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NEW SAV-PRESSURE CORRECTION METHODS FOR THE NAVIER-STOKES EQUATIONS: STABILITY AND ERROR ANALYSIS

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ABSTRACT. We construct new first- and second-order pressure correction schemes using the scalar auxiliary variable approach for the Navier-Stokes equations. These schemes are linear, decoupled and only require solving a sequence of Poisson type equations at each time step. Furthermore, they are unconditionally energy stable. We also establish rigorous error estimates in the two dimensional case for the velocity and pressure approximation of the first-order scheme without any condition on the time step.

1. INTRODUCTION

We consider numerical approximation of the time-dependent incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad in\ \Omega\times J, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad in\ \Omega\times J, \tag{1b}$$

$$\mathbf{u} = \mathbf{0} \quad on\ \partial\Omega\times J, \tag{1c}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a sufficiently smooth boundary $\partial\Omega$, $J = (0, T]$, $(\mathbf{u}, p)$ represent the unknown velocity and pressure, $\mathbf{f}$ is an external body force, $\nu > 0$ is the viscosity coefficient and $\mathbf{n}$ is the unit outward normal of the domain Ω.

The above system is one of the most fundamental systems in mathematical and physical science. Its numerical approximations play an eminent role in many branches of science and engineering, and an enormous amount of work has been devoted to the design and analysis of numerical schemes for its approximation; see, for instance, [8, 9, 15, 36] and the references therein.

Two of the main difficulties in numerically solving Navier-Stokes equations are: (i) the coupling of velocity and pressure by the incompressible condition $\nabla\cdot\mathbf{u} = 0$; and (ii) the treatment of nonlinear term. There are essentially two classes of numerical approaches to deal with the incompressible constraint: the coupled approach

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and the decoupled approach. The coupled approach requires solving a saddle point problem at each time step so it could be computationally expensive for dynamical simulations although many efficient solution techniques are available [3, 6, 8]. The decoupled approach, originated from the so called projection method [5, 34], leads to a sequence of Poisson type equations to solve at each time step, assuming that the nonlinear term is treated explicitly; hence it can be extremely efficient, particularly for dynamical simulations using finite difference or spectral methods. An enormous amount of work (cf., for instance, [10, 11, 14, 29, 38–40]) has been devoted to develop various projection type schemes. We refer to [13] for an overview of the decoupled approach before 2006. Some more recent work can be found in [12, 19, 20, 26, 27] and the references therein. For examples, Rebholz and Xiao [27] constructed a new alternative of Yosida algebraic splitting methods by applying the usual or pressure-corrected Yosida splitting techniques to the Navier-Stokes equations, see also [26], Guermond and Minev [12] proposed a high-order time stepping technique based on an artificial compressibility perturbation by using a Taylor series technique.

From a computational point of view, it is desirable to be able to treat the nonlinear term explicitly so that one only needs to solve simple linear equations with constant coefficients at each time step. This is especially beneficial if a decoupled approach is used so one only needs to solve a sequence of Poisson type equations at each time step. However, such an explicit treatment usually leads to a stability constraint on the time step. To the best of the authors' knowledge, apart from the recently developed schemes [25] based on the scalar auxiliary variable (SAV) approach [32, 33], there were no schemes with explicit treatment of nonlinear term that were unconditionally energy diminishing, an important property satisfied by the exact solution of the Navier-Stokes equations. We mention however that it is possible to prove that the numerical solution of a semi-implicit scheme remains to be bounded (but not energy diminishing) assuming the time step is sufficiently small, but independent of spatial discretization size; see for instance [16, 40]. In a recent work [25], Dong et al. constructed the following scheme: Find ($\mathbf{u}^{n+1}$, p^{n+1}, q^{n+1}) by solving

$$\begin{cases} \frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + \frac{q^{n+1}}{\sqrt{E(\mathbf{u}^n)+C_0}}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n - \nu\Delta\mathbf{u}^{n+1} + \nabla p^{n+1} = 0, \quad \mathbf{u}^{n+1}|_{\partial\Omega} = 0; & (2) \\ \nabla \cdot \mathbf{u}^{n+1} = 0, & (3) \\ 2q^{n+1}\frac{q^{n+1}-q^n}{\Delta t} = (\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + \frac{q^{n+1}}{\sqrt{E(\mathbf{u}^n)+C_0}}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n, \mathbf{u}^{n+1}), & (4) \end{cases}$$

where $E(\mathbf{u}) = \int_\Omega \frac{1}{2}|\mathbf{u}|^2$ is the total energy. It is shown in [25] that the above scheme satisfies the following property:

$$|q^{n+1}|^2 - |q^n|^2 \leq -\nu\|\nabla\mathbf{u}^{n+1}\|^2_{L^2(\Omega)}, \quad \forall n \geq 0. \tag{5}$$

Since q^n is an approximation of the energy $E(\mathbf{u}(t^n))$, the above scheme is unconditionally energy stable with a modified energy. It can be shown that the above scheme reduces to two generalized Stokes equations (with constant coefficient) plus a nonlinear algebraic equation for the auxiliary variable q^{n+1} at each time step. So the scheme is essentially as efficient as the usual semi-implicit scheme without the auxiliary variable. Moreover, one can also adopt a pressure-correction strategy so that the two generalized Stokes equations at each time step can be replaced by a sequence of Poisson-type equations. Ample numerical results presented in [25] shown that the above scheme is more efficient and robust than the usual semi-implicit schemes. However, there are also some theoretical and practical issues: (i) It only provides a bound for the scalar sequence $\{q^n\}$ which is intended as an approximation of the energy $E(\mathbf{u})$ but with no direct relation in the discrete case. (ii) The scheme requires solving a nonlinear algebraic equation. Hence, it is very difficult to show that the nonlinear algebraic equation always has a real positive solution and to derive an error estimate based just on (5).

The main purpose of this paper is to construct new SAV schemes for the Navier-Stokes equations and to carry out a rigorous error analysis. Our main contributions are:

- We construct new SAV schemes with first-order pressure-correction and second-order rotational pressure-correction. The new scheme enjoys the following additional advantages: (i) it is purely linear so it does not require solving nonlinear algebraic equation; (ii) it provides better stability: instead of (5), our first-order scheme satisfies

$$(\|\mathbf{u}^{n+1}\|^2_{L^2(\Omega)} + |q^{n+1}|^2) - (\|\mathbf{u}^n\|^2_{L^2(\Omega)} + |q^n|^2) \le -2\nu\|\nabla\mathbf{u}^{n+1}\|^2_{L^2(\Omega)}, \quad \forall n \ge 0,$$

  where the extra term $\|\mathbf{u}^{n+1}\|^2_{L^2(\Omega)}$ is essential to carry out an error analysis; (iii) it is coupled with a pressure-correction strategy [10,38] so only Poisson-type equations need to be solved at each time step.
- We prove that our new second-order scheme based on the second-order rotational pressure-correction is unconditionally energy stable. To the best of our knowledge, these are the first purely linear schemes for Navier-Stokes equations with explicit treatment of nonlinear terms with proven unconditional energy stability.
- We carry out a rigorous error analysis for our first-order scheme and derive optimal error estimates in the two-dimensional case for the velocity and pressure without any restriction on the time step.

The paper is organized as follows. In Section 2, we provide some preliminaries. In Section 3, we present first- and second-order pressure correction projection schemes based on the SAV approach, and describe the solution procedure. In Section 4, we derive the unconditional energy stability for both first- and second-order schemes. In Section 5, we carry out a rigorous error analysis to establish for the first-order SAV pressure-correction scheme. Numerical experiments are presented in Section 6 to validate our theoretical results.

2. Preliminaries

We describe below some notations and results which will be frequently used in this paper.

Throughout the paper, we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances.

Let Ω be an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$; we will use the standard notations $L^2(\Omega)$, $H^k(\Omega)$ and $H^k_0(\Omega)$ to denote the usual Sobolev spaces over Ω. The norm corresponding to $H^k(\Omega)$ will be denoted simply by $\|\cdot\|_k$. In particular, we use $\|\cdot\|$ to denote the norm in $L^2(\Omega)$. Besides, $(\cdot,\cdot)$ is used to denote the inner product in $L^2(\Omega)$. The vector functions and vector spaces will be indicated by boldface type.

We define

$$\mathbf{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0\}, \quad \mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \operatorname{div} \mathbf{v} = 0\},$$

and the Stokes operator

$$A\mathbf{u} = -P_H \Delta \mathbf{u}, \quad \forall\ \mathbf{u} \in D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V},$$

where P_H is the orthogonal projector in $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$ and the Stokes operator A is an unbounded positive self-adjoint closed operator in $\mathbf{H}$ with domain $D(A)$.

Let us recall the following inequalities which will be used in the sequel [17, 36]:

$$\|\nabla \mathbf{v}\| \le c_1 \|A^{\frac{1}{2}} \mathbf{v}\|, \quad \|\Delta \mathbf{v}\| \le c_1 \|A\mathbf{v}\|, \quad \forall \mathbf{v} \in D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}. \tag{6}$$

We then derive from the above and Poincaré inequality that

$$\|\mathbf{v}\| \le c_1 \|\nabla \mathbf{v}\|, \ \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad \|\nabla \mathbf{v}\| \le c_1 \|A\mathbf{v}\|, \quad \forall \mathbf{v} \in D(A), \tag{7}$$

where c_1 is a positive constant depending only on Ω.

Next we define the trilinear form $b(\cdot,\cdot,\cdot)$ by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} d\mathbf{x}.$$

We can easily obtain that the trilinear form $b(\cdot,\cdot,\cdot)$ is a skew-symmetric with respect to its last two arguments, i.e.,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad \forall\ \mathbf{u} \in \mathbf{H}, \quad \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega), \tag{8}$$

and

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \forall\ \mathbf{u} \in \mathbf{H}, \quad \mathbf{v} \in \mathbf{H}_0^1(\Omega). \tag{9}$$

By using a combination of integration by parts, Holder's inequality, and Sobolev inequalities [29, 35], we have that for $d \le 4$,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le \begin{cases} c_2 \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1, \\ c_2 \|\mathbf{u}\|_2 \|\mathbf{v}\| \|\mathbf{w}\|_1, \\ c_2 \|\mathbf{u}\|_2 \|\mathbf{v}\|_1 \|\mathbf{w}\|, \\ c_2 \|\mathbf{u}\|_1 \|\mathbf{v}\|_2 \|\mathbf{w}\|, \\ c_2 \|\mathbf{u}\| \|\mathbf{v}\|_2 \|\mathbf{w}\|_1, \end{cases} \tag{10}$$

and that for $d \le 2$,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le c_2 \|\mathbf{u}\|_1^{1/2} \|\mathbf{u}\|^{1/2} \|\mathbf{v}\|_1^{1/2} \|\mathbf{v}\|^{1/2} \|\mathbf{w}\|_1, \tag{11}$$

where c_2 is a positive constant depending only on Ω.

We will frequently use the following discrete version of the Gronwall lemma [16, 28]:

Lemma 2.1. *Let a_k, b_k, c_k, d_k, γ_k, Δt_k be nonnegative real numbers such that*

$$a_{k+1} - a_k + b_{k+1} \Delta t_{k+1} + c_{k+1} \Delta t_{k+1} - c_k \Delta t_k \le a_k d_k \Delta t_k + \gamma_{k+1} \Delta t_{k+1} \tag{12}$$

for all $0 \le k \le m$. Then

$$a_{m+1} + \sum_{k=0}^{m+1} b_k \Delta t_k \le \exp\left(\sum_{k=0}^{m} d_k \Delta t_k\right) \{a_0 + (b_0 + c_0)\Delta t_0 + \sum_{k=1}^{m+1} \gamma_k \Delta t_k\}. \tag{13}$$

3. The pressure-correction schemes based on the SAV approach

In this section, we construct first- and second-order pressure-correction schemes based on the SAV approach for the Navier-Stokes equations.

Set

$$\Delta t = T/N,\ t^n = n\Delta t,\ d_t g^{n+1} = \frac{g^{n+1} - g^n}{\Delta t},\ \text{for } n \le N,$$

and introduce a scalar function

$$q(t) = \exp(-\frac{t}{T}). \tag{14}$$

This function will serve as the scalar auxiliary variable (SAV). Then, we rewrite the governing system into the following equivalent form:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \frac{q(t)}{\exp(-\frac{t}{T})}(\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & (15) \\ \frac{dq}{dt} = -\frac{1}{T}q + \frac{1}{\exp(-\frac{t}{T})}\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}d\mathbf{x}, & (16) \\ \nabla\cdot\mathbf{u} = 0. & (17) \end{cases}$$

Note that the last term in (16) is zero thanks to (9). This term is added to balance the nonlinear term in (15) in the discretized case. It is clear that the above system is equivalent to the original system.

Remark 3.1. Note that in the case of inhomogeneous Dirichlet boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{g}$, (16) should be replaced by

$$\frac{dq}{dt} = -\frac{1}{T}q + \frac{1}{\exp(-\frac{t}{T})}(\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}d\mathbf{x} - \int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{g})\cdot\frac{1}{2}|\mathbf{g}|^2 ds). \tag{18}$$

Next we construct below linear, decoupled, first-order and second-order pressure-correction schemes for the above system (15)-(17).

Scheme I (first-order accuracy). The first-order semi-discrete version of the pressure-correction method can be written as follows: Find $(\tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, q^{n+1})$ by solving

$$\frac{\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\Delta t} + \frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n - \nu\Delta\tilde{\mathbf{u}}^{n+1} + \nabla p^n = \mathbf{f}^{n+1}, \qquad \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = 0; \tag{19}$$

$$\frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1}}{\Delta t} + \nabla(p^{n+1} - p^n) = 0; \tag{20}$$

$$\nabla\cdot\mathbf{u}^{n+1} = 0, \quad \mathbf{u}^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0; \tag{21}$$

$$\frac{q^{n+1} - q^n}{\Delta t} = -\frac{1}{T}q^{n+1} + \frac{1}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}). \tag{22}$$

We now describe how to solve the semi-discrete-in-time scheme (19)-(22) efficiently.

We denote $S^{n+1} = \exp(\frac{t^{n+1}}{T})q^{n+1}$ and set

$$\tilde{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}_1^{n+1} + S^{n+1}\tilde{\mathbf{u}}_2^{n+1}, \tag{23}$$

in (19); we can determine $\tilde{\mathbf{u}}_i^{n+1}$ $(i=1,2)$ from

$$\frac{\tilde{\mathbf{u}}_1^{n+1}-\mathbf{u}^n}{\Delta t}=\nu\Delta\tilde{\mathbf{u}}_1^{n+1}-\nabla p^n+\mathbf{f}^{n+1},\quad \tilde{\mathbf{u}}_1^{n+1}|_{\partial\Omega}=0; \tag{24}$$

$$\frac{\tilde{\mathbf{u}}_2^{n+1}}{\Delta t}+(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n=\nu\Delta\tilde{\mathbf{u}}_2^{n+1},\quad \tilde{\mathbf{u}}_2^{n+1}|_{\partial\Omega}=0. \tag{25}$$

Then, setting

$$\begin{cases} \mathbf{u}^{n+1}=\mathbf{u}_1^{n+1}+S^{n+1}\mathbf{u}_2^{n+1}, & (26)\\ p^{n+1}=p_1^{n+1}+S^{n+1}p_2^{n+1} & (27)\end{cases}$$

in (20)-(21), Then we can determine $\mathbf{u}_i^{n+1}$, p_i^{n+1} $(i=1,2)$ from

$$\begin{cases} \frac{\mathbf{u}_1^{n+1}-\tilde{\mathbf{u}}_1^{n+1}}{\Delta t}+\nabla(p_1^{n+1}-p^n)=0, & (28)\\ \frac{\mathbf{u}_2^{n+1}-\tilde{\mathbf{u}}_2^{n+1}}{\Delta t}+\nabla p_2^{n+1}=0, & (29)\\ \nabla\cdot\mathbf{u}_i^{n+1}=0,\quad \mathbf{u}_i^{n+1}\cdot\mathbf{n}|_{\partial\Omega}=0. & (30)\end{cases}$$

Once $\tilde{\mathbf{u}}_i^{n+1}$, $\mathbf{u}_i^{n+1}$, p_i^{n+1} $(i=1,2)$ are known, we can plug (23) and $S^{n+1}=\exp(\frac{t^{n+1}}{T})q^{n+1}$ in (22) to determine explicitly S^{n+1} from:

$$\begin{aligned}&\left(\frac{T+\Delta t}{T\Delta t}-\exp(\frac{2t^{n+1}}{T})((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{\mathbf{u}}_2^{n+1})\right)\exp(-\frac{t^{n+1}}{T})S^{n+1}\\ &\quad=\exp(\frac{t^{n+1}}{T})((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{\mathbf{u}}_1^{n+1})+\frac{1}{\Delta t}q^n.\end{aligned} \tag{31}$$

We observe that it is not clear that the first term

$$\left(\frac{T+\Delta t}{T\Delta t}-\exp(\frac{2t^{n+1}}{T})((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{\mathbf{u}}_2^{n+1})\right)$$

is non zero from the above. However, we can replace the term $((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{\mathbf{u}}_2^{n+1})$ in the above by taking the inner product of (25) with $\tilde{\mathbf{u}}_2^{n+1}$ to obtain

$$\begin{aligned}&\left(\frac{T+\Delta t}{T\Delta t}+\exp(\frac{2t^{n+1}}{T})(\frac{\|\tilde{\mathbf{u}}_2^{n+1}\|^2}{\Delta t}+\nu\|\nabla\tilde{\mathbf{u}}_2^{n+1}\|^2)\right)\exp(-\frac{t^{n+1}}{T})S^{n+1}\\ &\quad=\exp(\frac{t^{n+1}}{T})((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{\mathbf{u}}_1^{n+1})+\frac{1}{\Delta t}q^n.\end{aligned} \tag{32}$$

Hence, we can unique determine S^{n+1} from the above. Finally, we can obtain $\mathbf{u}^{n+1}$ and p^{n+1} from (26) to (27).

In summary, at each time step, we only need to solve two Poisson-type equations (24)-(25) and (28)–(30) which can be solved as two Poisson equations. Hence, the scheme is very efficient.

Scheme II (second-order accuracy). The second-order semi-discrete version of the rotational pressure-correction method [10] can be written as follows: Find $(\tilde{\mathbf{u}}^{n+1},\mathbf{u}^{n+1},p^{n+1},q^{n+1})$ by olving

(33)

$$\frac{3\tilde{\mathbf{u}}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\Delta t}+\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\bar{\mathbf{u}}^n\cdot\nabla)\bar{\mathbf{u}}^n-\nu\Delta\tilde{\mathbf{u}}^{n+1}+\nabla p^n=\mathbf{f}^{n+1},\tilde{\mathbf{u}}^{n+1}|_{\partial\Omega}=0;$$

$$\frac{3\mathbf{u}^{n+1}-3\tilde{\mathbf{u}}^{n+1}}{2\Delta t}+\nabla(p^{n+1}-p^n+\nu\nabla\cdot\tilde{\mathbf{u}}^{n+1})=0; \tag{34}$$

$$\nabla\cdot\mathbf{u}^{n+1}=0,\quad \mathbf{u}^{n+1}\cdot\mathbf{n}|_{\partial\Omega}=0; \tag{35}$$

$$\frac{3q^{n+1}-4q^n+q^{n-1}}{2\Delta t}=-\frac{1}{T}q^{n+1}+\frac{1}{\exp(-\frac{t^{n+1}}{T})}((\bar{\mathbf{u}}^n\cdot\nabla)\bar{\mathbf{u}}^n,\tilde{\mathbf{u}}^{n+1}), \tag{36}$$

where $\bar{\mathbf{u}}^n=2\mathbf{u}^n-\mathbf{u}^{n-1}$. For $n=0$, we can compute $(\tilde{\mathbf{u}}^1,\ \mathbf{u}^1,\ p^1,\ q^1)$ by the first-order scheme described above.

Implementation of the second-order scheme (33)-(36) is essentially the same as that of the first-order scheme (19)-(22).

4. Energy stability

In this section, we will demonstrate that the first- and second-order pressure-correction schemes (19)-(22) and (33)-(36) are unconditionally energy stable.

Theorem 4.1. *In the absence of the external force* $\mathbf{f}$*, the scheme* (19)-(22) *is unconditionally stable in the sense that*

$$E^{n+1}-E^n\le -2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2,\quad \forall\Delta t,\ n\ge 0,$$

where

$$E^{n+1}=\|\mathbf{u}^{n+1}\|^2+|q^{n+1}|^2+(\Delta t)^2\|\nabla p^{n+1}\|^2.$$

Proof. Taking the inner product of (19) with $\Delta t\tilde{\mathbf{u}}^{n+1}$ and using the identity

$$(a-b,a)=\frac{1}{2}(|a|^2-|b|^2+|a-b|^2), \tag{37}$$

we have

$$\begin{aligned}&\frac{\|\tilde{\mathbf{u}}^{n+1}\|^2-\|\mathbf{u}^n\|^2}{2}+\frac{\|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\|^2}{2}+\Delta t\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1})\\&=-\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2-\Delta t(\nabla p^n,\tilde{\mathbf{u}}^{n+1}).\end{aligned} \tag{38}$$

Recalling (20), we have

$$\mathbf{u}^{n+1}+\Delta t\nabla p^{n+1}=\tilde{\mathbf{u}}^{n+1}+\Delta t\nabla p^n. \tag{39}$$

Taking the inner product of (39) with itself on both sides and noticing $(\nabla p^{n+1},\mathbf{u}^{n+1})=-(p^{n+1},\nabla\cdot\mathbf{u}^{n+1})=0$, we have

$$\|\mathbf{u}^{n+1}\|^2+(\Delta t)^2\|\nabla p^{n+1}\|^2=\|\tilde{\mathbf{u}}^{n+1}\|^2+2\Delta t(\nabla p^n,\tilde{\mathbf{u}}^{n+1})+(\Delta t)^2\|\nabla p^n\|^2. \tag{40}$$

Combining (38) with (40) leads to

$$\begin{aligned}&\frac{\|\mathbf{u}^{n+1}\|^2-\|\mathbf{u}^n\|^2}{2}+\frac{\|\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n\|^2}{2}+\frac{(\Delta t)^2}{2}\|\nabla p^{n+1}\|^2\\&+\Delta t\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1})=\frac{(\Delta t)^2}{2}\|\nabla p^n\|^2-\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2.\end{aligned} \tag{41}$$

Multiplying (22) by $q^{n+1}\Delta t$ and using the above equation, we have

$$\begin{aligned}&\frac{1}{2}|q^{n+1}|^2-\frac{1}{2}|q^n|^2+\frac{1}{2}|q^{n+1}-q^n|^2\\&=-\frac{1}{T}\Delta t|q^{n+1}|^2+\Delta t\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{\mathbf{u}}^{n+1}).\end{aligned} \tag{42}$$

Then summing up (41) with (42) results in

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + |q^{n+1}|^2 - |q^n|^2 + \frac{2}{T}\Delta t|q^{n+1}|^2 + (\Delta t)^2\|\nabla p^{n+1}\|^2\\ &\quad - (\Delta t)^2\|\nabla p^n\|^2 + |q^{n+1} - q^n|^2 + \|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n\|^2 \leq -2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2,\end{aligned}$$

which implies the desired result. □

The energy stability for any rotational pressure-correction schemes is much more involved [10], particularly in the nonlinear case. Previously, the energy stability of second-order rotational pressure-correction schemes is only proved for the time dependent Stokes equations [4,10], and only very recently, an energy stability result is proved for the first-order rotational pressure-correction scheme for the Navier-Stokes equations in [4].

Theorem 4.2. *In the absence of the external force* $\mathbf{f}$*, the scheme* (33)-(36) *is unconditionally stable in the sense that*

$$E^{n+1} - E^n \leq -2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2, \quad \forall\Delta t,\ n \geq 0,$$

where

$$\begin{aligned}E^{n+1} =&\|\mathbf{u}^{n+1}\|^2 + \|2\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla(p^{n+1} + g^{n+1})\|^2\\ &+ 2\nu^{-1}\Delta t\|g^{n+1}\|^2 + |q^{n+1}|^2 + |2q^{n+1} - q^n|^2,\end{aligned}$$

where $\{g^{n+1}\}$ *is recursively defined by*

$$g^0 = 0, \quad g^{n+1} = \nu\nabla\cdot\tilde{\mathbf{u}}^{n+1} + g^n,\ n \geq 0. \tag{43}$$

Proof. Taking the inner product of (33) with $4\Delta t\tilde{\mathbf{u}}^{n+1}$ leads to

$$\begin{aligned}&2(3\tilde{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1}) + 4\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2\\ &= -4\Delta t\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\bar{\mathbf{u}}^n\cdot\nabla)\bar{\mathbf{u}}^n, \tilde{\mathbf{u}}^{n+1}) - 4\Delta t(\nabla p^n, \tilde{\mathbf{u}}^{n+1}).\end{aligned} \tag{44}$$

Using (34) and the identity

$$2(3a - 4b + c, a) = |a|^2 + |2a - b|^2 - |b|^2 - |2b - c|^2 + |a - 2b + c|^2, \tag{45}$$

we have

(46)

$$\begin{aligned}&2(3\tilde{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1}) = 2\left(3(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}) + 3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1}\right)\\ &\quad = 6(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}, \tilde{\mathbf{u}}^{n+1}) + 2(3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \mathbf{u}^{n+1})\\ &\qquad + 2(3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1})\\ &\quad = 3(\|\tilde{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{u}^{n+1}\|^2 + \|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}\|^2) + \|\mathbf{u}^{n+1}\|^2 + \|2\mathbf{u}^{n+1} - \mathbf{u}^n\|^2\\ &\qquad - \|\mathbf{u}^n\|^2 - \|2\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 + \|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2.\end{aligned}$$

Setting $H^{n+1} = p^{n+1} + g^{n+1}$, we can recast (34) as

$$\sqrt{3}\mathbf{u}^{n+1} + \frac{2}{\sqrt{3}}\Delta t\nabla H^{n+1} = \sqrt{3}\tilde{\mathbf{u}}^{n+1} + \frac{2}{\sqrt{3}}\Delta t\nabla H^n. \tag{47}$$

Taking the inner product of (47) with itself on both sides, we have

$$
\begin{aligned}
&3\|\mathbf{u}^{n+1}\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla H^{n+1}\|^2 \\
=&3\|\tilde{\mathbf{u}}^{n+1}\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla H^{n}\|^2 + 4\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla p^n) + 4\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla g^n).
\end{aligned} \tag{48}
$$

Thanks to (43), we have

$$
\begin{aligned}
&4\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla g^n) = -4\nu^{-1}\Delta t(g^{n+1} - g^n, g^n) \\
=&2\nu^{-1}\Delta t(\|g^n\|^2 - \|g^{n+1}\|^2 + \|g^{n+1} - g^n\|^2) \\
=&2\nu^{-1}\Delta t\|g^n\|^2 - 2\nu^{-1}\Delta t\|g^{n+1}\|^2 + 2\nu\Delta t\|\nabla\cdot\tilde{\mathbf{u}}^{n+1}\|^2.
\end{aligned} \tag{49}
$$

Using the identity

$$
\|\nabla\times\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2 = \|\nabla\mathbf{v}\|^2, \quad \forall\ \mathbf{v}\in\mathbf{H}_0^1(\Omega), \tag{50}
$$

we have

$$
\begin{aligned}
4\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla g^n) =& 2\nu^{-1}\Delta t\|g^n\|^2 - 2\nu^{-1}\Delta t\|g^{n+1}\|^2 \\
&+ 2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 - 2\nu\Delta t\|\nabla\times\mathbf{u}^{n+1}\|^2.
\end{aligned} \tag{51}
$$

Then combining (44) with (45)-(51) results in

$$
\begin{aligned}
&\|\mathbf{u}^{n+1}\|^2 + \|2\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla H^{n+1}\|^2 + 2\nu^{-1}\Delta t\|g^{n+1}\|^2 \\
&+ 3\|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}\|^2 + 2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 + 2\nu\Delta t\|\nabla\times\mathbf{u}^{n+1}\|^2 \\
\le&\|\mathbf{u}^n\|^2 + \|2\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla H^n\|^2 + 2\nu^{-1}\Delta t\|g^n\|^2 \\
&- 4\Delta t\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\bar{\mathbf{u}}^n\cdot\nabla)\bar{\mathbf{u}}^n, \tilde{\mathbf{u}}^{n+1}).
\end{aligned} \tag{52}
$$

Multiplying (36) by $4\Delta t q^{n+1}$ and using (45), we have

$$
\begin{aligned}
&|q^{n+1}|^2 + |2q^{n+1} - q^n|^2 - |q^n|^2 - |2q^n - q^{n-1}|^2 + |q^{n+1} - 2q^n + q^{n-1}|^2 \\
=&-\frac{4}{T}\Delta t|q^{n+1}|^2 + 4\Delta t\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\bar{\mathbf{u}}^n\cdot\nabla)\bar{\mathbf{u}}^n, \tilde{\mathbf{u}}^{n+1}).
\end{aligned} \tag{53}
$$

Then summing up (52) with (53) results in

$$
\begin{aligned}
&\|\mathbf{u}^{n+1}\|^2 + \|2\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla H^{n+1}\|^2 + 2\nu^{-1}\Delta t\|g^{n+1}\|^2 \\
&+ |q^{n+1}|^2 + |2q^{n+1} - q^n|^2 + |q^{n+1} - 2q^n + q^{n-1}|^2 + \frac{4}{T}\Delta t|q^{n+1}|^2 \\
&+ 3\|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}\|^2 + 2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 + 2\nu\Delta t\|\nabla\times\mathbf{u}^{n+1}\|^2 \\
\le&\|\mathbf{u}^n\|^2 + \|2\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla H^n\|^2 \\
&+ 2\nu^{-1}\Delta t\|g^n\|^2 + |q^n|^2 + |2q^n - q^{n-1}|^2,
\end{aligned} \tag{54}
$$

which implies the desired result. □

5. Error analysis

In this section, we carry out a rigorous error analysis for the first-order semi-discrete scheme (19)-(22). We shall only consider the two-dimensional case. Error estimates for the three dimensional case are still elusive.

There exists a large body of work devoted to the error analysis of various numerical schemes for the Navier-Stokes equations (1); we refer to, e.g., [2, 16, 18, 21, 37] for different schemes with coupled approach, and [11, 13, 29, 30, 39, 40] for different schemes with decoupled approach. On the other hand, for the SAV approach, some error analysis has been carried out for various gradient flows [1, 24, 31]. In a recent attempt [23], we considered a MAC discretization to a second-order version of the scheme (2)-(4) and proved corresponding error estimates. However, due to the difficulty associated with the nonlinear algebraic equation, we had to assume that there is a numerical solution satisfying $q^{n+1}/E(\mathbf{u}^n) \geq c_0 > 0$. Since our new scheme is purely linear, we shall prove optimal error estimates below without any assumption on the numerical solution.

Let $(\tilde{\mathbf{u}}^{n+1},\ \mathbf{u}^{n+1},\ p^{n+1},\ q^{n+1})$ be the solution of (19)-(22). Then we derive immediately from Theorem 4.1 that

$$\|\mathbf{u}^{m+1}\| \leq k_0, \quad |q^{m+1}| \leq k_1, \quad \forall\ 0 \leq m \leq N-1, \tag{55}$$

$$\Delta t \sum_{n=0}^{m} \|\tilde{\mathbf{u}}^{n+1}\|_1^2 \leq k_2, \quad \forall\ 0 \leq m \leq N-1, \tag{56}$$

where the constants k_i $(i = 0, 1, 2)$ are independent of Δt.

We set

$$\begin{cases} \tilde{e}_{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}^{n+1} - \mathbf{u}(t^{n+1}), & e_{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}(t^{n+1}), \\ e_p^{n+1} = p^{n+1} - p(t^{n+1}), & e_q^{n+1} = q^{n+1} - q(t^{n+1}). \end{cases}$$

5.1. **Error estimates for the velocity.** The main result of this section is stated in Theorem 5.1.

Theorem 5.1. *Assuming* $d = 2$ *and* $\mathbf{u} \in H^3(0,T;\mathbf{L}^2(\Omega)) \bigcap H^1(0,T;\mathbf{H}_0^2(\Omega)) \bigcap W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$, $p \in H^2(0,T;H^1(\Omega))$, *then for the first-order scheme* (19)-(22), *we have*

$$\begin{aligned} &\|e_{\mathbf{u}}^{m+1}\|^2 + \Delta t \sum_{n=0}^{m} \|\nabla \tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \sum_{n=0}^{m} \|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^{n}\|^2 + (\Delta t)^2 \|\nabla e_p^{m+1}\|^2 \\ &\quad + |e_q^{m+1}|^2 + \Delta t \sum_{n=0}^{m} |d_t e_q^{n+1}|^2 \leq C(\Delta t)^2, \quad \forall\ 0 \leq m \leq N-1, \end{aligned} \tag{57}$$

where C *is a positive constant independent of* Δt.

Proof. We shall follow the steps in the stability proof of Theorem 4.1.

Step 1. We start by establishing an error equation corresponding to (41). Let $\mathbf{R}_{\mathbf{u}}^{n+1}$ be the truncation error defined by

$$\mathbf{R}_{\mathbf{u}}^{n+1} = \frac{\partial \mathbf{u}(t^{n+1})}{\partial t} - \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (t^n - t) \frac{\partial^2 \mathbf{u}}{\partial t^2} dt. \tag{58}$$

Subtracting (15) at t^{n+1} from (19), we obtain

$$\begin{aligned}\frac{\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^{n}}{\Delta t}-\nu\Delta\tilde{e}_{\mathbf{u}}^{n+1}=&\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\\&-\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n}-\nabla(p^{n}-p(t^{n+1}))+\mathbf{R}_{\mathbf{u}}^{n+1}.\end{aligned}\tag{59}$$

We obtain from (20) that

$$\frac{e_{\mathbf{u}}^{n+1}-\tilde{e}_{\mathbf{u}}^{n+1}}{\Delta t}+\nabla(p^{n+1}-p^{n})=0.\tag{60}$$

Taking the inner product of (59) with $\tilde{e}_{\mathbf{u}}^{n+1}$, we obtain

$$\begin{aligned}&\frac{\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2-\|e_{\mathbf{u}}^{n}\|^2}{2\Delta t}+\frac{\|\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^{n}\|^2}{2\Delta t}+\nu\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2\\&=\left(\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})-\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n},\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&-(\nabla(p^{n}-p(t^{n+1})),\tilde{e}_{\mathbf{u}}^{n+1})+(\mathbf{R}_{\mathbf{u}}^{n+1},\tilde{e}_{\mathbf{u}}^{n+1}).\end{aligned}\tag{61}$$

Taking the inner product of (60) with $\frac{e_{\mathbf{u}}^{n+1}+\tilde{e}_{\mathbf{u}}^{n+1}}{2}$, we derive

$$\frac{\|e_{\mathbf{u}}^{n+1}\|^2-\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2}{2\Delta t}+\frac{1}{2}\left(\nabla(p^{n+1}-p^{n}),\tilde{e}_{\mathbf{u}}^{n+1}\right)=0.\tag{62}$$

Adding (61) and (62), we have

$$\begin{aligned}&\frac{\|e_{\mathbf{u}}^{n+1}\|^2-\|e_{\mathbf{u}}^{n}\|^2}{2\Delta t}+\frac{\|\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^{n}\|^2}{2\Delta t}+\nu\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2\\&=\left(\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})-\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n},\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&-\frac{1}{2}\left(\nabla(p^{n+1}+p^{n}-2p(t^{n+1})),\tilde{e}_{\mathbf{u}}^{n+1}\right)+(\mathbf{R}_{\mathbf{u}}^{n+1},\tilde{e}_{\mathbf{u}}^{n+1}).\end{aligned}\tag{63}$$

For the first term on the right hand side of (63), we have

$$\begin{aligned}&\left(\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})-\frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n},\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&=\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\left(((\mathbf{u}(t^{n+1})-\mathbf{u}^{n})\cdot\nabla)\mathbf{u}(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&+\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^{n}\cdot\nabla)(\mathbf{u}(t^{n+1})-\mathbf{u}^{n}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\&-\frac{e_{q}^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n},\tilde{e}_{\mathbf{u}}^{n+1}\right).\end{aligned}\tag{64}$$

Thanks to (55) and (10), the first term on the right hand side of (64) can be estimated by
(65)

$$
\begin{aligned}
&\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\left(((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla)\mathbf{u}(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\qquad\le c_2(1+c_1)\exp(1)\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|\|\mathbf{u}(t^{n+1})\|_2\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|\\
&\qquad\le\frac{\nu}{6}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+C\|\mathbf{u}(t^{n+1})\|_2^2\|e_{\mathbf{u}}^n\|^2+C\|\mathbf{u}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|^2dt.
\end{aligned}
$$

Using Cauchy-Schwarz inequality and recalling (55), the second term on the right hand side of (64) can be bounded using (10) and (11) by
(66)

$$
\begin{aligned}
&\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^n\cdot\nabla)(\mathbf{u}(t^{n+1})-\mathbf{u}^n),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad=\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^n\cdot\nabla)(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)),\tilde{e}_{\mathbf{u}}^{n+1}\right)-\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\left((e_{\mathbf{u}}^n\cdot\nabla)e_{\mathbf{u}}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\qquad-\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}(t^n)\cdot\nabla)e_{\mathbf{u}}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad\le c_2(1+c_1)\exp(1)\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|(\|\mathbf{u}^n\|\|\int_{t^n}^{t^{n+1}}\mathbf{u}_tdt\|_2+\|e_{\mathbf{u}}^n\|\|\mathbf{u}(t^n)\|_2)\\
&\qquad+c_2(1+c_1)\exp(1)\|e_{\mathbf{u}}^n\|^{1/2}\|e_{\mathbf{u}}^n\|_1^{1/2}\|e_{\mathbf{u}}^n\|^{1/2}\|e_{\mathbf{u}}^n\|_1^{1/2}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|\\
&\quad\le\frac{\nu}{6}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+C(\|\mathbf{u}(t^n)\|_2^2+\|e_{\mathbf{u}}^n\|_1^2)\|e_{\mathbf{u}}^n\|^2\\
&\qquad+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_2^2dt.
\end{aligned}
$$

Next we estimate the second term on the right hand side of (63). Recalling (60), we have
(67)

$$
\begin{aligned}
&-\frac{1}{2}\left(\nabla(p^{n+1}+p^n-2p(t^{n+1})),\tilde{e}_{\mathbf{u}}^{n+1}\right)=-\frac{1}{2}\left(\nabla(e_p^{n+1}+e_p^n-p(t^{n+1})+p(t^n)),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad=-\frac{1}{2}\big(\nabla(e_p^{n+1}+e_p^n-p(t^{n+1})+p(t^n)),\\
&\qquad\qquad e_{\mathbf{u}}^{n+1}+\Delta t(\nabla(e_p^{n+1}-e_p^n)+\nabla(p(t^{n+1})-p(t^n))\big)\\
&\quad=-\frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2-\|\nabla e_p^n\|^2)-\Delta t(\nabla(p(t^{n+1})-p(t^n)),\nabla e_p^n)\\
&\qquad+\frac{\Delta t}{2}\|\nabla(p(t^{n+1})-p(t^n))\|^2\\
&\quad\le-\frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2-\|\nabla e_p^n\|^2)+(\Delta t)^2\|\nabla e_p^n\|^2\\
&\qquad+C\left(\Delta t+(\Delta t)^2\right)\int_{t^n}^{t^{n+1}}\|\nabla p_t(t)\|^2dt.
\end{aligned}
$$

For the last term on the right hand side of (63), we have

$$(\mathbf{R}_{\mathbf{u}}^{n+1}, \tilde{e}_{\mathbf{u}}^{n+1}) \leq \frac{\nu}{6}\|\nabla \tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\Delta t \int_{t^n}^{t^{n+1}} \|\mathbf{u}_{tt}\|_{-1}^2 dt. \tag{68}$$

Finally, combining (63) with (64)-(68), we obtain

$$\begin{aligned}
&\frac{\|e_{\mathbf{u}}^{n+1}\|^2 - \|e_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\nu}{2}\|\nabla \tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2) \\
&\quad \leq -\frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^n \cdot \nabla)\mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}\right) + C(\|\mathbf{u}(t^n)\|_2^2 + \|e_{\mathbf{u}}^n\|_1^2)\|e_{\mathbf{u}}^n\|^2 \\
&\qquad + (\Delta t)^2\|\nabla e_p^n\|^2 + C\|\mathbf{u}(t^{n+1})\|_2^2 \Delta t \int_{t^n}^{t^{n+1}} \|\mathbf{u}_t\|^2 dt \\
&\qquad + C\Delta t \int_{t^n}^{t^{n+1}} \|\mathbf{u}_t\|_2^2 dt + C\Delta t \int_{t^n}^{t^{n+1}} \|\mathbf{u}_{tt}\|_{-1}^2 dt \\
&\qquad + C\left(\Delta t + (\Delta t)^2\right) \int_{t^n}^{t^{n+1}} \|\nabla p_t(t)\|^2 dt, \quad \forall\ 0 \leq m \leq N-1.
\end{aligned} \tag{69}$$

Step 2. Note that the first term on the right hand side cannot be easily bounded. As in the stability proof, we shall balance it with a term from the error equation for q corresponding to (42). We proceed as follows.

Subtracting (16) from (22) leads to

$$\begin{aligned}
&\frac{e_q^{n+1} - e_q^n}{\Delta t} + \frac{1}{T} e_q^{n+1} \\
&= \frac{1}{\exp(-\frac{t^{n+1}}{T})}\left(((\mathbf{u}^n \cdot \nabla)\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}) - ((\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}))\right) + \mathbf{R}_q^{n+1},
\end{aligned} \tag{70}$$

where

$$\mathbf{R}_q^{n+1} = \frac{dq(t^{n+1})}{dt} - \frac{q(t^{n+1}) - q(t^n)}{\Delta t} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (t^n - t)\frac{\partial^2 q}{\partial t^2} dt. \tag{71}$$

Multiplying both sides of (70) by e_q^{n+1} yields

$$\begin{aligned}
&\frac{|e_q^{n+1}|^2 - |e_q^n|^2}{2\Delta t} + \frac{|e_q^{n+1} - e_q^n|^2}{2\Delta t} + \frac{1}{T}|e_q^{n+1}|^2 \\
&= \frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n \cdot \nabla)\mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}) - \frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^n \cdot \nabla)(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), \mathbf{u}(t^{n+1})\right) \\
&\quad - \frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left(((\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \cdot \nabla)\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1})\right) + \mathbf{R}_q^{n+1} e_q^{n+1}.
\end{aligned} \tag{72}$$

Thanks to (10) and (56), the second term on the right hand side of (72) can be bounded by

$$
\begin{aligned}
&-\frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^n\cdot\nabla)(\mathbf{u}(t^{n+1})-\mathbf{u}^n),\mathbf{u}(t^{n+1})\right)\\
&\qquad\le c_2\exp(1)\|\mathbf{u}^n\|_1\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|_0\|\mathbf{u}(t^{n+1})\|_2|e_q^{n+1}|\\
&\qquad\le\frac{1}{4k_2}\|\mathbf{u}^n\|_1^2|e_q^{n+1}|^2+C\|e_{\mathbf{u}}^n\|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_0^2dt.
\end{aligned}
\tag{73}
$$

The third term on the right hand side of (72) can be bounded by

$$
\begin{aligned}
&-\frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left(((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla)\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1})\right)\\
&\qquad\le c_2\exp(1)\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|\|\mathbf{u}(t^{n+1})\|_1\|\mathbf{u}(t^{n+1})\|_2|e_q^{n+1}|\\
&\qquad\le C\|e_{\mathbf{u}}^n\|^2+\frac{1}{4T}|e_q^{n+1}|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|^2dt.
\end{aligned}
\tag{74}
$$

For the last term on the right hand side of (72), we have

$$
\mathbf{R}_q^{n+1}e_q^{n+1}\le\frac{1}{4T}|e_q^{n+1}|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|q_{tt}\|^2dt.
\tag{75}
$$

Combining (72) with (73)-(75) results in

$$
\begin{aligned}
&\frac{|e_q^{n+1}|^2-|e_q^n|^2}{2\Delta t}+\frac{|e_q^{n+1}-e_q^n|^2}{2\Delta t}+\frac{1}{2T}|e_q^{n+1}|^2\\
&\qquad\le\frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1})+\frac{1}{4k_2}\|\mathbf{u}^n\|_1^2|e_q^{n+1}|^2+C\|e_{\mathbf{u}}^n\|^2\\
&\qquad\quad+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_0^2dt+C\Delta t\int_{t^n}^{t^{n+1}}\|q_{tt}\|^2dt.
\end{aligned}
\tag{76}
$$

Note that the first term on the right hand side above is what we need to balance the first term on the right hand side of (69).

Step 3. Summing up (76) with (69) leads to

$$
\begin{aligned}
&\frac{|e_q^{n+1}|^2-|e_q^n|^2}{2\Delta t}+\frac{|e_q^{n+1}-e_q^n|^2}{2\Delta t}+\frac{1}{2T}|e_q^{n+1}|^2+\frac{\|e_{\mathbf{u}}^{n+1}\|^2-\|e_{\mathbf{u}}^n\|^2}{2\Delta t}\\
&\quad+\frac{\|\tilde{e}_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^n\|^2}{2\Delta t}+\frac{\nu}{2}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+\frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2-\|\nabla e_p^n\|^2)\\
&\le\frac{1}{4k_2}\|\mathbf{u}^n\|_1^2|e_q^{n+1}|^2+C(\|\mathbf{u}(t^n)\|_2^2+\|e_{\mathbf{u}}^n\|_1^2)\|e_{\mathbf{u}}^n\|^2\\
&\quad+(\Delta t)^2\|\nabla e_p^n\|^2+C\|\mathbf{u}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|^2dt\\
&\quad+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_2^2dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_{tt}\|_{-1}^2dt\\
&\quad+C\left(\Delta t+(\Delta t)^2\right)\int_{t^n}^{t^{n+1}}\|\nabla p_t(t)\|^2dt+C\Delta t\int_{t^n}^{t^{n+1}}\|q_{tt}\|^2dt.
\end{aligned}
\tag{77}
$$

Multiplying (77) by $2\Delta t$ and summing over n, $n = 0, 2, \ldots, m^*$, where m^* is the time step at which $|e_q^{m^*+1}|$ achieves its maximum value, we can obtain

(78)

$$
\begin{aligned}
&|e_q^{m^*+1}|^2 + \frac{\Delta t}{T}\sum_{n=0}^{m^*}|e_q^{n+1}|^2 + \|e_{\mathbf{u}}^{m^*+1}\|^2 + \nu\Delta t\sum_{n=0}^{m^*}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + (\Delta t)^2\|\nabla e_p^{m^*+1}\|^2 \\
&\quad\le \frac{1}{2k_2}|e_q^{m^*+1}|^2\Delta t\sum_{n=0}^{m^*}\|\mathbf{u}^n\|_1^2 + C\Delta t\sum_{n=0}^{m^*}(\|\mathbf{u}(t^n)\|_2^2 + \|e_{\mathbf{u}}^n\|_1^2)\|e_{\mathbf{u}}^n\|^2 \\
&\quad + (\Delta t)^3\sum_{n=0}^{m^*}\|\nabla e_p^n\|^2 + C\|\mathbf{u}(t^{n+1})\|_2^2(\Delta t)^2\int_{t^0}^{t^{m^*+1}}\|\mathbf{u}_t\|^2 dt \\
&\quad + C(\Delta t)^2\int_{t^0}^{t^{m^*+1}}\|\mathbf{u}_t\|_2^2 dt + C(\Delta t)^2\int_{t^0}^{t^{m^*+1}}\|\mathbf{u}_{tt}\|_{-1}^2 dt \\
&\quad + C\left((\Delta t)^2 + (\Delta t)^3\right)\int_{t^0}^{t^{m^*+1}}\|\nabla p_t(t)\|^2 dt + C(\Delta t)^2\int_{t^0}^{t^{m^*+1}}\|q_{tt}\|^2 dt.
\end{aligned}
$$

Thanks to (56), the first term on the right hand side is bounded by $\frac{1}{2}|e_q^{m^*+1}|^2$. Then, applying the discrete Gronwall lemma 2.1, we obtain

$$
\begin{aligned}
&|e_q^{m^*+1}|^2 + \Delta t\sum_{n=0}^{m^*}|e_q^{n+1}|^2 + \|e_{\mathbf{u}}^{m^*+1}\|^2 + \nu\Delta t\sum_{n=0}^{m^*}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + (\Delta t)^2\|\nabla e_p^{m^*+1}\|^2 \\
&\quad\le C(\|\mathbf{u}\|^2_{H^1(0,T;H^2(\Omega))} + \|\mathbf{u}\|^2_{H^2(0,T;H^{-1}\Omega))} + \|q\|^2_{H^2(0,T)})(\Delta t)^2.
\end{aligned}
$$

Since $|e_q^{m^*+1}| = \max_{0\le m\le N-1}|e_q^{m+1}|$, the above also implies

$$
\begin{aligned}
&|e_q^{m+1}|^2 + \Delta t\sum_{n=0}^{m}|e_q^{n+1}|^2 + \|e_{\mathbf{u}}^{m+1}\|^2 + \nu\Delta t\sum_{n=0}^{m}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 \\
&\quad\le C(\|\mathbf{u}\|^2_{H^1(0,T;H^2(\Omega))} + \|\mathbf{u}\|^2_{H^2(0,T;H^{-1}\Omega))} + \|q\|^2_{H^2(0,T)})(\Delta t)^2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\forall 0\le m\le N-1.
\end{aligned}
\tag{79}
$$

Next multiplying both sides of (70) with $d_t e_q^{n+1}$ leads to

(80)

$$
\begin{aligned}
&|d_t e_q^{n+1}|^2 + \frac{|e_q^{n+1}|^2 - |e_q^n|^2}{2T\Delta t} + \frac{|e_q^{n+1} - e_q^n|^2}{2T\Delta t} \\
&\quad= \frac{d_t e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}) \\
&\quad - \frac{d_t e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^n\cdot\nabla)(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), \mathbf{u}(t^{n+1})\right) \\
&\quad - \frac{d_t e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left(((\mathbf{u}(t^{n+1}) - \mathbf{u}^n)\cdot\nabla)\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1})\right) + \mathbf{R}_q^{n+1} d_t e_q^{n+1}.
\end{aligned}
$$

Thanks to (79), we have

$$
\Delta t\sum_{n=0}^{m}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 \le C(\Delta t)^2, \tag{81}
$$

which implies that

$$\|\mathbf{u}^{n+1}\|_1 \le C\|\tilde{\mathbf{u}}^{n+1}\|_1 \le C\left((\Delta t)^{1/2} + \|\nabla \mathbf{u}(t^{n+1})\|\right). \tag{82}$$

The above inequality holds thanks to the fact that [36]

$$\|\mathbf{u}^{n+1}\|_{\mathbf{H}^1(\Omega)} = \|P_H \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)} \le c(\Omega)\|\tilde{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}.$$

Then, thanks to (11), the first term on the right hand side of (80) can be estimated by

$$\begin{aligned} &\frac{d_t e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}) \\ &\le (1+c_1)c_2\|\mathbf{u}^n\|^{1/2}\|\mathbf{u}^n\|_1^{1/2}\|\mathbf{u}^n\|^{1/2}\|\mathbf{u}^n\|_1^{1/2}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\||d_t e_q^{n+1}| \\ &\le \frac{1}{6}|d_t e_q^{n+1}|^2 + C(\Delta t + \|\nabla\mathbf{u}(t^{n+1})\|^2)\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2. \end{aligned} \tag{83}$$

The second and third terms on the right hand side of (80) can be bounded by

$$\begin{aligned} &-\frac{d_t e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left((\mathbf{u}^n\cdot\nabla)(\mathbf{u}(t^{n+1})-\mathbf{u}^n), \mathbf{u}(t^{n+1})\right) \\ &-\frac{d_t e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}\left(((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla)\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1})\right) \\ &\qquad \le \frac{1}{6}|d_t e_q^{n+1}|^2 + C\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C(\Delta t)^2. \end{aligned} \tag{84}$$

The last term on the right hand side of (80) can be bounded by

$$\mathbf{R}_q^{n+1} d_t e_q^{n+1} \le \frac{1}{6}|d_t e_q^{n+1}|^2 + C\Delta t\int_{t^n}^{t^{n+1}}\|q_{tt}\|^2 dt. \tag{85}$$

Finally combining (80) with (81)-(84) results in

$$\begin{aligned} &|d_t e_q^{n+1}|^2 + \frac{|e_q^{n+1}|^2 - |e_q^n|^2}{2T\Delta t} + \frac{|e_q^{n+1}-e_q^n|^2}{2T\Delta t} \\ \le &\frac{1}{2}|d_t e_q^{n+1}|^2 + C(\Delta t + \|\nabla\mathbf{u}(t^{n+1})\|^2)\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 \\ &+ C\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\Delta t\int_{t^n}^{t^{n+1}}\|q_{tt}\|^2 dt + C(\Delta t)^2. \end{aligned} \tag{86}$$

Multiplying (86) by $2T\Delta t$ and summing up for n from 0 to m, we obtain

$$\begin{aligned} &T\Delta t\sum_{n=0}^{m}|d_t e_q^{n+1}|^2 + |e_q^{m+1}|^2 \\ &\qquad \le C\Delta t\sum_{n=0}^{m}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\Delta t\sum_{n=0}^{m}\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C(\Delta t)^2. \end{aligned}$$

Combining the above with (79), we obtain the desired result. □

5.2. **Error estimates for the pressure.** The main result in this subsection is the following error estimate for the pressure which requires additional regularities.

Theorem 5.2. *Assuming* $d = 2$ *and* $\mathbf{u} \in H^3(0,T;\mathbf{L}^2(\Omega)) \bigcap H^1(0,T;\mathbf{H}_0^2(\Omega)) \bigcap W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$, $p \in H^2(0,T;H^1(\Omega))$, *then for the first-order scheme* (19)-(22), *we have*

$$\Delta t \sum_{n=0}^{m} \|e_p^{n+1}\|^2_{L^2(\Omega)/R} \le C(\Delta t)^2, \quad \forall\ 0 \le m \le N-1, \tag{87}$$

where C *is a positive constant independent of* Δt.

Proof. In order to prove the above results, we need to first establish an estimate on $\|e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|$.

Adding (59) and (60) leads to

$$\begin{aligned} \frac{e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n}{\Delta t} - \nu\Delta\tilde{e}_{\mathbf{u}}^{n+1} = & \frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) \\ & - \frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n - \nabla e_p^{n+1} + \mathbf{R}_{\mathbf{u}}^{n+1}. \end{aligned} \tag{88}$$

Define $d_{tt}e_{\mathbf{u}}^{n+1} = \frac{d_t e_{\mathbf{u}}^{n+1} - d_t e_{\mathbf{u}}^n}{\Delta t}$. Then taking the difference of two consecutive steps in (88), we have

$$d_{tt}e_{\mathbf{u}}^{n+1} - \nu\Delta d_t\tilde{e}_{\mathbf{u}}^{n+1} = d_t\mathbf{R}_{\mathbf{u}}^{n+1} - \nabla d_t e_p^{n+1} + \sum_{i=1}^{3} S_i, \tag{89}$$

where

$$\begin{aligned} S_1 = & d_t(q^{n+1}\exp(\frac{t^{n+1}}{T}))((\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \cdot \nabla)\mathbf{u}(t^{n+1}) \\ & + q^n\exp(\frac{t^n}{T})((d_t\mathbf{u}(t^{n+1}) - d_t\mathbf{u}^n) \cdot \nabla)\mathbf{u}(t^{n+1}) \\ & + q^n\exp(\frac{t^n}{T})((\mathbf{u}(t^n) - \mathbf{u}^{n-1}) \cdot \nabla)d_t\mathbf{u}(t^{n+1}), \end{aligned} \tag{90}$$

$$\begin{aligned} S_2 = & d_t(q^{n+1}\exp(\frac{t^{n+1}}{T}))(\mathbf{u}^n \cdot \nabla)(\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \\ & + q^n\exp(\frac{t^n}{T})(d_t e_{\mathbf{u}}^n \cdot \nabla)(\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \\ & + q^n\exp(\frac{t^n}{T})(d_t\mathbf{u}(t^n) \cdot \nabla)(\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \\ & + q^n\exp(\frac{t^n}{T})(\mathbf{u}^{n-1} \cdot \nabla)(d_t\mathbf{u}(t^{n+1}) - d_t\mathbf{u}^n), \end{aligned} \tag{91}$$

and

$$\begin{aligned} S_3 = & - d_t(e_q^{n+1}\exp(\frac{t^{n+1}}{T}))(\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) \\ & - e_q^n\exp(\frac{t^n}{T})(d_t\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) \\ & - e_q^n\exp(\frac{t^n}{T})(\mathbf{u}(t^n) \cdot \nabla)d_t\mathbf{u}(t^{n+1}). \end{aligned} \tag{92}$$

Taking the inner product of (89) with $d_t\tilde{e}_{\mathbf{u}}^{n+1}$, we find

$$(d_{tt}e_{\mathbf{u}}^{n+1}, d_t\tilde{e}_{\mathbf{u}}^{n+1}) + \nu\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 = (d_t\mathbf{R}_{\mathbf{u}}^{n+1}, d_t\tilde{e}_{\mathbf{u}}^{n+1}) - (\nabla d_t e_p^{n+1}, d_t\tilde{e}_{\mathbf{u}}^{n+1}) + \sum_{i=1}^{3}(S_i, d_t\tilde{e}_{\mathbf{u}}^{n+1}). \tag{93}$$

For the first term on the left hand side, we have

$$(d_{tt}e_{\mathbf{u}}^{n+1}, d_t\tilde{e}_{\mathbf{u}}^{n+1}) = \frac{\|d_t e_{\mathbf{u}}^{n+1}\|^2 - \|d_t e_{\mathbf{u}}^{n}\|^2}{2\Delta t} + \frac{\|d_t e_{\mathbf{u}}^{n+1} - d_t e_{\mathbf{u}}^{n}\|^2}{2\Delta t}. \tag{94}$$

We bound the terms on the right hand side as follows.

$$(d_t\mathbf{R}_{\mathbf{u}}^{n+1}, d_t\tilde{e}_{\mathbf{u}}^{n+1}) \leq \frac{\nu}{8}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\Delta t\int_{t^{n-1}}^{t^{n+1}}\|\mathbf{u}_{ttt}(t)\|^2 dt. \tag{95}$$

The second term on the right hand of (93) can be transformed into

$$-(\nabla d_t e_p^{n+1}, d_t\tilde{e}_{\mathbf{u}}^{n+1}) = -(\nabla d_t e_p^{n}, d_t\tilde{e}_{\mathbf{u}}^{n+1}) - (\nabla(d_t e_p^{n+1} - d_t e_p^{n}), d_t\tilde{e}_{\mathbf{u}}^{n+1}). \tag{96}$$

Since we can derive from (60) that

$$d_t\tilde{e}_{\mathbf{u}}^{n+1} = d_t e_{\mathbf{u}}^{n+1} + \nabla(p^{n+1} - 2p^n + p^{n-1}). \tag{97}$$

The first term on the right hand of (96) can be estimated by

$$\begin{aligned}
&-(\nabla d_t e_p^{n}, d_t\tilde{e}_{\mathbf{u}}^{n+1}) = -(\nabla d_t e_p^{n}, \nabla(p^{n+1} - 2p^n + p^{n-1}))\\
=&-\Delta t(\nabla d_t e_p^{n}, \nabla(d_t e_p^{n+1} - d_t e_p^{n}))\\
&-(\Delta t)^2\left(\nabla d_t e_p^{n}, \frac{1}{(\Delta t)^2}\nabla(p(t^{n+1}) - 2p(t^n) + p(t^{n-1}))\right)\\
\leq&-\frac{\Delta t}{2}(\|\nabla d_t e_p^{n+1}\|^2 - \|\nabla d_t e_p^{n}\|^2 - \|\nabla d_t e_p^{n+1} - \nabla d_t e_p^{n}\|^2)\\
&+(\Delta t)^2\|\nabla d_t e_p^{n}\|^2 + C\Delta t\int_{t^{n-1}}^{t^{n+1}}\|\nabla p_{tt}\|^2 dt.
\end{aligned} \tag{98}$$

The second term on the right hand of (96) can be bounded by

(99)

$$\begin{aligned}
&-(\nabla(d_t e_p^{n+1} - d_t e_p^{n}), d_t\tilde{e}_{\mathbf{u}}^{n+1}) = -\big(\nabla(d_t e_p^{n+1} - d_t e_p^{n}), \nabla(p^{n+1} - 2p^n + p^{n-1})\big)\\
=&-\Delta t\left(\nabla(d_t e_p^{n+1} - d_t e_p^{n}), \frac{1}{\Delta t}\nabla(p(t^{n+1}) - 2p(t^n) + p(t^{n-1}))\right)\\
&-\Delta t\left(\nabla(d_t e_p^{n+1} - d_t e_p^{n}), \nabla(d_t e_p^{n+1} - d_t e_p^{n})\right)\\
\leq&-\frac{\Delta t}{2}\|\nabla d_t e_p^{n+1} - \nabla d_t e_p^{n}\|^2 + C(\Delta t)^2\int_{t^{n-1}}^{t^{n+1}}\|\nabla p_{tt}\|^2 dt.
\end{aligned}$$

Recalling (10)-(11), (55) and (82) and using Young inequality, we have

(100)

$$(S_1, d_t\tilde{e}_{\mathbf{u}}^{n+1}) \leq \frac{\nu}{8}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\|d_t e_{\mathbf{u}}^{n}\|^2 + C\|e_{\mathbf{u}}^{n}\|^2 + C\|e_{\mathbf{u}}^{n-1}\|^2 + C(\Delta t)^2,$$

$$
\begin{aligned}
(S_2, d_t\tilde{e}_{\mathbf{u}}^{n+1}) \le& C|\left((d_te_{\mathbf{u}}^n\cdot\nabla)e_{\mathbf{u}}^n, d_t\tilde{e}_{\mathbf{u}}^{n+1}\right)| + C|\left((\mathbf{u}^{n-1}\cdot\nabla)d_te_{\mathbf{u}}^n, d_t\tilde{e}_{\mathbf{u}}^{n+1}\right)| \\
&+ \frac{\nu}{32}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\|e_{\mathbf{u}}^n\|_1^2 + C(\Delta t)^2 \\
\le& C|\left((d_te_{\mathbf{u}}^n\cdot\nabla)e_{\mathbf{u}}^n, d_t\tilde{e}_{\mathbf{u}}^{n+1}\right)| + C|\left((e_{\mathbf{u}}^{n-1}\cdot\nabla)d_te_{\mathbf{u}}^n, d_t\tilde{e}_{\mathbf{u}}^{n+1}\right)| \\
&+ \frac{\nu}{16}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\|e_{\mathbf{u}}^n\|_1^2 + C\|d_te_{\mathbf{u}}^n\|^2 + C(\Delta t)^2 \\
\le& C\|d_te_{\mathbf{u}}^n\|^{1/2}\|\nabla d_t\tilde{e}_{\mathbf{u}}^n\|^{1/2}\|e_{\mathbf{u}}^n\|^{1/2}\|e_{\mathbf{u}}^n\|_1^{1/2}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\| \\
&+ C\|e_{\mathbf{u}}^{n-1}\|^{1/2}\|e_{\mathbf{u}}^{n-1}\|_1^{1/2}\|d_te_{\mathbf{u}}^n\|^{1/2}\|\nabla d_t\tilde{e}_{\mathbf{u}}^n\|^{1/2}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\| \\
&+ \frac{\nu}{16}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\|e_{\mathbf{u}}^n\|_1^2 + C\|d_te_{\mathbf{u}}^n\|^2 + C(\Delta t)^2 \\
\le& \frac{\nu}{8}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C(\|e_{\mathbf{u}}^{n-1}\|_0^2\|e_{\mathbf{u}}^{n-1}\|_1^2 + \|e_{\mathbf{u}}^n\|_0^2\|e_{\mathbf{u}}^n\|_1^2)\|\nabla d_t\tilde{e}_{\mathbf{u}}^n\|^2 \\
&+ C\|d_te_{\mathbf{u}}^n\|^2 + C\|e_{\mathbf{u}}^n\|_1^2 + C(\Delta t)^2,
\end{aligned} \tag{101}
$$

and

$$
(S_3, d_t\tilde{e}_{\mathbf{u}}^{n+1}) \le \frac{\nu}{8}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C|d_te_q^{n+1}|^2 + C|e_q^{n+1}|^2 + C(\Delta t)^2. \tag{102}
$$

Then combining (93) with (97)-(102), we have

$$
\begin{aligned}
&\frac{\|d_te_{\mathbf{u}}^{n+1}\|^2 - \|d_te_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\|d_te_{\mathbf{u}}^{n+1} - d_te_{\mathbf{u}}^n\|^2}{2\Delta t} \\
&+ \frac{\nu}{2}\|\nabla d_t\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \frac{\Delta t}{2}(\|\nabla d_te_p^{n+1}\|^2 - \|\nabla d_te_p^n\|^2 + \|\nabla d_te_p^{n+1} - \nabla d_te_p^n\|^2) \\
\le& (\Delta t)^2\|\nabla d_te_p^n\|^2 + C(\|e_{\mathbf{u}}^{n-1}\|_0^2\|e_{\mathbf{u}}^{n-1}\|_1^2 + \|e_{\mathbf{u}}^n\|_0^2\|e_{\mathbf{u}}^n\|_1^2)\|\nabla d_t\tilde{e}_{\mathbf{u}}^n\|^2 \\
&+ C\|d_te_{\mathbf{u}}^n\|^2 + C\|e_{\mathbf{u}}^n\|_1^2 + C\|e_{\mathbf{u}}^{n-1}\|^2 + C|d_te_q^{n+1}|^2 \\
&+ C|e_q^{n+1}|^2 + C(\Delta t)^2.
\end{aligned} \tag{103}
$$

Recalling Theorem 5.1, we have

$$
\|\nabla d_t\tilde{e}_{\mathbf{u}}^n\|^2 \le (\Delta t)^{-2}\|\nabla\tilde{e}_{\mathbf{u}}^n\|^2 \le C(\Delta t)^{-1},\ \forall 1 \le n \le N.
$$

Then multiplying (103) by $2\Delta t$, summing up for n from 1 to m and applying the discrete Gronwall lemma 2.1, we can obtain

$$
\begin{aligned}
&\|d_te_{\mathbf{u}}^{m+1}\|^2 + (\Delta t)^2\|\nabla d_te_p^{m+1}\|^2 \\
\le& \|d_te_{\mathbf{u}}^1\|^2 + (\Delta t)^3\sum_{n=1}^m \|\nabla d_te_p^{n+1}\|^2 + (\Delta t)^2\|\nabla d_te_p^1\|^2 + C(\Delta t)^2.
\end{aligned} \tag{104}
$$

It remains to estimate $\|d_te_{\mathbf{u}}^1\|^2$ and $(\Delta t)^2\|\nabla d_te_p^1\|^2$. Using (59), we have

$$
\begin{aligned}
\tilde{e}_{\mathbf{u}}^1 - \nu\Delta t\tilde{e}_{\mathbf{u}}^1 =& \Delta t\frac{q(t^1)}{\exp(-\frac{t^1}{T})}(\mathbf{u}(t^1)\cdot\nabla)\mathbf{u}(t^1) - \Delta t\frac{q^1}{\exp(-\frac{t^1}{T})}\mathbf{u}^0\cdot\nabla\mathbf{u}^0 \\
&- \Delta t\nabla(p^0 - p(t^1)) + \Delta t\mathbf{R}_{\mathbf{u}}^1.
\end{aligned} \tag{105}
$$

Taking the inner product of (105) with $\tilde{e}_{\mathbf{u}}^1$ leads to

(106)

$$\begin{aligned}\|\tilde{e}_{\mathbf{u}}^1\|^2 + \nu\Delta t\|\nabla\tilde{e}_{\mathbf{u}}^1\|^2 =&\Delta t\left(\frac{q(t^1)}{\exp(-\frac{t^1}{T})}(\mathbf{u}(t^1)\cdot\nabla)\mathbf{u}(t^1) - \frac{q^1}{\exp(-\frac{t^1}{T})}(\mathbf{u}^0\cdot\nabla)\mathbf{u}^0, \tilde{e}_{\mathbf{u}}^1\right)\\ &-\Delta t(\nabla(p(t^0)-p(t^1)), \tilde{e}_{\mathbf{u}}^1) + \Delta t(\mathbf{R}_{\mathbf{u}}^1, \tilde{e}_{\mathbf{u}}^1)\\ \le&\frac{1}{2}\|\tilde{e}_{\mathbf{u}}^1\|^2 + C(\Delta t)^4,\end{aligned}$$

from which we obtain

$$\|d_t e_{\mathbf{u}}^1\|^2 \le \|d_t\tilde{e}_{\mathbf{u}}^1\|^2 = (\Delta t)^{-2}\|\tilde{e}_{\mathbf{u}}^1\|^2 \le C(\Delta t)^2.$$

We can derive from (60) with $n = 1$ that

$$(\Delta t)^2\|\nabla d_t e_p^1\|^2 \le (\Delta t)^{-2}(\|e_{\mathbf{u}}^1\|^2 + \|\tilde{e}_{\mathbf{u}}^1\|^2) + (\Delta t)^2\|\nabla d_t p(t^1)\|^2 \le C(\Delta t)^2. \tag{107}$$

Combining the above estimates with (104), we finally obtain

$$\|d_t e_{\mathbf{u}}^{m+1}\|^2 + (\Delta t)^2\|\nabla d_t e_p^{m+1}\|^2 \le C(\Delta t)^2, \tag{108}$$

which implies in particular

$$\|e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\| \le C(\Delta t)^2. \tag{109}$$

We are now in position to prove the pressure estimate. Taking the inner product of (88) with $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, we have

$$\begin{aligned}&(\nabla e_p^{n+1}, \mathbf{v}) = -(\frac{e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n}{\Delta t}, \mathbf{v}) + \nu(\Delta\tilde{e}_{\mathbf{u}}^{n+1}, \mathbf{v}) + (\mathbf{R}_{\mathbf{u}}^{n+1}, \mathbf{v})\\ &+\left(\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1}) - \frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, \mathbf{v}\right).\end{aligned} \tag{110}$$

Taking notice of the fact that

$$\|e_p^{n+1}\|_{L^2(\Omega)/\mathbb{R}} \le \sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{(\nabla e_p^{n+1}, \mathbf{v})}{\|\nabla\mathbf{v}\|}. \tag{111}$$

By using (64)-(66) and (82), we can derive that, for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$,

$$\begin{aligned}&\left(\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1}) - \frac{q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, \mathbf{v}\right)\\ =&\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\Big(((\mathbf{u}(t^{n+1}) - \mathbf{u}^n)\cdot\nabla)\mathbf{u}(t^{n+1}), \mathbf{v}\Big)\\ &+\frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})}\Big((\mathbf{u}^n\cdot\nabla)(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), \mathbf{v}\Big)\\ &-\frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, \mathbf{v})\\ \le&C(\|e_{\mathbf{u}}^n\| + \|\nabla\tilde{e}_{\mathbf{u}}^n\| + \|\int_{t^n}^{t^{n+1}}\mathbf{u}_t dt\|_1 + |e_q^{n+1}|)\|\nabla\mathbf{v}\|.\end{aligned} \tag{112}$$

Hence thanks to Theorem 5.1 and (68), (108), we can derive from the above that

$$\begin{aligned}\Delta t \sum_{n=0}^{m} \|e_p^{n+1}\|^2_{L^2(\Omega)/\mathbb{R}} \le & C\Delta t \sum_{n=0}^{m} \big(\|d_t e_{\mathbf{u}}^{n+1}\|^2 + \|\nabla \tilde{e}_{\mathbf{u}}^{n+1}\|^2 \\ & + \|\nabla \tilde{e}_{\mathbf{u}}^{n}\|^2 + \|e_{\mathbf{u}}^{n}\|^2 + |e_q^{n+1}|^2\big) \\ & + C(\Delta t)^2 \int_{t^0}^{t^{m+1}} (\|\mathbf{u}_t\|_1^2 + \|\mathbf{u}_{tt}\|_{-1}^2) dt \le C(\Delta t)^2. \end{aligned} \tag{113}$$

The proof is complete. □

6. Numerical experiments

In this section, we carry out some numerical experiments to verify the accuracy and stability of the first- and second-order SAV schemes with pressure correction for the Navier-Stokes equations. In all examples below, we take $\Omega = (0,1)\times(0,1)$.

6.1. **Convergence test.** In this subsection, we present two examples to verify the convergence rates for the first- and second-order SAV schemes with pressure correction for the Navier-Stokes equations. We set $T = 1$, $\nu = 0.1$ and the spatial discretization is based on the MAC scheme on the staggered grid with $N_x = N_y = 250$ so that the spatial discretization error is negligible compared to the time discretization error for the time steps used in the experiments.

Example 1. The right hand side of the equations is computed according to the analytic solution given by:

$$\begin{cases} p(x,y,t) = \sin(t)(\sin(\pi y) - 2/\pi), \\ u_1(x,y,t) = \sin(t)\sin^2(\pi x)\sin(2\pi y), \\ u_2(x,y,t) = -\sin(t)\sin(2\pi x)\sin^2(\pi y). \end{cases}$$

Example 2. The right hand side of the equations is computed according to the analytic solution given by:

$$\begin{cases} p(x,y,t) = t^2(x-0.5), \\ u_1(x,y,t) = -128t^2x^2(x-1)^2y(y-1)(2y-1), \\ u_2(x,y,t) = 128t^2y^2(y-1)^2x(x-1)(2x-1). \end{cases}$$

Numerical results for Examples 1 and 2 with first- and second-order schemes are presented in Tables 1-4. We observe that the results for the first-order scheme are consistent with the error estimates in Theorems 5.1 and 5.2. While second-order convergence rates for the velocity and SAV variable in L^∞ norm, and nearly second-order convergence rates for the pressure in L^2 norm were observed for the second-order scheme.

TABLE 1. Errors and convergence rates for Example 1 with the first-order scheme (19)-(22)

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^2}$	Rate	$\|e_q\|_\infty$	Rate
$\frac{1}{10}$	5.77E-3	—	2.20E-2	—	2.26E-2	—
$\frac{1}{20}$	2.25E-3	1.36	1.06E-2	1.06	1.02E-2	1.14
$\frac{1}{40}$	1.04E-3	1.11	5.13E-3	1.04	4.87E-3	1.07
$\frac{1}{80}$	5.01E-4	1.05	2.54E-3	1.01	2.37E-3	1.04

TABLE 2. Errors and convergence rates for Example 1 with the second-order scheme (33)-(36)

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^2}$	Rate	$\|e_q\|_\infty$	Rate
$\frac{1}{10}$	1.99E-3	—	7.83E-3	—	4.69E-3	—
$\frac{1}{20}$	5.25E-4	1.92	2.47E-3	1.66	1.24E-3	1.92
$\frac{1}{40}$	1.36E-4	1.95	7.20E-4	1.78	3.17E-4	1.97
$\frac{1}{80}$	3.95E-5	1.78	1.99E-4	1.85	7.97E-5	1.99

TABLE 3. Errors and convergence rates for Example 2 with the first-order scheme (19)-(22)

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^2}$	Rate	$\|e_q\|_\infty$	Rate
$\frac{1}{10}$	1.14E-2	—	2.13E-2	—	2.03E-2	—
$\frac{1}{20}$	5.08E-3	1.17	1.07E-2	0.99	9.44E-3	1.11
$\frac{1}{40}$	2.46E-3	1.05	5.30E-3	1.01	4.61E-3	1.03
$\frac{1}{80}$	1.23E-3	1.00	2.63E-3	1.01	2.30E-3	1.01

TABLE 4. Errors and convergence rates for Example 2 with the second-order scheme (33)-(36)

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^2}$	Rate	$\|e_q\|_\infty$	Rate
$\frac{1}{10}$	3.95E-3	—	5.95E-3	—	1.82E-3	—
$\frac{1}{20}$	1.06E-3	1.90	1.66E-3	1.84	4.09E-4	2.16
$\frac{1}{40}$	2.77E-4	1.94	4.51E-4	1.88	9.82E-5	2.06
$\frac{1}{80}$	8.09E-5	1.78	1.21E-4	1.89	2.42E-5	2.02

As a comparison, we also implemented the following pressure-correction version of the scheme (2)-(4).

Scheme III. Find ($\tilde{\mathbf{u}}^{n+1}$, $\mathbf{u}^{n+1}$,p^{n+1},q^{n+1}) by solving

$$
\begin{cases}
\frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\Delta t} + \frac{q^{n+1}}{\sqrt{E(\mathbf{u}^n)+C_0}}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n - \nu\Delta\tilde{\mathbf{u}}^{n+1} + \nabla p^n = \mathbf{f}^{n+1}, \\
\qquad\qquad\qquad\qquad\qquad\qquad \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = 0; & (114)\\
\frac{\mathbf{u}^{n+1}-\tilde{\mathbf{u}}^{n+1}}{\Delta t} + \nabla(p^{n+1}-p^n) = 0; & (115)\\
\nabla\cdot\mathbf{u}^{n+1} = 0, \quad \mathbf{u}^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0; & (116)\\
2q^{n+1}\frac{q^{n+1}-q^n}{\Delta t} = \left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + \frac{q^{n+1}}{\sqrt{E(\mathbf{u}^n)+C_0}}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}\right), & (117)
\end{cases}
$$

where $E(\mathbf{u}) = \int_\Omega \frac{1}{2}|\mathbf{u}|^2$ is the total energy. Numerical results with the MAC discretization of the above scheme are listed in Tables 5 and 6. It is observed that the results with this scheme are essentially the same as the results by our new first-order SAV scheme in Tables 1 and 3. Note that the above scheme requires solving a nonlinear algebraic equation at each time step.

6.2. **Energy dissipation.** The new schemes are unconditionally energy dissipative with a modified energy. In the following example, we show that the original energy computed by the new schemes is also dissipative. We set

$$T = 1,\ \Delta t = 0.001,\ N_x = N_y = 100,\ \mathbf{f} = 0,$$

TABLE 5. Errors and convergence rates for Example 1 with the first-order scheme (114)-(117)

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^2}$	Rate
$\frac{1}{10}$	5.75E-3	—	2.10E-2	—
$\frac{1}{20}$	2.24E-3	1.36	9.55E-3	1.13
$\frac{1}{40}$	1.04E-3	1.11	4.46E-3	1.10
$\frac{1}{80}$	5.01E-4	1.05	2.17E-3	1.04

TABLE 6. Errors and convergence rates for Example 2 with the first-order scheme (114)-(117)

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^2}$	Rate
$\frac{1}{10}$	1.14E-2	—	1.88E-2	—
$\frac{1}{20}$	5.05E-3	1.17	8.91E-3	1.08
$\frac{1}{40}$	2.44E-3	1.05	4.30E-3	1.05
$\frac{1}{80}$	1.22E-3	1.00	2.11E-3	1.03

and use as initial velocity $u_1(x,y)=\sin^2(\pi x)\sin(2\pi y),\ u_2(x,y)=-\sin(2\pi x)\sin^2(\pi y)$. Evolutions of original energy $\frac{1}{2}\|\mathbf{u}\|^2$ with different Reynolds numbers $Re = \frac{1}{\nu} = 1000, 3000, 5000, 8000, 10000$ are presented in Figure 1. We observe that the original energy is dissipative in all cases.

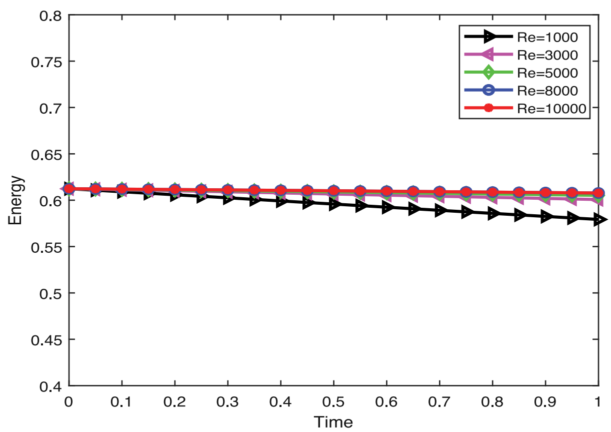

FIGURE 1. Evolutions of the original energy with different Reynolds numbers $Re = 1000,\ 3000,\ 5000,\ 8000,\ 10000$, respectively

6.3. **Lid-driven cavity.** As the last example, we demonstrate that our new schemes are also robust for a real physical simulation, the well-known lid-driven cavity benchmark problem [7, 22].

We take $Re = 5000$, and use the first-order SAV scheme with pressure correction with $N_x = N_y = 128$ MAC discretization and $\Delta t = 2e-3$. We plot the snapshots at $2000,\ 3000$ and 5000 time steps in Figure 2, and at the final steady state with the velocity at the center line compared with the benchmark results in Figure 3.

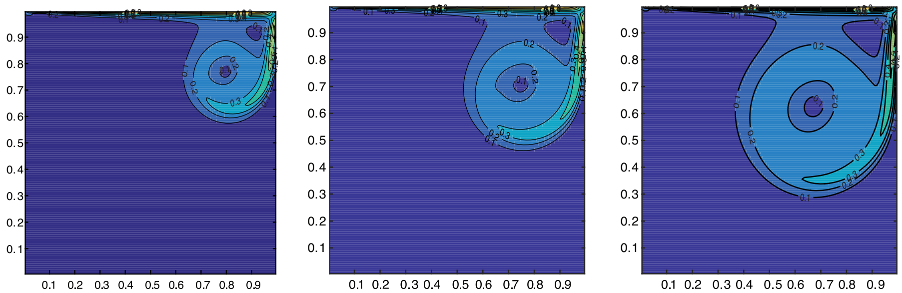

FIGURE 2. The combined velocity field for $Re = 5000$ at different time steps 2000, 3000 and 5000

We observe that our numerical simulation captures the dynamical evolution of the velocity field, and leads to identical final steady state compared with the benchmark results.

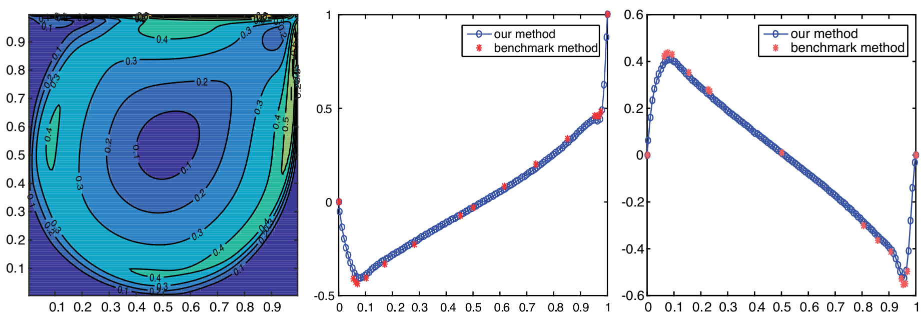

FIGURE 3. The combined velocity field and the velocity at the center line with x-component velocity at $x = 0.5$ and y-component velocity at $y = 0.5$ for $Re = 5000$ at the final steady state

7. Concluding remarks

We constructed novel first- and second-order linear and decoupled pressure correction schemes based on the SAV approach for the Navier-Stokes equations, and proved that they are unconditionally energy stable. Compared with the previous version of SAV scheme (2)-(4), the new schemes possess two distinct advantages: (i) they are purely linear, eliminating the numerical and theoretical difficulties associated with the nonlinear algebraic equation in (4), and (ii) they lead to a much stronger stability result with a uniform bound on the L^2-norm of the numerical solution, which is essential for the error analysis, and enable us to derive optimal error estimates for the first-order scheme without any restriction on the time step. Another main contribution is that we proved unconditional energy stability for the new SAV scheme based on the second-order rotational pressure-correction scheme. To the best of the authors' knowledge, these schemes are the first of such kind for the Navier-Stokes equations with unconditional energy stability while treating the nonlinear term explicitly.

We only carried out a rigorous error analysis for the first-order scheme in the two dimensional case. Due to the rotational form of the pressure correction in the second-order scheme, its error analysis will be much more involved, as indicated by the technicality in its analysis without the nonlinear term [10]. Also, we encounter an essential difficulty for the error analysis of the pressure-correction scheme in the three dimensional case so that our error analysis is limited to the two-dimensional case. The error estimates for the second-order scheme and/or for the three-dimensional case will be left for a future endeavor.

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