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A Rational Approximation and Its Applications to Differential Equations on the Half Line

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An orthogonal system of rational functions is introduced. Some results on rational approximations based on various orthogonal projections and interpolations are established. These results form the mathematical foundation of the related spectral method and pseudospectral method for solving differential equations on the half line. The error estimates of the rational spectral method and rational pseudospectral method for two model problems are established. The numerical results agree well with the theoretical estimates and demonstrate the effectiveness of this approach.

KEY WORDS: Legendre rational polynomials; rational approximation; spectral method; pseudospectral method.

1. INTRODUCTION

Many science and engineering problems of current interest are set in unbounded domains. In the context of spectral methods, a number of approaches for treating unbounded domains have been proposed and investigated. A direct approach is to use spectral method associated with some orthogonal systems in unbounded domains, such as the Hermite spectral method and the Laguerre method, see, e.g., Maday *et al.* [12], Guo [8], and Guo and Shen [11]. It is also possible to reformulate original problems in unbounded domains to certain singular/degenerate problems in bounded domains by variable transformations, and then use

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the Jacobi polynomials to approximate the resulting singular problems, see, e.g., Guo [6, 10]. Another effective method for solving such problems is based on rational approximations. For instance, Christov [5] and Boyd [3, 4] developed some spectral methods on infinite intervals by using mutually orthogonal systems of rational functions. But the convergences and error estimates for those rational spectral methods are still not available.

In this paper, we investigate the spectral method and pseudospectral method on the half line by using a new mutually orthogonal system of rational functions, with the weight function $(x+1)^{-2}$. We also give a framework for theoretical analysis of rational approximation in weighted Sobolev space. Although we may make variable transformations to change differential equations on the half line into certain singular/degenerate problems on a finite interval, it is preferable to approximate the differential equations on the half line directly using rational approximations in certain cases, such as in exterior problems where the obstacles may become too complicated after variable transformations. Indeed, for this type of exterior problems, we can choose a circle/sphere which encloses the obstacle, and then use a combined finite-element, Fourier-rational approximation in which the geometric complexity is handled by a finite-element method and the domain outside the circle/sphere is handled by a rational approximation in the radial direction and Fourier approximation in other direction(s). The details of this approach is beyond the scope of this paper and will be considered in a future work.

This paper is organized as follows. In the next section, we introduce the system of rational functions induced by the Legendre polynomials and its basic properties. In Sec. 3, we study various orthogonal projections and establish some results on the rational approximation. In Sec. 4, we consider two kinds of rational interpolations. The results in these two sections form the mathematical foundation for the related spectral method and pseudospectral method. In Sec. 5, we analyze rational spectral method and rational pseudospectral method for two model problems. In Sec. 6, we discuss numerical implementations and present some numerical results which agree well with the theoretical analysis and which demonstrate the effectiveness of this new approach.

2. LEGENDRE RATIONAL FUNCTIONS

We first introduce some notations. Let $\Lambda = \{x \mid 0 < x < \infty\}$ and $\chi(x)$ be a positive weight function. For $1 \leqslant p \leqslant \infty$, let

$$L^p_\chi(\Lambda) = \{v \mid v \text{ is measurable and } \|v\|_{L^p_\chi} < \infty\}$$

where

$$\|v\|_{L^p_\chi}=\begin{cases}\left(\int_\Lambda |v(x)|^p\,\chi(x)\,dx\right)^{1/p} & 1\leqslant p<\infty\\ \operatorname*{ess\,sup}_{x\in\Lambda}|v(x)| & p=\infty\end{cases}\tag{2.1}$$

We denote by $(u, v)_\chi$ and $\|v\|_\chi$ respectively the inner product and the norm of the space $L^2_\chi(\Lambda)$, i.e.,

$$(u, v)_\chi=\int_\Lambda u(x)\,v(x)\,\chi(x)\,dx, \quad \|v\|_\chi=(v, v)_\chi^{1/2}$$

For any non-negative integer m, we set

$$H^m_\chi(\Lambda)=\left\{v \,\middle|\, \partial^k_x v=\frac{d^k v}{dx^k}\in L^2_\chi(\Lambda), 0\leqslant k\leqslant m\right\}$$

equipped with the inner product, the semi-norm and the norm as follows,

$$(u, v)_{m,\chi}=\sum_{k=0}^m (\partial^k_x u, \partial^k_x v)_\chi, \quad |v|_{m,\chi}=\|\partial^m_x v\|_\chi, \quad \|v\|_{m,\chi}=(v, v)^{1/2}_{m,\chi}$$

For any real number $r>0$, we define the space $H^r_\chi(\Lambda)$ with the norm $\|v\|_{r,\chi}$ by space interpolation as in Adams [1]. As usual χ will be omitted from the notations if $\chi(x)\equiv 1$.

Let $L_l(x)$ be the Legendre polynomial of degree l. We recall that $L_l(x)$ is the eigenfunction of the singular Sturm–Liouville problem

$$\partial_x((1-x^2)\,\partial_x L_l(x))+l(l+1)\,L_l(x)=0, \qquad l=0, 1, 2,... \tag{2.2}$$

We define the Legendre rational function of degree l by

$$R_l(x)=\sqrt{2}\,L_l\left(\frac{x-1}{x+1}\right)$$

Thus, $R_l(x)$ is the lth eigenfunction of the singular Sturm–Liouville problem

$$(x+1)^2\,\partial_x(x\,\partial_x v)+\lambda v=0, \quad x\in\Lambda \tag{2.3}$$

with the corresponding eigenvalue $\lambda=l(l+1)$.

Clearly, we have

$$\lim_{x\to\infty} R_l(x)=\sqrt{2}, \qquad \lim_{x\to\infty} x\,\partial_x R_l(x)=\lim_{x\to\infty}\frac{2\sqrt{2}\,x}{(x+1)^2}\,L_l'\left(\frac{x-1}{x+1}\right)=0$$

We also recall that the recurrence formulas and the orthogonality of the Legendre polynomials lead to

$$R_{l+1}(x)=\frac{2l+1}{l+1}\cdot\frac{x-1}{x+1}\,R_l(x)-\frac{l}{l+1}\,R_{l-1}(x), \qquad l\geqslant 1$$

$$2(2l+1)\,R_l(x)=(x+1)^2\,(\partial_x R_{l+1}(x)-\partial_x R_{l-1}(x)), \qquad l\geqslant 1$$

and

$$\int_\Lambda R_l(x)\,R_m(x)\,\omega(x)\,dx=(l+\tfrac{1}{2})^{-1}\,\delta_{l,m} \tag{2.4}$$

where $\omega(x)=(x+1)^{-2}$ and $\delta_{l,m}$ is the Kronecker function. Thus, the Legendre rational expansion of a function $v\in L^2_\omega(\Lambda)$ is

$$v(x)=\sum_{l=0}^{\infty}\hat{v}_l R_l(x), \qquad \text{with} \quad \hat{v}_l=(l+\tfrac{1}{2})\int_\Lambda v(x)\,R_l(x)\,\omega(x)\,dx$$

Next, let $\omega_1(x)=x$. By virtue of (2.3), (2.4) and the asymptotic behaviors of $R_l(x)$ and $x\,\partial_x R_l(x)$ at infinity, we find that $\{\partial_x R_l(x)\}$ are mutually orthogonal in $L^2_{\omega_1}(\Lambda)$, namely,

$$\int_\Lambda \partial_x R_l(x)\,\partial_x R_m(x)\,\omega_1(x)\,dx=l(l+1)(l+\tfrac{1}{2})^{-1}\,\delta_{l,m} \tag{2.5}$$

We shall now derive some inverse inequalities and embedding inequalities. Let N be any positive integer, and

$$\mathscr{R}_N=\operatorname{span}\{R_0,R_1,\ldots,R_N\}$$

Hereafter, we denote by c a generic positive constant independent of any function and N.

Theorem 2.1. For any $\phi\in\mathscr{R}_N$ and $1\leqslant p\leqslant q\leqslant\infty$,

$$\|\phi\|_{L^q_\omega}\leqslant(2(p+1)\,N^2)^{1/p-1/q}\,\|\phi\|_{L^p_\omega}$$

Proof. Let $y\in\tilde{\Lambda}=(-1,1)$, $x=(1+y)/(1-y)$. For any $\phi\in\mathscr{R}_N$, we set $\psi(y)=\phi((1+y)/(1-y))$. By the definition of $\mathscr{R}_N$, we have $\psi(y)\in\mathscr{P}_N$, which is the set of polynomials of degree at most N. By an inverse inequality in $\mathscr{P}_N$ (see, e.g., Theorem 2.7 of Guo [7]),

$$\left(\int_{\tilde{\Lambda}} |\psi(y)|^q \, dy\right)^{1/q} \leqslant ((p+1)\, N^2)^{1/p-1/q} \left(\int_{\tilde{\Lambda}} |\psi(y)|^p \, dy\right)^{1/p}$$

Therefore,

$$\begin{aligned}\|\phi\|_{L^q_\omega} &= 2^{-1/q} \left(\int_{\tilde{\Lambda}} |\psi(y)|^q \, dy\right)^{1/q} \\ &\leqslant 2^{-1/q}((p+1)\, N^2)^{1/p-1/q} \left(\int_{\tilde{\Lambda}} |\psi(y)|^p \, dy\right)^{1/p} \\ &= (2(p+1)\, N^2)^{1/p-1/q} \, \|\phi\|_{L^p_\omega}\end{aligned}$$

Theorem 2.2. Let m be any non-negative integer and $2 \leqslant p < \infty$. Then, for any $\phi \in \mathscr{R}_N$,

$$\|\partial_x^m \phi\|_{L^p_\omega} \leqslant cN^{2m} \, \|\phi\|_{L^p_\omega}$$

Also for any $r \geqslant 0$,

$$\|\phi\|_{r,\,\omega} \leqslant cN^{2r} \, \|\phi\|_\omega$$

Proof. Let $y \in \tilde{\Lambda}$, $\mathscr{P}_N$ and $\psi(y)$ be the same as in the proof of the last theorem. Then (see, e.g., Theorem 2.8 of Guo [7]),

$$\|\partial_y^m \psi\|_{L^p(\tilde{\Lambda})} \leqslant cN^{2m} \, \|\psi\|_{L^p(\tilde{\Lambda})}$$

Thus

$$\begin{aligned}\|\partial_x \phi\|_{L^p_\omega} &= \left(\frac{1}{2^{p+1}} \int_{\tilde{\Lambda}} |\partial_y \psi(y)|^p (y-1)^{2p} \, dy\right)^{1/p} \leqslant 2 \left(\int_{\tilde{\Lambda}} |\partial_y \psi(y)|^p \, dy\right)^{1/p} \\ &\leqslant cN^2 \left(\int_{\tilde{\Lambda}} |\psi(y)|^p \, dy\right)^{1/p} = cN^2 \, \|\phi\|_{L^p_\omega}\end{aligned}$$

By repeating the above procedure, we deduce that for any non-negative integer m,

$$\|\partial_x^m \phi\|_{L^p_\omega} \leqslant cN^{2m} \, \|\phi\|_{L^p_\omega}$$

The second result follows from the above inequality with $p = 2$ and space interpolation. □

Remark 2.1. In particular, for any $\psi \in \mathscr{P}_N$,

$$\|\partial_y \psi\|_{L^2(\tilde{\Lambda})} \leqslant \tfrac{3}{2} N^2 \|\psi\|_{L^2(\tilde{\Lambda})}$$

which leads to that

$$\|\partial_x \phi\|_\omega \leqslant 3N^2 \|\phi\|_\omega$$

Theorem 2.3. If $v \in L^2_{\omega^2}(\Lambda)$, $\partial_x v \in L^2_\omega(\Lambda)$ and $v(0) = 0$, then

$$\|v\|_{\omega^2} \leqslant \tfrac{2}{3} |v|_{1,\omega}$$

If $v \in L^2_\omega(\Lambda)$, $\partial_x v \in L^2(\Lambda)$ and $v(0) = 0$, then

$$\|v\|_\omega \leqslant 2\sqrt{2}\, |v|_1$$

Proof. Let $u(y) = v((1+y)/(1-y))$. Then, in order to prove the first result, it suffices to prove that

$$\int_{\tilde{\Lambda}} u^2(y)(1-y)^2\, dy \leqslant \tfrac{4}{9} \int_{\tilde{\Lambda}} (\partial_y u(y))^2 (1-y)^4\, dy$$

Since $u(-1) = 0$, we have that for any $y \in \tilde{\Lambda}$,

$$u^2(y)(1-y)^3 = \int_{-1}^{y} \partial_z (u^2(z)(1-z)^3)\, dz$$

Hence,

$$\begin{aligned} u^2(y)(1-y)^3 &+ 3\int_{-1}^{y} u^2(z)(1-z)^2\, dz \\ &= 2\int_{-1}^{y} u(z)\, \partial_z u(z)(1-z)^3\, dz \\ &\leqslant 2\left(\int_{\tilde{\Lambda}} u^2(z)(1-z)^2\, dz\right)^{1/2} \left(\int_{\tilde{\Lambda}} (\partial_z u(z))^2 (1-z)^4\, dz\right)^{1/2} \end{aligned}$$

Letting $y \to 1$, we obtain that

$$\int_{\tilde{\Lambda}} u^2(y)(1-y)^2\, dy \leqslant \tfrac{4}{9} \int_{\tilde{\Lambda}} (\partial_y u)^2 (1-y)^4\, dy$$

This proves the first result. The second result follows from the first result applied to $(x+1)\, v(x)$. □

3. LEGENDRE RATIONAL POLYNOMIAL APPROXIMATIONS

In this section, we investigate various orthogonal projections.

We define the $L^2_\omega(\Lambda)$-orthogonal projection $P_N : L^2_\omega(\Lambda) \to \mathscr{R}_N$ by

$$(P_n v - v, \phi)_\omega = 0, \qquad \forall \phi \in \mathscr{R}_N$$

In order to estimate $\|P_N v - v\|_\omega$, we need to introduce the space

$$H^r_{\omega, A}(\Lambda) = \{v \mid v \text{ is a measurable and } \|v\|_{r, \omega, A} < \infty\}$$

where for non-negative integer r,

$$\|v\|_{r, \omega, A} = \left(\sum_{k=0}^{r} \|(x+1)^{(r/2)+k}\, \partial_x^k v\|_\omega^2 \right)^{1/2}$$

For any real $r > 0$, the space $H^r_{\omega, A}(\Lambda)$ is defined by space interpolation.

Let A be the Sturm–Liouville operator in (2.3), namely,

$$Av(x) = -\omega^{-1}(x)\, \partial_x (x\, \partial_x v(x))$$

By induction,

$$A^m v(x) = \sum_{k=1}^{2m} (x+1)^{m+k}\, p_k(x)\, \partial_x^k v(x) \tag{3.1}$$

where $p_k(x)$ are some rational functions which are bounded uniformly on the whole interval Λ. So A^m is a continuous mapping from $H^{2m}_{\omega, A}(\Lambda)$ to $L^2_\omega(\Lambda)$.

Theorem 3.1. For any $v \in H^r_{\omega, A}(\Lambda)$ and $r \geqslant 0$,

$$\|P_N v - v\|_\omega \leqslant c N^{-r}\, \|v\|_{r, \omega, A}$$

Proof. We first assume that $r = 2m$. By virtue of (2.3), (2.4) and integration by parts,

$$\begin{aligned}
\hat{v} &= \frac{1}{2}(2l+1) \int_\Lambda v(x)\, R_l(x)\, \omega(x)\, dx = \frac{2l+1}{2l(l+1)} \int_\Lambda v(x)\, AR_l(x)\, \omega(x)\, dx \\
&= -\frac{2l+1}{2l(l+1)} \int_\Lambda v(x)\, \partial_x (x\, \partial_x R_l(x))\, dx = \frac{2l+1}{2l(l+1)} \int_\Lambda x\, \partial_x v(x)\, \partial_x R_l(x)\, dx \\
&= -\frac{2l+1}{2l(l+1)} \int_\Lambda \partial_x (x\, \partial_x v(x))\, R_l(x)\, dx = \frac{2l+1}{2l(l+1)} \int_\Lambda Av(x)\, R_l(x)\, \omega(x)\, dx \\
&= \cdots = \frac{2l+1}{2l^m (l+1)^m} \int_\Lambda A^m v(x)\, R_l(x)\, \omega(x)\, dx
\end{aligned} \tag{3.2}$$

Therefore, we derive from (3.1), (3.2) and the definition of $H^r_{\omega,A}(\Lambda)$ that

$$\|P_N v - v\|^2_\omega = \sum_{l=N+1}^{\infty} \hat{v}_l^2 \, \|R_l\|^2_\omega$$
$$\leqslant cN^{-4m} \sum_{l=N+1}^{\infty} \left(\frac{\int_\Lambda A^m v(x)\, R_l(x)\, \omega(x)\, dx}{\|R_l\|^2_\omega} \right)^2 \|R_l\|^2_\omega$$
$$\leqslant cN^{-4m} \, \|A^m v\|^2_\omega \leqslant cN^{-4m} \, \|v\|^2_{r,\omega,A}$$

Next, let $r = 2m+1$. By (2.3) and integration by parts,

$$\hat{v}_l = \frac{2l+1}{2l^m(l+1)^m} \int_\Lambda A^m v(x)\, R_l(x)\, \omega(x)\, dx$$
$$= -\frac{2l+1}{2l^{m+1}(l+1)^{m+1}} \int_\Lambda A^m v(x)\, \partial_x(x\, \partial_x R_l(x))\, dx$$
$$= -\frac{2l+1}{2l^{m+1}(l+1)^{m+1}} \int_\Lambda \partial_x(A^m v(x))\, \partial_x R_l(x)\, \omega_1(x)\, dx$$

Thanks to (2.5) and (3.1),

$$\|P_N v - v\|^2_\omega = \sum_{l=N+1}^{\infty} \hat{v}_l^2 \, \|R_l\|^2_\omega$$
$$= \sum_{l=N+1}^{\infty} \frac{2l+1}{2(l(l+1))^{2m+2}} \left(\int_\Lambda \partial_x(A^m v)\, \partial_x R_l(x)\, \omega_1(x)\, dx \right)^2$$
$$= \sum_{l=N+1}^{\infty} \frac{(2l+1)\, \|\partial_x R_l\|^2_{\omega_1}}{2(l(l+1))^{2m+2}}$$
$$\times \left(\frac{\int_\Lambda \partial_x(A^m v)\, \partial_x R_l(x)\, \omega_1(x)\, dx}{\|\partial_x R_l\|^2_{\omega_1}} \right)^2 \|\partial_x R_l\|^2_{\omega_1}$$
$$\leqslant cN^{-2(2m+1)} \sum_{l=N+1}^{\infty} \left(\frac{\int_\Lambda \partial_x(A^m v)\, \partial_x R_l(x)\, \omega_1(x)\, dx}{\|\partial_x R_l\|^2_{\omega_1}} \right)^2 \|\partial_x R_l\|^2_{\omega_1}$$
$$\leqslant cN^{-2(2m+1)} \, \|\partial_x(A^m v)\|^2_{\omega_1} \leqslant cN^{-2(2m+1)} \, \|\partial_x(A^m v)(x+1)^{3/2}\|^2_\omega$$
$$\leqslant cN^{-2(2m+1)} \, \|v\|^2_{r,\omega,A}$$

The general result follows from the previous results and space interpolation. □

The $H^1_\omega(\Lambda)$-orthogonal projection $P^1_N : H^1_\omega(\Lambda) \to \mathscr{R}_N$ is a mapping such that for any $v \in H^1_\omega(\Lambda)$,

$$(P^1_N v - v, \phi)_{1,\omega} = 0, \qquad \forall \phi \in \mathscr{R}_N$$

In order to estimate $\|P^1_N v - v\|_{1,\omega}$, we recall some approximation results on Jacobi polynomials established in [9]. Let us define

$$L^2_{\alpha,\beta}(\tilde{\Lambda}) = \left\{ u \,\middle|\, \|u\|_{L^2_{\alpha,\beta}} = \left(\int_{\tilde{\Lambda}} u^2(y)(1-y)^\alpha (1+y)^\beta \, dy \right)^{1/2} < +\infty \right\} \tag{3.3}$$

and

$$\begin{aligned} a_{\alpha,\beta,\gamma,\delta}(u,w) = & \int_{\tilde{\Lambda}} \partial_y u \, \partial_y w (1-y)^\alpha (1+y)^\beta \, dy \\ & + \int_{\tilde{\Lambda}} u(y)\, w(y)(1-y)^\gamma (1+y)^\delta \, dy \end{aligned} \tag{3.4}$$

We also denote $H^0_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda}) = L^2_{\gamma,\delta}(\tilde{\Lambda})$ and

$$H^1_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda}) = \{ u \mid u \text{ is measurable on } \tilde{\Lambda} \text{ and } \|u\|_{1,\alpha,\beta,\gamma,\delta} < +\infty \} \tag{3.5}$$

where $\|u\|_{1,\alpha,\beta,\gamma,\delta} = a^{1/2}_{\alpha,\beta,\gamma,\delta}(u,u)$. For $0 < \mu < 1$, $H^\mu_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda})$ and its norm $\|u\|_{\mu,\alpha,\beta,\gamma,\delta}$ are defined by space interpolation. We also define

$$H^r_{\alpha,\beta,*}(\tilde{\Lambda}) = \{ u \mid u \text{ is measurable on } \tilde{\Lambda} \text{ and } \|u\|_{r,\alpha,\beta,*} < +\infty \} \tag{3.6}$$

where for non-negative integer r,

$$\|u\|^2_{r,\alpha,\beta,*} = A^{(1)}_{r,\alpha,\beta}(u) + A^{(2)}_{r,\alpha,\beta}(u) \tag{3.7}$$

with

$$\begin{aligned} A^{(1)}_{r,\alpha,\beta}(u) &= \sum_{k=r-[r/2]+1}^{r} \int_{\tilde{\Lambda}} (\partial^k_y u(y))^2 (1-y^2)^{-r+2k-1} (1-y)^\alpha (1+y)^\beta \, dy \\ A^{(2)}_{r,\alpha,\beta}(u) &= \sum_{k=1}^{[(r+1)/2]} \int_{\tilde{\Lambda}} (\partial^k_y u(y))^2 (1-y)^\alpha (1+y)^\beta \, dy \end{aligned} \tag{3.8}$$

The space $H^r_{\alpha,\beta,*}(\tilde{\Lambda})$ and its norm $\|u\|_{r,\alpha,\beta,*}$ for real positive r are defined by space interpolation.

Let $\tilde{P}^1_{N,\alpha,\beta,\gamma,\delta}: H^1_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda}) \to \mathscr{P}_N$ be orthogonal projection operator defined by

$$a_{\alpha,\beta,\gamma,\delta}(\tilde{P}^1_{N,\alpha,\beta,\gamma,\delta}u - u, \psi) = 0, \qquad \forall \psi \in \mathscr{P}_N \tag{3.9}$$

By using the notations on pp. 380–381 and Theorem 2.5 in [9], we know that for $\alpha \leqslant \gamma + 2$, $\beta \leqslant \delta + 2$, and for any $u \in H^r_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda})$ with $r \geqslant 1$, we have

$$\|\tilde{P}^1_{N,\alpha,\beta,\gamma,\delta}u - u\|^2_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{2-2r} \|u\|^2_{r,\alpha,\beta,*} \tag{3.10}$$

If in addition, $\alpha \leqslant \gamma + 1$, $\beta \leqslant \delta + 1$ and $0 \leqslant \mu \leqslant 1$, then

$$\|\tilde{P}^1_{N,\alpha,\beta,\gamma,\delta}u - u\|^2_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{2\mu-2r} \|u\|^2_{r,\alpha,\beta,*} \tag{3.11}$$

In order to estimate $\|P^1_N v - v\|_{1,\omega}$ we need to introduce another space. For any non-negative integer r,

$$H^r_{\omega,B}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{r,\omega,B} < +\infty\} \tag{3.12}$$

where

$$\|v\|_{r,\omega,B} = \left(\sum_{k=1}^{r} \|(x+1)^{r/2+k-1/2}\, \partial_x^k v\|^2_\omega \right)^{1/2} \tag{3.13}$$

As usual, for any $r > 0$, the space $H^r_{\omega,B}(\Lambda)$ and its norm are defined by space interpolation.

Theorem 3.2. For any $v \in H^r_{\omega,B}(\Lambda)$ with $r \geqslant 1$,

$$\|P^1_N v - v\|_{1,\omega} \leqslant cN^{1-r} \|v\|_{r,\omega,B}$$

Proof. By definition, $\|P^1_N v - v\|_{1,\omega} \leqslant \|\phi - v\|_{1,\omega}$ for any $\phi \in \mathscr{R}_N$. Let $y = (x-1)/(x+1)$, $u(y) = v((1+y)/(1-y))$. By taking $\phi = \tilde{P}^1_{N,2,0,0,0}u(y)|_{y=(x-1)/(x+1)}$, a direct computation together with (3.10) ($\alpha = 2$, $\beta = \gamma = \delta = 0$) leads to

$$\begin{aligned} \|\phi - v\|^2_{1,\omega} &= \tfrac{1}{8} \int_{\tilde{\Lambda}} (\partial_y \tilde{P}^1_{N,2,0,0,0}u(y) - \partial_y u(y))^2 (y-1)^4\, dy \\ &\quad + \tfrac{1}{2} \int_{\tilde{\Lambda}} (\tilde{P}^1_{N,2,0,0,0}u(y) - u(y))^2\, dy \\ &\leqslant c\, \|\tilde{P}^1_{N,2,0,0,0}u - u\|^2_{1,2,0,0,0} \leqslant cN^{2-2r} \|u\|^2_{r,2,0,*} \end{aligned}$$

Note that $1-y=2/(x+1)$, $1-y^2=4x/(x+1)^2$ and one can show easily by induction that

$$\partial_y^k u(y)=\sum_{j=1}^{k} q_j(x)(x+1)^{k+j}\,\partial_x^j v(x) \tag{3.14}$$

where $q_j(x)$ are some rational polynomials which are uniformly bounded on Λ. Thus, for any non-negative integer r,

$$A^{(1)}_{r,2,0}(u)\leqslant c\sum_{k=r-[r/2]+1}^{r}\ \sum_{j=1}^{k}\|(x+1)^{(r/2)+j-(1/2)}\,\partial_x^j v\|_\omega^2\leqslant c\,\|v\|^2_{r,\omega,B}$$

Similarly, we have

$$A^{(2)}_{r,2,0}(u)\leqslant c\sum_{k=1}^{[(r+1)/2]}\ \sum_{j=1}^{k}\|(x+1)^{k+j-1}\,\partial_x^j v\|_\omega^2\leqslant c\,\|v\|^2_{r,\omega,B}$$

This fact together with space interpolation complete the proof. □

When we apply the Legendre rational spectral method to partial differential equations with Dirichlet boundary conditions at $x=0$, we need another orthogonal projection. Let us denote

$$H^1_{0,\omega}(\Lambda)=\{v\mid v\in H^1_\omega(\Lambda),\ v(0)=0 \text{ and } v(x)(x+1)^{-3/2}\to 0, \text{ as } x\to\infty\}$$

$$\mathscr{R}^0_N=\{\phi\in\mathscr{R}_N\mid\phi(0)=0\} \tag{3.15}$$

$$a^\nu_\omega(u,v)=(\partial_x u,\partial_x(v\omega))+\nu(u,v)_\omega$$

We define the $H^1_{0,\omega}(\Lambda)$-orthogonal projection $P^{1,0}_N: H^1_{0,\omega}(\Lambda)\to\mathscr{R}^0_N$ by

$$a^\nu_\omega(P^{1,0}_N v-v,\phi)=0,\qquad \forall\phi\in\mathscr{R}^0_N$$

Lemma 3.1. For any $u,\ v\in H^1_{0,\omega}(\Lambda)$ and $\nu>\frac{3}{4}$,

$$a^\nu_\omega(v,v)\geqslant c\,\|v\|^2_{1,\omega}$$

$$|a^\nu_\omega(u,v)|\leqslant c\,\|u\|_{1,\omega}\,\|v\|_{1,\omega}$$

Proof. Let $\nu=\frac{3}{4}+\varepsilon$ and $\varepsilon>0$. By integrating by parts and using Theorem 2.3, we find that

$$
\begin{aligned}
a_\omega^\nu(v, v) &= (\partial_x v, \partial_x(v\omega)) + \nu(v, v)_\omega \\
&= |v|_{1,\omega}^2 + \nu\, \|v\|_\omega^2 + (\partial_x v, v\partial_x \omega) \\
&= |v|_{1,\omega}^2 + \nu\, \|v\|_\omega^2 + \frac{1}{2}\int_\Lambda \partial_x(v^2(x))\, \partial_x\omega(x)\, dx \\
&= |v|_{1,\omega}^2 + \nu\, \|v\|_\omega^2 - 3\int_\Lambda v^2(x)(x+1)^{-4}\, dx \\
&= |v|_{1,\omega}^2 + \nu\, \|v\|_\omega^2 - 3\, \|v\|_{\omega^2}^2 \\
&= |v|_{1,\omega}^2 - \left(\frac{9}{4} - \frac{\varepsilon}{2}\right) \|v\|_{\omega^2}^2 + \left(\frac{3}{4} + \varepsilon\right) \|v\|_\omega^2 - \left(\frac{3}{4} + \frac{\varepsilon}{2}\right) \|v\|_{\omega^2}^2 \\
&\geqslant \frac{2}{9}\varepsilon\, |v|_{1,\omega}^2 + \frac{\varepsilon}{2}\, \|v\|_\omega^2
\end{aligned} \tag{3.16}
$$

This leads to the first result. Next, by the Cauchy–Schwartz inequality and Theorem 2.3,

$$
\begin{aligned}
|(\partial_x u, v\,\partial_x\omega)| &= \left|\int_\Lambda \partial_x u(x)\, v(x)\, \partial_x\omega(x)\, dx\right| \\
&\leqslant 2\, \|\partial_x u\|_\omega\, \|v\|_{\omega^2} \leqslant c\, |u|_{1,\omega}\, |v|_{1,\omega}
\end{aligned}
$$

which implies the second result. □

In order to estimate $\|P_N^{1,0} v - v\|_{1,\omega}$, we need another result in [9]. Let us define

$$
H^{1,L}_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda}) = \{u \in H^1_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda}) \mid u(-1) = 0\}, \qquad \mathscr{P}_N^L = \{u \in \mathscr{P}_N \mid u(-1) = 0\}
$$

and the orthogonal projection $\tilde{P}^{1,L}_{N,\alpha,\beta,\gamma,\delta} : H^{1,L}_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda}) \to \mathscr{P}_N^L$ by

$$
a_{\alpha,\beta,\gamma,\delta}(\tilde{P}^{1,L}_{N,\alpha,\beta,\gamma,\delta} u - u, \psi) = 0, \qquad \forall \phi \in \mathscr{P}_N^L
$$

Thanks to Theorem 2.6 in [9], we know that if $\alpha \leqslant \gamma + 2$, $\beta \leqslant 0$ and $\delta \geqslant 0$, then for any $u \in H^r_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda}) \cap H^{1,L}_{\alpha,\beta,\gamma,\delta}(\tilde{\Lambda})$, we have

$$
\|\tilde{P}^{1,L}_{N,\alpha,\beta,\gamma,\delta} u - u\|_{1,\alpha,\beta,\gamma,\delta}^2 \leqslant cN^{2-2r}\, \|u\|_{r,\alpha,\beta,*}^2 \tag{3.17}
$$

and if in addition, $\alpha \leqslant \gamma + 1$, $\beta \leqslant \delta + 1$ and $0 \leqslant \mu \leqslant 1$, then

$$
\|\tilde{P}^{1,L}_{N,\alpha,\beta,\gamma,\delta} u - u\|_{\mu,\alpha,\beta,\gamma,\delta}^2 \leqslant cN^{2\mu-2r}\, \|u\|_{r,\alpha,\beta,*}^2 \tag{3.18}
$$

Theorem 3.3. For any $v \in H^r_{\omega,B}(\Lambda) \cap H^1_{0,\omega}(\Lambda)$, $v > \frac{3}{4}$ and $r \geqslant 1$,

$$\|P_N^{1,0} v - v\|_{1,\omega} \leqslant cN^{1-r} \|v\|_{r,\omega,B}$$

Proof. By Lemma 3.1, for any $\phi \in \mathscr{R}_N^0$,

$$\begin{aligned}\|P_N^{1,0} v - v\|_{1,\omega}^2 &\leqslant c a_\omega^v(P_N^{1,0} v - v, P_N^{1,0} v - v)\\ &= c a_\omega^v(P_N^{1,0} v - v, \phi - v)\\ &\leqslant c \|P_N^{1,0} v - v\|_{1,\omega} \|\phi - v\|_{1,\omega}\end{aligned}$$

Therefore

$$\|P_N^{1,0} v - v\|_{1,\omega} \leqslant c \inf_{\phi \in \mathscr{R}_N^0} \|\phi - v\|_{1,\omega} \tag{3.19}$$

Next, let $x = (1+y)/(1-y)$, $u(y) = v((1+y)/(1-y))$ and take $\phi = \tilde{P}^{1,L}_{N,2,0,0,0} u(y)|_{y=(x-1)/(x+1)}$ in (3.19). Then, the desired result follows from (3.17) and a similar argument as in the proof of Theorem 3.2. □

We now consider yet another orthogonal projection which will be used in the Legendre rational interpolation approximations and in the Legendre rational pseudospectral method. Let

$$\hat{a}_\omega(u, v) = \tfrac{1}{2} \int_\Lambda \partial_x u(x)\, \partial_x v(x)(x+1)\, dx + \int_\Lambda u(x)\, v(x)\, \omega(x)\, dx \tag{3.20}$$

The orthogonal projection $\hat{P}_N^1 : H^1_{\omega,A}(\Lambda) \to \mathscr{R}_N$ is a mapping such that for any $v \in H^1_{\omega,A}(\Lambda)$,

$$\hat{a}_\omega(\hat{P}_N^1 v - v, \phi) = 0, \qquad \forall \phi \in \mathscr{R}_N \tag{3.21}$$

Theorem 3.4. For any $v \in H^r_{\omega,A}(\Lambda)$ and $r \geqslant 1$,

$$\|\hat{P}_N^1 v - v\|_\omega \leqslant cN^{-r} \|v\|_{r,\omega,A}$$

and

$$\|(x+1)^{3/2} \partial_x(\hat{P}_N^1 v - v)\|_\omega \leqslant cN^{1-r} \|v\|_{r,\omega,A}$$

Proof. Let us denote

$$u(y) = v\left(\frac{1+y}{1-y}\right), \qquad u_N^*(y) = \hat{P}_N^1 v(x)|_{x=(1+y)/(1-y)}$$

By definition, we have

$$\int_{\tilde{\Lambda}} \partial_y(u_N^*(y)-u(y))\,\partial_y\psi(y)(1-y)\,dy$$

$$+\int_{\tilde{\Lambda}} (u_N^*(y)-u(y))\,\psi(y)\,dy=0, \qquad \forall\psi\in\mathscr{P}_N \tag{3.22}$$

Thus, $u_N^*(y)=\tilde{P}^1_{N,1,0,0,0}u(y)$. Under the transform $x=(1+y)/(1-y)$, we have

$$\int_{\Lambda} (\hat{P}^1_N v-v)^2\,\omega(x)\,dx=\tfrac{1}{2}\int_{\tilde{\Lambda}} (u_N^*-u)^2\,dy$$

$$\int_{\Lambda} (x+1)^3\,(\partial_x(\hat{P}^1_N v-v))^2\,\omega(x)\,dx=\int_{\tilde{\Lambda}} (\partial_y(u_N^*-u))^2\,(1-y)\,dy$$

Therefore, we derive from (3.11) with $\alpha=1$, $\beta=\gamma=\delta=0$ that

$$\|\hat{P}^1_N v-v\|_\omega\leqslant \|u_N^*-u\|^2_{0,1,0,0,0}\leqslant cN^{-2r}\,\|u\|^2_{r,1,0,*} \tag{3.23}$$

and

$$\|(x+1)^{3/2}\,\partial_x(\hat{P}^1_N v-v)\|_\omega\leqslant c\,\|u_N^*-u\|^2_{1,1,0,0,0}\leqslant cN^{2-2r}\,\|u\|^2_{r,1,0,*} \tag{3.24}$$

A direction computation leads to

$$A^{(1)}_{r,1,0}(u)\leqslant c\sum_{k=r-[r/2]+1}^{r}\ \sum_{j=1}^{k}\|(x+1)^{(r/2)+j}\,\partial_x^j v\|^2_\omega\leqslant c\,\|v\|^2_{r,\omega,\Lambda}$$

Similarly,

$$A^{(2)}_{r,1,0}(u)\leqslant c\sum_{k=0}^{[(r+1)/2]}\ \sum_{j=1}^{k}\|(x+1)^{k+j-(1/2)}\,\partial_x^j v\|^2_\omega\leqslant c\,\|v\|^2_{r,\omega,\Lambda} \qquad \square$$

We now give an estimate for the L^∞-norm of the projection operator $\hat{P}^1_N$ which will be useful for analyzing nonlinear problems. Similar estimates for other projection operators can also be established. To do this, we introduce the following Hilbert space. For any non-negative integer r,

$$H^r_{\omega,C}(\Lambda)=\{v\mid v \text{ is measurable and } \|v\|_{r,\omega,C}<\infty\}$$

with the norm

$$\|v\|_{r,\omega,C}=\left(\sum_{k=0}^{r}\|(x+1)^{r+k}\,\partial_x^k v\|^2_\omega\right)^{1/2}$$

For any real $r > 0$, the space $H^r_{\omega, C}(\Lambda)$ and its norm are defined by space interpolation.

Theorem 3.5. For any $v \in H^r_{\omega, C}(\Lambda)$ with $r > 1$, we have

$$\|\hat{P}^1_N v\|_{L^\infty(\Lambda)} \leqslant c\, \|v\|_{r, w, C}$$

Proof. Let $u(y)$, u^*_N and $\tilde{P}^1_{N, 1, 0, 0, 0} u(y)$ be the same as defined above. Then

$$\|\hat{P}^1_N v\|_{L^\infty(\Lambda)} = \|u^*_N\|_{L^\infty(\tilde{\Lambda})} = \|\tilde{P}^1_{N, 1, 0, 0, 0} u(y)\|_{L^\infty(\tilde{\Lambda})}$$

Thanks to Theorem 2.11 in [9], we have for $r > 1$,

$$\|\tilde{P}^1_{N, 1, 0, 0, 0} u(y)\|_{L^\infty(\tilde{\Lambda})} \leqslant c(\|u\|_{r, 1, 0, *} + \|u\|_{H^r(\tilde{\Lambda})})$$

Moreover, by (3.14),

$$\begin{aligned}
\|u\|^2_{H^r(\tilde{\Lambda})} &= \sum_{k=0}^{r} \sum_{j=1}^{k} \int_\Lambda (x+1)^{2k+2j} (\partial^j_x v(x))^2\, \omega(x)\, dx \\
&\leqslant c \sum_{j=0}^{r} \int_\Lambda (x+1)^{2r+2j}\, (\partial^j_x v(x))^2\, \omega(x)\, dx \leqslant c\, \|v\|^2_{r, \omega, C}
\end{aligned}$$

Then the desired result follows. □

4. LEGENDRE RATIONAL INTERPOLATION APPROXIMATION

In actual computations, it is convenient to use interpolations. We first consider the Legendre–Gauss rational interpolation. We denote by $\zeta_{N, j}$ the $N+1$ distinct real zeros of $R_{N+1}(x)$, $0 \leqslant j \leqslant N$. Indeed,

$$\zeta_{N, j} = (1 + \sigma_{N, j})(1 - \sigma_{N, j})^{-1} \tag{4.1}$$

where $\sigma_{N, j}$ are the zeros of $L_{N+1}(x)$. Let $\omega_{N, j}$ be the corresponding Christoffel numbers, $0 \leqslant j \leqslant N$, such that

$$\int_\Lambda \phi(x)\, \omega(x)\, dx = \sum_{j=0}^{N} \phi(\zeta_{N, j})\, \omega_{N, j}, \qquad \forall \phi \in \mathscr{R}_{2N+1} \tag{4.2}$$

As we know, the weights of the Legendre–Gauss quadrature are

$$\rho_{N, j} = \frac{2}{(1 - \sigma^2_{N, j})(\partial_x L_{N+1}(\sigma_{N, j}))^2}, \qquad 0 \leqslant j \leqslant N$$

Thus

$$\omega_{N,j} = \frac{2}{\zeta_{N,j}(\zeta_{N,j}+1)^2 \, (\partial_x R_{N+1}(\zeta_{N,j}))^2} \tag{4.3}$$

Moreover, by virtue of (15.3.10) in Szegö [15],

$$\omega_{N,j} \sim \frac{2\pi\zeta_{N,j}^{1/2}}{(N+1)(\zeta_{N,j}+1)} \tag{4.4}$$

We next introduce the discrete inner product and the discrete norm associated with the Legendre–Gauss rational interpolation points,

$$(u, v)_{\omega,N} = \sum_{j=0}^{N} u(\zeta_{N,j}) \, v(\zeta_{N,j}) \, \omega_{N,j}, \quad \|v\|_{\omega,N} = (v, v)_{\omega,N}^{1/2}$$

Thanks to (4.2), we have

$$(\phi, \psi)_{\omega,N} = (\phi, \psi)_{\omega}, \qquad \forall \phi\psi \in \mathscr{R}_{2N+1} \tag{4.5}$$

For any $v \in C(\Lambda)$, the Legendre–Gauss rational interpolant $I_N v \in \mathscr{R}_N$ such that

$$I_N v(\zeta_{N,j}) = v(\zeta_{N,j}), \qquad 0 \leqslant j \leqslant N$$

or equivalently,

$$(I_N v - v, \phi)_{\omega,N} = 0, \qquad \forall \phi \in \mathscr{R}_N$$

The following theorem is related to the stability of the Legendre–Gauss rational interpolation.

Theorem 4.1. For any $v \in H^1_{\omega,A}(\Lambda)$,

$$\|I_N v\|_{\omega} \leqslant c(\|v\|_{\omega} + N^{-1} \, \|x^{1/2} \, \partial_x v\|)$$

Proof. By (4.4) and (4.5),

$$\begin{aligned} \|I_N v\|_{\omega}^2 = \|I_N v\|_{\omega,N}^2 &= \sum_{j=0}^{N} v^2(\zeta_{N,j}) \, \omega_{N,j} \\ &\leqslant cN^{-1} \sum_{j=0}^{N} v^2(\zeta_{N,j}) \, \zeta_{N,j}^{1/2}(\zeta_{N,j}+1)^{-1} \end{aligned}$$

Let $x=(1+\cos\theta)/(1-\cos\theta)$ and $\hat{v}(\theta)=v((1+\cos\theta)/1-\cos\theta))$. Then

$$\|I_N v\|_\omega^2 \leqslant cN^{-1} \sum_{j=0}^{N} \hat{v}^2(\theta_{N,j}) \left(\frac{1+\cos\theta_{N,j}}{1-\cos\theta_{N,j}}\right)^{1/2} (1-\cos\theta_{N,j})$$

According to (4.1) and Theorem 8.9.1 in Szegö,[15]

$$\theta_{N,j} = \frac{1}{N+1}(j\pi + \mathcal{O}(1)), \qquad 0 \leqslant j \leqslant N \tag{4.6}$$

where $\mathcal{O}(1)$ is bounded uniformly for all $0\leqslant j\leqslant N$. Now, let $a_0=\mathcal{O}(1)/(N+1)$ and $a_1=(N\pi+\mathcal{O}(1))/(N+1)$. Then $\theta_{N,j}\in K_j\subset[a_0, a_1]$, K_j being of size $c/(N+1)$. Consequently,

$$\|I_N v\|_\omega^2 \leqslant cN^{-1} \sum_{j=0}^{N} \sup_{\theta\in K_j} |\hat{v}^2(\theta)\sin\theta|$$

By using an inequality of space interpolation (see (13.7) in Bernardi and Maday [2]), we know that for any $f\in H^1(a,b)$,

$$\max_{a\leqslant x\leqslant b} |f(x)| \leqslant c\left(\frac{1}{\sqrt{b-a}}\|f\|_{L^2(a,b)} + \sqrt{b-a}\,\|\partial_x f\|_{L^2(a,b)}\right) \tag{4.7}$$

Hence

$$\begin{aligned}
\|I_N v\|_\omega^2 &\leqslant c\sum_{j=0}^{N} (\|\hat{v}(\theta)\sin^{1/2}\theta\|_{L^2(K_j)}^2 + N^{-2}\,\|\partial_\theta(\hat{v}(\theta)\sin^{1/2}\theta)\|_{L^2(K_j)}^2)\\
&\leqslant c(\|\hat{v}(\theta)\sin^{1/2}\theta\|_{L^2(0,\pi)}^2 + N^{-2}\,\|\partial_\theta(\hat{v}(\theta)\sin^{1/2}\theta)\|_{L^2(a_0,a_1)}^2)\\
&\leqslant c(\|\hat{v}(\theta)\sin^{1/2}\theta\|_{L^2(0,\pi)}^2 + N^{-2}\,\|\partial_\theta\hat{v}(\theta)\sin^{1/2}\theta)\|_{L^2(0,\pi)}^2\\
&\quad + \left(\sup_{a_0\leqslant\theta\leqslant a_1}\frac{\cos^2\theta}{N^2\sin\theta}\right)\|\hat{v}(\theta)\sin^{1/2}\theta\|_{L^2(0,\pi)}^2)\\
&\leqslant c(\|\hat{v}(\theta)\sin^{1/2}\theta\|_{L^2(0,\pi)}^2 + N^{-2}\,\|\partial_\theta\hat{v}(\theta)\sin^{1/2}\theta\|_{L^2(0,\pi)}^2)\\
&\leqslant c(\|v(x)\|_{L^2_\omega(\Lambda)}^2 + N^{-2}\,\|x^{1/2}\,\partial_x v\|^2)
\end{aligned}$$

This completes the proof. □

Theorem 4.2. For any $v\in H^r_{\omega,\Lambda}(\Lambda)$ and $0\leqslant\mu\leqslant 1\leqslant r$,

$$\|I_N v - v\|_{\mu,\omega} \leqslant cN^{2\mu-r}\,\|v\|_{r,\omega,\Lambda}$$

Proof. Since $I_N(\hat{P}^1_N v)$ coincides with $\hat{P}^1_N v$, we derive from Theorems 3.4 and 4.1 that

$$\begin{aligned}\|I_N v-\hat{P}^1_N v\|_\omega &\leqslant (\|\hat{P}^1_N v-v\|_\omega+N^{-1}\|x^{1/2}\,\partial_x(\hat{P}^1_N v-v)\|)\\ &\leqslant cN^{-r}\,\|v\|_{r,\omega,A} \end{aligned}\tag{4.8}$$

Using Theorem 3.4 again,

$$\|I_N v-v\|_\omega\leqslant \|\hat{P}^1_N v-v\|_\omega+\|I_N v-\hat{P}^1_N v\|_\omega\leqslant cN^{-r}\,\|v\|_{r,\omega,A}\tag{4.9}$$

Furthermore, by (4.8) and Theorems 2.2 and 3.4,

$$\begin{aligned}|I_N v-v|_{1,\omega} &\leqslant |\hat{P}^1_N v-v|_{1,\omega}+|I_N v-\hat{P}^1_N v|_{1,\omega}\\ &\leqslant \|\hat{P}^1_N v-v\|_{1,\omega}+cN^2\,\|I_N v-\hat{P}^1_N v\|_\omega\\ &\leqslant c\,\|\hat{P}^1_N v-v\|_{1,\omega}+cN^{2-r}\,\|v\|_{r,\omega,A}\\ &\leqslant cN^{2-r}\,\|v\|_{r,\omega,A}\end{aligned}\tag{4.10}$$

Finally, we get the desired result by (4.9), (4.10) and space interpolation. □

We now deal with the Legendre–Gauss–Radau rational interpolation. We denote by $\hat{\zeta}_{N,j}$ the $N+1$ distinct zeros of $R_N(x)+R_{N+1}(x)$, $0\leqslant j\leqslant N$. Indeed,

$$\hat{\zeta}_{N,j}=(1+\hat{\sigma}_{N,j})(1-\hat{\sigma}_{N,j})^{-1}\tag{4.11}$$

where $\hat{\sigma}_{N,j}$ are the zeros of $L_N(x)+L_{N+1}(x)$. In particular, $\hat{\zeta}_{N,N}=0$. Let $\hat{\omega}_{N,j}$ be the corresponding Christoffel numbers, $0\leqslant j\leqslant N$, such that

$$\int_A \phi(x)\,\omega(x)\,dx=\sum_{j=0}^{N}\phi(\hat{\zeta}_{N,j})\,\hat{\omega}_{N,j},\qquad \forall\phi\in\mathscr{R}_{2N}\tag{4.12}$$

As we know, the weights of the Legendre–Gauss–Radau quadrature are

$$\hat{\rho}_{N,j}=\frac{1}{(N+1)^2}\frac{1-\hat{\sigma}_{N,j}}{(L_N(\hat{\sigma}_{N,j}))^2},\qquad 0\leqslant j\leqslant N-1$$

$$\hat{\rho}_{N,N}=\frac{2}{(N+1)^2}$$

Thus

$$\hat{\omega}_{N,j} = \frac{2}{(N+1)^2} \frac{1}{(\hat{\zeta}_{N,j}+1)(R_N(\hat{\zeta}_{N,j}))^2}, \qquad 0 \leqslant j \leqslant N-1$$
$$\hat{\omega}_{N,N} = \frac{1}{(N+1)^2} \tag{4.13}$$

Moreover, by virtue of (15.3.10) in Szegö [15],

$$\hat{\omega}_{N,j} \sim \frac{4\pi}{N} \frac{\hat{\zeta}_{N,j}^{1/2}}{(\hat{\zeta}_{N,j}+1)}, \qquad 0 \leqslant j \leqslant N-1 \tag{4.14}$$

The discrete inner product and the discrete norm associated with the Legendre–Gauss–Radau rational interpolation points are,

$$(u, v)_{\omega, N, \sim} = \sum_{j=0}^{N} u(\hat{\zeta}_{N,j})\, v(\hat{\zeta}_{N,j})\, \hat{\omega}_{N,j}, \qquad \|v\|_{\omega, N, \sim} = (v, v)_{\omega, N, \sim}^{1/2}$$

Thanks to (4.12),

$$(\phi, \psi)_{\omega, N, \sim} = (\phi, \psi)_{\omega}, \qquad \forall \phi\psi \in \mathscr{R}_{2N} \tag{4.15}$$

For any $v \in C(\bar{\Lambda})$, the Legendre–Gauss–Radau rational interpolant $\hat{I}_N v(x) \in \mathscr{R}_N$, satisfying

$$\hat{I}_N v(\hat{\zeta}_{N,j}) = v(\hat{\zeta}_{N,j}), \qquad 0 \leqslant j \leqslant N$$

or equivalently,

$$(\hat{I}_N v - v, \phi)_{\omega, N, \sim} = 0, \qquad \forall \phi \in \mathscr{R}_N$$

The following theorem is related to the stability of the Legendre–Gauss–Radau rational interpolation.

Theorem 4.3. For any $v \in H^1_{\omega, A}(\Lambda)$,

$$\|\hat{I}_N v\|_{\omega} \leqslant c(\|v\|_{\omega} + N^{-1}\, \|(x+1)^{1/2}\, \partial_x v\|)$$

Proof. By (4.13), (4.14) and (4.15),

$$\|\hat{I}_N v\|_{\omega}^2 = \|\hat{I}_N v\|_{\omega, N, \sim}^2 = \sum_{j=0}^{N} v^2(\hat{\zeta}_{N,j})\, \hat{\omega}_{N,j}$$
$$\leqslant cN^{-1} \sum_{j=0}^{N-1} v^2(\hat{\zeta}_{N,j})\, \hat{\zeta}_{N,j}^{1/2}(1+\hat{\zeta}_{N,j})^{-1} + (N+1)^{-2}\, v^2(0)$$

By the trace theorem,

$$|v(0)|^2 \leqslant c\, \|v\|^2_{H^{1/2}(0,1)} \leqslant c\, \left\| \frac{1}{x+1}\, v \right\|^2_{H^{1/2}(0,1)} \leqslant c\, \|v\|^2_{1/2,\omega} \leqslant c\, \|v\|^2_{1,\omega}$$

Let $x = (1+\cos\theta)/(1-\cos\theta)$ and $\hat{v}(\theta) = v((1+\cos\theta)/(1-\cos\theta))$. Then

$$\|\hat{I}_N v\|^2_\omega \leqslant cN^{-1} \sum_{j=0}^{N-1} \hat{v}^2(\hat{\theta}_{N,j}) \left(\frac{1+\cos\hat{\theta}_{N,j}}{1-\cos\hat{\theta}_{N,j}}\right)^{1/2} (1-\cos\hat{\theta}_{N,j}) + cN^{-2}\, \|v\|^2_{1,\omega}$$

According to (4.11), Theorem 8.9.1 of Szegö [15], and the relation between $\sigma_{N,j}$ and $\hat{\sigma}_{N,j}$, we assert that

$$\hat{\theta}_{N,j} = \frac{1}{N}(j\pi + \mathcal{O}(1)), \qquad 0 \leqslant j \leqslant N-1$$

Then, the conclusion follows from an similar argument as in the proof of Theorem 4.1. □

Theorem 4.4. For any $v \in H^r_{\omega,A}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\|\hat{I}_N v - v\|_{\mu,\omega} \leqslant cN^{2\mu - r}\, \|v\|_{r,\omega,A}$$

Proof. Since $\hat{I}_N(\hat{P}^1_N v)$ coincides with $\hat{P}^1_N v$, we have from Theorems 3.4 and 4.3 that

$$\begin{aligned} \|\hat{I}_N v - \hat{P}^1_N v\|_\omega &\leqslant c(\|\hat{P}^1_N v - v\|_\omega + N^{-1}\, \|(x+1)^{1/2}\, \partial_x(\hat{P}^1_N v - v)\|) \\ &\leqslant cN^{-r}\, \|v\|_{r,\omega,A} \end{aligned} \tag{4.16}$$

Using Theorem 3.4 again,

$$\|\hat{I}_N v - v\|_\omega \leqslant \|\hat{P}^1_N v - v\|_\omega + \|\hat{I}_N v - \hat{P}^1_N v\|_\omega \leqslant cN^{-r}\, \|v\|_{r,\omega,A} \tag{4.17}$$

Furthermore, by (4.16), and Theorems 2.2 and 3.4,

$$\begin{aligned} |\hat{I}_N v - v|_{1,\omega} &\leqslant |\hat{P}^1_N v - v|_{1,\omega} + |\hat{I}_N v - \hat{P}^1_N v|_{1,\omega} \\ &\leqslant \|\hat{P}^1_N v - v\|_{1,\omega} + cN^2\, \|\hat{I}_N v - \hat{P}^1_N v\|_\omega \\ &\leqslant c\, \|\hat{P}^1_N v - v\|_{1,\omega} + cN^{2-r}\, \|v\|_{r,\omega,A} \\ &\leqslant cN^{2-r}\, \|v\|_{r,\omega,A} \end{aligned} \tag{4.18}$$

Finally, we get the desired result by (4.16), (4.18) and space interpolation. □

5. APPLICATIONS

We consider first the following model problem

$$\begin{cases} -\partial_x^2 U(x) + \nu U(x) = f(x), & 0 < x < \infty \\ U(0) = 0 \\ (x+1)^{-3/2} U(x) \to 0, & \text{as} \quad x \to \infty \end{cases} \tag{5.1}$$

where $\nu > 0$ and $f(x)$ is a given function. For simplicity, we assume $\nu > \frac{3}{4}$. Otherwise, we can use the variable transformation $x = \alpha y$, $\alpha > \frac{1}{2}\sqrt{3/\nu}$. A weak formulation of (5.1) with $\nu > \frac{3}{4}$ is to find $U \in H^1_{0,\omega}(\Lambda)$ such that

$$\alpha^\nu_\omega(U, v) = (f, v)_\omega, \qquad \forall v \in H^1_{0,\omega}(\Lambda) \tag{5.2}$$

If $f \in (H^1_{0,\omega}(\Lambda))'$, then by Lemma 3.1 and the Lax–Milgram Lemma, (5.2) with $\nu > \frac{3}{4}$ has a unique solution.

The Legendre rational spectral scheme for (5.1) is to find $u_N \in \mathscr{R}^0_N$, such that

$$a^\nu_\omega(u_N, \phi) = (f, \phi)_\omega, \qquad \forall \phi \in \mathscr{R}^0_N \tag{5.3}$$

Theorem 5.1. If $U \in H^r_{\omega,B}(\Lambda) \cap H^1_{0,\omega}(\Lambda)$, $\nu > \frac{3}{4}$ and $r \geqslant 1$, then

$$\|u_N - U\|_{1,\omega} \leqslant cN^{1-r} \|U\|_{r,\omega,B}$$

Proof. Let $U_N = P^{1,0}_N U$. By (5.2),

$$a^\nu_\omega(U_N, \phi) = (f, \phi)_\omega, \qquad \forall \phi \in \mathscr{R}^0_N \tag{5.4}$$

Let $\hat{U}_N = u_N - U_N$. Then, by (5.3) and (5.4),

$$a^\nu_\omega(\hat{U}_N, \phi) = 0, \qquad \forall \phi \in \mathscr{R}^0_N \tag{5.5}$$

Thus, $u_N = U_N$ and the desired result follows from Theorem 3.3. □

We now consider the Legendre–Gauss–Radau rational pseudospectral scheme for (5.1). Let

$$a^\nu_{\omega,N}(u, v) = (\partial_x u, \partial_x v - 2v(1+x)^{-1})_{\omega,N,\sim} + \nu(u, v)_{\omega,N,\sim}$$

Since

$$\partial_x R_l(x) = \frac{2\sqrt{2}}{(x+1)^2} L_l'\left(\frac{x-1}{x+1}\right) = \frac{\sqrt{2}}{2}(1-y)^2 L_l'(y)\Big|_{y=(x-1)/(x+1)} \in \mathscr{R}_{l+1}$$

and

$$R_l(x)(1+x)^{-1} = \frac{\sqrt{2}}{2}(1-y)\,L_l(y)\Big|_{y=(x-1)/(x+1)} \in \mathscr{R}_{l+1}$$

we know from (4.15) that for any $\phi, \psi \in \mathscr{R}_{N-1}$,

$$a^{\nu}_{\omega,N}(\phi,\psi) = a^{\nu}_{\omega}(\phi,\psi) \tag{5.6}$$

A legendre rational pseudospectral method for (5.1) is to find $u_N \in \mathscr{R}^0_{N-1}$ such that

$$a^{\nu}_{\omega,N}(u_N,\phi) = (f,\phi)_{\omega,N,\sim}, \qquad \forall \phi \in \mathscr{R}^0_{N-1} \tag{5.7}$$

Theorem 5.2. If $U \in H^r_{\omega,B}(\Lambda) \cap H^1_{0,\omega}(\Lambda)$, $f \in H^{r-1}_{\omega,A}(\Lambda)$, $\nu > \frac{3}{4}$ and $r \geqslant 1$, then

$$\|u_N - U\|_{1,\omega} \leqslant cN^{1-r}(\|U\|_{r,\omega,B} + \|f\|_{r-1,\omega,A}) \tag{5.8}$$

Proof. By (4.15) and Theorem 2.3, we have for any $\phi \in \mathscr{R}_{N-1}$,

$$\begin{aligned} |(f,\phi)_{\omega,N,\sim}| &= \left|\left(\hat{I}_N((x+1)\,f), \frac{\phi}{x+1}\right)_{\omega,N,\sim}\right| = \left|\left(\hat{I}_N(x+1)\,f), \frac{\phi}{x+1}\right)_{\omega}\right| \\ &\leqslant \|\hat{I}_N((x+1)\,f)\|_{\omega}\,\|\phi\|_{\omega^2} \\ &\leqslant \|\hat{I}_N((x+1)\,f)\|_{\omega}\,\|\phi\|_{1,\omega} \end{aligned} \tag{5.9}$$

Hence, by the Lax–Milgram Lemma, (5.7) has a unique solution such that

$$\|u_N\|_{1,\omega} \leqslant c\,\|\hat{I}_N((x+1)\,f)\|_{\omega}$$

Let $U_N = P^{1,0}_{N-1}U$. Then by (5.6) and (5.7), we have for any $\phi \in \mathscr{R}^0_{N-1}$,

$$\begin{aligned} a^{\nu}_{\omega}(U_N,\phi) &= (f,\phi)_{\omega} \\ a^{\nu}_{\omega}(u_N,\phi) &= (I_N f,\phi)_{\omega} \end{aligned} \tag{5.10}$$

Therefore,

$$a^{\nu}_{\omega}(U_N - u_N,\phi) = (f - I_N f,\phi)_{\omega}, \qquad \phi \in \mathscr{R}^0_{N-1}$$

Let $\varepsilon = \nu - \frac{3}{4} > 0$. Taking $\phi = U_N - u_N$ and using (3.16), we obtain

$$\begin{aligned}\frac{\varepsilon}{2}\|u_N - U_N\|_\omega^2 + \frac{2\varepsilon}{9}|u_N - U_N|_{1,\omega}^2 &\\ \leqslant a_{\omega,N}^\nu(u_N - U_N, u_N - U_N)&\\ = (f - I_N f, U_N - u_N)_\omega \leqslant \|f - I_N f\|_\omega \|U_N - u_N\|_\omega&\\ \leqslant \varepsilon \|f - I_N f\|_\omega^2 + \frac{\varepsilon}{4}\|U_N - u_N\|_\omega^2 & \qquad (5.11)\end{aligned}$$

Therefore, by Theorems 3.3 and 4.4,

$$\begin{aligned}\|u_N - U\|_{1,\omega} &\leqslant \|U_N - U\|_{1,\omega} + \|u_N - U_N\|_{1,\omega}\\ &\leqslant cN^{1-r}(\|U\|_{r,\omega,B} + \|f\|_{r-1,\omega,A}) \qquad \square\end{aligned}$$

Next, we consider a time-dependent model problem

$$\begin{cases}\partial_t U(x,t) - \partial_x^2 U(x,t) = f(x,t) & 0 < x < \infty, \quad 0 < t \leqslant T\\ \partial_x U(0,t) = 0 & 0 \leqslant t \leqslant T\\ (x+1)^{-3/2} U(x,t) \to 0 & \text{as} \quad x \to \infty, \quad 0 \leqslant t \leqslant T\\ U(x,0) = U_0(x) & 0 < x < \infty\end{cases} \qquad (5.12)$$

A weak formulation of (5.12) is to find $U \in L^2(0, T; H_\omega^1(\Lambda)) \cap L^\infty(0, T; L_\omega^2(\Lambda))$ such that

$$(\partial_t U(t), v)_\omega + (\partial_x U(t), \partial_x v)_\omega + (\partial_x U(t), v\,\partial_x \omega) = (f, v)_\omega, \qquad \forall v \in H_\omega^1(\Lambda) \qquad (5.13)$$

If $U_0 \in L_\omega^2(\Lambda)$ and $f \in L^2(0, T; L_\omega^2(\Lambda))$, then (5.12) has a unique solution. The Legendre rational spectral scheme for (5.12) is to find $u_N(t) \in \mathscr{R}_N$ for all $0 \leqslant t \leqslant T$ such that

$$\begin{cases}(\partial_t u_N(t), \phi)_\omega + (\partial_x u_N(t), \partial_x \phi)_\omega + (\partial_x u_N(t), \phi\,\partial_x \omega) = (f, \phi)_\omega, & \forall \phi \in \mathscr{R}_N\\ u_N(0) = P_N U_0\end{cases} \qquad (5.14)$$

Theorem 5.3. If $U \in H^1(0, T; H_{\omega,B}^r(\Lambda))$ and $U_0 \in H_{\omega,A}^{r-1}(\Lambda) \cap H_{\omega,B}^r(\Lambda)$ with $r \geqslant 1$, then for all $0 \leqslant t \leqslant T$,

$$\|u_N(t) - U(t)\|_\omega^2 + \int_0^t \|u_N(s) - U(s)\|_{1,\omega}^2\, ds \leqslant c^* e^{ct} N^{2-2r}$$

where c^* is a positive constant depending only on the norms of U and U_0 in the mentioned spaces.

Proof. Let $U_N = P_N^1 U$. By (5.13),

$$\begin{cases} (\partial_t U_N(t), \phi)_\omega + (U_N(t), \phi)_{1,\omega} + (\partial_x U_N(t), \phi\, \partial_x \omega) - (U_N(t), \phi)_\omega \\ \qquad = (f, \phi)_\omega + \sum_{i=1}^{3} G_i(t, \phi), \qquad \forall \phi \in \mathscr{R}_N \\ U_N(0) = P_N^1 U_0 \end{cases} \tag{5.15}$$

where

$$G_1(t, \phi) = (\partial_t U_n(t) - \partial_t U(t), \phi)_\omega$$
$$G_2(t, \phi) = (\partial_x U_N(t) - \partial_x U(t), \phi\, \partial_x \omega)$$
$$G_3(t, \phi) = (U(t) - U_N(t), \phi)_\omega$$

Now, let $\hat{U}_N = u_N - U_N$. Then by (5.14) and (5.15),

$$\begin{cases} (\partial_t \hat{U}_N(t), \phi)_\omega + (\hat{U}_N(t), \phi)_{1,\omega} + (\partial_x \hat{U}_N(t), \phi\, \partial_x \omega) - (\hat{U}_N(t), \phi)_\omega \\ \qquad = -\sum_{i=1}^{3} G_i(t, \phi) \\ \hat{U}_N(0) = P_N U_0 - P_N^1 U_0 \end{cases} \tag{5.16}$$

Taking $\phi = \hat{U}_N$ in (5.16), we find

$$\frac{d}{dt} \|\hat{U}_N(t)\|_\omega^2 + 2\|\hat{U}_N(t)\|_{1,\omega}^2 + 2(\partial_x \hat{U}_N(t)\, \partial_x \omega) - 2\, \|\hat{U}_N(t)\|_\omega^2$$
$$\leqslant 2 \sum_{i=1}^{3} |G_i(t, \hat{U}_N(t))| \tag{5.17}$$

It is easy to see that

$$2\, |(\partial_x \hat{U}_N(t), \hat{U}_N(t)\, \partial_x \omega)| + 2\, \|\hat{U}_N(t)\|_\omega^2 \leqslant |\hat{U}_N(t)|_{1,\omega}^2 + 6\, \|\hat{U}_N(t)\|_\omega^2$$

By virtue of Theorems 3.1 and 3.2,

$$2\, |G_1(t, \hat{U}_N(t))| \leqslant \|\hat{U}_N(t)\|^2{}_\omega + cN^{2-2r}\, \|\partial_t U(t)\|_{r,\omega,B}^2$$
$$2\, |G_2(t, \hat{U}_N(t))| \leqslant \|\hat{U}_N(t)\|^2{}_\omega + cN^{2-2r}\, \|U(t)\|_{r,\omega,B}^2$$
$$2\, |G_3(t, \hat{U}_N(t))| \leqslant \|\hat{U}_N(t)\|^2{}_\omega + cN^{2-2r}\, \|U(t)\|_{r,\omega,B}^2$$
$$\|\hat{U}_N(0)\|_\omega^2 \leqslant cN^{2-2r}(\|U_0\|_{r-1,\omega,A}^2 + \|U_0\|_{r,\omega,B}^2)$$
$$\leqslant cN^{2-2r}\, \|U_0\|_{r,\omega,B}^2$$

Thus,

$$\frac{d}{dt}\|\hat{U}_N(t)\|_\omega^2+\|\hat{U}_N(t)\|_{1,\omega}^2$$
$$\leqslant 8\,\|\hat{U}_N(t)\|_\omega^2+cN^{2-2r}(\|\partial_t U(t)\|_{r,\omega,B}^2+\|U(t)\|_{r,\omega,B}^2) \tag{5.18}$$

Let us denote

$$E(v,t)=\|v(t)\|_\omega^2+\int_0^t\|v(s)\|_{1,\omega}^2\,ds$$
$$\rho(t)=cN^{2-2r}\left(\|U_0\|_{r-1,\omega,A}^2+\|U_0\|_{r,\omega,B}^2+\int_0^t(\|\partial_s U(s)\|_{r,\omega,B}^2)\,ds\right)$$

Integrating (5.18) with respect to t, we obtain that

$$E(\hat{U}_N,t)\leqslant c\int_0^t E(\hat{U}_N,s)\,ds+\rho(t)$$

Finally, we use the above estimate and Theorem 3.2 to get the following result. □

We now consider the Legendre–Gauss rational pseudospectral scheme for (5.13): Find $u_N(t)\in\mathscr{R}_{N-1}$ for all $0\leqslant t\leqslant T$ such that

$$\begin{cases}(\partial_t u_N(t),\phi)_{\omega,N}+(\partial_x u_N(t),\partial_x\phi-2\phi(x+1)^{-1})_{\omega,N}\\ \qquad=(f(t),\phi)_{\omega,N}, \qquad \forall\phi\in\mathscr{R}_{N-1}\\ u_N(0)=I_{N-1}U_0\end{cases} \tag{5.19}$$

Theorem 5.4. If $U\in H^1(0,T;H^r_{\omega,B}(\Lambda))$, $U_0\in H^{r-1}_{\omega,A}(\Lambda)\cap H^r_{\omega,B}(\Lambda)$ and $f\in H^{r-1}_{\omega,A}(\Lambda)$ with $r\geqslant 1$, then for all $0\leqslant t\leqslant T$,

$$\|u_N(t)-U(t)\|_\omega^2+\int_0^t\|u_N(s)-U(s)\|_{1,\omega}^2\,ds\leqslant c^*e^{ct}N^{2-2r}$$

where c^* is a positive constant depending only on the norms of U, U_0 and f in the mentioned spaces.

Proof. Let $U_N=P^1_{N-1}U$. We get from (5.13) that

$$(\partial_t U_N(t),\phi)_{\omega,N}+(\partial_x U_N(t),\partial_x\phi-2\phi(x+1)^{-1})_{\omega,N}$$
$$=(f(t),\phi)_\omega+\sum_{i=4}^{6}G_i(t,\phi), \qquad \forall\phi\in\mathscr{R}_{N-1}, \quad t\in(0,T] \tag{5.20}$$

where

$$G_4 = (\partial_t U_N(t), \phi)_{\omega, N} - (\partial_t U(t), \phi)_\omega$$
$$G_5 = (\partial_x U_N(t), \partial_x \phi)_{\omega, N} - (\partial_x U(t), \partial_x \phi)_\omega$$
$$G_6 = 2(\partial_x U(t), \phi(x+1)^{-1})_\omega - 2(\partial_x U_N(t), \phi(x+1)^{-1})_{\omega, N}$$

Further, let $\hat{U}_N = u_N - U_N$. then by subtracting (5.20) from (5.19), we obtain that for any $\phi \in \mathscr{R}_{N-1}$ and $t \in (0, T]$,

$$\begin{aligned}(\partial_t \hat{U}_N(t), \phi)_{\omega, N} &+ (\partial_x \hat{U}_N(t), \partial_x \phi - 2\phi(x+1)^{-1})_{\omega, N} \\ &= (f(t), \phi)_{\omega, N} - (f(t), \phi)_\omega - \sum_{i=4}^{6} G_i(t, \phi) \\ \hat{U}_N(0) &= I_{N-1} U_0 - P^1_{N-1} U_0\end{aligned} \tag{5.21}$$

Taking $\phi = \hat{U}_N(t)$ in (5.21), we get from (4.5) that

$$\begin{aligned}\frac{d}{dt} \|\hat{U}_N(t)\|^2_\omega &+ 2\, |\hat{U}_N(t)|^2_{1, \omega} \\ &\leqslant 2 \sum_{i=4}^{6} |G_i(t, \hat{U}_N(t))| + 4\, |(\partial_x \hat{U}_N(t), \hat{U}_N(t)(x+1)^{-1})_\omega| \\ &\quad + 2\, |(f(t), \hat{U}_N(t))_{\omega, N} - (f(t), \hat{U}_N(t))_\omega|\end{aligned} \tag{5.22}$$

Next, we estimate the terms at the right side of (5.22). By virtue of (4.5) and Theorem 3.2,

$$2\, |G_4(t, \hat{U}_N(t))| \leqslant \|\hat{U}_N(t)\|^2{}_\omega + cN^{2-2r}\, \|\partial_t U(t)\|^2_{r, \omega, B}$$
$$2\, |G_5(t, \hat{U}_N(t))| \leqslant \tfrac{1}{2}\, |\hat{U}_N(t)|^2_{1, \omega} + cN^{2-2r}\, \|U(t)\|^2_{r, \omega, B}$$
$$2\, |G_6(t, \hat{U}_N(t))| \leqslant \|\hat{U}_N(t)\|^2_\omega + cN^{2-2r}\, \|U(t)\|^2_{r, \omega, B}$$

Clearly,

$$4\, |(\partial_x \hat{U}_N(t), \hat{U}_N(t)(x+1)^{-1})_\omega| \leqslant \tfrac{1}{2}\, |\hat{U}_N(t)|^2_{1, \omega} + 8\, \|\hat{U}_N(t)\|^2_\omega$$

Thanks to Theorem 4.2,

$$\begin{aligned}2\, |(f(t), \hat{U}_N(t))_{\omega, N} - (f(t), \hat{U}_N(t))_\omega| &= 2\, |(I_N f(t) - f(t), \hat{U}_N(t))_\omega| \\ &\leqslant \|\hat{U}_N(t)\|^2_\omega + cN^{2-2r}\, \|f(t)\|^2_{r-1, \omega, A}\end{aligned} \tag{5.23}$$

Moreover, Theorems 3.2 and 4.2 imply that

$$\|\hat{U}_N(0)\|_\omega^2 \leqslant cN^{2-2r}(\|U_0\|_{r-1,\omega,A}^2 + \|U_0\|_{r,\omega,B}^2)$$

Thus,

$$\frac{d}{dt}\|\hat{U}_N(t)\|_\omega^2 + \|\hat{U}_N(t)\|_{1,\omega}^2 \leqslant 12\,\|\hat{U}_N(t)\|_\omega^2 + cN^{2-2r}(\|\partial_t U(t)\|_{r,\omega,B}^2 + \|U(t)\|_{r,\omega,B}^2 + \|f(t)\|_{r-1,\omega,A}^2) \tag{5.24}$$

Let

$$\rho_1(t) = cN^{2-2r}\left(\|U_0\|_{r-1,\omega,A}^2 + \|U_0\|_{r,\omega,B}^2 + \int_0^t (\|\partial_s U(s)\|_{r,\omega,B}^2 + \|U(s)\|_{r,\omega,B}^2 + \|f(s)\|_{r-1,\omega,A}^2)\,ds\right)$$

Integrating (5.24) with respect to t, we obtain that

$$E(\hat{U}_N, t) \leqslant c\int_0^t E(\hat{U}_N, s)\,ds + \rho_1(t)$$

which implies the desired result. □

6. NUMERICAL IMPLEMENTATIONS

We now present an efficient algorithm for solving (5.1) by using the rational pseudospectral scheme (5.7). As is shown in [14] (see also [13]), one need to use *compact combinations* of rational Legendre polynomials as basis functions. Indeed, setting $\psi_j(x) = 1/\sqrt{2}(R_j(x) + R_{j+1}(x))$, we have $\psi(0) = 0$ and

$$\mathscr{R}_{N-1}^0 = \text{span}\{\psi_j : j = 0, 1, ..., N-2\} \tag{6.1}$$

Hence, setting

$$b_{kj} = (\psi_j, \psi_k)_{\omega, N, \sim} = (\psi_j, \psi_k)_\omega, \qquad a_{kj} = a_{\omega,N}^\nu(\psi_j, \psi_k) = -(\psi_j'', \psi_k)_\omega$$

$$u_N = \sum_{j=0}^{N-2} x_j \psi_j(x), \qquad \bar{x} = (x_0, x_1, ..., x_{N-2})^t \tag{6.2}$$

$$\bar{f} = (f_0, f_1, ..., f_{N-2})^t \qquad \text{with} \quad f_k = (f, \psi_j)_\omega$$

the Rational Legendre Galerkin approximation (5.7) is reduced to:

$$(\nu B + A)\,\bar{x} = \bar{f} \tag{6.3}$$

One verify easily by the transform $x = (1+y)/(1-y)$ that

$$b_{kj} = \int_{-1}^{1} (L_j(y) + L_{j+1}(y))(L_k(y) + L_{k+1}(y))\,dy$$

$$\begin{aligned} a_{kj} &= -\tfrac{1}{4}\int_{-1}^{1} (1-y)^2\,\{(1-y)^2(L_j'(y) + L_{j+1}'(y))\}'\,(L_k(y) + L_{k+1}(y))\,dy \\ &= -\tfrac{1}{4}\int_{-1}^{1} \{(1-y)^2\,[(1-y)^2(L_k(y) + L_{k+1}(y))]'\}'\,(L_j(y) + L_{j+1}(y))\,dy \end{aligned} \tag{6.4}$$

By using the orthogonality of Legendre polynomials, one find immediately that $B = (b_{kj})$ is a symmetric tridiagonal matrix whose nonzero entries are

$$b_{kk} = \frac{2}{2k+1} + \frac{2}{2k+3}, \qquad b_{k,\,k+1} = b_{k+1,\,k} = \frac{2}{2k+3}$$

Similarly, one find that $A = (a_{kj})$ is a non-symmetric seven diagonal matrix whose nonzero entries can be directly computed, although a little tedious, using the properties of Legendre polynomials. Hence, (6.3) can be efficiently solved.

We now present some numerical experiments using the above scheme to solve (5.1) with $\nu = 2$. Three illustrative examples are considered.

Example 1. $U(x) = \sin kxe^{-x}$.

Here, the function decays exponentially at infinity, so Theorem 5.2 predicts that errors of rational pseudospectral approximation will decrease faster than any algebraic rate. In Fig. 1, we plot the $\log_{10}$ of H^1_ω-errors vs. $\sqrt{N}$. The two near straight lines corresponding to $k = 1, 2$ and 4 indicate that the errors decay like $e^{-c\sqrt{N}}$.

Example 2. $U(x) = x/(1+x)^h$.

The second example decays algebraicly at infinity without essential singularity. One can check directly that $\|U\|_{r,\omega,B} + \|f\|_{r-1,\omega,A}$ is finite for $r < 2h$. Hence, according to Theorem 5.2, we can expect a convergence rate for the H^1_ω-norm to be of the order $2h - 1 - \varepsilon$ for any $\varepsilon > 0$. The observed convergence rate for the H^1_ω-norm plotted in Fig. 2 is about $2h$. Note that when h is a positive integer, the exact solution will be a rational polynomial so its pseudospectral approximation with $N \geqslant h + 2$ will be exact.

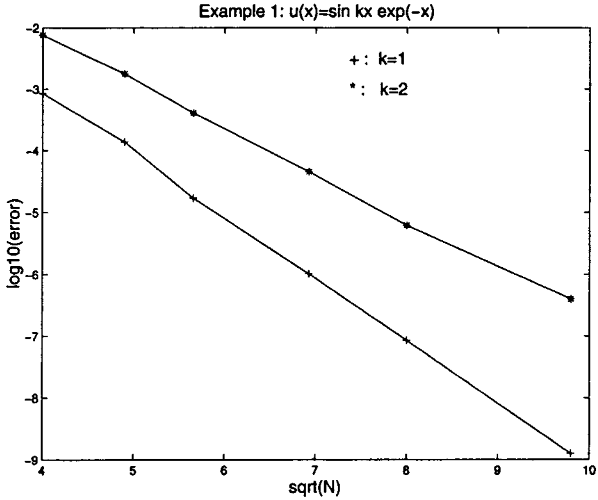

Fig. 1. Convergence rates of the rational pseudospectral approximation: Example 1.

Example 3. $U(x) = (\sin 2x)/(1+x)^h$.

The third example decays algebraicly at infinity but it also has an essential singularity at infinity. One can check directly that $\|U\|_{r,\omega,B} + \|f\|_{r-1,\omega,A}$ is finite for $r < \frac{2}{3}(h+1)$. Hence, according to Theorem 5.1, we can expect a convergence rate of order $\frac{2}{3}h - \frac{1}{3} - \varepsilon$ with any $\varepsilon > 0$ for the

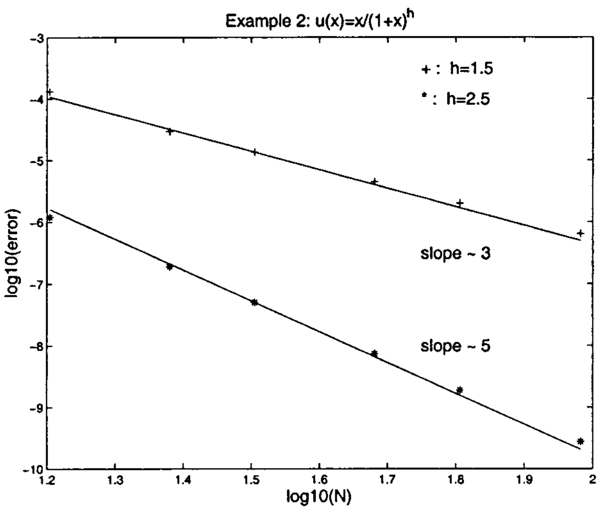

Fig. 2. Convergence rates of the rational pseudospectral approximation: Example 2.

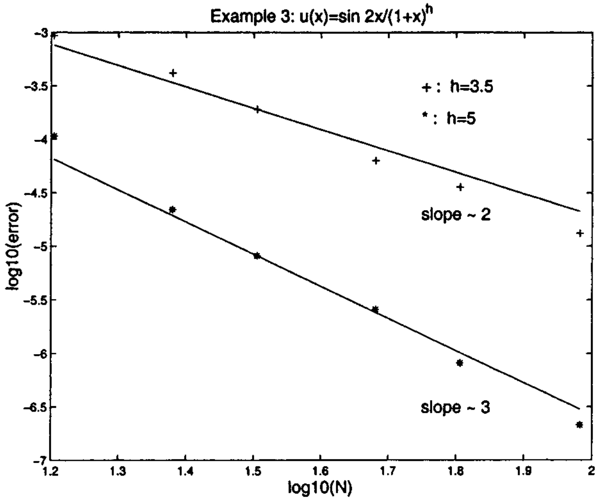

Fig. 3. Convergence rates of the Legendre rational approximation: Example 3.

H^1_ω-error. The observed convergence rate plotted in Fig. 3 agrees well with the theoretical result.

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REFERENCES

1. Adams, R. A. (1975). *Soblov Spaces*, Academic Press, New York.
2. Bernardi, C., and Maday, Y. (1997). In *Spectral Method*, Ciarlet, P. G., and Lions, L. L. (eds.), Handbook of Numerical Analysis, Vol. 5 (Part 2), North-Holland.
3. Boyd, J. P. (1987). Orthogonal rational functions on a semi-infinite interval, *J. Comput. Phys.* **70**, 63–88.
4. Boyd, J. P. (1987). Spectral methods using rational basis functions on an infinite interval, *J. Comput. Phys.* **69**, 112–142.
5. Christov, C. I. (1982). A complete orthogonal system of functions in $l^2(-\infty, \infty)$ space, *SIAM J. Appl. Math.* **42**, 1337–1344.
6. Guo, Benuy (1998). Gegenbauer approximation and its applications to differential equations on the whole line, *J. Math. Anal. Appl.* **226**, 180–206.
7. Guo, Benyu (1998). *Spectral Methods and Their Applications*, World Scientific Publishing Co. Inc., River Edge, New Jersey.
8. Guo, Benyu (1999). Error estimation of Hermite spectral method for nonlinear partial differential equations, *Math. Comp.* **68**(227), 1067–1078.

9. Guo, Benyu (2000). Jacobi approximations in certain Hilbert spaces and their applications to singular differential equations, *J. Math. Anal. Appl.* **243**, 373–408.
10. Guo, Benyu (2000). Jacobi spectral approximation and its applications to differential equations on the half line, *J. Comput. Math.* **18**, 95–112.
11. Guo, Benyu and Shen, Jie (2000), Laguerre–Galerkin method for nonlinear partial differential equations on a semiinfinite interval, *Numer. Math.* **86**, 635–654.
12. Maday, Y., Pernaud-Thomas, B., and Vandeven, H. (1985). Reappraisal of Laguerre type spectral methods, *La Recherche Aerospatiale* **6**, 13–35.
13. Shen, Jie (1994). Efficient spectral-Galerkin method I. direct solvers for second- and fourth-order equations by using Legendre polynomials, *SIAM J. Sci. Comput.* **15**, 1489–1505.
14. Shen, Jie (1996). In *Efficient Chebyshev–Legendre Galerkin Methods for Elliptic Problems*, Ilin, A. V., and Scott, R. (eds.), *Proceedings of ICOSAHOM'95*, Houston J. Math., pp. 233–240.
15. Szegö, G. (1975). *Orthogonal Polynomials*, 4th ed., Vol. 23, AMS Coll. Publ.