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Efficient Spectral Methods for Transmission Eigenvalues and Estimation of the Index of Refraction

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Abstract. An important step in estimating the index of refraction of electromagnetic scattering problems is to compute the associated transmission eigenvalue problem. We develop in this paper efficient and accurate spectral methods for computing the transmission eigenvalues associated to the electromagnetic scattering problems. We present ample numerical results to show that our methods are very effective for computing transmission eigenvalues (particularly for computing the smallest eigenvalue), and together with the linear sampling method, provide an efficient way to estimate the index of refraction of a non-absorbing inhomogeneous medium.

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1 Introduction

The inverse electromagnetic scattering problem plays an important role in many applications, and is notoriously difficult. Recently a new method using transmission eigenvalues to estimate the index of refraction of a non-absorbing inhomogeneous medium is proposed in [4, 5, 9]. The method consists of several steps. First, the support of the scattering obstacle can be recovered by using the measured scattering data and the linear sampling method [12], and the transmission eigenvalues can be identified from the far field data. Then, the bounds for smallest and largest eigenvalues of the (matrix) index of refraction can be obtained in terms of the support of the scattering obstacle and the first

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transmission eigenvalue of the anisotropic media [3]. Finally, reconstructions of the electric permittivity (if it is a scalar constant) or an estimate of the eigenvalues of the matrix in the case of anisotropic permittivity can be obtained [5].

The effectiveness of the above method rests on having an efficient and robust algorithm for computing transmission eigenvalues for a scalar permittivity. In this paper, we develop efficient spectral methods for computing the transmission eigenvalues in circular and rectangular domains. In particular, for circular domains with stratified media, our method reduces the problem to a sequence of one-dimensional transmission eigenvalue problems that can be solved efficiently and accurately by a spectral-element methods. An error estimate for convergence of the transmission eigenvalues is also provided in this case.

We present ample numerical results to show that our methods are very effective, particularly for computing the few smallest eigenvalues. By using this together with the linear sampling method, we can effectively estimate the (matrix) index of refraction of a non-absorbing inhomogeneous medium.

The organization of the paper is as follows: In §2, we describe the general approach introduced in [4, 5, 9] for estimation of the index of refraction of a non-absorbing inhomogeneous medium. In §3, we derive a weak formulation, construct a Fourier-spectral-element method and derive error estimates for transmission eigenvalues in circular domains. In §4, we describe a spectral method for computing transmission eigenvalues in rectangular domains. We present some numerical results to validate our numerical algorithms in §5.

2 Description of the general approach

In this section, we describe briefly the method introduced in [4, 5, 9] for estimation of the index of refraction of a non-absorbing inhomogeneous medium. We first show how to obtain transmission eigenvalues from far field data, and then describe an algorithm to reconstruct/estimate the index of refraction.

2.1 Transmission eigenvalue problem

Let $D\subset R^d$ $(d=2,3)$ be a bounded, simply connected open set with a piecewise smooth boundary ∂D. We assume that the domain D is the support of an anisotropic dielectric object, and the incident field is a time-harmonic electromagnetic plane wave with frequency ω. Then, the scattering by the anisotropic medium leads to the following problem for the interior electric and magnetic fields E^{int}, H^{int}, and the scattered electric and magnetic field

E^{ext},H^{ext} [5]:

$$\nabla\times E^{ext}-ikH^{ext}=0, \qquad in \quad R^3\backslash\overline{D}, \tag{2.1}$$
$$\nabla\times H^{ext}+ikE^{ext}=0, \qquad in \quad R^3\backslash\overline{D}, \tag{2.2}$$
$$\nabla\times E^{int}-ikH^{int}=0, \qquad in \quad D, \tag{2.3}$$
$$\nabla\times H^{int}+ikN(x)E^{int}=0, \qquad in \quad D, \tag{2.4}$$
$$E^{ext}\times\nu-E^{int}\times\nu=0, \qquad on \quad \partial D, \tag{2.5}$$
$$H^{ext}\times\nu-H^{int}\times\nu=0, \qquad on \quad \partial D. \tag{2.6}$$

where ν is the unit outward normal to ∂D and $N(x)$ is a 3×3 symmetric matrix representing the electric permittivity. We write

$$E^{ext}=E^s+E^i, \qquad H^{ext}=H^s+H^i,$$

where E^i and H^i are the incident electric and magnetic fields given by

$$E^i(x):=\frac{i}{k}\nabla\times\nabla\times pe^{ikx\cdot d}, \qquad H^i(x):=\frac{1}{k^2}\nabla\times pe^{ikx\cdot d},$$

with d being a unit vector giving the direction of propagation and p the polarization vector. The scattered electric and magnetic fields E^s and H^s satisfy the Silver-Müller radiation condition

$$\lim_{r=|x|\to\infty}(H^s\times x-rE^s)=0,$$

uniformly in $\hat{x}=x/|x|$.

In this paper, we consider the two-dimensional scattering problem from an orthotropic medium. More precisely, we assume that the scatterer is an infinitely long dielectric cylinder with axis in the z-direction, and denote by D the cross section of the cylinder in (x_1,x_2) plane. We assume that the dielectric cylinder is orthotropic, i.e. the matrix $N(x)$ takes the form

$$N=\begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33}\end{pmatrix},$$

with $n_{11}n_{22}-n_{12}n_{21}\neq0$. If we consider incident waves such that the electric field is perpendicular to the z axis, then the magnetic fields have only one non-zero component in the z direction, i.e.

$$H^{ext}=(0,0,\omega), \qquad H^i=(0,0,v^i), \qquad H^s=(0,0,v^s).$$

Assuming that N^{-1} exists, and expressing the electric fields in terms of magnetic fields, Eqs. (2.1)$-$(2.6) now lead to the following transmission problem for ω and v [2]:

$$\nabla\cdot A\nabla\omega+k^2\omega=0, \qquad in \quad D, \tag{2.7}$$

$$\Delta v+k^2 v=0, \qquad in \quad D_e, \tag{2.8}$$

$$\omega-v=0, \quad \frac{\partial\omega}{\partial\nu_A}-\frac{\partial v}{\partial\nu}=0, \qquad on \quad \partial D, \tag{2.9}$$

$$v=v^s+v^i, \quad \lim_{r\to\infty}\sqrt{r}\Big(\frac{\partial v^s}{\partial r}-ikv^s\Big)=0, \tag{2.10}$$

where v^s is the scattered field and v^i is the given incident field. In the case of plane waves, the incident field is given by $v^i:=e^{ikx\cdot d}$ with $|x|=1$. Moreover

$$\frac{\partial\omega}{\partial\nu_A}(x):=\nu(x)\cdot A(x)\nabla\omega(x), x\in\partial D,$$

$$A=\frac{1}{n_{11}n_{22}-n_{12}n_{21}}\begin{pmatrix} n_{11} & n_{12}\\ n_{21} & n_{22}\end{pmatrix},$$

and the radiation condition (2.10) holds uniformly with respect to $\hat{x}=x/|x|$. The existence of a unique solution to (2.7)$-$(2.10) can be established by using a variational procedure [6,13].

We recall that the scatterer D can be determined with the linear sampling method by solving the far field equation

$$\int_\Omega v_\infty(\hat{x},d)g(d)ds(d)=\Phi_\infty(\hat{x},z),$$

where $v_\infty(\hat{x},d)$ is the far field pattern, Φ_∞ is the far field pattern of the radiating fundamental solution

$$\Phi(x,y):=\frac{i}{4}H_0^{(1)}(k|x-y|),$$

and $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero. Indeed, it is shown in [6] that the far field operator $F:L^2(\Omega)\to L^2(\Omega)$ defined by

$$(Fg)(\hat{x}):=\int_\Omega v_\infty(\hat{x},d)g(d)ds(d),$$

is injective with dense range, provided that k is not a transmission eigenvalue, i.e., a value of k for which the (homogeneous) interior transmission problem

$$\nabla\cdot A\nabla\omega+k^2\omega=0, \qquad in \quad D, \tag{2.11}$$

$$\Delta v+k^2 v=0, \qquad in \quad D, \tag{2.12}$$

$$\omega-v=0, \qquad on \quad \partial D, \tag{2.13}$$

$$\frac{\partial\omega}{\partial\nu_A}-\frac{\partial v}{\partial\nu}=0, \qquad on \quad \partial D \tag{2.14}$$

has a nontrivial solution $\omega, v \in H^1(D)$. Due to the lack of injectivity and the denseness of the range of the far field operator F, the L^2-norm of the (regularized) solution to

$$(Fg)(\hat{x}) := \Phi_\infty(\hat{x}, z_0), \qquad z_0 \in D$$

can be expected to be large when k is a transmission eigenvalue [2]. This provides a method to determine the transmission eigenvalues using the far field data.

2.2 Estimation of the index of refraction

We denote the smallest real transmission eigenvalue of (2.11)-(2.14) by $k_1(D,A)$. Since $k_1(D,A)$ can be computed from far field data [5] as shown above, we now describe an algorithm similar to that used in [15] to estimate the index of refraction. by using $k_1(D,A)$.

We shall restrict our attention to the case $A = I/n(x)$, more general case can be treated similarly as in [5]. Denoting

$$\mathbf{w} = A\nabla\omega \quad \text{and} \quad \mathbf{v} = \nabla v,$$

we can rewrite Eqs. (2.11)−(2.14) as

$$\nabla(\nabla\cdot\mathbf{w}) + k^2 n\mathbf{w} = 0, \qquad \text{in} \quad D, \tag{2.15}$$

$$\nabla(\nabla\cdot\mathbf{v}) + k^2\mathbf{v} = 0, \qquad \text{in} \quad D, \tag{2.16}$$

$$\nu\cdot\mathbf{w} = \nu\cdot\mathbf{v}, \qquad \text{on} \quad \partial D, \tag{2.17}$$

$$\nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{v}, \qquad \text{on} \quad \partial D. \tag{2.18}$$

The suitable functional spaces to analyze this problem are (see [2] and [7] for details)

$$\begin{aligned} &H(\mathrm{div},D) := \{\mathbf{u}\in(L^2(D))^d : \nabla\cdot\mathbf{u}\in L^2(D)\}, \qquad d=2,3 \\ &H_0(\mathrm{div},D) := \{\mathbf{u}\in H(\mathrm{div},D) : \nu\cdot\mathbf{u} = 0 \quad \text{on} \quad \partial D\}, \end{aligned}$$

and

$$\begin{aligned} &\widetilde{H}(D) := \{\mathbf{u}\in H(\mathrm{div},D) : \nabla\cdot\mathbf{u}\in H^1(D)\}, \\ &\widetilde{H}_0(D) := \{\mathbf{u}\in H_0(\mathrm{div},D) : \nabla\cdot\mathbf{u}\in H_0^1(D)\}, \end{aligned}$$

equipped with the scalar product $(\mathbf{u},\mathbf{v})_{\widetilde{H}(D)} := (\mathbf{u},\mathbf{v})_{L_2(D)} + (\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v})_{H^1(D)}$ and corresponding norm $\|\cdot\|_{\widetilde{H}}$. Hence a solution $\mathbf{u},\mathbf{v}$ of the interior transmission eigenvalue problem (2.15)−(2.18) is such that $\mathbf{w}\in(L^2(D))^d, \mathbf{v}\in(L^2(D))^d$ and $\mathbf{w}-\mathbf{v}\in\widetilde{H}_0(D)$. Let $\mathbf{u} = \mathbf{w}-\mathbf{v}\in \widetilde{H}_0(D)$, then we derive from (2.15)-(2.18) that

$$(\nabla\nabla\cdot + k^2 n)\frac{1}{n-1}(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}) = 0 \quad \text{in} \quad D, \tag{2.19}$$

which can be written in the following variational form

$$\int_D \frac{1}{n-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})(\nabla\nabla\cdot\overline{\mathbf{v}}+k^2 n\overline{\mathbf{v}})dx=0 \quad \text{for all} \quad \mathbf{v}\in\widetilde{H}_0(D). \tag{2.20}$$

We note that the eigenvalue problem (2.20) can be written as an operator equation

$$\mathbb{A}_\tau\mathbf{u}-\tau\mathbb{B}\mathbf{u}=0 \ \text{ and } \ \tilde{\mathbb{A}}_\tau\mathbf{u}-\tau\mathbb{B}\mathbf{u}=0, \quad \text{for} \quad \mathbf{u}\in\widetilde{H}_0(D), \tag{2.21}$$

where $\tau=k^2$. Here the bounded linear operators $A_\tau:\widetilde{H}_0(D)\to\widetilde{H}_0(D)$, $\tilde{A}_\tau:\widetilde{H}_0(D)\to\widetilde{H}_0(D)$ and $B:\widetilde{H}_0(D)\to\widetilde{H}_0(D)$ are the operators defined using the Riesz representation theorem associated with the sesquilinear form $\mathcal{A}_\tau$,$\tilde{\mathcal{A}}_\tau$ and $\mathcal{B}$ which are defined by (see [7] for more details)

$$\mathcal{A}_\tau(\mathbf{u},\mathbf{v}):=\left(\frac{1}{n-1}(\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}),(\nabla\nabla\cdot\mathbf{v}+\tau\mathbf{v})\right)_D+\tau^2(\mathbf{u},\mathbf{v})_D,$$
$$\tilde{\mathcal{A}}_\tau(\mathbf{u},\mathbf{v}):=\left(\frac{n}{1-n}(\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}),(\nabla\nabla\cdot\mathbf{v}+\tau\mathbf{v})\right)_D+(\nabla\nabla\cdot\mathbf{u},\nabla\nabla\cdot\mathbf{v})_D,$$

and

$$\mathcal{B}(\mathbf{u},\mathbf{v}):=(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_D,$$

respectively.
Now we consider the following generalized eigenvalue problem:

$$\mathcal{A}_\tau(\mathbf{u},\mathbf{v})-\lambda(\tau)\mathcal{B}(\mathbf{u},\mathbf{v})=0, \qquad \text{for all} \quad \mathbf{v}\in\widetilde{H}_0(D) \tag{2.22}$$
$$\tilde{\mathcal{A}}_\tau(\mathbf{u},\mathbf{v})-\lambda(\tau)\mathcal{B}(\mathbf{u},\mathbf{v})=0, \qquad \text{for all} \quad \mathbf{v}\in\widetilde{H}_0(D). \tag{2.23}$$

From (2.21), a transmission eigenvalue is the root of

$$f(\tau):=\lambda(\tau)-\tau. \tag{2.24}$$

Theorem 2.1. Let $k_1(D,n)$ be the first transmission eigenvalue for (2.15)-(2.18) and let α and β be positive constants. Denote by $k_1(D,n_*)$ and $k_1(D,n^*)$ the first transmission eigenvalue for (2.15)-(2.18) for $n(x)\equiv n_*$ and $n(x)\equiv n^*$ respectively. Then, we have the following results:

(i) if $n^*\geq n(x)\geq n_*\geq\alpha>1$,then $0<k_1(D,n^*)\leq k_1(D,n)\leq k_1(D,n_*)$.
(ii) if $0<n_*\leq n(x)\leq n^*\leq 1-\beta$,then $0<k_1(D,n_*)\leq k_1(D,n)\leq k_1(D,n^*)$.

Proof. The proof is similar to that of Theorem 3.3 in [5], so we shall only sketch the proof for the case of $n^*\geq n(x)\geq n_*\geq\alpha>1$.

Obviously, for any $\mathbf{u}\in\widetilde{H}_0(D)$, we have

$$\begin{aligned}&\frac{\frac{1}{n^*-1}\|\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}\|_D^2+\tau^2\|\mathbf{u}\|_D^2}{\|\nabla\cdot\mathbf{u}\|_D^2}\\ \le&\frac{(\frac{1}{n-1}(\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}),(\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}))_D+\tau^2\|\mathbf{u}\|_D^2}{\|\nabla\cdot\mathbf{u}\|_D^2}\\ &\frac{\frac{1}{n_*-1}\|\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}\|_D^2+\tau^2\|\mathbf{u}\|_D^2}{\|\nabla\cdot\mathbf{u}\|_D^2}.\end{aligned}$$

Therefore, for an arbitrary $\tau>0$, we have

$$\lambda_1(\tau,D,n^*)-\tau\le\lambda_1(\tau,D,n(x))-\tau\le\lambda_1(\tau,D,n_*)-\tau, \tag{2.25}$$

where $\lambda_1(\tau,D,n^*)$, $\lambda_1(\tau,D,n(x))$ and $\lambda_1(\tau,D,n_*)$ are the first transmission eigenvalues of (2.22) corresponding to the index of refraction n^*, $n(x)$ and n_*, respectively. Now using (2.25) for $\tau_1=k_1^2(D,n^*)$, we have $\lambda_1(\tau_1,D,n(x))-\tau_1\ge 0$. Again using (2.25) for $\tau_2=k_1^2(D,n_*)$, we have $\lambda_1(\tau_2,D,n(x))-\tau_2\le 0$. Then by continuity of the mapping $\tau\to\lambda_1(\tau,D,n(x))$, there is an eigenvalue corresponding to $(D,n(x))$ between $k_1(D,n^*)$ and $k_1(D,n_*)$.

To complete the proof, we need to show that this is the first eigenvalue for $(D,n(x))$. Indeed, if $k_1(D,n(x))<k_1(D,n^*)$, then from (2.25), we have $\lambda_1(\tau_3,D,n(x))-\tau_3\le 0$ for $\tau_3=k_1^2(D,n(x))$. On the other hand, it is shown in [8] that for $\tau_0>0$ sufficiently small, we have $\lambda_1(\tau_0,D,n(x))-\tau_0\ge 0$, which means that there is a transmission eigenvalue for (D,n^*) less than the first one, a contradiction to the assumption. □

Let $\mu_D: L^\infty(D)\to R$ which maps a given index of refraction $n(x)$ onto the smallest transmission eigenvalue of D, namely, $\mu_D(n)=k_1(D,n)$. Assuming that $k_1(D,n)$ is obtained from far field data, we seek a constant n_0 minimizing the difference between $\mu_D(n)$ and $k_1(D,n)$. when $n>1$ (the case of $0<n<1$ can be treated the same way), Theorem (2.1) shows that the transmission eigenvalues for n being a constant are monotonically decreasing with respect to n. Since $k_1(D,n)$ is a continuous function of n, we can estimate n_0 using the following algorithm (cf. [15]) such that the computed lowest transmission eigenvalue $\mu(D)$ coincides with the value $k_1(D,n)$ obtained from the far field data:

AlgorithmN $n_0=algorithmN(k_1(D,n),tol)$
(i) estimate an interval a and b, such that $k_1(D)$ lie between $k_1(D,a)$ and $k_1(D,b)$
(ii) compute $k_1(D,a)$ and $k_1(D,b)$
    while abs$(a-b)>tol$
        $c=(a+b)/2$ and compute $k_1(D,c)$
      if $k_1(D,c)>k_1(D,n)$ then
        $a=c$
      else

$b = c$
end if
end
$n_0 = c$.

We observe that a key component in the above algorithm is to have an efficient and robust numerical method for computing the smallest real transmission eigenvalue. Next, we will develop efficient and accurate spectral methods for computing the transmission eigenvalues.

3 An efficient spectral method for transmission eigenvalues in circular domains

We consider in this section the domain D being a disk with radius R, and assume that the index of reflection n is stratified along the radial direction r, namely, $n = n_0(r)$.

3.1 Separation of variables

We start by employ a classical technique, separation of variables, to reduce the problem to a sequence of one-dimensional problems.

The equation (2.11)−(2.14) can be restated as

$$\Delta\omega + k^2 n_0 \omega = 0, \qquad in \quad D, \tag{3.1}$$

$$\Delta v + k^2 v = 0, \qquad in \quad D, \tag{3.2}$$

$$\omega - v = 0, \qquad on \quad \partial D, \tag{3.3}$$

$$\frac{\partial\omega}{\partial\nu} - n_0 \frac{\partial v}{\partial\nu} = 0, \qquad on \quad \partial D. \tag{3.4}$$

Applying the polar transformation $x = r\cos\theta, y = r\sin\theta$ to (3.1)−(3.4), and denoting

$$\widetilde{w}(r,\theta) = w(r\cos\theta, r\sin\theta), \quad \widetilde{v}(r,\theta) = v(r\cos\theta, r\sin\theta),$$

we obtain that for all $\theta \in [0,2\pi)$,

$$\widetilde{w}_{rr} + \frac{1}{r}\widetilde{w}_r + \frac{1}{r^2}\widetilde{w}_{\theta\theta} + k^2 n_0 \widetilde{w} = 0, \qquad r \in (0,R), \tag{3.5}$$

$$\widetilde{v}_{rr} + \frac{1}{r}\widetilde{v}_r + \frac{1}{r^2}\widetilde{v}_{\theta\theta} + k^2 \widetilde{v} = 0, \qquad r \in (0,R), \tag{3.6}$$

$$\widetilde{w} = \widetilde{v}, \qquad r = R, \tag{3.7}$$

$$\widetilde{w}_r = n_0 \widetilde{v}_r, \qquad r = R, \tag{3.8}$$

Since $\widetilde{w}$ and $\widetilde{v}$ are all 2π-periodic in θ, we can write

$$\widetilde{w}(r,\theta)=\sum_{|m|=0}^{\infty} w_m(r)e^{im\theta}, \qquad \widetilde{v}(r,\theta)=\sum_{|m|=0}^{\infty} v_m(r)e^{im\theta}.$$

Substituting these expansions in (3.5)$-$(3.8), we obtain a sequence of one-dimensional problems for each Fourier mode m.

• **Case** $m\neq 0$**:**

$$\frac{d^2w_m}{dr^2}+\frac{1}{r}\frac{dw_m}{dr}-\frac{m^2}{r^2}w_m+k^2n_0(r)w_m=0, \quad r\in(0,R), \tag{3.9}$$

$$\frac{d^2v_m}{dr^2}+\frac{1}{r}\frac{dv_m}{dr}-\frac{m^2}{r^2}v_m+k^2v_m=0, \qquad r\in(0,R), \tag{3.10}$$

$$w_m(0)=v_m(0)=0; w_m(R)=v_m(R), \quad \frac{dw_m}{dr}(R)=n_0(R)\frac{dv_m}{dr}(R). \tag{3.11}$$

• **Case** $m=0$**:**

$$\frac{d^2w_0}{dr^2}+\frac{1}{r}\frac{dw_0}{dr}+k^2n_0(r)w_0=0, \qquad r\in(0,R), \tag{3.12}$$

$$\frac{d^2v_0}{dr^2}+\frac{1}{r}\frac{dv_0}{dr}+k^2v_0=0, \qquad r\in(0,R), \tag{3.13}$$

$$w_0(R)=v_0(R), \frac{dw_0}{dr}(R)=n_0(R)\frac{dv_0}{dr}(R). \tag{3.14}$$

Next, we show a simple result linking the transmission eigenvalues of the original problem (3.1)-(3.4) to that of the one-dimensional problems (3.9)-(3.11) or (3.12)-(3.14).

Proposition 3.1. Let $D=\{(x,y): x^2+y^2<R\}$. Then,

1. any transmission eigenvalue k, of the one-dimensional problems (3.9)-(3.11) or (3.12)-(3.14), is a transmission eigenvalue of the problem (3.1)-(3.4);

2. for any transmission eigenvalue k of the the problem (3.1)-(3.4), there exists at least one m such that k is a transmission eigenvalue of (3.9)-(3.11) or (3.12)-(3.14).

Proof. It is clear that if (w_m,v_m,k), with $\|w_m\|+\|v_m\|\neq 0$ and $k\neq 0$, solves (3.9)-(3.11) or (3.12)-(3.14), then it also solves (3.1)-(3.4). Hence, k is a transmission eigenvalue of the original problem (3.1)-(3.4).

Conversely, if (w,v,k), with $\|w\|+\|v\|\neq 0$ and $k\neq 0$, solves the original problem (3.1)-(3.4), then, there exists at least one m such that (w_m,v_m,k), with $\|w_m\|+\|v_m\|\neq 0$ where (w_m,v_m) are the m-th Fourier expansion coefficient of (v,m), solves (3.9)-(3.11) or (3.12)-(3.14). Hence, all transmission eigenvalues of the original problem (3.1)-(3.4) can be obtained by solving (3.9)-(3.11) for all $m\neq 0$ and (3.12)-(3.14). □

3.2 Weak formulation

We describe below a weak formulation which is convenient for numerical computation of the transmission eigenvalues. To fix the idea, we shall only describe our approach for the case $m\neq 0$ in detail, since the case with $m=0$ can be treated similarly. We first transform the domain $r\in(0,R)$ to $t\in I:=(-1,1)$ in which a spectral method is usually applied to. Let

$$r=\frac{R(t+1)}{2},\quad \widetilde{w}_m(t)=w_m(\frac{R(t+1)}{2}),\quad \widetilde{v}_m(t)=v_m(\frac{R(t+1)}{2}),\quad \widetilde{n}_0(t)=n_0(\frac{R(t+1)}{2}).$$

Then, (3.9)-(3.11) become

$$\widetilde{w}''_m+\frac{1}{t+1}\widetilde{w}'_m-\frac{m^2}{(t+1)^2}\widetilde{w}_m+\frac{R^2}{4}k^2\widetilde{n}_0(t)\widetilde{w}_m=0,\quad t\in(-1,1), \tag{3.15}$$

$$\widetilde{v}''_m+\frac{1}{t+1}\widetilde{v}'_m-\frac{m^2}{(t+1)^2}\widetilde{v}_m+\frac{R^2}{4}k^2\widetilde{v}_m=0,\quad t\in(-1,1), \tag{3.16}$$

$$\widetilde{w}_m(-1)=\widetilde{v}_m(-1)=0;\quad \widetilde{w}_m(1)=\widetilde{v}_m(1),\quad \widetilde{w}'_m(1)=\widetilde{n}_0(1)\widetilde{v}'_m(1). \tag{3.17}$$

Let us denote $Y=\{v\in H^1(I): v(-1)=0\}$. Writing $\widetilde{w}_m=w_{m0}+h$, $\widetilde{v}_m=v_{m0}+h$ with $w_{m0},v_{m0}\in X:=H_0^1(I)$, $h\in X_b$ where X_b is the complement of $H_0^1(I)$ in Y, namely $Y=X\oplus X_b$. By definition, we have $\widetilde{w}_m(1)=\widetilde{v}_m(1)$.

Taking the inner product of (3.15) and (3.16) with $\tilde{w}_{m0}$, $\tilde{v}_{m0}\in H_0^1(I)$ respectively, we obtain

$$\begin{aligned}&\big((t+1)(w_{m0}+h)',\tilde{w}'_{m0}\big)+m^2\left(\frac{1}{t+1}(w_{m0}+h),\tilde{w}_{m0}\right)\\ =&\frac{R^2}{4}k^2\big(\widetilde{n}_0(t)(t+1)(w_{m0}+h),\tilde{w}_{m0}\big),\\ &\big((t+1)(v_{m0}+h)',\tilde{v}'_{m0}\big)+m^2\left(\frac{1}{t+1}(v_{m0}+h),\tilde{v}_{m0}\right)\\ =&\frac{R^2}{4}k^2\big((t+1)(v_{m0}+h),\tilde{v}_{m0}\big).\end{aligned}$$

Next, Taking the inner product of (3.15) and (3.16) with $\tilde{h}\in X_b$ respectively, we get

$$\begin{aligned}&\big((t+1)(w_{m0}+h)',\tilde{h}'\big)+m^2\left(\frac{1}{t+1}(w_{m0}+h),\tilde{h}\right)-2\tilde{w}'_m(1)\tilde{h}(1)\\ =&\frac{R^2}{4}k^2\big(\widetilde{n}_0(t)(t+1)(w_{m0}+h),\tilde{h}\big),\\ &\big((t+1)(v_{m0}+h)',\tilde{h}'\big)+m^2\left(\frac{1}{t+1}(v_{m0}+h),\tilde{h}\right)-2\tilde{v}'_m(1)\tilde{h}(1)\\ =&\frac{R^2}{4}k^2\big((t+1)(v_{m0}+h),\tilde{h}\big).\end{aligned}$$

We derive from the last two equations and the condition $\widetilde{w}_m'(1)=\tilde{n}_0(1)\widetilde{v}_m'(1)$ that

$$\begin{aligned}&\left((t+1)(w_{m0}+h)',\tilde{h}'\right)+m^2\left(\frac{1}{t+1}(w_{m0}+h),\tilde{h}\right)\\&\quad-\tilde{n}_0(1)\left(\left((t+1)(v_{m0}+h)',\tilde{h}'\right)+m^2\left(\frac{1}{t+1}(v_{m0}+h),\tilde{h}\right)\right)\\=&\frac{R^2}{4}k^2\left(\widetilde{n}_0(t)(t+1)(w_{m0}+h)-\tilde{n}_0(1)(t+1)(v_{m0}+h),\tilde{h}\right).\end{aligned}$$

Hence, a weak formulation for (3.15)-(3.17) is: find $\widetilde{w}_m=w_{m0}+h$, $\widetilde{v}_m=v_{m0}+h$ with $w_{m0},v_{m0}\in X$ and $h\in X_b$, $k\in C$, such that $\forall\tilde{w}_{m0},\tilde{v}_{m0}\in X$ and $\tilde{h}\in X_b$, we have

$$\begin{aligned}&\left((t+1)(w_{m0}+h)',\tilde{w}_{m0}'\right)+m^2\left(\frac{1}{t+1}(w_{m0}+h),\tilde{w}_{m0}\right)\\=&\frac{R^2}{4}k^2\left(\widetilde{n}_0(t)(t+1)(w_{m0}+h),\tilde{w}_{m0}\right),\qquad(3.18)\\&\left((t+1)(v_{m0}+h)',\tilde{v}_{m0}'\right)+m^2\left(\frac{1}{t+1}(v_{m0}+h),\tilde{v}_{m0}\right)\\=&\frac{R^2}{4}k^2\left((t+1)(v_{m0}+h),\tilde{v}_{m0}\right),\qquad(3.19)\\&\left((t+1)(w_{m0}+h)',\tilde{h}'\right)+m^2\left(\frac{1}{t+1}(w_{m0}+h),\tilde{h}\right)\\&\quad-\tilde{n}_0(1)\left(\left((t+1)(v_{m0}+h)',\tilde{h}'\right)+m^2\left(\frac{1}{t+1}(v_{m0}+h),\tilde{h}\right)\right)\\=&\frac{R^2}{4}k^2\left(\widetilde{n}_0(t)(t+1)(w_{m0}+h)-\tilde{n}_0(1)(t+1)(v_{m0}+h),\tilde{h}\right).\qquad(3.20)\end{aligned}$$

3.3 An efficient spectral-element method

In many applications, $n_0(r)$ is a piecewise smooth function. In order to deal with the piecewise smoothness, we shall use a spectral-element method.

Let $I_i=(t_{i-1},t_i)$, $1\leq i\leq M$ with $-1=t_0<t_1<\cdots<t_M=1$. Let P_N be the set of polynomials of degree less than or equal to N, and define the spectral-element approximation to X:

$$X_N=\{v\in C(I):v(\pm 1)=0,v|_{I_i}\in P_N,1\leq i\leq M\}.$$

To deal with X_b, we define

$$h_M(t)=\begin{cases}0, & t\in[-1,t_{M-1}],\\ \dfrac{t-t_{M-1}}{1-t_{M-1}}, & t\in(t_{M-1},1].\end{cases}\qquad(3.21)$$

Then, the spectral-element method for (3.18)-(3.20) is: Find $w_N^m = w_N^{m0} + \alpha h_M(t)$, $v_N^m = v_N^{m0} + \alpha h_M(t)$ with $w_N^{m0}, v_N^{m0} \in X_N$, $\alpha \in \mathbb{R}$, and $k_N \in \mathbb{C}$, such that for $\forall \tilde{w}_N^{m0}, \tilde{v}_N^{m0} \in X_N$, we have

$$\begin{aligned}
&\big((t+1)(w_N^{m0}+\alpha h_M)',(\tilde{w}_N^{m0})'\big)+m^2\big(\frac{1}{t+1}(w_N^{m0}+\alpha h_M),\tilde{w}_N^{m0}\big)\\
=&\frac{R^2}{4}k_N^2\big(\tilde{n}_0(t)(t+1)(w_N^{m0}+\alpha h_M),\tilde{w}_N^{m0}\big), \qquad (3.22)\\
&\big((t+1)(v_N^{m0}+\alpha h_M)',(\tilde{v}_N^{m0})'\big)+m^2\big(\frac{1}{t+1}(v_N^{m0}+\alpha h_M),\tilde{v}_N^{m0}\big)\\
=&\frac{R^2}{4}k_N^2\big((t+1)(v_N^{m0}+\alpha h_M),\tilde{v}_N^{m0}\big), \qquad (3.23)\\
&\big((t+1)(w_N^{m0}+\alpha h_M)',h_M'\big)+m^2\big(\frac{1}{t+1}(w_N^{m0}+\alpha h_M),h_M\big)\\
&\quad-\tilde{n}_0(1)\Big(\big((t+1)(v_N^{m0}+\alpha h_M)',h_M'\big)+m^2\big(\frac{1}{t+1}(v_N^{m0}+\alpha h_M),h_M\big)\Big)\\
=&\frac{R^2}{4}k_N^2\big(\tilde{n}_0(t)(t+1)(w_N^{m0}+\alpha h_M)-\tilde{n}_0(1)(t+1)(v_N^{m0}+\alpha h_M),h_M\big). \qquad (3.24)
\end{aligned}$$

We now construct a set of basis functions for X_N which will consist of interior basis functions and interface basis functions.

Let $\phi_i(x) = L_i(x) - L_{i+2}(x)$, $i = 0,1,\cdots,N-2$, where L_k is the Legendre polynomial of degree k. It is clear that $\{\phi_k\}_{k=0}^{N-2}$ form a basis for $P_N \cap H_0^1(I)$ [14]. For each subinterval $I_j = (t_{j-1}, t_j)$, we define a set of interior basis functions by

$$\varphi_{i+1,j}(t) := \begin{cases} \phi_i(x_j(t)), & t \in I_j, \\ 0, & \text{otherwise}, \end{cases}$$

where $x_j(t) = \frac{2}{t_j - t_{j-1}} t - \frac{t_j + t_{j-1}}{t_j - t_{j-1}}$, $t \in I_j$. Then, a set of all interior basis functions in X_N is

$$X_N^0 = \cup_{j=1,\cdots,M} \mathrm{span}\{\varphi_{1,j}, \varphi_{2,j}, \cdots, \varphi_{N-1,j}\}.$$

Next, we define the following interface basis functions

$$\xi_j(t) := \begin{cases} \frac{1}{2}(x_j(t)+1), & t \in I_j \\ -\frac{1}{2}(x_{j+1}(t)-1), & t \in I_{j+1} \\ 0, & \text{otherwise}. \end{cases} \quad ,j = 1,2,\cdots,M-1.$$

Then, it is clear that

$$X_N = X_N^0 \bigoplus \mathrm{span}\{\xi_1, \xi_2, \cdots, \xi_{M-1}\}.$$

Hence, we look for

$$\begin{aligned} w_N^m &= \Big(\sum_{j=1}^{M}\sum_{i=1}^{N-1} w_{ij}^m \varphi_{ij} + \sum_{j=1}^{M-1} w_j^m \xi_j\Big) + \alpha h_M, \\ v_N^m &= \Big(\sum_{j=1}^{M}\sum_{i=1}^{N-1} v_{ij}^m \varphi_{ij} + \sum_{j=1}^{M-1} v_j^m \xi_j\Big) + \alpha h_M, \end{aligned} \tag{3.25}$$

where h_M is defined in (3.21).

Let us denote

$$\begin{aligned} \bar{w} &= (w_{1,1}^m,\cdots,w_{N-1,1}^m;\cdots;w_{1,M}^m\cdots,w_{N-1,M}^m), \\ \bar{v} &= (v_{1,1}^m,\cdots,v_{N-1,1}^m;\cdots;v_{1,M}^m\cdots,v_{N-1,M}^m), \\ \bar{\xi} &= (w_1^m,\cdots,w_{M-1}^m;v_1^m,\cdots,v_{M-1}^m,\alpha). \end{aligned}$$

Then, the vector contains all the unknowns is

$$\bar{u} = (\bar{w},\bar{v},\bar{\xi})^T := (\bar{u}_0,\bar{\xi})^T. \tag{3.26}$$

Now, plugging the expressions of (3.25) in (3.22)-(3.24), and taking $\tilde{w}_N^{m0}$ and $\tilde{w}_N^{m0}$ through all the basis functions in X_N, we will arrive at the following linear eigenvalue system:

$$A\bar{u} = k_N^2 B\bar{u}, \tag{3.27}$$

where A and B are corresponding "stiffness" and "mass" matrices, which are sparse if $n_0(r)$ is piecewise constant, and their non zero entries can be explicitly computed using the properties of Legendre polynomials. Hence, the above one-dimensional eigenvalue problem can be efficiently solved by using a standard procedure [10].

The system for $m=0$, (3.12)-(3.14), can be treated similarly.

3.4 Error analysis

We establish below an error estimate for the transmission eigenvalues in terms of the errors for the corresponding eigenfunctions. In particular, we show that the convergence rate of the eigenvalue is twice of that of the eigenfunctions in the energy norm, as in the case of usual eigenvalue problems.

For any positive weight function ω, we denote the weighted L^2-norm and weighted H^1 semi-norm by

$$\|u\|_\omega^2 = \int_{-1}^1 u^2 \omega dt, \qquad |u|_{1,\omega}^2 = \int_{-1}^1 (u_t)^2 \omega dt. \tag{3.28}$$

Denoting $w_1 = t+1, w_2 = 1/(t+1)$ and $w_3 = \tilde{n}_0(t)(t+1)$, we define two energy norms associated with (3.18)-(3.20) by

$$\begin{aligned} \|w\|_{E_1,m}^2 &= |w|_{1,\omega_1}^2 + m^2\|w\|_{\omega_2}^2 + \frac{R^2}{4}k^2\|w\|_{\omega_3}^2, \\ \|v\|_{E_2,m}^2 &= |v|_{1,\omega_1}^2 + m^2\|v\|_{\omega_2}^2 + \frac{R^2}{4}k^2\|v\|_{\omega_1}^2. \end{aligned} \tag{3.29}$$

In order to describe errors more precisely, we also define two related pseudo norms

$$\begin{aligned}\|w\|^2_{\triangle,m} &= \left| |w|^2_{1,\omega_1} + m^2\|w\|^2_{\omega_2} - \frac{R^2}{4}k^2\|w\|^2_{\omega_3} \right|, \\ \|v\|^2_{\square,m} &= \left| |v|^2_{1,\omega_1} + m^2\|v\|^2_{\omega_2} - \frac{R^2}{4}k^2\|v\|^2_{\omega_1} \right|.\end{aligned} \tag{3.30}$$

Using a argument similar to the proof of Theorem 2.2 in [1], we can prove the following result:

Theorem 3.1. Let $(k,\widetilde{w}_m,\widetilde{v}_m),(k_N,w_N^m,v_N^m)$ be the solutions of (3.18)-(3.20) and (3.22)-(3.24), respectively. Then the following inequality holds:

$$\begin{aligned}|k_N - k| &\le C_1(\|\widetilde{w}_m - w_N^m\|^2_{\triangle,m} + \tilde{n}_0(1)\|\widetilde{v}_m - v_N^m\|^2_{\square,m}) \\ &\le C_1(\|\widetilde{w}_m - w_N^m\|^2_{E_1,m} + \tilde{n}_0(1)\|\widetilde{v}_m - v_N^m\|^2_{E_2,m}),\end{aligned}$$

where

$$C_1 = 1/|\frac{R^2}{4}(k_N+k)(||w_N^m||^2_{w_3} - ||\sqrt{\tilde{n}_0(1)}v_N^m||^2_{w_1})|.$$

Similar results can be derived for the case $m=0$. We omit the detail for brevity.

4 Rectangular domains

We consider in this section the domain D being a rectangular. Without loss of generality, we take $D=(-1,1)^2$. We also assume $A = I/n(x)$.

4.1 Weak formulation

Let $X = H^1(D)$. Writing $w = w_0 + h_b$, $v = v_0 + h_b$ with $w_0, v_0 \in H_0^1(D)$ and $h_b \in X_b$, where X_b is the complement of $H_0^1(D)$ in X, namely $X = H_0^1(D) \oplus X_b$.

Taking the inner product of (2.11) and (2.12) with $\tilde{w}_0$, $\tilde{v}_0 \in H_0^1(D)$ respectively, we obtain

$$\begin{aligned}&(\nabla(w_0+h_b),\nabla\tilde{w}_0) - k^2(n(w_0+h_b),\tilde{w}_0) = 0, \\ &(\nabla(v_0+h_b),\nabla\tilde{v}_0) - k^2((v_0+h_b),\tilde{v}_0) = 0.\end{aligned}$$

Next, taking the inner product of (2.11) and (2.12) with $\tilde{h}_b \in X_b$ and $n\tilde{h}_b$ respectively, we find

$$\begin{aligned}&(\nabla(w_0+h_b),\nabla\tilde{h}_b) - k^2(n(w_0+h_b),\tilde{h}_b) - \int_{\partial D}\frac{\partial w}{\partial \nu}\mathrm{conj}(\tilde{h}_b)ds = 0, \\ &(\nabla(v_0+h_b),\nabla(n\tilde{h}_b)) - k^2(n(v_0+h_b),\tilde{h}_b) - \int_{\partial D} n\frac{\partial v}{\partial \nu}\mathrm{conj}(\tilde{h}_b)ds = 0,\end{aligned}$$

where conj$(\tilde{h}_b)$ is the complex conjugate of $\tilde{h}_b$. Note that the condition (2.14) leads to $\frac{\partial w}{\partial \nu}|_{\partial D} = n\frac{\partial v}{\partial \nu}|_{\partial D}$. Hence, we derive from the last three equations that

$$(\nabla(w_0+h_b),\nabla\tilde{h}_b)-(\nabla(v_0+h_b),\nabla(n\tilde{h}_b))=k^2(n(w_0-v_0),\tilde{h}_b).$$

Thus, a weak formulation of (2.11)-(2.14) is: Find $w=w_0+h_b$, $v=v_0+h_b$ with $w_0,v_0\in H_0^1(D)$ and $h_b\in X_b$, such that $\forall\tilde{w}_0,\tilde{v}_0\in H_0^1(D)$ and $\tilde{h}_b\in X_b$, $k\in C$ we have

$$(\nabla(w_0+h_b),\nabla\tilde{w}_0)-k^2(n(w_0+h_b),\tilde{w}_0)=0, \tag{4.1}$$

$$(\nabla(v_0+h_b),\nabla\tilde{v}_0)-k^2((v_0+h_b),\tilde{v}_0)=0, \tag{4.2}$$

$$(\nabla(w_0+h_b),\nabla\tilde{h}_b)-(\nabla(v_0+h_b),\nabla(n\tilde{h}_b))=k^2(n(w_0-v_0),\tilde{h}_b). \tag{4.3}$$

4.2 Legendre-Galerkin method

Let $X_N=P_N^2$, $X_N^0=X_N\cap H_0^1(D)$ and X_N^b be the complement of X_N^0 in X_N, namely $X_N=X_N^0\oplus X_N^b$. Then, the Legendre-Galerkin approximation of (4.1)-(4.3) is:

Find $w_N=w_N^0+h_N^b$, $v_N=v_N^0+h_N^b$, with $w_N^0,v_N^0\in X_N^0$ and $h_N^b\in X_N^b$, such that $\forall\tilde{w}_N^0,\tilde{v}_N^0\in X_N^0$ and $\tilde{h}_N^b\in X_N^b$, $k_N\in C$ we have

$$(\nabla(w_N^0+h_N^b),\nabla\tilde{w}_N^0)-k_N^2(n(w_N^0+h_N^b),\tilde{w}_N^0)=0, \tag{4.4}$$

$$(\nabla(v_N^0+h_N^b),\nabla\tilde{v}_N^0)-k_N^2((v_N^0+h_N^b),\tilde{v}_N^0)=0, \tag{4.5}$$

$$(\nabla(w_N^0+h_N^b),\nabla\tilde{h}_N^b)-(\nabla(v_N^0+h_N^b),\nabla(n\tilde{h}_N^b))=k_N^2(n(w_N^0-v_N^0),\tilde{h}_N^b). \tag{4.6}$$

Let $\varphi_i(x)=L_i(x)-L_{i+2}(x), i=0,1,\cdots,N-2$, $\varphi_{N-1}=L_0(x), \varphi_N(x)=L_1(x)$. It is clear that $\{\varphi_k\}_{k=0}^N$ (reap. $\{\varphi_k\}_{k=0}^{N-2}$) form a basis for P_N (reap. $P_N\cap H_0^1(I)$). Hence, we have

$$\begin{aligned}
X_N &= \mathrm{span}\{\varphi_i(x)\varphi_j(y): i,j=0,1,\cdots,N\},\\
X_N^0 &= \mathrm{span}\{\varphi_i(x)\varphi_j(y): i,j=0,1,\cdots,N-2\},\\
X_N^b &= \mathrm{span}\{\varphi_i(x)\varphi_j(y): 0\le i,j\le N \text{ with } i \text{ or } j>N-2\}.
\end{aligned}$$

We write

$$\begin{aligned}
w_N &= \sum_{i,j=0}^N w_{ij}\varphi_i(x)\varphi_j(y)=w_N^0+h_N^b,\\
v_N &= \sum_{i,j=0}^N v_{ij}\varphi_i(x)\varphi_j(y)=v_N^0+h_N^b,
\end{aligned} \tag{4.7}$$

with $v_{ij}=w_{ij}$ when i or $j>N-2$. By taking $\tilde{w}_N^0$ in (4.4) and $\tilde{v}_N^0$ in (4.5) through all basis functions of X_N^0, and $\tilde{h}_N^b$ in (4.6) through all basis functions of X_N^b, we can arrive at the following linear eigenvalue problem:

$$A\bar{u}=k_N^2B\bar{u}, \tag{4.8}$$

where $\bar{u}$ is the vector consisting of all unknown coefficients in (4.7). Note that in the AlgorithmN (see Section 2), we only need to compute the transmission eigenvalues for n being a constant. In this case, the matrices A and B are sparse and their entries can be computed by using the properties of Legendre polynomials. The smallest real eigenvalue can then be computed efficiently by using a robust real eigenvalue searching strategy (cf. Section 3.3 in [11]).

5 Numerical results and discussions

We now present the results of numerical experiments which exhibit the stability and accuracy of our new algorithm.

5.1 Approximation of transmission eigenvalues

5.1.1 Circular domain

We now perform a sequence of tests to study the convergence behavior of our algorithm in the case of circular domains.

We first consider n being a constant, and fix $R=\frac{1}{2}$ and $n=4$. The numerical approximations of the first real eigenvalue with different mode m are given in Table 1.

In some applications, it is also useful to compute complex transmission eigenvalues. Our method is obviously not restricted to real eigenvalues. As an example, we list in Table 2 approximations of the first pair complex transmission eigenvalue.

We then consider n being a piecewise constant. In Table 3, we report the results with $R=1$, $n=\frac{1}{2}, r\in(0,\frac{1}{2})$ and$n=4, r\in(\frac{1}{2},1)$; while in Table 4, we report the results with $R=1$ and $n=1/4, r\in(0,\frac{1}{4})$, $n=3/2, r\in(\frac{1}{4},\frac{1}{2})$, $n=2, r\in(\frac{1}{2},\frac{3}{4})$, $n=3, r\in(\frac{3}{4},1)$.

Table 1: First real transmission eigenvalues corresponding to different m's on a disk with $R=1/2$ and n=4.

N	5	10	15
m=0	5.805219223	5.80521611	5.80521611
m=1	6.800813225	6.800758979	6.800758979
m=2	7.566054144	7.565971624	7.565971624
m=3	7.606703264	7.606625373	7.606625373
m=4	8.445983269	8.445797645	8.445797645

Table 2: The first pair complex transmission eigenvalues corresponding to different m's on a disk with R=1/2 and n=4.

N	5	10	15
m=0	$7.94874077\pm1.132667679*i$	$7.94871275\pm1.132503138*i$	$7.94871275\pm1.132503138*i$
m=1	$3.828048255\pm1.64765092*i$	$3.828048314\pm1.647650894*i$	$3.828048314\pm1.647650894*i$
m=2	$6.219969318\pm1.724299411*i$	$6.219976219\pm1.724289409*i$	$6.219976219\pm1.724289409*i$

Table 3: First real transmission eigenvalues corresponding to different m's on a disk with n being a piecewise constant on two intervals.

N	5	10	15
m=0	1.07840749	1.078407482	1.078407482
m=1	10.70612944	10.05805766	10.05805623
m=2	5.36858867	5.365092908	5.365092908
m=3	5.019065419	5.018721077	5.018721077
m=4	4.4353025	4.435092461	4.435092461
m=5	4.837488844	4.837233024	4.837233024

Table 4: First real transmission eigenvalues corresponding to different m's on a disk with n being a piecewise constant on four intervals.

N	5	10	15
m=0	1.447476728	1.447476727	1.447476727
m=1	11.01591027	11.00060386	11.00060386
m=2	8.670488103	8.67032666	8.67032666
m=3	7.093716561	7.093701478	7.093701478
m=4	7.530640992	7.530619461	7.530619461

We observe that in all cases, we can obtain 10-digit accuracy with $N=10$. Notice also that the smallest real eigenvalue (for all m) occurs at $m=0$ in all these cases.

5.1.2 Rectangular domain

We take $D=(-1/2,1/2)^2$ and n being a constant. Approximations of the first real transmission eigenvalue for different n are reported in Table 5. For comparison, the corresponding numerical results of [2] are listed in the last row. We observe that our method achieves much higher accuracy than those reported in [2].

Table 5: first real transmission eigenvalue for different n and $D=[-1/2,1/2]^2$.

N	$n=2$	$n=3$	$n=6$	$n=9$	$n=12$
N=15	13.18670237	7.640564026	4.002117352	3.064742348	2.34700105
N=20	13.1866814	7.640563733	4.002117352	3.06474221	2.347001026
N=25	13.18668137	7.640563703	4.002117352	3.064742196	2.347001024
Results in [2]	13.0	8.0	4.4	3.14	2.7

5.2 Estimation of index of refraction

Let D be a given domain and A be the matrix of the index of refraction, and let $k_1^{app}(D,A)$ be the approximate first real transmission eigenvalue estimated from the far field data.

Table 6: Numerical results for estimation of scalar a.

Domain	*Matrix*	*Eigenvalues*	$k_1^{app}(D,A)$	predicted a
Circle	A_{iso}	1/4,1/4	5.8	0.24971756
	A_1	1/2,1/8	4.81	0.18981380
	A_2	1/6,1/8	3.95	0.13539387
	A_{2r}	1/6,1/8	3.95	0.13539387
Square	A_{iso}	1/4,1/4	5.3	0.25007646
	A_1	1/2,1/8	4.1	0.17341806
	A_2	1/6,1/8	3.55	0.13579291
	A_{2r}	1/6,1/8	3.7	0.14593559

Table 7: Numerical results in [5] for estimation of scalar a.

Domain	*Matrix*	*Eigenvalues*	$k_1^{app}(D,A)$	predicted a
Circle	A_{iso}	1/4,1/4	5.8	0.248
	A_1	1/2,1/8	4.81	0.188
	A_2	1/6,1/8	3.95	0.134
	A_{2r}	1/6,1/8	3.95	0.134
Square	A_{iso}	1/4,1/4	5.3	0.248
	A_1	1/2,1/8	4.1	0.172
	A_2	1/6,1/8	3.55	0.135
	A_{2r}	1/6,1/8	3.7	0.145

Our goal is to use the AlgorithmN in Section 2 to find a constant a such that the isotropic matrix $B=aI$ will lead to the same first real eigenvalue, i.e., $k_1(D,B)=k_1^{app}(D,A)$.

We consider the following four different A's used in [5]: An isotropic case with

$$A_{iso}=\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix},$$

and three anisotropic cases

$$A_1=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}, \qquad A_2=\begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}, \qquad A_{2r}=\begin{pmatrix} 0.1372 & 0.0189 \\ 0.0189 & 0.1545 \end{pmatrix}.$$

Note that A_{2r} can be obtained by rotating A_2 by 1 radian. Thus, A_2 and A_{2r} have the same eigenvalues.

From Theorem 2.1 and the continuity of eigenvalues, we expect that a should lie between the upper and lower eigenvalues of A. The numerical results are reported in Table 6. We observe that in the isotropic case, the predicted a reconstructs the diagonal value of the A_{iso}, while in the anisotropic case, the predicted a lies between the eigenvalues of the matrix A.

As a comparison, the corresponding numerical results in [5] are listed in Table 7. We observe that our method leads to much more accurate results.

5.3 Summary

A key step in estimating the index of refraction is to compute efficiently and accurately the first real transmission eigenvalues of the associated transmission eigenvalue problems. We developed in this paper efficient spectral methods for computing the transmission eigenvalue problems, and derived, in the case of circular domain, an error estimate for the transmission eigenvalues in terms of the error for the corresponding eigenfunctions. We presented numerical results to show that our methods can efficiently compute very accurate approximations of transmission eigenvalues, and can be effectively used, along with the linear sampling method [12], to estimate the index of refraction for electromagnetic scattering problems.

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