2$ yields

$$\|\phi - v\|^2_{\omega_2,\Lambda} \leq 8\|\partial_\rho(\phi - v)\|^2_{\omega_2,\Lambda} + 8\|\phi - v\|^2_{\omega,\Lambda} \leq cN^{2-r}|v|^2_{A^r,\Lambda}.$$

The proof is thus complete.

Next, we derive an approximation result in the $L^\infty(\Lambda)$-norm. To this end, we need the following embedding inequality.

**Lemma 3.2.** *For any $v \in H^1_\eta(\Lambda)$,*

$$\|(1+\rho)\mathrm{e}^{-\frac{\rho}{2}}v\|_{L^\infty(\Lambda)} \le 2\|v\|_{1,\eta,\Lambda}.$$

*Proof.* For any $\rho \in \Lambda$, we have from integration by parts that

$$\begin{aligned}(\rho^2+2\rho)\mathrm{e}^{-\rho}v^2(\rho) &= \int_0^\rho \partial_\xi((\xi^2+2\xi)\mathrm{e}^{-\xi}v^2(\xi))d\xi \\ &= 2\int_0^\rho (\xi^2+2\xi)\mathrm{e}^{-\xi}v(\xi)\partial_\xi v(\xi)d\xi + \int_0^\rho (2-\xi^2)\mathrm{e}^{-\xi}v^2(\xi)d\xi \\ &\le \int_0^\rho (\xi^2+2\xi)\mathrm{e}^{-\xi}(\partial_\xi v(\xi))^2d\xi + \int_0^\rho (2\xi+2)\mathrm{e}^{-\xi}v^2(\xi)d\xi \\ &\le \int_\Lambda (\rho^2+2\rho)\mathrm{e}^{-\rho}(\partial_\rho v(\rho))^2d\rho + \int_\Lambda (2\rho+2)\mathrm{e}^{-\rho}v^2(\rho)d\rho.\end{aligned} \tag{3.4}$$

By (2.3) of Xu and Guo [20],

$$\mathrm{e}^{-\rho}v^2(\rho) \le 2\int_\Lambda \mathrm{e}^{-\rho}(v^2(\rho)+(\partial_\rho v(\rho))^2)d\rho.$$

Adding the above to (3.4) yields that

$$\begin{aligned}(\rho+1)^2\mathrm{e}^{-\rho}v^2(\rho) &= \int_\Lambda (\rho^2+2\rho+2)\mathrm{e}^{-\rho}(\partial_\rho v(\rho))^2d\rho + \int_\Lambda (2\rho+4)\mathrm{e}^{-\rho}v^2(\rho)d\rho \\ &\le 4\,\|v\|^2_{1,\eta,\Lambda}.\end{aligned}$$

Combining Lemmas 3.1 and 3.2, we obtain the following result:

**Lemma 3.3.** *For any $v \in A^r(\Lambda)$ and integer $r \ge 2$,*

$$\|(\rho+1)\mathrm{e}^{-\frac{\rho}{2}}({}_0\Pi^1_N v - v)\|_{L^\infty(\Lambda)} \le cN^{1-\frac{r}{2}}|v|_{A^r,\Lambda}.$$

Since we will expand functions in the azimuthal direction by a Fourier series, we recall a basic result on the Fourier approximation in one-dimension. Let $I=(0,2\pi)$ and $H^r(I)$ be the Sobolev space with norm $\|\cdot\|_{r,I}$ and semi-norm $|\cdot|_{r,I}$. For any non-negative integer $m$, $H^m_p(I)$ denotes the subspace of $H^m(I)$, consisting of all functions whose derivatives of order up to $m-1$ are periodic with the period $2\pi$. For any real $r>0$, the space $H^r_p(I)$ is defined as in Adams [1]. In particular, $L^2_p(I)=H^0_p(I)$. Let $M$ be any positive integer, and $\tilde V_M(I) = \mathrm{span}\left\{\, \mathrm{e}^{il\theta} \mid |l| \le M \right\}$. We denote by $V_M(I)$ the subset of $\tilde V_M(I)$ consisting of all real-valued functions.

As usual, the $L^2_p(I)$-orthogonal projection $P_M : L^2_p(I) \to V_M(I)$ is defined by

$$\int_I (P_M v(\theta) - v(\theta))\phi(\theta)d\theta = 0, \qquad \forall \phi \in V_M(I).$$

The next lemma can be found in Canuto, Hussaini, Quarteroni and Zang [4].

**Lemma 3.4.** *Let integer $r \ge 0$ and $\mu \le r$. Then for any $v \in H^r_p(I)$,*

$$\|P_M v - v\|_{\mu,I} \le cM^{\mu-r}|v|_{r,I}.$$

We are now in position to study the mixed Laguerre-Fourier approximation.

Let $\Omega = \Lambda \times I$ and $L^2_\chi(\Omega)$ be the weighted Sobolev space with the following inner product and norm,

$$(u,v)_\chi = \int_\Omega u(\rho,\theta)v(\rho,\theta)\chi(\rho)d\rho d\theta, \quad \|v\| = (v,v)_\chi^{\frac{1}{2}}.$$

The weighted Sobolev spaces $H^r_\chi(\Omega)$ and its norm $\|v\|_{r,\chi}$ and semi-norm $|v|_{r,\chi}$ are defined in the usual manner. In particular, we set

$${}_0H^1_{p,\omega}(\Omega) = \left\{\ v \in H^1_\omega(\Omega) \mid v(\rho,\theta+2\pi) = v(\rho,\theta) \text{ and } v(0,\theta) = 0, \text{ for } \theta \in I,\ \rho \in \Lambda \right\}.$$

Next, we define the non-isotropic space

$${}_0H^1_{p,\eta,\omega}(\Omega) = \{\ v \mid v \text{ is measurable on } \Omega \text{ and } \|v\|_{1,\eta,\omega} < \infty\}$$

where

$$|v|_{1,\eta,\omega} = (\|\partial_\rho v\|^2_\eta + \|\partial_\theta v\|^2_\omega)^{\frac{1}{2}}, \qquad \|v\|_{1,\eta,\omega} = (|v|^2_{1,\eta,\omega} + \|v\|^2_\omega)^{\frac{1}{2}}.$$

Let use denote

$$V_{N,M}(\Omega) = {}_0\mathcal{P}_N(\Lambda) \otimes V_M(I).$$

We define an orthogonal projector ${}_0P^1_{N,M}\colon {}_0H^1_{p,\eta,\omega}(\Omega) \to V_{N,M}$ by

$$\left(\partial_\rho({}_0P^1_{N,M}v - v), \partial_\rho\phi\right)_\eta + \left(\partial_\theta({}_0P^1_{N,M}v - v), \partial_\theta\phi\right)_\omega = 0, \quad \forall \phi \in V_{N,M}. \tag{3.5}$$

In order to describe the approximation results related to this projection operator, we introduce the non-isotropic space

$$\mathcal{B}^{r,s} = A^r(\Lambda, H^1_p(I)) \cap A^2(\Lambda, H^s_p(I)) \cap H^1_\eta(\Lambda, H^{s-1}_p(I)),$$

equipped with the norm

$$\|v\|_{\mathcal{B}^{r,s}} = \left(\|v\|^2_{A^r(\Lambda,H^1(I))} + \|v\|^2_{A^2(\Lambda,H^s(I))} + \|v\|^2_{H^1_\eta(\Lambda,H^{s-1}(I))}\right)^{\frac{1}{2}}$$

where the space $A^r(\Lambda)$ and its norm are the same as in (3.1).

**Theorem 3.1.** *For any $v \in \mathcal{B}^{r,s} \cap {}_0H^1_{p,\eta,\omega}(\Omega)$ and integers $r \geq 2, s \geq 1$, we have*

$$\|v - {}_0P^1_{N,M}v\|_{1,\eta,\omega} \leq c(N^{1-\frac{r}{2}} + M^{1-s})\|v\|_{\mathcal{B}^{r,s}}.$$

*Proof.* By the projection theorem,

$$|v - {}_0P^1_{N,M}v|_{1,\eta,\omega} \leq |v - \phi|_{1,\eta,\omega}, \qquad \forall \phi \in V_{N,M}(\Omega). \tag{3.6}$$

Let $\phi = {}_0\Pi^1_N(P_M v)$. We use Lemmas 3.1 and 3.4 to deduce that

$$\begin{aligned}
\|\partial_\rho(v - {}_0\Pi^1_N(P_M v))\|_\eta &\leq \|\partial_\rho v - P_M(\partial_\rho v)\|_\eta + \|\partial_\rho(P_M v - {}_0\Pi^1_N(P_M v))\|_\eta \\
&\leq cM^{1-s}\|\partial_\rho v\|_{L^2_\eta(\Lambda,H^{s-1}(I))} + cN^{1-\frac{r}{2}}\|P_M v\|_{A^r(\Lambda,L^2(I))} \\
&\leq cM^{1-s}\|\partial_\rho v\|_{L^2_\eta(\Lambda,H^{s-1}(I))} + cN^{1-\frac{r}{2}}\|v\|_{A^r(\Lambda,L^2(I))} \\
&\leq c(M^{1-s} + N^{1-\frac{r}{2}})\|v\|_{\mathcal{B}^{r,s}}.
\end{aligned} \tag{3.7}$$

Using Lemmas 3.1 and 3.4 again, we obtain that

$$\begin{aligned}
\|\partial_\theta(v - {}_0\Pi_N^1(P_M v))\|_\omega &\le \|{}_0\Pi_N^1\partial_\theta v - \partial_\theta v\|_\omega + \|{}_0\Pi_N^1(\partial_\theta(P_M v - \partial_\theta v))\|_\omega \\
&\le \|{}_0\Pi_N^1\partial_\theta v - \partial_\theta v\|_\eta + \|{}_0\Pi_N^1(\partial_\theta(P_M v - v))\|_\eta \\
&\le cN^{1-\frac{r}{2}}\|\partial_\theta v\|_{A^r(\Lambda,L^2(I))} + c\|\partial_\theta(P_M v - v)\|_{A^2(\Lambda,L^2(I))} \qquad (3.8) \\
&\le cN^{1-\frac{r}{2}}\|\partial_\theta v\|_{A^r(\Lambda,L^2(I))} + cM^{1-s}\|v\|_{A^2(\Lambda,H^s(I))} \\
&\le c(N^{1-\frac{r}{2}} + M^{1-s})\|v\|_{\mathcal{B}^{r,s}}.
\end{aligned}$$

The combination of (3.6)-(3.8) leads to

$$|v - {}_0P_{N,M}^1 v|_{1,\eta,\omega} \le c(N^{1-\frac{r}{2}} + M^{1-s})\|v\|_{\mathcal{B}^{r,s}}.$$

Finally, by Lemma 2.2 with $\alpha = 0$,

$$\begin{aligned}
\|v - {}_0P_{N,M}^1 v\|_\omega &\le c\|\partial_\rho(v - {}_0P_{N,M}^1 v)\|_\omega \le c\|\partial_\rho(v - {}_0P_{N,M}^1 v)\|_\eta \\
&\le c|v - {}_0P_{N,M}^1 v|_{1,\eta,\omega} \le c(N^{1-\frac{r}{2}} + M^{1-s})\|v\|_{\mathcal{B}^{r,s}}.
\end{aligned}$$

## 4. Mixed Laguerre-Fourier Spectral Method for Exterior Problems

In this section, we take a model problem as an example to show how to construct and analyze the mixed Laguerre-Fourier schemes for exterior problems.

Let $x = (x_1, x_2), |x| = \sqrt{x_1^2 + x_2^2}$ and $\tilde{\Omega} = \{\ x \mid |x| > 1\}$. We consider the following model problem

$$\begin{cases} -\Delta U + \beta U = F, & \text{in } \tilde{\Omega}, \\ U(x)|_{\partial\tilde{\Omega}} = g, \ \lim\limits_{|x|\to\infty} U(x) = 0, & \end{cases} \tag{4.1}$$

where $\beta$ is a positive constant, and $F$ and $g$ are given functions. For simplicity, we assume that $g \equiv 0$.

Under the following polar transformation:

$$x_1 = (\rho+1)\cos\theta,\ x_2 = (\rho+1)\sin\theta,\ \bar{U}(\rho,\theta) = U(x_1,x_2),\ \bar{F}(\rho,\theta) = F(x_1,x_2),$$

the problem (4.1) becomes

$$\begin{cases} -\dfrac{1}{\rho+1}\partial_\rho((\rho+1)\partial_\rho\bar{U}) - \dfrac{1}{(\rho+1)^2}\partial_\theta^2\bar{U} + \beta\bar{U} = \bar{F}, & \text{in } \Omega, \\ \bar{U}(0,\theta) = 0,\ \ \bar{U}(\rho,\theta+2\pi) = \bar{U}(\rho,\theta),\ \ \lim\limits_{\rho\to\infty}\bar{U}(\rho,\theta) = 0, & \theta\in I. \end{cases} \tag{4.2}$$

Since Problem (4.1) is well-posed in the standard functional space, it is not appropriate to consider (4.2) in a weighted Sobolev space with the Laguerre weight $\omega(\rho)$. Hence, we use the following change of variables

$$u(\rho,\theta) = (\rho+1)^{-\frac{1}{2}}\mathrm{e}^{\frac{1}{2}\rho}\bar{U}(\rho,\theta), \quad f(\rho,\theta) = (\rho+1)^{\frac{3}{2}}\mathrm{e}^{\frac{1}{2}\rho}\bar{F}(\rho,\theta),$$

to transform (4.2) to

$$\begin{cases} -(\rho+1)^2\partial_\rho^2 u + (\rho^2-1)\partial_\rho u - \partial_\theta^2 u + (\beta(\rho+1)^2 + \frac{1}{2} + \frac{1}{2}\rho - \frac{1}{4}\rho^2)u = f, & \text{in } \ \Omega, \\ u(0,\theta) = 0,\ \ u(\rho,\theta+2\pi) = u(\rho,\theta),\ \ \lim\limits_{\rho\to\infty}(\rho+1)^{\frac{1}{2}}e^{-\frac{1}{2}\rho}u(\rho,\theta) = 0, & \theta\in I. \end{cases} \tag{4.3}$$

We now consider the existence and regularity of solutions for the problem (4.3). For this purpose, let us denote

$$\begin{aligned}\mathcal{A}(u,v)= & \int_\Omega (\rho+1)^2 \mathrm{e}^{-\rho}\partial_\rho u \partial_\rho v d\rho d\theta + \int_\Omega \mathrm{e}^{-\rho}\partial_\theta u \partial_\theta v d\rho d\theta \\ & + \int_\Omega \mathrm{e}^{-\rho}\left(\beta(\rho+1)^2 - \frac{1}{4}\rho^2 + \frac{1}{2}\rho + \frac{1}{2}\right) uv d\rho d\theta.\end{aligned}$$

A weighted (with $\omega(\rho) = e^{-\rho}$ and $\eta(\rho) = (\rho+1)^2 e^{-\rho})$) weak formulation of (4.3) is to find $u \in {}_0H^1_{p,\eta,\omega}(\Omega)$ such that

$$\mathcal{A}(u,v) = (f,v)_\omega, \qquad \forall v \in {}_0H^1_{p,\eta,\omega}(\Omega). \tag{4.4}$$

**Lemma 4.1.** *For any $u$, $v \in {}_0H^1_{p,\eta,\omega}(\Omega)$,*

$$\begin{aligned}\mathcal{A}(v,v) \geq & \int_\Omega (\rho+1)^2 \mathrm{e}^{-\rho}(\partial_\rho v(\rho,\theta))^2 d\rho d\theta + \int_\Omega \mathrm{e}^{-\rho}(\partial_\theta v(\rho,\theta))^2 d\rho d\theta \\ & +(\beta - \tfrac{1}{4})\int_\Omega (\rho+1)^2 \mathrm{e}^{-\rho} v^2(\rho,\theta) d\rho d\theta + \frac{3}{4}\int_\Omega \mathrm{e}^{-\rho} v^2(\rho,\theta) d\rho d\theta,\end{aligned}$$

*and*

$$|\mathcal{A}(u,v)| \leq c\|u\|_{1,\eta,\omega}\|v\|_{1,\eta,\omega}.$$

*Proof.* Obviously

$$-\frac{1}{4}\rho^2 + \frac{1}{2}\rho + \frac{1}{2} \geq -\frac{1}{4}(\rho+1)^2 + \frac{3}{4},$$

which leads to the first result. The second result follows from Lemma 2.2 with $\alpha = 2$.

**Theorem 4.1.** *If $\beta \geq \frac{1}{4}$ and $(\rho+1)^{-1}f \in L^2_\omega(\Omega)$, then, (4.4) admits a unique solution $u(\rho,\theta)$ with $\|u\|_{1,\eta,\omega} \leq c\|(\rho+1)^{-1}f\|_\omega$.*

*Proof.* Due to $\beta \geq \frac{1}{4}$ and Lemma 4.1, $\mathcal{A}(u,v)$ is coercive on ${}_0H^1_{p,\eta,\omega}(\Omega) \times {}_0H^1_{p,\eta,\omega}(\Omega)$. Moreover, by the result (iii) of Lemma 2.2,

$$|(f,v)_\omega| \leq c\|v\|_{1,\eta,\omega}\|(\rho+1)^{-1}f\|_\omega.$$

Thus, the conclusion follows from the Lax-Milgram Lemma.

**Remark 4.1.** The condition on $f$ in Theorem 4.1 means $(\rho+1)^{\frac{1}{2}}\bar{F} \in L^2_\omega(\Omega)$, and equivalently $F \in L^2(\tilde{\Omega})$.

Next, we consider the mixed Laguerre-Fourier approximation for (4.4): find $u_{N,M} \in V_{N,M}$ such that

$$\mathcal{A}(u_{N,M},\phi) = (f,\phi)_\omega, \qquad \forall \phi \in V_{N,M}. \tag{4.5}$$

The following result is a direct consequence of Lemmas 2.1 and 4.1, and the Lax-Milgram Lemma.

**Theorem 4.2.** *For $\beta \geq \frac{1}{4}$, the problem (4.5) admits a unique solution $u_{N,M}$. Moreover,*

$$\|\partial_\rho u_{N,M}\|^2_\eta + \|\partial_\theta u_{N,M}\|^2_\omega + (\beta - \frac{1}{4})\|u_{N,M}\|^2_\eta + \frac{1}{2}\|u_{N,M}\|^2_\omega \leq \|(\rho+1)^{-1}f\|^2_\omega. \tag{4.6}$$

We now turn our attention to the error analysis.

**Theorem 4.3.** *Let $\beta \geq \frac{1}{4}$ and integers $r \geq 2, s \geq 1$. For $u \in \mathcal{B}^{r,s}(\Omega)$, we have*

$$\|\partial_\rho(u - u_{N,M})\|^2_\eta + (\beta - \frac{1}{4})\|u - u_{N,M}\|^2_\eta + \|\partial_\theta(u - u_{N,M})\|^2_\omega \leq c(N^{1-\frac{r}{2}} + M^{1-s})^2\|u\|^2_{\mathcal{B}^{r,s}}.$$

*Proof.* Let $u^*_{N,M} = {}_0P^1_{N,M}u$. We obtain from (3.5) and (4.4) that

$$\begin{aligned}&\int_\Omega (\rho+1)^2 e^{-\rho}\partial_\rho u^*_{N,M}\partial_\rho\phi d\rho d\theta + \int_\Omega e^{-\rho}\partial_\theta u^*_{N,M}\partial_\theta\phi d\rho d\theta\\ &+\int_\Omega e^{-\rho}\left(\beta(\rho+1)^2-\frac{1}{4}\rho^2+\frac{1}{2}\rho+\frac{1}{2}\right)u\phi d\rho d\theta = \int_\Omega e^{-\rho}f\phi d\rho d\theta, \quad \forall\phi\in V_{N,M}.\end{aligned} \tag{4.7}$$

Now setting $\tilde{u}_{N,M} = u_{N,M} - u^*_{N,M}$ and subtracting (4.7) from (4.5), we obtain

$$\mathcal{A}(\tilde{u}_{N,M},\phi) = G(u,u^*_{N,M};\phi), \qquad \forall\phi\in V_{N,M} \tag{4.8}$$

where

$$G(u,u^*_{N,M};\phi) = \int_\Omega e^{-\rho}\left(\beta(\rho+1)^2-\frac{1}{4}\rho^2+\frac{1}{2}\rho+\frac{1}{2}\right)(u-u^*_{N,M})\phi d\rho d\theta.$$

By Lemma 2.2 with $\alpha = 2$,

$$\|\tilde{u}_{N,M}\|^2_\eta \le c(\|\partial_\rho\tilde{u}_{N,M}\|^2_\eta + \|\tilde{u}_{N,M}\|^2_\omega).$$

Therefore, we deduce that for any $\delta > 0$,

$$|G(u,u^*_{N,M};\tilde{u}_{N,M})| \quad \le \delta(\|\partial_\rho\tilde{u}_{N,M}\|^2_\eta + \|\tilde{u}_{M,N}\|^2_\omega) + \tfrac{c}{\delta}(\|\partial_\rho(u-u^*_{N,M})\|^2_\eta + \|u-u^*_{N,M}\|^2_\omega). \tag{4.9}$$

Taking $\phi = \tilde{u}_{N,M}$ in (4.8), we use (4.9), Lemma 4.1 and Theorem 3.1 to obtain that

$$\|\partial_\rho\tilde{u}_{N,M}\|^2_\eta + (\beta-\frac{1}{4})\|\tilde{u}_{N,M}\|^2_\eta + \|\partial_\theta\tilde{u}_{N,M}\|^2_\omega \le c(N^{1-\frac{r}{2}}+M^{1-s})^2\|u\|^2_{\mathcal{B}^{r,s}}$$

which completes the proof.

**Remark 4.2.** The numerical solution of the original problem (4.1) is

$$U_{N,M} = (1+\rho)^{\frac{1}{2}}e^{-\frac{\rho}{2}}u_{N,M}.$$

Thanks to Theorem 4.3 and the fact that $U = (1+\rho)^{\frac{1}{2}}e^{-\frac{1}{2}\rho}u$, we can obtain the following estimate:

$$\|U-U_{N,M}\|_{H^1(\tilde{\Omega})} \le c(N^{1-\frac{r}{2}}+M^{1-s})\|(1+\rho)^{-\frac{1}{2}}e^{\frac{\rho}{2}}U\|_{\mathcal{B}^{r,s}}.$$

**Remark 4.3.** It can be shown, by using a suitable transformation, that the results of Theorems 4.1-4.3 are also valid for any $\beta > 0$. However, how to extend the convergence result to the more interesting case, $\beta < 0$, is still an open question. Nevertheless, the algorithm developed here can still be used to approximate the solution of (4.1) in the case of $\beta < 0$.

## 5. Implementation Details and Numerical Results

Let us first describe in some details an efficient implementation for scheme (4.5). For simplicity, we denote $\mathcal{L}^{(0)}_l(\rho)$ by $\mathcal{L}_l(\rho)$ and set

$$\psi_l(\rho) = \mathcal{L}_{l-1}(\rho) - \mathcal{L}_l(\rho), \qquad 1\le l\le N,$$

and

$$\begin{aligned}\phi^1_{lm}(\rho,\theta) &= \psi_l(\rho)\cos m\theta, &&1\le l\le N,\ 0\le m\le M.\\ \phi^2_{lm}(\rho,\theta) &= \psi_l(\rho)\sin m\theta, &&1\le l\le N,\ 1\le m\le M.\end{aligned}$$

Since $\psi_l(1)=0$, $\phi_{lm}$ can be used as basis functions for $V_{NM}$. Hence, we can expand $u_{N,M}$ as

$$u_{N,M}(\rho,\theta)=\sum_{l=1}^{N}\left(\sum_{m=0}^{M}u_{lm}^1\phi_{lm}^1(\rho,\theta)+\sum_{m=1}^{M}u_{lm}^2\phi_{lm}^2(\rho,\theta)\right).$$

On the other hand, we write

$$f(\rho,\theta)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(f_{l,m}^1\mathcal{L}_l(\rho)\cos m\theta+f_{l,m}^2\mathcal{L}_l(\rho)\sin m\theta).$$

We note that in actual computation, the Fourier-Laguerre Gauss-Radau quadrature should be used to approximate the values of $\{f_{k,n}^q\}$.

Let us denote

$$Z_M=\{(q,n):q=1,\ n=0,1,\cdots,M;q=2,n=1,2,\cdots,M\}.$$

Taking $\phi(\rho,\theta)=\phi_{kn}^q(\rho,\theta)$ in (4.5) for $(q,n)\in Z_M$, we derive by using the orthogonality of the trigonometric functions that (4.5) is equivalent to the following $2M+1$ linear systems:

$$\begin{aligned}\sum_{l=1}^{N}\left(\int_\Lambda(\rho+1)^2\mathrm{e}^{-\rho}\partial_\rho\psi_l\partial_\rho\psi_k d\rho+\int_\Lambda \mathrm{e}^{-\rho}(\beta(\rho+1)^2+n^2-\frac{1}{4}\rho^2+\frac{1}{2}\rho+\frac{1}{2})\psi_l\psi_k d\rho\right)u_{ln}^q\\ =g_{k,n}^q,\ 1\le k\le N,\end{aligned}\tag{5.1}$$

where $g_{k,n}^q=f_{k-1,n}^q-f_{k,n}^q,\ 1\le k\le N$.

Let us denote

$$\begin{aligned}
&\bar{x}_n^q=(u_{1,n}^q,u_{2,n}^q,\cdots,u_{N,n}^q)^T, && \bar{g}_n^q=(g_{1,n}^q,g_{2,n}^q,\cdots,g_{N,n}^q)^T,\\
&a_{lk}=\int_\Lambda(\rho+1)^2\mathrm{e}^{-\rho}\partial_\rho\psi_l(\rho)\partial_\rho\psi_k(\rho)d\rho, && A=(a_{kl})_{k,l=1,2,\cdots,N},\\
&b_{lk}=\int_\Lambda \mathrm{e}^{-\rho}\psi_l(\rho)\psi_k(\rho)d\rho, && B=(b_{kl})_{k,l=1,2,\cdots,N},\\
&c_{lk}=\int_\Lambda \rho\mathrm{e}^{-\rho}\psi_l(\rho)\psi_k(\rho)d\rho, && C=(c_{kl})_{k,l=1,2,\cdots,N},\\
&d_{lk}=\int_\Lambda \rho^2\mathrm{e}^{-\rho}\psi_l(\rho)\psi_k(\rho)d\rho, && D=(d_{kl})_{k,l=1,2,\cdots,N}.
\end{aligned}$$

Then, (5.1) becomes

$$(A+(\beta+n^2+\frac{1}{2})B+(2\beta+\frac{1}{2})C+(\beta-\frac{1}{4})D)\bar{x}_n^q=\bar{g}_n^q,\ (q,n)\in Z_M.\tag{5.2}$$

Using the orthogonality relations of Laguerre polynomials, one can easily derive that

$$a_{kl}=\begin{cases}6k^2-2k+1, & l=k,\\ -4k^2+2k-1\pm(1-4k), & l=k\pm1,\\ k^2-k+1\pm(2k-1), & l=k\pm2,\\ 0, & \text{otherwise},\end{cases}$$

$$b_{kl}=\begin{cases}2, & l=k,\\ -1, & l=k\pm1,\\ 0, & \text{otherwise},\end{cases}$$

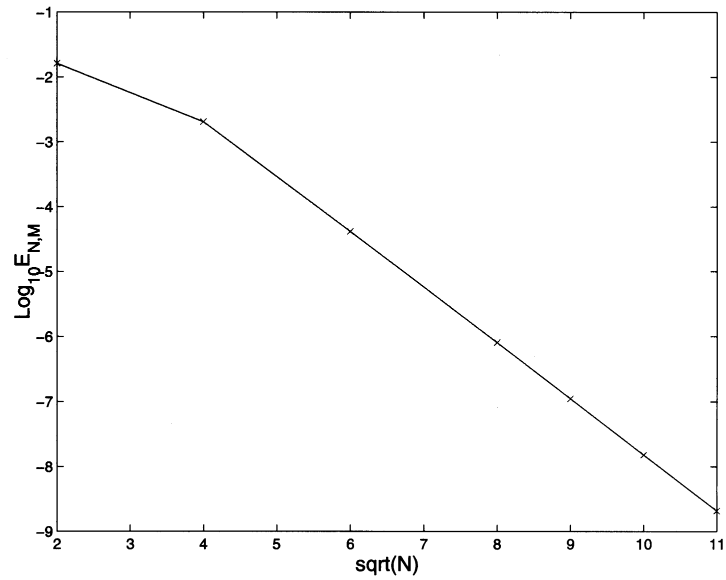


Figure 5.1. Convergence rate for $N = M^2$

$$c_{kl} = \begin{cases} 6k, & l = k, \\ -2(2k \pm 1), & l = k \pm 1, \\ k \pm 1, & l = k \pm 2, \\ 0, & \text{otherwise}, \end{cases}$$

$$d_{kl} = \begin{cases} 4(5k^2 + 1), & l = k, \\ -(15k^2 \pm 15k + 6), & l = k \pm 1, \\ 6(k \pm 1)^2, & l = k \pm 2, \\ -(k \pm 1)(k \pm 2), & l = k \pm 3, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the matrices in the linear system (5.2) are symmetric with five or seven non-zero diagonals. Hence, the system (5.2) can be efficiently solved. Note that an efficient algorithm based on the Laguerre functions was proposed in [18]. However, the corresponding linear system there, although sparse, was not symmetric. Hence, the algorithm presented here is advantageous in this regard.

We now present an illustrative numerical result. We take the exact solution of (4.3) to be

$$u(\rho, \theta) = \frac{\rho^2}{\rho + 1.0} \mathrm{e}^{-\rho} \sin\theta$$

and use the scheme (4.5) to obtain the numerical solution $u_{N,M}$. We set $E_{N,M} = \|u - u_{N,M}\|_{L^2_\omega(\Omega)}$. It can be easily checked that $\|u\|_{\mathcal{B}^{r,s}}$ is finite for any $r, s > 0$. Hence, Theorem 4.3 indicates that $E_{N,M}$ converges to zero faster than any algebraic power.

Note that Theorem 4.3 indicates that at least for smooth functions, a proper relation between $N$ and $M$ is: $N \sim M^2$. In Figure 1, we plot the convergence rates of the scheme (4.5) with $N = M^2$. The straight line in Figure 1 indicates that the error $E_{N,M}$ behaves like $exp(-c\sqrt{N})$, i.e., it converges sub-geometrically.

## 6. Concluding Remarks

In the first part of this paper, we studied the generalized Laguerre approximations and established error estimates in the non-uniformly weighted spaces for various orthogonal projections. These estimates improve previously published results for the special case $\alpha = 0$ and are valid for the generalized Laguerre approximations with $\alpha > -1$.

In the second part, we proposed a mixed Laguerre-Fourier spectral method for the Helmholtz equation in a two dimensional exterior domain. We obtained sharp error estimates for the proposed method by transforming the original system, which is not well-posed in the desired weighted Sobolev spaces, to a system which is well-posed in a suitable functional space. We have also constructed an efficient numerical algorithm and presented an illustrative numerical result.

Note that in terms of numerical algorithm, the effect of the change of variable is equivalent to using an approximation by Laguerre functions as in [18]. However, to carry out the analysis for the approximation using Laguerre functions, one needs to develop corresponding approximation results which are beyond the scope of this paper.

Although we only considered a simple model problem in this paper, but the results developed here will be useful for the numerical analysis of more complicated equations in fluid dynamics and electromagnetics.

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