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Wavenumber explicit analysis for time-harmonic Maxwell equations in a spherical shell and spectral approximations

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This article is devoted to wavenumber explicit analysis of the electric field satisfying the second-order time-harmonic Maxwell equations in a spherical shell and, hence, for variant scatterers with ϵ-perturbation of the inner ball radius. The spherical shell model is obtained by assuming that the forcing function is zero outside a circumscribing ball and replacing the radiation condition with a transparent boundary condition involving the capacity operator. Using the divergence-free vector spherical harmonic expansions for two components of the electric field, the Maxwell system is reduced to two sequences of decoupled one-dimensional boundary value problems in the radial direction. The reduced problems naturally allow for truncated vector spherical harmonic spectral approximation of the electric field and one-dimensional global polynomial approximation of the boundary value problems. We analyse the error in the resulting spectral approximation for the spherical shell model. Using a perturbation transformation, we generalize the approach for ϵ-perturbed nonspherical scatterers by representing the resulting field in ϵ-power series expansion with coefficients being spherical shell electric fields.

Keywords: Maxwell equations; Helmholtz equation; wavenumber explicit analysis; Dirichlet-to-Neumann boundary conditions; divergence-free vector spherical harmonic expansions.

1. Introduction

This article is concerned with wavenumber explicit analysis and spectral-Galerkin approximation of the time-harmonic Maxwell equations:

$$\nabla \times \nabla \times \boldsymbol{E}^{a,b} - k^2\boldsymbol{E}^{a,b} = \boldsymbol{F}^{a,b}, \quad \text{in } \ \Omega = B \setminus \bar{D}\,; \tag{1.1}$$

$$\boldsymbol{E}^{a,b} \times \boldsymbol{e}_r = \boldsymbol{0}, \quad \text{on } \ \partial D; \quad (\nabla \times \boldsymbol{E}^{a,b}) \times \boldsymbol{e}_r - \mathrm{i}k\mathscr{T}_b[\boldsymbol{E}_S^{a,b}] = \boldsymbol{h} \quad \text{at } \ r = b, \tag{1.2}$$

where $\Omega = \{a < r = |\boldsymbol{r}| < b\}$ is the spherical shell formed by two concentric spheres D and B of radii a and b, respectively, $\boldsymbol{E}^{a,b}$ is the electric field, $\boldsymbol{E}_S^{a,b} = -\boldsymbol{E}^{a,b} \times \boldsymbol{e}_r \times \boldsymbol{e}_r$ (with $\boldsymbol{e}_r = \boldsymbol{r}/r$) is the tangential field,

$k > 0$ is the wavenumber, $\mathrm{i} = \sqrt{-1}$ and $\mathscr{T}_b$ is the capacity operator (see Nédélec, 2001, (5.3.88) and (3.13) below). Here, the source term $\boldsymbol{F}^{a,b}$ is assumed to be compactly supported in Ω, and the function $\boldsymbol{h}$ in (1.2) is added for potentially inhomogeneous boundary conditions.

We also consider (1.1) and (1.2) with the spherical shell Ω being replaced with $B \setminus \bar{D}$, where D is an ϵ-perturbation of the inner ball of radius a:

$$D = \Big\{(r,\theta,\phi) \,:\, 0 < r < a + \epsilon f(\theta,\phi),\ \theta \in [0,\pi],\ \phi \in [0,2\pi)\Big\}. \tag{1.3}$$

Based on the transformed field expansion (TFE) (David & Fernando, 2004), the electric field in $\hat{\Omega} = \{a + \epsilon f(\theta,\phi) < r < b\}$ can be represented as an ϵ-power series with the expansion coefficients being spherical shell electric fields. Therefore, the spectral algorithm and analysis for the Maxwell equations in the spherical shell are essential for such a variant.

The study of the above model problems is motivated by the exterior Maxwell system:

$$\begin{aligned} &-\mathrm{i}\omega\mu\boldsymbol{H} + \nabla\times\boldsymbol{E} = \boldsymbol{0}, \quad -\mathrm{i}\omega\varepsilon\boldsymbol{E} - \nabla\times\boldsymbol{H} = \boldsymbol{J}, \quad \text{in} \quad \mathbb{R}^3\setminus\bar{D}; \\ &(\boldsymbol{E}\times\boldsymbol{n})|_{\partial D} = \boldsymbol{0}; \quad \lim_{r\to\infty} r\Big(\sqrt{\mu/\varepsilon}\,\boldsymbol{H}\times\hat{\boldsymbol{x}} - \boldsymbol{E}\Big) = \boldsymbol{0}, \end{aligned} \tag{1.4}$$

where the scatterer D is a simply connected, bounded, perfect conductor, $\boldsymbol{E}, \boldsymbol{H}$ are respectively the electric and magnetic fields, μ is the magnetic permeability, ε is the electric permittivity, ω is the frequency of the harmonic wave, $\boldsymbol{n}$ is the outward normal and $\hat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$. The boundary condition at infinity in (1.4) is known as the Silver–Müller radiation condition. Typically, the electric current density $\boldsymbol{J}$ is localized in space; for example, it is restricted to flow on an antenna (cf. Orfanidis, 2002). The Maxwell system (1.4) plays an important role in many scientific and engineering applications, including in particular electromagnetic wave scattering, and is also of mathematical interest (see, e.g., Nédélec, 2001; Monk, 2003; Colton & Kress, 2013b). Despite its seeming simplicity, the system (1.4) is notoriously difficult to solve numerically. Some of the main challenges include (i) the indefiniteness when ω is not sufficiently small; (ii) highly oscillatory solutions when ω is large; (iii) the incompressibility (i.e., $\mathrm{div}(\mu\boldsymbol{H}) = \mathrm{div}(\varepsilon\boldsymbol{E}) = 0$), which is implicitly implied by (1.4) and (iv) the unboundedness of the domain. On the one hand, one needs to construct approximation spaces such that the discrete problems are well posed and lead to good approximations for a wide range of wavenumbers. On the other hand, one needs to develop efficient algorithms for solving the indefinite linear system, particularly for large wavenumber, resulted from a given discretization. We refer to Monk (2003) and the references therein for various contributions with respect to numerical approximations of the time-harmonic Maxwell equations. The methods of choice for dealing with unbounded domains include the perfectly matched layer technique (Berenger, 1994), boundary integral method (Jin *et al.*, 1991; Lin *et al.*, 2009; Sauter & Schwab, 2011; Chandler-Wilde *et al.*, 2012; Colton & Kress, 2013a; Kirsch & Hettlich, 2015) and the artificial boundary condition (Engquist & Majda, 1977; Grote & Keller, 1995; Hagstrom, 1999). The last approach is to enclose the obstacles and inhomogeneities (and nonlinearities at times) with an artificial boundary. A suitable boundary condition is then imposed, leading to a numerically solvable boundary value problem (BVP) in a finite domain. The artificial boundary condition is known as a transparent (or nonreflecting) boundary condition (TBC), if the solution of the reduced problem coincides with the solution of the original problem restricted to the finite domain.

The TBC characterized by the capacity operator $\mathscr{T}_b$ (cf. Nédélec, 2001) can reduce the exterior Maxwell equations to an equivalent BVP. With this, we obtain the second-order problems (1.1) and (1.2) by eliminating the magnetic field $\boldsymbol{H}$ and adding $\boldsymbol{h}$ in (1.2) to deal with potentially inhomogeneous boundary conditions. Note that the wavenumber $k = \omega\sqrt{\mu\varepsilon}$ and we denote $\eta = \sqrt{\mu/\varepsilon}$. In Nédélec (2001) and other related works (e.g., Ma *et al.*, 2015), the usual vector spherical harmonics (VSH) are used to expand the electric field $\boldsymbol{E}^{a,b}$. Then the problems (1.1) and (1.2) can be reduced to a coupled system of two components of $\boldsymbol{E}^{a,b}$, whereas the other component satisfies the same equation reduced from the Helmholtz equation (cf. Ma *et al.*, 2015):

$$-\Delta U^{a,b} - k^2 U^{a,b} = F^{a,b}, \qquad \text{in } \Omega = B\backslash\bar{D}, \tag{1.5}$$

$$U^{a,b}|_{\partial D} = 0; \quad \partial_r U^{a,b} - \mathcal{T}_b[U^{a,b}] = H, \quad \text{at } r = b, \tag{1.6}$$

where $\mathcal{T}_b$ is the Dirichlet-to-Neumann (DtN) operator (Nédélec, 2001) (see (2.1) below). The wavenumber explicit analysis for the above Helmholtz equation has been carried out in Shen & Wang (2007) (also see Chandler-Wilde & Monk, 2008 for starlike scatterers), but the analysis for two coupled components appears very difficult. In fact, only the result on well posedness of (1.1) and (1.2) was obtained in Ma *et al.* (2015). However, if we use divergence-free vector spherical harmonics (Morse & Feshbach, 1953; Bullard & Gellman, 1954), the Maxwell systems (1.1) and (1.2), in the case D is a sphere, can be reduced to two sequences of one-dimensional problems, which are completely decoupled and the same as those obtained from the Helmholtz equations (1.5) (note: one sequence is with the boundary conditions (1.6), but the other is with a slightly different boundary condition at $r = a$). Therefore, we can carry out wavenumber explicit analysis for these decoupled problems, leading to wavenumber explicit estimates for the Maxwell equations in a spherical shell with exact TBC.

There has been a longstanding research interest in wavenumber explicit estimates for the Helmholtz and Maxwell equations. In particular, much effort has been devoted to the Helmholtz problems (see, e.g., Douglas *et al.*, 1993; Ihlenburg & Babuška, 1995; Babuška & Sauter, 2000; Demkowicz & Ihlenburg, 2001; Ainsworth, 2004; Shen & Wang, 2005; Cummings & Feng, 2006; Hetmaniuk, 2007; Shen & Wang, 2007; Chandler-Wilde & Monk, 2008; Ganesh & Hawkins, 2008, 2009; Feng & Wu, 2011; Melenk & Sauter, 2011; Moiola & Spence, 2014; Spence, 2014; Baskin *et al.*, 2016 as a partial list of literature). The Rellich identities played an essential role in obtaining wavenumber explicit estimates for the Helmholtz equation in a star-shaped domain (cf. Melenk, 1995; Shen & Wang, 2005; Cummings & Feng, 2006; Hetmaniuk, 2007; Shen & Wang, 2007; Chandler-Wilde & Monk, 2008; Melenk & Sauter, 2011). In this article, we shall also use a Rellich-type identity on one-dimensional equations reduced from the Helmholtz or Maxwell equations to derive wavenumber explicit estimates.

It is noteworthy that most of the results were established for the Helmholtz equation with an approximate boundary condition: $\partial_r U - \mathrm{i}kU = 0$. However, as shown in Shen & Wang (2007) and Chandler-Wilde & Monk (2008), the presence of the exact DtN boundary condition brought about significant challenges for the analysis. It is also important to point out that some new estimates for more general settings were recently obtained in Moiola & Spence (2014), Spence (2014) and Baskin *et al.* (2016). On the other hand, Hiptmair *et al.* (2011) (and Feng, 2011 independently) extended the argument based on the Rellich identities to the time-harmonic Maxwell equations and derived for the first time the wavenumber explicit estimates, but with the approximate boundary condition: $(\nabla \times \boldsymbol{E}) \times \boldsymbol{e}_r - \mathrm{i}k\boldsymbol{E}_S = \boldsymbol{h}$.

The main purposes of this article are to extend the analysis in Shen & Wang (2007) to the Maxwell equations (1.1) and (1.2), and in the meantime, provide an essential improvement, which is critical to obtaining the desired estimate for the Maxwell equations, to an estimate for the Helmholtz equation in Shen & Wang (2007). We demonstrate that the spectral algorithm and analysis for the Maxwell equations in the spherical shell are essential for dealing with the perturbed scattering problem by using the TFE approach (David & Fernando, 2004).

The rest of the article is organized as follows. In Section 2, we conduct a delicate study of the DtN kernel in (2.2) and use the new estimates to improve the estimates for the Helmholtz equation (cf. Lemma 2.3 and Theorem 2.4), by removing the factor $k^{1/3}$ in Shen & Wang (2007, Theorem 3.1). Using the divergence-free VSH expansion of the electric field, we reduce in Section 3 the Maxwell systems (1.1) and (1.2) in the spherical shell to two sequences of decoupled one-dimensional BVPs in the radial direction. This is essential to derive the wavenumber explicit bounds in Theorem 3.10. In Section 4, we study a spectral approximation of the reduced Maxwell equations and derive the corresponding wavenumber explicit error estimates for the one-dimensional problems (cf. Lemmas 4.3 and 4.5), which finally lead to the wavenumber explicit error estimates for the Maxwell system (cf. Theorem 4.6). In Section 5, we apply the TFE (David & Fernando, 2004) to deal with an ϵ-perturbed scatterer, and using the general framework derived in Nicholls & Shen (2009), we obtain rigorous wavenumber explicit error estimates for the complete algorithm. Some concluding remarks are presented in the last section.

2. Improved wavenumber explicit estimates for the Helmholtz equation

In this section, we improve the *a priori* estimates for the Helmholtz equations (1.5) and (1.6) in Shen & Wang (2007, Theorem 3.1), where the DtN operator is defined by

$$\mathcal{T}_b[U^{a,b}] = \sum_{l=1}^{\infty}\sum_{|m|=0}^{l} k\frac{h_l^{(1)'}(kb)}{h_l^{(1)}(kb)}\,\widehat{U}_l^m\,Y_l^m, \quad \text{where}\quad \widehat{U}_l^m = \int_S U^{a,b}\big|_{r=b}\,\overline{Y_l^m}\,\mathrm{d}S, \tag{2.1}$$

and $\{Y_l^m\}$ are spherical harmonics (SPH) defined on the unit spherical surface S (cf. Appendix A).

2.1 *Properties of the DtN kernel*

The key is to conduct a delicate analysis of the DtN kernel:

$$\mathcal{T}_{l,\kappa} =: \frac{h_l^{(1)'}(\kappa)}{h_l^{(1)}(\kappa)} \quad l \geq 1,\ \ \kappa > 0. \tag{2.2}$$

Recall that (cf. Shen & Wang, 2007, (2.16))

$$\mathrm{Re}(\mathcal{T}_{l,\kappa}) = -\frac{1}{2\kappa} + \frac{J_\nu(\kappa)J_\nu'(\kappa)+Y_\nu(\kappa)Y_\nu'(\kappa)}{J_\nu^2(\kappa)+Y_\nu^2(\kappa)}, \quad \mathrm{Im}(\mathcal{T}_{l,\kappa}) = \frac{2}{\pi\kappa}\frac{1}{J_\nu^2(\kappa)+Y_\nu^2(\kappa)} \tag{2.3}$$

for $\nu = l+1/2$, where J_ν and Y_ν are Bessel functions of the first and second kinds, respectively, of order ν (cf. Abramowitz & Stegun, 1964). Alternatively, we can formulate

$$\mathrm{Re}(\mathcal{T}_{l,\kappa}) = \frac{l}{\kappa} - \frac{Y_{\nu+1}(\kappa)}{Y_\nu(\kappa)} - \mathrm{Im}(\mathcal{T}_{l,\kappa})\frac{J_\nu(\kappa)}{Y_\nu(\kappa)} = -\frac{1}{2\kappa} + \frac{Y_\nu'(\kappa)}{Y_\nu(\kappa)} - \mathrm{Im}(\mathcal{T}_{l,\kappa})\frac{J_\nu(\kappa)}{Y_\nu(\kappa)}, \tag{2.4}$$

which can be derived from (2.3) and the properties of Bessel functions. Recall that (see Nédélec, 2001, Page 87):

$$-\frac{l+1}{\kappa} \leq \mathrm{Re}(\mathcal{T}_{l,\kappa}) < -\frac{1}{\kappa}, \quad 0 < \mathrm{Im}(\mathcal{T}_{l,\kappa}) < 1. \tag{2.5}$$

In what follows, let $0 < \theta_0 < 1$ be a prescribed constant, and let

$$\kappa_0 = \sqrt{\theta_0/2}\,(1-\theta_0)^{-3/2} \quad (\text{e.g., } \kappa_0 \approx 21.21, \text{ if } \theta_0 = 0.9). \tag{2.6}$$

On the basis of asymptotic properties of Bessel functions, we shall carry out the analysis separately for four cases (note: in the course of the analysis, we shall show how these arise (see (B.10))):

$$\rho := \frac{\nu}{\kappa} \in (0,\theta_0) \cup [\theta_0,\vartheta_1] \cup (\vartheta_1,\vartheta_2) \cup [\vartheta_2,\infty) \quad \text{for} \quad \nu = l + 1/2, \quad l \geq 1, \tag{2.7}$$

where $\kappa > \kappa_0$ is fixed, and

$$\begin{aligned}
\vartheta_1 := \vartheta_1(\kappa) &= \frac{1}{2}\left(\sqrt[3]{1+\sqrt{1+\frac{2}{27\kappa^2}}} + \sqrt[3]{1-\sqrt{1+\frac{2}{27\kappa^2}}}\right)^3, \\
\vartheta_2 := \vartheta_2(\kappa) &= \frac{1}{2}\left(\sqrt[3]{1+\sqrt{1-\frac{2}{27\kappa^2}}} + \sqrt[3]{1-\sqrt{1-\frac{2}{27\kappa^2}}}\right)^3.
\end{aligned} \tag{2.8}$$

LEMMA 2.1 Let $\theta_0, \kappa_0, \vartheta_1$ and ϑ_2 be the same as in (2.6) and (2.8). Then we have

$$0 < \vartheta_1 < 1 < \vartheta_2, \quad \forall \kappa > \sqrt{2/27}, \tag{2.9}$$

and

$$\vartheta_1 = 1 - \frac{1}{\sqrt[3]{2}\,\kappa^{2/3}} + O(\kappa^{-4/3}), \quad \vartheta_2 = 1 + \frac{1}{\sqrt[3]{2}\,\kappa^{2/3}} + O(\kappa^{-4/3}). \tag{2.10}$$

Moreover, if $\kappa > \kappa_0$ then we have $\theta_0 < \vartheta_1$.

Proof. We examine the function $f(t) := \sqrt[3]{1+t} + \sqrt[3]{1-t}$, $t \geq 0$, associated with (2.8). One verifies readily that $f'(t) < 0$ for all $t > 0, t \neq 1$. Thus, $f(t)$ is monotonically decreasing, and

$$\sqrt[3]{2\vartheta_1} = f\big(\sqrt{1+2/(27\kappa^2)}\big) < f(1) < f\big(\sqrt{1-2/(27\kappa^2)}\big) = \sqrt[3]{2\vartheta_2}, \tag{2.11}$$

which implies (2.9). It is evident that

$$t_1 := \sqrt{1+\frac{2}{27\kappa^2}} = 1 + \frac{1}{27\kappa^2} + O(\kappa^{-4}). \tag{2.12}$$

A direct calculation from (2.8) yields

$$2\vartheta_1 = 2 + 3\{(1+t_1)^{2/3}(1-t_1)^{1/3} + (1+t_1)^{1/3}(1-t_1)^{2/3}\} = 2 - \frac{\sqrt[3]{2}}{\kappa^{2/3}}\big(\sqrt[3]{1+t_1} + \sqrt[3]{1-t_1}\big)$$
$$= 2 - \frac{\sqrt[3]{2}}{\kappa^{2/3}}\left(\sqrt[3]{2 + \frac{1}{27\kappa^2}} - \frac{1}{3\kappa^{2/3}}\right) + O(\kappa^{-2}) = 2 - \frac{\sqrt[3]{2}}{\kappa^{2/3}}\left(\sqrt[3]{2} - \frac{1}{3\kappa^{2/3}} + O(\kappa^{-2})\right) + O(\kappa^{-2}),$$

which implies the asymptotic estimate of ϑ_1 in (2.10). Similarly, we can derive the estimate of ϑ_2.

We now show that $\theta_0 < \vartheta_1$, for all $\kappa > \kappa_0$ with κ_0 given by (2.6). Observe from (2.11)–(2.12) that $\sqrt[3]{2\vartheta_1} = f(t_1)$, so it suffices to show $\sqrt[3]{2\vartheta_0} < \sqrt[3]{2\vartheta_1} = f(t_1)$. Using the monotonic decreasing property of f, we just require $f^{-1}(\sqrt[3]{2\theta_0}) > t_1 = \sqrt{1 + 2/(27\kappa^2)}$, so working out f^{-1}, we can obtain κ_0 in (2.6). □

In what follows, the expression '$A \lesssim B$' means that there exists a positive constant C, only depending on the domain (but independent of k and the related unknowns or functions), such that $A \leq CB$. As with Abramowitz & Stegun (1964) and Olver *et al.* (2010), the notation '$A \sim B$' stands for $A(\nu) = B(\nu) + \mathrm{LH}(\nu)$ or $A(\nu) = B(\nu)(1 + \mathrm{LH}(\nu))$, where for sufficiently small or large parameter ν, $\mathrm{LH}(\nu)$ is some insignificant lower-order or higher-order term to be dropped in the bound or estimate.

We have the following estimates of $\mathrm{Re}(\mathcal{T}_{l,\kappa})$ and the refined estimates of $\mathrm{Im}(\mathcal{T}_{l,\kappa})$ in Shen & Wang (2007, (2.35)).

THEOREM 2.2 Let $\theta_0, \vartheta_1, \vartheta_2$ and κ_0 be the same as in (2.6) and (2.8). Denote $\nu = l + 1/2$ and $\rho = \nu/\kappa$. Then for any $\kappa > \kappa_0$, we have the approximation

$$\mathrm{Re}(\mathcal{T}_{l,\kappa}) \sim E^R_{l,\kappa}, \quad \mathrm{Im}(\mathcal{T}_{l,\kappa}) \sim E^I_{l,\kappa} \quad \forall\, l \geq 1, \tag{2.13}$$

where

(i) for $\rho = \nu/\kappa \in (0, \theta_0)$,

$$E^R_{l,\kappa} = -\frac{1}{2\kappa}\left(1 + \frac{1}{1-\rho^2}\right), \quad E^I_{l,\kappa} = \sqrt{1-\rho^2}\,; \tag{2.14}$$

(ii) for $\rho = \nu/\kappa \in [\theta_0, \vartheta_1]$,

$$E^R_{l,\kappa} = -\frac{1}{2\kappa}\left(1 + \frac{1}{2(1-\rho)}\right), \quad E^I_{l,\kappa} = \sqrt{2\rho(1-\rho)}\,; \tag{2.15}$$

(iii) for $\rho = \nu/\kappa \in (\vartheta_1, \vartheta_2)$,

$$E^R_{l,\kappa} = -\frac{1}{c_1}\left(\frac{2}{\nu}\right)^{1/3}\big(1 + 2c_1 t + c_2 t^2\big) - \frac{1}{2\kappa}, \quad E^I_{l,\kappa} = \sqrt{3}c_1\rho\big(1 - 2c_1 t\big)\left(\frac{2}{\nu}\right)^{1/3}, \tag{2.16}$$

where $t = -\sqrt[3]{2}\,(\kappa - \nu)/\sqrt[3]{\nu}$ (note: $|t| < 1$), and

$$c_1 = \frac{3^{\frac{1}{3}}}{2}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \approx 0.3645, \quad c_2 = \frac{1 - 16c_1^3}{2c_1} \approx 0.3088; \tag{2.17}$$

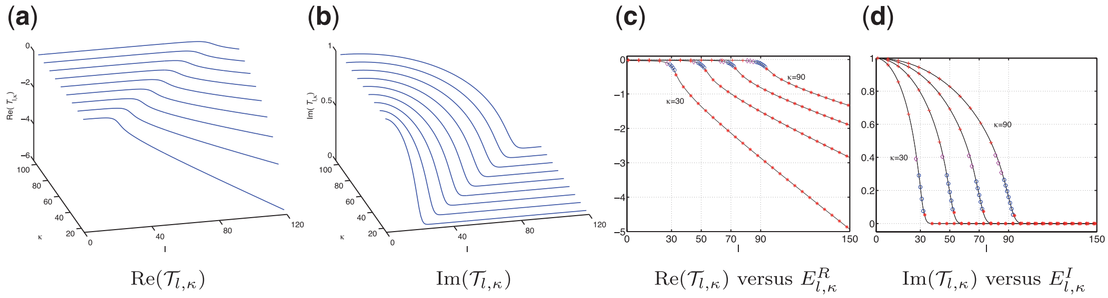

FIG. 1. (a,b) Real and imaginary parts of $\mathcal{T}_{l,\kappa}$ with various samples $(l,\kappa) \in [0,120] \times [1,100]$. (c) $\mathrm{Re}(\mathcal{T}_{l,\kappa})$ (solid line) against $E^R_{l,\kappa}$. (d) $\mathrm{Im}(\mathcal{T}_{l,\kappa})$ (solid line) against $E^I_{l,\kappa}$ with $\kappa = 30, 50, 70, 90$ (note in (c–d), '+' for $\rho = \nu/\kappa \in (0,\theta_0)$, '$\diamond$' for $\rho \in [\theta_0, \vartheta_1]$, '$\circ$' for $\rho \in (\vartheta_1, \vartheta_2)$ and '$*$' for $\rho \in [\vartheta_2, \infty)$).

(iv) for $\rho = \nu/\kappa \in [\vartheta_2, \infty)$,

$$E^R_{l,\kappa} = -\sqrt{\rho^2 - 1} - \frac{1}{2\kappa}\left(1 - \frac{1}{\rho^2 - 1}\right), \quad E^I_{l,\kappa} = \sqrt{\rho^2 - 1}\, e^{-2\nu\Psi}, \quad \text{where} \tag{2.18}$$

$$\Psi = \ln(\rho + \sqrt{\rho^2 - 1}) - \frac{\sqrt{\rho^2 - 1}}{\rho} \quad \rho > 1. \tag{2.19}$$

We provide the proof of this theorem in Appendix B. In Fig. 1, we depict in (a,b) the graphs of $\mathrm{Re}(\mathcal{T}_{l,\kappa})$ and $\mathrm{Im}(\mathcal{T}_{l,\kappa})$ for various l and κ, and in (c,d), the exact value and approximations in Theorem 2.2 for various samples of κ.

2.2 *Improved estimates for the Helmholtz equation*

We first introduce some notation. Let $I := (a,b)$ and $\omega > 0$ be a generic weight function defined on a generic domain Λ. The weighted Sobolev space $H^s_\omega(\Lambda)$ with $s \geq 0$ is defined as usual in Adams (1975). In particular, $L^2_\omega(\Lambda)$ is the weighted L^2-space with the inner product and norm:

$$(u,v)_{\omega,\Lambda} = \int_\Lambda u \cdot \bar{v}\omega \, \mathrm{d}\boldsymbol{x} \quad \|u\|_{\omega,\Lambda} = (u,u)^{\frac{1}{2}}_{\omega,\Lambda},$$

which also apply to vector-valued functions. If $\omega \equiv 1$ or $\Lambda = I = (a,b)$, we drop ω or Λ in the notation.

The inner product of $L^2(S)$ is defined as

$$\langle U, V\rangle_S = \int_0^{2\pi}\int_0^{\pi} u\bar{v} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi.$$

We also use the anisotropic Sobolev spaces, e.g., $H^{s'}_p(S; H^s_\omega(I))$ (where 'p' stands for the periodicity) with the norm characterized by the spherical harmonic expansion coefficients $\widehat{U}^m_l$ of U (cf. Shen & Wang, 2007, (1.8)).

Denote ${}_0H^1(I) = \{v \in H^1(I) : v(a) = 0\}$ and $\varpi = r^2$. A weak form of (1.5)–(1.6) is to find $U^{a,b} \in H^1_p(S; {}_0H^1(I))$ such that (cf. Shen & Wang, 2007, (3.9)):

$$\begin{aligned}\mathbb{B}(U^{a,b}, V) &= (\partial_r U^{a,b}, \partial_r V)_{\varpi,\Omega} + (\nabla_S U^{a,b}, \nabla_S V)_\Omega - k^2 (U^{a,b}, V)_{\varpi,\Omega} - b^2 \langle \mathcal{T}_b U^{a,b}, V\rangle_S \\ &= (F^{a,b}, V)_{\varpi,\Omega} + \langle H, V\rangle_S \quad \forall\, V \in H^1_p(S; {}_0H^1(I)).\end{aligned} \tag{2.20}$$

We expand $U^{a,b}, F^{a,b}, H$ in SPH series as

$$\left\{U^{a,b}, F^{a,b}, H\right\} = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \left\{\widehat{U}_l^m(r), \widehat{F}_l^m(r), \widehat{H}_l^m\right\} Y_l^m(\theta,\varphi). \tag{2.21}$$

Taking $V = \widehat{V}_{l'}^{m'}(r) Y_{l'}^{m'}$ in (2.20) and using the property of SPH (cf. Appendix A), we obtain the corresponding weak form for each mode (l, m) : find $u = \widehat{U}_l^m \in {}_0H^1(I)$ such that

$$\begin{aligned}\mathbb{B}_l^m(u, v) &:= (u', v')_\varpi + \beta_l (u, v) - k^2 (u, v)_\varpi - kb^2 \mathcal{T}_{l,kb} u(b) \bar{v}(b) \\ &= (f, v)_\varpi + b^2 h \bar{v}(b) \quad \forall v \in {}_0H^1(I),\end{aligned} \tag{2.22}$$

where $\beta_l = l(l+1), f = \widehat{F}_l^m$ and $h = \widehat{H}_l^m$. Here, we drop the weight function ϖ in the space ${}_0H^1(I)$ as it is uniformly bounded below away from 0 on I.

We have the following improved estimate in the sense that $k^{1/3}$ is removed from Shen & Wang (2007, Lemma 3.1).

LEMMA 2.3 Let u be the solutions of (2.22). If $f \in L^2(I)$ then we have that for all $k \geq k_0 > 0$ (for some fixed constant k_0) and for $l \geq 1, 0 \leq |m| \leq l$,

$$\|u'\|_\varpi^2 + \beta_l \|u\|^2 + k^2 \|u\|_\varpi^2 \lesssim \|f\|_\varpi^2 + |h|^2. \tag{2.23}$$

Proof. Taking $v = u$ in (2.22), we obtain

$$\|u'\|_\varpi^2 + \beta_l \|u\|^2 - k^2 \|u\|_\varpi^2 - kb^2 \operatorname{Re}(\mathcal{T}_{l,kb}) |u(b)|^2 = \operatorname{Re}(f, u)_\varpi + b^2 \operatorname{Re}(h \bar{u}(b)), \tag{2.24a}$$

$$-kb^2 \operatorname{Im}(\mathcal{T}_{l,kb}) |u(b)|^2 = \operatorname{Im}(f, u)_\varpi + b^2 \operatorname{Im}(h \bar{u}(b)). \tag{2.24b}$$

Next taking $v = 2(r-a)u'$ in (2.22) and following the derivations in (Shen & Wang, 2007, (3.26)–(3.28)), we obtain

$$\begin{aligned}&b^2 |I| |u'(b)|^2 + \beta_l |I| |u(b)|^2 + 2a \|\sqrt{r} u'\|^2 + k^2 \int_a^b \left(3 - \frac{2a}{r}\right) |u|^2 r^2 \, \mathrm{d}r \\ &\quad = \|u'\|_\varpi^2 + \beta_l \|u\|^2 + k^2 b^2 |I| |u(b)|^2 + 2 \operatorname{Re}(f, (r-a)u')_\varpi \\ &\qquad + 2b^2 |I| \operatorname{Re}\big(h \bar{u}'(b)\big) + 2kb^2 |I| \operatorname{Re}\big\{\mathcal{T}_{l,kb}\, u(b) \bar{u}'(b)\big\},\end{aligned} \tag{2.25}$$

where $|I| = b - a$. Substituting $\|u'\|_{\varpi}^2 + \beta_l \|u\|^2$ in the identity (2.24a) into the above and collecting the terms, we obtain

$$
\begin{aligned}
& b^2|I||u'(b)|^2 + \left\{\beta_l |I| - kb^2 \operatorname{Re}(\mathcal{T}_{l,kb})\right\}|u(b)|^2 + 2a\|\sqrt{r}u'\|^2 + 2k^2 \int_a^b \left(1 - \frac{a}{r}\right)|u|^2 r^2 \,\mathrm{d}r \\
&\quad = k^2 b^2 |I||u(b)|^2 + 2kb^2|I| \operatorname{Re}\left\{\mathcal{T}_{l,kb}\, u(b)\, \bar{u}'(b)\right\} + 2b^2|I| \operatorname{Re}\left(h\bar{u}'(b)\right) \\
&\qquad + 2\operatorname{Re}(f, (r-a)u')_{\varpi} + b^2 \operatorname{Re}(h\bar{u}(b)) + \operatorname{Re}(f, u)_{\varpi}.
\end{aligned}
\tag{2.26}
$$

Hereafter, let C and $\{C_i, \varepsilon_i\}$ be generic constants independent of k, l, m, and any function. Using the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
& 2kb^2|I| \left|\operatorname{Re}\left\{\mathcal{T}_{l,kb}\, u(b)\, \bar{u}'(b)\right\}\right| \le \varepsilon_1 b^2 |I||u'(b)|^2 + \varepsilon_1^{-1} k^2 b^2 |I||\mathcal{T}_{l,kb}|^2 |u(b)|^2; \\
& 2b^2|I| \left|\operatorname{Re}\left(h\bar{u}'(b)\right)\right| \le \varepsilon_2 b^2 |I||u'(b)|^2 + \varepsilon_2^{-1} b^2 |I||h|^2; \\
& b^2 \left|\operatorname{Re}(h\bar{u}(b))\right| \le \varepsilon_3 kb^2 \,|\operatorname{Re}(\mathcal{T}_{l,kb})||u(b)|^2 + \frac{b^2}{\varepsilon_3 k\, |\operatorname{Re}(\mathcal{T}_{l,kb})|}|h|^2; \\
& 2\left|\operatorname{Re}(f, (r-a)u')_{\varpi}\right| \le \varepsilon_4 \|\sqrt{r}u'\|^2 + \varepsilon_4^{-1} b|I|^2 \|f\|_{\varpi}^2; \\
& \left|\operatorname{Re}(f, u)_{\varpi}\right| \le \varepsilon_5 \|u\|_{\varpi}^2 + (4\varepsilon_5)^{-1}\|f\|_{\varpi}^2.
\end{aligned}
\tag{2.27}
$$

Thus, by choosing suitable $\{\varepsilon_i\}$, we obtain from (2.26)–(2.27) that

$$
C_1 b^2 |I||u'(b)|^2 + D_{l,k}|I||u(b)|^2 + C_2 a\|\sqrt{r}u'\|^2 + C_3 k^2 \|u\|_{\varpi}^2 \lesssim \|f\|_{\varpi}^2 + \left(1 + \frac{1}{k\, |\operatorname{Re}(\mathcal{T}_{l,kb})|}\right)|h|^2,
\tag{2.28}
$$

where $C_1 = 1 - (\varepsilon_1 + \varepsilon_2)$, $C_2 = 2 - \varepsilon_4$, $C_3 = 2(1 - a/\xi) - \varepsilon_5/k^2$ with $\xi \in (a, b)$ and

$$
D_{l,k} = \beta_l - (1 - \varepsilon_3)kb^2|I|^{-1} \operatorname{Re}(\mathcal{T}_{l,kb}) - k^2 b^2 \left(1 + \varepsilon_1^{-1}|\mathcal{T}_{l,kb}|^2\right).
\tag{2.29}
$$

It remains to estimate $D_{l,k}$, which can be negative for small l. According to the estimates in Theorem 2.2, we conduct the analysis for four different cases as in (2.7).

(i) If $\rho = \frac{\nu}{kb} \in (0, \theta_0]$ for fixed $0 < \theta_0 < 1$, we obtain from (2.24b) that

$$
\begin{aligned}
k^2 b^2 \,|u(b)|^2 &\le \frac{k}{|\operatorname{Im}(\mathcal{T}_{l,kb})|}\left\{|\operatorname{Im}(f, u)_{\varpi}| + b^2 \,|\operatorname{Im}(h\overline{u(b)})|\right\} \\
&\le \frac{\varepsilon_7}{2} k^2 \|u\|_{\varpi}^2 + \frac{\|f\|_{\varpi}^2}{2\varepsilon_7 |\operatorname{Im}(\mathcal{T}_{l,kb})|^2} + \frac{k^2 b^2}{2}|u(b)|^2 + \frac{|h|^2}{2|\operatorname{Im}(\mathcal{T}_{l,kb})|^2}.
\end{aligned}
\tag{2.30}
$$

By (2.14), $\operatorname{Im}(\mathcal{T}_{l,kb})$ in this range behaves like a constant, so (2.30) implies

$$
k^2 b^2 \,|u(b)|^2 \le \varepsilon_7 k^2 \|u\|_{\varpi}^2 + C\left(\|f\|_{\varpi}^2 + |h|^2\right).
\tag{2.31}
$$

By (2.14), $|\mathcal{T}_{l,kb}|^2 \le C$, so $D_{l,k} \le -Ck^2b^2$. Therefore, using (2.28) and (2.31) leads to

$$\|\sqrt{r}u'\|^2 + k^2\|u\|_{\varpi}^2 + k^2|u(b)|^2 \le C\big(\|f\|_{\varpi}^2 + |h|^2\big). \tag{2.32}$$

Thus, we derive the desired estimate in this case from (2.24a) and (2.32).

(ii) For $\rho = \frac{\nu}{kb} \in (\theta_0, \vartheta_1]$, we first show that for any $\bar{c}_0 \in (1-\theta_0, 1/\sqrt[3]{2})$ and $kb > 1$, there exists a unique $\gamma_0 \in [1/3, 1)$ such that

$$\rho = 1 - \bar{c}_0(kb)^{\gamma_0 - 1}, \quad \text{i.e.,} \quad \gamma_0 = 1 + \frac{\ln((1-\rho)/\bar{c}_0)}{\ln(kb)}. \tag{2.33}$$

Apparently, γ_0 decreases with respect to ρ, so by (2.10),

$$\frac{1}{3} - \frac{\ln(\sqrt[3]{2}\bar{c}_0)}{\ln(kb)} + \frac{\ln(1+O((kb)^{-2/3})}{\ln(kb)} = 1 + \frac{\ln((1-\vartheta_1)/\bar{c}_0)}{\ln(kb)} \le \gamma_0 < 1 + \frac{\ln((1-\theta_0)/\bar{c}_0)}{\ln(kb)}. \tag{2.34}$$

Then one verifies readily that for $\bar{c}_0 \in (1-\theta_0, 1/\sqrt[3]{2})$, we have $\gamma_0 \in [1/3, 1)$. In view of (2.33), we can write

$$\nu = kb - \bar{c}_0(kb)^{\gamma_0}. \tag{2.35}$$

Thus, by (2.15),

$$\mathrm{Re}(\mathcal{T}_{l,kb}) \sim -\frac{1}{2\bar{c}_0}(kb)^{-\gamma_0}, \quad \mathrm{Im}(\mathcal{T}_{l,kb}) \sim \sqrt{2\bar{c}_0}(kb)^{(\gamma_0-1)/2} \quad |\mathcal{T}_{l,kb}|^2 \sim 2\bar{c}_0(kb)^{\gamma_0-1}, \tag{2.36}$$

which implies

$$D_{l,k} \sim \nu^2 - \frac{1}{4} + (1-\varepsilon_3)\frac{b}{2|I|\bar{c}_0}(kb)^{1-\gamma_0} - k^2b^2\big(1 + \varepsilon_1^{-1}2\bar{c}_0(kb)^{\gamma_0-1}\big) \sim -2\bar{c}_0(1+\varepsilon_1^{-1})(kb)^{\gamma_0+1}. \tag{2.37}$$

By (2.24b) and the Cauchy–Schwarz inequality,

$$\begin{aligned}(kb)^{\gamma_0+1}|u(b)|^2 &\le \frac{(kb)^{\gamma_0}}{|\mathrm{Im}(\mathcal{T}_{l,kb})|}\big\{|\mathrm{Im}(f,u)_{\varpi}| + b^2\,|\mathrm{Im}(h\bar{u}(b))|\big\} \\ &\le \frac{\varepsilon_7}{2}k^2\|u\|_{\varpi}^2 + \frac{(kb)^{2\gamma_0-2}}{2\varepsilon_7|\mathrm{Im}(\mathcal{T}_{l,kb})|^2}\|f\|_{\varpi}^2 + \frac{(kb)^{\gamma_0+1}}{2}|u(b)|^2 + \frac{(kb)^{\gamma_0-1}}{2|\mathrm{Im}(\mathcal{T}_{l,kb})|^2}|h|^2.\end{aligned} \tag{2.38}$$

Then by (2.36) and (2.38),

$$(kb)^{\gamma_0+1}\,|u(b)|^2 \le \varepsilon_7k^2\|u\|_{\varpi}^2 + C\big((kb)^{\gamma_0-1}\|f\|_{\varpi}^2 + |h|^2\big). \tag{2.39}$$

Thus, we derive from (2.28) that

$$\|\sqrt{r}u'\|^2 + k^2\|u\|^2_{\varpi} + (kb)^{\gamma_0+1}|u(b)|^2 \le C\big(\|f\|^2_{\varpi} + |h|^2\big). \tag{2.40}$$

Therefore, we obtain (2.23) from (2.24a) and (2.40).

(iii) If $\rho = \frac{\nu}{kb} \in (\vartheta_1, \vartheta_2]$, we find from (2.10) that

$$kb - \sqrt[3]{\frac{kb}{2}} + O(k^{-1/3}) < \nu \le kb + \sqrt[3]{\frac{kb}{2}} + O(k^{-1/3}). \tag{2.41}$$

By (2.16),

$$\mathrm{Re}(\mathcal{T}_{l,kb}) \sim -\bar{c}_1(kb)^{-1/3}, \quad \mathrm{Im}(\mathcal{T}_{l,kb}) \sim \bar{c}_2(kb)^{-1/3} \quad |\mathcal{T}_{l,kb}|^2 \sim \bar{c}_3(kb)^{-2/3}, \tag{2.42}$$

where $\{\bar{c}_i\}$ are some positive constants independent of k, l. We can follow the same procedure as for Case (ii) (but with $\gamma_0 = 1/3$) to derive

$$\|\sqrt{r}u'\|^2 + k^2\|u\|^2_{\varpi} + (kb)^{4/3}|u(b)|^2 \le C\big(\|f\|^2_{\varpi} + |h|^2\big). \tag{2.43}$$

Similarly, (2.23) follows from (2.24a) and (2.43).

(iv) If $\rho = \frac{\nu}{kb} \in (\vartheta_2, \infty)$, we find from (2.18) that $\mathrm{Im}(\mathcal{T}_{l,kb})$ decays exponentially with respect to l, so we cannot get a useful bound of $|u(b)|$ from (2.24b). We therefore consider two cases:

$$\text{(a)} \quad \nu = kb + \bar{c}_5(kb)^{\gamma_1} \quad \text{with} \quad 1/3 < \gamma_1 < 1; \quad \text{(b)} \quad \nu \ge \eta\, kb, \tag{2.44}$$

for constant $\bar{c}_5 \in (\eta - 1, 1/\sqrt[3]{2})$ and $1 < \eta < 1 + 1/\sqrt[3]{2}$. Here, we show that Case (a) can cover $\rho \in (\vartheta_2, \eta)$. Indeed, similar to (2.33)–(2.34), we have $\rho = 1 + \bar{c}_5(kb)^{\gamma_1 - 1}$, and

$$\frac{1}{3} - \frac{\ln(\sqrt[3]{2}\bar{c}_5)}{\ln(kb)} + \frac{\ln(1 + O((kb)^{-2/3})}{\ln(kb)} = 1 + \frac{\ln((\vartheta_2 - 1)/\bar{c}_5)}{\ln(kb)} < \gamma_1 < 1 + \frac{\ln((\eta - 1)/\bar{c}_5)}{\ln(kb)}. \tag{2.45}$$

This implies if $\bar{c}_5 \in (\eta - 1, 1/\sqrt[3]{2})$ and $1 < \eta < 1 + 1/\sqrt[3]{2}$, then $1/3 < \gamma_1 < 1$ and we can write ν in the form of (a).

In the first case, we derive from (2.18) that

$$\mathrm{Re}(\mathcal{T}_{l,kb}) \sim \sqrt{2\bar{c}_5}(kb)^{(\gamma_1 - 1)/2} \quad |\mathcal{T}_{l,kb}|^2 \sim 2\bar{c}_5(kb)^{\gamma_1 - 1}, \quad D_{l,k} \sim -2\bar{c}_5(\varepsilon_1^{-1} - 1)(kb)^{\gamma_1 + 1}, \tag{2.46}$$

where we recall that $\varepsilon_1 < 1$. Noticing that

$$\beta_l\|u\|^2 - k^2\|u\|^2_{\varpi} \ge (\beta_l - k^2b^2)\|u\|^2 \ge 0 \tag{2.47}$$

and $\mathrm{Re}(\mathcal{T}_{l,kb}) < 0$, we deduce from (2.24a) that

$$-kb^2\,\mathrm{Re}(\mathcal{T}_{l,kb})|u(b)|^2 \le |\mathrm{Re}(f, u)_{\varpi}| + b^2\,|\mathrm{Re}(h\bar{u}(b))|. \tag{2.48}$$

Using (2.46), (2.48) and following the derivation of (2.38), we can get

$$(kb)^{\gamma_1+1}\,|u(b)|^2 \le \varepsilon_8 k^2 \|u\|_{\varpi}^2 + C\big((kb)^{\gamma_1-1}\|f\|_{\varpi}^2 + |h|^2\big). \tag{2.49}$$

We then derive from (2.28) that

$$\|\sqrt{r}u'\|^2 + k^2\|u\|_{\varpi}^2 + (kb)^{\gamma_1+1}|u(b)|^2 \le C\big(\|f\|_{\varpi}^2 + |h|^2\big). \tag{2.50}$$

Thus, we derive (2.23) for this case from (2.24a) and (2.50).

In the second case of (2.44), we observe from (2.18) that

$$\mathrm{Re}(\mathcal{T}_{l,kb}) \sim -\frac{\nu}{kb} \quad |\mathcal{T}_{l,kb}|^2 \sim \frac{\nu^2}{k^2b^2}, \tag{2.51}$$

which implies

$$D_{l,k} \sim \nu^2 - \frac{1}{4} + (1-\varepsilon_3)\frac{b\nu}{|I|} - k^2b^2 - \varepsilon_1^{-1}\nu^2 \sim -\bar{c}_6\,\beta_l. \tag{2.52}$$

Then, by (2.51) and (2.48),

$$\beta_l|u(b)|^2 \le \varepsilon_8\beta_l\|u\|^2 + C\big(\|f\|_{\varpi}^2 + |h|^2\big). \tag{2.53}$$

We then derive from (2.28) that

$$\|\sqrt{r}u'\|^2 + k^2\|u\|_{\varpi}^2 + \beta_l|u(b)|^2 \le C\big(\|f\|_{\varpi}^2 + |h|^2\big). \tag{2.54}$$

Finally, we obtain (2.23) from (2.24a) and (2.54). □

Thanks to the above lemma and the orthogonality of SPH, one can easily derive the following improved result, where a factor of $k^{1/3}$ is removed from the upper bound of Shen & Wang (2007, Theorem 3.1).

THEOREM 2.4 Let $U^{a,b}$ be the solution of (2.20). If $F^{a,b} \in L^2(\Omega)$ and $H \in L^2(S)$ then we have

$$\|\nabla U^{a,b}\|_{\Omega} + k\|U^{a,b}\|_{\Omega} \lesssim \|F^{a,b}\|_{\Omega} + \|H\|_{L^2(S)}. \tag{2.55}$$

REMARK 2.5 Similar wavenumber explicit estimate was derived by Chandler-Wilde & Monk (2008, Lemma 3.8) for general starlike scatterers and $H = 0$, together with an explicit constant in the upper bound. However, the result therein does not imply the mode-by-mode estimate in Lemma 2.3. The analysis in this article essentially relies on the estimates bounded by the corresponding mode of the data.

3. *A priori* estimates for the reduced Maxwell equations

In this section, we perform the wavenumber explicit *a priori* estimates for the Maxwell equations (1.1) and (1.2). The key is to employ a divergence-free vector harmonic expansion of the fields and reduce the problem of interest into two sequences of *decoupled one-dimensional* Helmholtz problems. This decoupling not only leads to a more efficient numerical algorithm, but also greatly simplifies its analysis.

3.1 *Dimension reduction via divergence-free VSH expansions*

Introduce the spaces

$$\mathbb{H}(\mathrm{div};\Omega)=\big\{\boldsymbol{E}\in\boldsymbol{L}^2(\Omega):\mathrm{div}\boldsymbol{E}\in L^2(\Omega)\big\},\quad \mathbb{H}(\mathrm{div}^0;\Omega)=\big\{\boldsymbol{E}\in\mathbb{H}(\mathrm{div};\Omega):\mathrm{div}\boldsymbol{E}=0\big\},\tag{3.1}$$

where $\mathbb{H}(\mathrm{div};\Omega)$ is equipped with the graph norm as defined in Monk (2003, p. 52).

Built upon the SPH $\{Y_l^m\}$, the VSH $\big\{Y_l^m\boldsymbol{e}_r,\nabla_S Y_l^m,\boldsymbol{T}_l^m=\nabla_S Y_l^m\times\boldsymbol{e}_r\big\}$ forms a complete, orthogonal system of $(L^2(S))^3$ and refer to Appendix A for some relevant properties. The following VSH expansion of a solenoidal (or divergence free) field plays an important role in our analysis and spectral algorithm.

Proposition 3.1 For any $\boldsymbol{E}\in(L^2(\Omega))^3$, we expand it as

$$\boldsymbol{E}=v_{2,0}^0\,Y_0^0\boldsymbol{e}_r+\sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\big\{v_{1,l}^m\,\boldsymbol{T}_l^m+v_{2,l}^m\,Y_l^m\boldsymbol{e}_r+v_{3,l}^m\,\nabla_S Y_l^m\big\},\tag{3.2}$$

where

$$v_{1,l}^m=\beta_l^{-1}\langle\boldsymbol{E},\boldsymbol{T}_l^m\rangle_S,\quad v_{2,l}^m=\langle\boldsymbol{E},Y_l^m\boldsymbol{e}_r\rangle_S,\quad v_{3,l}^m=\beta_l^{-1}\langle\boldsymbol{E},\nabla_S Y_l^m\rangle_S,\quad \beta_l=l(l+1).\tag{3.3}$$

If $\boldsymbol{E}\in\mathbb{H}(\mathrm{div}^0;\Omega)$ then we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{2}{r}\right)v_{2,0}^0=0,\quad \frac{r}{\beta_l}\left(\frac{d}{dr}+\frac{2}{r}\right)v_{2,l}^m=v_{3,l}^m,\tag{3.4}$$

and we can write

$$\boldsymbol{E}=u_0^0\,Y_0^0\boldsymbol{e}_r+\sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\big\{u_{1,l}^m\,\boldsymbol{T}_l^m+\nabla\times\big(u_{2,l}^m\,\boldsymbol{T}_l^m\big)\big\},\tag{3.5}$$

where

$$u_0^0=v_{2,0}^0=\frac{c}{r^2},\quad u_{1,l}^m=v_{1,l}^m,\quad u_{2,l}^m=\beta_l^{-1}rv_{2,l}^m,\tag{3.6}$$

with c being an arbitrary constant.

Proof. Since $\mathrm{div}(v_{1,l}^m\boldsymbol{T}_l^m)=0$ (cf. (A.4)), we obtain from (4.8) and (A.6)–(A.7) that

$$\mathrm{div}\,\boldsymbol{E}=\left(\frac{d}{dr}+\frac{2}{r}\right)v_{2,0}^0+\sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{\left(\frac{d}{dr}+\frac{2}{r}\right)v_{2,l}^m-\frac{\beta_l}{r}v_{3,l}^m\right\}Y_l^m.\tag{3.7}$$

Then the identities in (3.4) follow from $\mathrm{div}\,\boldsymbol{E}=0$ immediately.

Note that the equation of $v_{2,0}^0$ in (3.4) has the general solution: $v_{2,0}^0 = c/r^2$. To derive (3.5) under (3.6), it suffices to show that

$$\beta_l^{-1}\nabla \times \left(rv_{2,l}^m \boldsymbol{T}_l^m\right) = v_{2,l}^m Y_l^m \boldsymbol{e}_r + v_{3,l}^m \nabla_S Y_l^m. \tag{3.8}$$

It follows from a direct calculation using (A.4), that is,

$$\beta_l^{-1}\nabla \times \left(rv_{2,l}^m \boldsymbol{T}_l^m\right) = v_{2,l}^m Y_l^m \boldsymbol{e}_r + \beta_l^{-1}\hat{\partial}_r(rv_{2,l}^m)\,\nabla_S Y_l^m = v_{2,l}^m Y_l^m \boldsymbol{e}_r + \frac{r}{\beta_l}\left(\frac{d}{dr}+\frac{2}{r}\right) v_{2,l}^m \nabla_S Y_l^m. \tag{3.9}$$

Therefore, the expansion (3.5) is a direct consequence of (3.2), (3.4) and (3.6). □

REMARK 3.2 Equivalently, we can reformulate (3.5) as

$$\boldsymbol{E} = u_0^0 Y_0^0 \boldsymbol{e}_r + \sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{u_{1,l}^m \boldsymbol{T}_l^m + \hat{\partial}_r u_{2,l}^m \nabla_S Y_l^m + \frac{\beta_l}{r} u_{2,l}^m Y_l^m \boldsymbol{e}_r\right\} \quad \hat{\partial}_r = \frac{d}{dr} + \frac{1}{r}, \tag{3.10}$$

which allows for exact imposition of the divergence-free condition. Such a VSH expansion turns out to be a very useful analytic and numerical tool for, e.g., Maxwell equations and Navier–Stokes equations in spherical geometry (see, e.g., Morse & Feshbach, 1953; Bullard & Gellman, 1954; Nédélec, 2001; Monk, 2003; Ganesh *et al.*, 2011; Colton & Kress, 2013b).

Denote by $\boldsymbol{L}_T^2(S)$ the space of tangential components of vector fields in $(L^2(S))^3$. Then we can expand $\boldsymbol{E}_S^{a,b} \in \boldsymbol{L}_T^2(S)$ as

$$\boldsymbol{\Psi} = \boldsymbol{E}_S^{a,b}|_{r=b} = \sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{\psi_{T,l}^m \boldsymbol{T}_l^m + \psi_{Y,l}^m \nabla_S Y_l^m\right\}, \tag{3.11}$$

where the expansion coefficients

$$\psi_{T,l}^m = \beta_l^{-1}\langle \boldsymbol{\Psi}, \boldsymbol{T}_l^m\rangle_S, \quad \psi_{Y,l}^m = \beta_l^{-1}\langle \boldsymbol{\Psi}, \nabla_S Y_l^m\rangle_S. \tag{3.12}$$

Recall that the capacity operator in (1.2) is defined by (cf. Nédélec, 2001, (5.3.88)):

$$\mathscr{T}_b[\boldsymbol{\Psi}] := \eta \boldsymbol{H} \times \boldsymbol{e}_r\big|_{r=b} = \sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{-\mathrm{i}\frac{\hat{\partial}_r h_l^{(1)}(kb)}{h_l^{(1)}(kb)}\psi_{T,l}^m \boldsymbol{T}_l^m + \mathrm{i}\frac{h_l^{(1)}(kb)}{\hat{\partial}_r h_l^{(1)}(kb)}\psi_{Y,l}^m \nabla_S Y_l^m\right\}, \tag{3.13}$$

where $\eta = \sqrt{\mu/\varepsilon}$, $h_l^{(1)}$ is the spherical Bessel function of the first kind (cf. Abramowitz & Stegun, 1964), and

$$\hat{\partial}_r h_l^{(1)}(kb) = \left(\frac{d}{dr} + \frac{1}{r}\right) h_l^{(1)}(r)\Big|_{r=kb}. \tag{3.14}$$

As $\boldsymbol{F}^{a,b}$ in (1.1) is a solenoidal field, we can expand it as (3.5) with the coefficients f_0^0 and $\{f_{1,l}^m, f_{2,l}^m\}$. We also expand the data $\boldsymbol{h} \in \boldsymbol{L}_T^2(S)$ in (1.6) as

$$\boldsymbol{h} = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \left\{ h_{T,l}^m \, \boldsymbol{T}_l^m + h_{Y,l}^m \, \nabla_S Y_l^m \right\}, \tag{3.15}$$

where the expansion coefficients are given by (3.12) with $\boldsymbol{h}$ in place of $\boldsymbol{\Psi}$.

Proposition 3.3 Denote

$$u_1 = u_{1,l}^m, \quad u_2 = u_{2,l}^m, \quad f_1 = f_{1,l}^m, \quad f_2 = f_{2,l}^m, \quad h_1 = h_{T,l}^m, \quad h_2 = k^{-1}\big(\mathcal{T}_{l,kb} + (kb)^{-1}\big) h_{Y,l}^m, \tag{3.16}$$

for $l \ge 1$. Then the Maxwell equations (1.1) and (1.2) reduce to $-k^2 u_0^0 = f_0^0$, and the following two sequences of one-dimensional problems:

$$-\frac{1}{r^2}(r^2 u_i')' + \frac{\beta_l}{r^2} u_i - k^2 u_i = f_i \quad r \in I = (a,b); \quad u_i'(b) - k\, \mathcal{T}_{l,kb}\, u_i(b) = h_i \quad i = 1,2, \tag{3.17}$$

but with different boundary conditions at $r = a$:

$$u_1(a) = 0, \quad u_2'(a) + a^{-1} u_2(a) = 0. \tag{3.18}$$

Proof. We first consider (1.1). Recall that if $\operatorname{div} \boldsymbol{u} = 0$ then we have $\nabla \times \nabla \times \boldsymbol{u} = -\Delta \boldsymbol{u}$. Since $\operatorname{div}\big(\nabla \times (f \boldsymbol{T}_l^m)\big) = 0$ (cf. (A.4)), we derive from (3.5) and (A.4)–(A.5) that

$$\begin{aligned} &\nabla \times \nabla \times \big(u_{1,l}^m \boldsymbol{T}_l^m\big) = -\Delta\big(u_{1,l}^m \boldsymbol{T}_l^m\big) = -\mathcal{L}_l\big(u_{1,l}^m\big) \boldsymbol{T}_l^m, \\ &\nabla \times \nabla \times \nabla \times \big(u_{2,l}^m \boldsymbol{T}_l^m\big) = -\nabla \times \big(\Delta\big(u_{2,l}^m \boldsymbol{T}_l^m\big)\big) = -\nabla \times \big(\mathcal{L}_l\big(u_{2,l}^m\big) \boldsymbol{T}_l^m\big), \end{aligned} \tag{3.19}$$

where the Bessel operator $\mathcal{L}_l$ is given in (A.3). Thus, using the expansions (3.5), we can reduce (1.1) to

$$-(\mathcal{L}_l + k^2) w(r) = f(r) \quad \text{for} \quad \{w, f\} = \{u_{1,l}^m, f_{1,l}^m\} \quad \text{or} \quad \{u_{2,l}^m, f_{2,l}^m\}, \tag{3.20}$$

for $l \ge 1$ and $r \in I$. In addition, we have

$$-k^2 u_0^0 = f_0^0, \quad \text{as} \;\; \nabla \times (u_0^0\, Y_0^0 \boldsymbol{e}_r) = \nabla \times (f_0^0\, Y_0^0 \boldsymbol{e}_r) = \boldsymbol{0}, \tag{3.21}$$

since $\boldsymbol{E}^{a,b}$ and $\boldsymbol{F}^{a,b}$ are solenoidal. This leads to the mode u_0^0, so we only consider the modes with $l \ge 1$ and $0 \le |m| \le l$. A direct calculation using (A.2)–(A.3) and (A.4)–(A.5) leads to the reduction of the boundary condition (1.2):

$$u_{1,l}^m(a) = 0, \quad \hat{\partial}_r u_{2,l}^m(a) = 0, \quad \text{where} \quad \hat{\partial}_r := \frac{d}{dr} + \frac{1}{r}. \tag{3.22}$$

We now turn to the DtN boundary condition (1.2). By (3.5) and (3.19),

$$\nabla \times \boldsymbol{E}^{a,b} = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \left\{ \nabla \times \big(u_{1,l}^m \boldsymbol{T}_l^m\big) - \mathcal{L}_l(u_{2,l}^m) \boldsymbol{T}_l^m \right\}. \tag{3.23}$$

Again from (A.2)–(A.3) and (A.4)–(A.5), we derive

$$\left(\nabla\times\boldsymbol{E}^{a,b}\right)\times\boldsymbol{e}_r\Big|_{r=b}=\sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{\left(\hat{\partial}_r u_{1,l}^m\right)\boldsymbol{T}_l^m+\mathcal{L}_l(u_{2,l}^m)\nabla_S Y_l^m\right\}\Big|_{r=b},$$

$$\boldsymbol{E}_S^{a,b}\Big|_{r=b}=\sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{u_{1,l}^m\boldsymbol{T}_l^m+\hat{\partial}_r u_{2,l}^m\,\nabla_S Y_l^m\right\}\Big|_{r=b}. \tag{3.24}$$

Then, by (3.13) and (3.24),

$$-\mathrm{i}k\mathscr{T}_b[\boldsymbol{E}_S^{a,b}]=\sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{-k\frac{\hat{\partial}_r h_l^{(1)}(kb)}{h_l^{(1)}(kb)}\,u_{1,l}^m(b)\,\boldsymbol{T}_l^m+k\frac{h_l^{(1)}(kb)}{\hat{\partial}_r h_l^{(1)}(kb)}\,\hat{\partial}_r u_{2,l}^m(b)\,\nabla_S Y_l^m\right\}. \tag{3.25}$$

Consequently, by (3.15) and (3.24), the DtN boundary condition (1.2) reduces to

$$\hat{\partial}_r u_{1,l}^m(b)-k\frac{\hat{\partial}_r h_l^{(1)}(kb)}{h_l^{(1)}(kb)}\,u_{1,l}^m(b)=h_{T,l}^m;\quad \mathcal{L}_l(u_{2,l}^m)(b)+k\frac{h_l^{(1)}(kb)}{\hat{\partial}_r h_l^{(1)}(kb)}\,\hat{\partial}_r u_{2,l}^m(b)=h_{Y,l}^m. \tag{3.26}$$

By the equation (3.20) (note: $f_{2,l}^m(b)=0$ as the source field is assumed to be compact supported), we have $\mathcal{L}_l(u_{2,l}^m)(b)=-k^2u_{2,l}^m(b)$, so we can simplify (3.26) as

$$\hat{\partial}_r u_{2,l}^m(b)-k\frac{\hat{\partial}_r h_l^{(1)}(kb)}{h_l^{(1)}(kb)}\,u_{2,l}^m(b)=\frac{1}{k}\frac{\hat{\partial}_r h_l^{(1)}(kb)}{h_l^{(1)}(kb)}\,h_{Y,l}^m. \tag{3.27}$$

This ends the derivation. □

3.2 A priori *estimates for* $\{u_{1,l}^m,u_{2,l}^m\}$

A weak form of (3.17)–(3.18) is to find $u_1\in{}_0H^1(I)$ such that

$$\mathbb{B}_l^m(u_1,w)=(f_1,w)_\varpi+b^2h_1\bar{w}(b)\quad\forall\, w\in{}_0H^1(I), \tag{3.28}$$

and to find $u_2\in H^1(I)$ such that

$$\mathbb{B}_l^m(u_2,w)-au_2(a)\bar{w}(a)=(f_2,w)_\varpi+b^2h_2\bar{w}(b)\quad\forall\, w\in H^1(I), \tag{3.29}$$

where the sesquilinear form $\mathbb{B}_l^m(\cdot,\cdot)$ is defined in (2.22).

Observe that the weak form for u_1 is the same as that of the Helmholtz equation in (2.22), whereas (3.29) differ from (3.28) with an extra term $-au_2(a)\bar{w}(a)$. As a result, we can obtain the *a priori* estimates like Lemma 2.3 by using the same argument.

THEOREM 3.4 Let u_1 and u_2 be solutions of (3.28) and (3.29), respectively. If $f_1,f_2\in L^2(\Lambda)$ then for all $k\geq k_0>0$ (for some fixed constant k_0), and $l\geq 1, 0\leq|m|\leq l$, we have

$$\|u_i'\|_\varpi^2+\beta_l\|u_i\|^2+k^2\|u_i\|_\varpi^2\lesssim\|f_i\|_\varpi^2+|h_i|^2\quad i=1,2. \tag{3.30}$$

Proof. The estimates in Lemma 2.3 carry over to u_1, so it suffices to consider u_2 and deal with the extra term herein. Following the proof of Lemma 2.3, we take two test functions: $w = u_2$ and $w = 2(r-a)u_2'$, and note that the term '$-au_2(a)\bar{w}(a)$' vanishes for the second test function. Thus, we only need to deal with the contribution from this extra term as follows:

$$\|u_2'\|_{\varpi}^2 + \beta_l\|u_2\|^2 + k^2\|u_2\|_{\varpi}^2 - a|u_2(a)|^2 \lesssim \|f_2\|_{\varpi}^2 + |h_2|^2. \tag{3.31}$$

Using the Sobolev inequality (see, e.g., Shen *et al.*, 2011, (B.33)), we obtain

$$\begin{aligned} a|u_2(a)|^2 &\le a\left(2+\frac{1}{b-a}\right)\|u_2\|\|u_2\|_1 \le a\left(2+\frac{1}{b-a}\right)\left(\|u_2\|^2+\|u_2\|\|u_2'\|\right) \\ &\le a^{-3}\left(2+\frac{1}{b-a}\right)\left(\|u_2\|_{\varpi}^2+\|u_2\|_{\varpi}\|u_2'\|_{\varpi}\right), \end{aligned} \tag{3.32}$$

where we used the simple inequality $\sqrt{A^2+B^2} \le |A|+|B|$, and the fact $\varpi/a^2 \ge 1$. Thus,

$$a|u_2(a)|^2 \le \frac{1}{2}\|u_2'\|_{\varpi}^2 + C\|u_2\|_{\varpi}^2. \tag{3.33}$$

Thus, by (3.31) and (3.33),

$$\frac{1}{2}\|u_2'\|_{\varpi}^2 + \beta_l\|u_2\|^2 + k^2\big(1-Ck^{-1}\big)\|u_2\|_{\varpi}^2 \lesssim \|f_2\|_{\varpi}^2 + |h_2|^2. \tag{3.34}$$

This leads to the desired estimate. □

It is important to point out that as the expansion in (3.10) involves $\{\hat{\partial}_r u_{2,l}^m\}$, the direct use of Theorem 3.4 and the orthogonality of VSH only leads to an overly pessimistic estimate: $\|\boldsymbol{E}^{a,b}\|_{\Omega} = O(1)$. However, the expected optimal estimate should be $\|\boldsymbol{E}^{a,b}\|_{\Omega} = O(k^{-1})$. In view of this, we next derive an 'auxiliary' equation of $\hat{\partial}_r u_{2,l}^m$ and apply the analysis similar to that for $\{u_{1,l}^m, u_{2,l}^m\}$ in the previous subsection.

3.3 A priori *estimates for* $\hat{\partial}_r u_{2,l}^m$

3.3.1 *Equation of* $\hat{\partial}_r u_{2,l}^m$.

Denote

$$\begin{aligned} &v_2 = \beta_l u_{2,l}^m/r = \beta_l u_2/r, \quad v_3 = \hat{\partial}_r u_{2,l}^m = \hat{\partial}_r u_2, \quad h_Y = -k\mathcal{S}_{l,kb}h_2 = h_{Y,l}^m, \\ &g_2 = \beta_l f_{2,l}^m/r = \beta_l f_2/r, \quad g_3 = \hat{\partial}_r f_{2,l}^m = \hat{\partial}_r f_2, \end{aligned} \tag{3.35}$$

where the DtN kernel pertinent to (3.13) is defined by

$$\mathcal{S}_{l,\kappa} := -\frac{h_l^{(1)}(\kappa)}{\hat{\partial}_r h_l^{(1)}(\kappa)} = -\frac{h_l^{(1)}(\kappa)}{h_l^{(1)'}(\kappa)+\kappa^{-1}h_l^{(1)}(\kappa)} = -\frac{1}{\mathcal{T}_{l,\kappa}+\kappa^{-1}} \quad l \ge 1, \ \ \kappa > 0. \tag{3.36}$$

Recall that $\mathcal{T}_{l,\kappa}$ is defined in (2.2).

From the equation of u_2 in Proposition 3.3, we can derive the following 'auxiliary' equation.

PROPOSITION 3.5 Let $v_3 = \hat{\partial}_r u_2$. Then we have

$$-\frac{1}{r^2}(r^2 v_3')' + \frac{\beta_l}{r^2}v_3 - k^2 v_3 - \frac{2}{r^2}v_2 = g_3 \quad r \in I,$$
$$v_3(a) = 0, \quad v_3'(b) - k\big(\mathcal{S}_{l,kb} - (kb)^{-1}\big)\, v_3(b) - b^{-1}v_2(b) = h_Y. \tag{3.37}$$

Alternatively, we can replace the boundary condition at $r = b$ in (3.37) by

$$v_3'(b) - \frac{\sigma_{l,kb}}{b}\, v_2(b) = \frac{h_Y}{kb\,\mathcal{S}_{l,kb}} = -\frac{h_2}{b}, \tag{3.38}$$

where

$$\sigma_{l,kb} := 1 - \frac{k^2 b^2}{\beta_l}\left(1 - \frac{1}{kb\mathcal{S}_{l,kb}}\right) = 1 - \frac{k^2 b^2}{\beta_l}\left(1 + \frac{\mathcal{T}_{l,kb}}{kb} + \frac{1}{k^2 b^2}\right). \tag{3.39}$$

Proof. One verifies readily that $\hat{\partial}_r v_3 = \hat{\partial}_r(\hat{\partial}_r u_2) = r^{-2}(r^2 u_2')'$, so by (3.17),

$$-\hat{\partial}_r v_3 + \frac{\beta_l}{r^2}u_2 - k^2 u_2 = f_2 \quad r \in I. \tag{3.40}$$

Applying $\hat{\partial}_r$ to both sides of the above equation, we obtain the first equation in (3.37) by a direct calculation. Since $v_3(a) = \hat{\partial}_r u_2(a)$, the boundary condition $v_3(a) = 0$ is a direct consequence of (3.18). Noting that $u_2'(b) = v_3(b) - u_2(b)/b$, we obtain from (3.36) and the boundary condition in (3.17) that

$$u_2(b) + \frac{\mathcal{S}_{l,kb}}{k}v_3(b) = \frac{\mathcal{S}_{l,kb}}{k}h_2 = -\frac{h_Y}{k^2}. \tag{3.41}$$

Taking $r = b$ in (3.40) (note: $f_2(b) = 0$), we obtain

$$u_2(b) = -k^{-2}\big(v_3'(b) + b^{-1}v_3(b) - b^{-1}v_2(b)\big). \tag{3.42}$$

Inserting (3.42) into (3.41) yields the boundary condition at $r = b$ in (3.37).

The alternative boundary condition (3.38) can be obtained by eliminating $v_3(b)$ in (3.37). More precisely, solving out $v_3(b)$ from (3.41) and using the fact $u_2(b) = bv_2(b)/\beta_l$, we can obtain (3.38)–(3.39) from (3.37). □

3.3.2 *Properties of the DtN kernel* $\mathcal{S}_{l,\kappa}$. By (3.36), we have that for integer $l \geq 1$ and real $\kappa > 0$,

$$\mathrm{Re}(\mathcal{S}_{l,\kappa}) = -\frac{\mathrm{Re}(\mathcal{T}_{l,\kappa}) + \kappa^{-1}}{(\mathrm{Re}(\mathcal{T}_{l,\kappa}) + \kappa^{-1})^2 + (\mathrm{Im}(\mathcal{T}_{l,\kappa}))^2}; \quad \mathrm{Im}(\mathcal{S}_{l,\kappa}) = \frac{\mathrm{Im}(\mathcal{T}_{l,\kappa})}{(\mathrm{Re}(\mathcal{T}_{l,\kappa}) + \kappa^{-1})^2 + (\mathrm{Im}(\mathcal{T}_{l,\kappa}))^2}, \tag{3.43}$$

which, together with (2.5), implies

$$\mathrm{Re}(\mathcal{S}_{l,\kappa}) > 0, \quad \mathrm{Im}(\mathcal{S}_{l,\kappa}) > 0 \quad \text{for} \quad l \geq 1, \quad \kappa > 0. \tag{3.44}$$

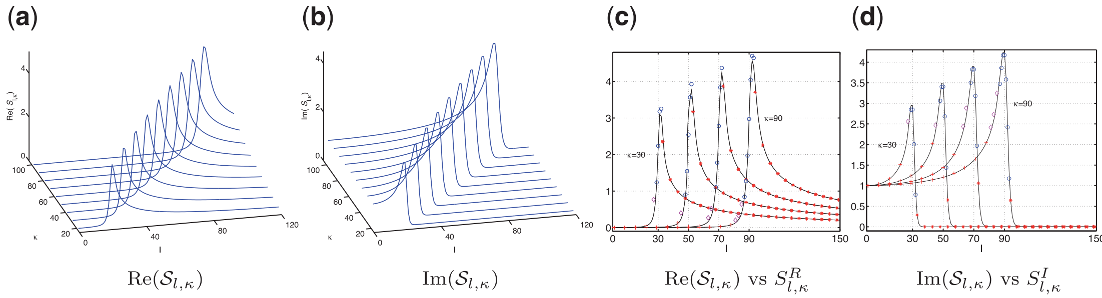

FIG. 2. (a,b) graphs of real and imaginary parts of $\mathcal{S}_{l,\kappa}$ for various $(l,\kappa) \in [0,120] \times [1,100]$. (c) $\mathrm{Re}(\mathcal{S}_{l,\kappa})$ (solid line) against $S^R_{l,\kappa}$. (d) $\mathrm{Im}(\mathcal{S}_{l,\kappa})$ (solid line) against $S^I_{l,\kappa}$ with $\kappa = 30, 50, 70, 90$ (note: '+' for $\rho = \nu/\kappa \in (0,\theta_0)$, '$\diamond$' for $\rho \in [\theta_0, \vartheta_1]$, '$\circ$' for $\rho \in (\vartheta_1, \vartheta_2)$ and '$*$' for $\rho \in [\vartheta_2, \infty)$).

In Fig. 2 (a,b), we depict the graphs of $\mathrm{Re}(\mathcal{S}_{l,\kappa})$ and $\mathrm{Im}(\mathcal{S}_{l,\kappa})$ for various samples $(l,\kappa) \in [0,120] \times [1,100]$, which shows a quite different behaviour, compared with that of $\mathcal{T}_{l,\kappa}$ in Fig. 1.

Thanks to (3.43) and the estimates in Theorem 2.2, we can analyse the behaviour of $\mathcal{S}_{l,\kappa}$. In Fig. 2(c,d), we plot the exact value and approximations in Theorem 3.6 below for various samples of κ.

THEOREM 3.6 Let $\theta_0, \vartheta_1, \vartheta_2$ and κ_0 be the same as in (2.6) and (2.8). Denote $\nu = l + 1/2$ and $\rho = \nu/\kappa$. Then for any $\kappa > \kappa_0$,

$$\mathrm{Re}(\mathcal{S}_{l,\kappa}) \sim S^R_{l,\kappa}, \quad \mathrm{Im}(\mathcal{S}_{l,\kappa}) \sim S^I_{l,\kappa} \quad \forall\, l \geq 1, \quad \text{where} \tag{3.45}$$

(i) for $\rho = \nu/\kappa \in (0, \theta_0)$,

$$S^R_{l,k} = \frac{1}{2\kappa}\left(\frac{\rho}{1-\rho^2}\right)^2, \quad S^I_{l,k} = \frac{1}{\sqrt{1-\rho^2}}; \tag{3.46}$$

(ii) for $\rho = \nu/\kappa \in [\theta_0, \vartheta_1]$,

$$S^R_{l,\kappa} = \frac{1}{4\rho(1-\rho)\kappa}\left(1 + \frac{1}{2(1-\rho)}\right), \quad S^I_{l,\kappa} = \frac{1}{\sqrt{2\rho(1-\rho)}}; \tag{3.47}$$

(iii) for $\rho = \nu/\kappa \in (\vartheta_1, \vartheta_2)$,

$$S^R_{l,\kappa} = \frac{1}{4c_1}\left(\frac{\nu}{2}\right)^{1/3} H^R(t), \quad S^I_{l,\kappa} = \frac{\sqrt{3}}{4c_1}\left(\frac{\nu}{2}\right)^{1/3} H^I(t), \tag{3.48}$$

where $t = -\sqrt[3]{2}\,(\kappa - \nu)/\sqrt[3]{\nu}$ (note: $|t| < 1$) and

$$H^R(t) = \frac{1 + 2c_1 t + c_2 t^2}{1 - 2c_1 t + (4c_1^2 + c_2/2)t^2 + c_1 c_2 t^3 + c_2^2 t^4/4},$$
$$H^I(t) = \frac{1 - 2c_1 t}{1 - 2c_1 t + (4c_1^2 + c_2/2)t^2 + c_1 c_2 t^3 + c_2^2 t^4/4}, \tag{3.49}$$

with c_1, c_2 given by (2.17);

(iv) for $\rho = \nu/\kappa \in [\vartheta_2, \infty)$,

$$S^R_{l,\kappa} = \frac{1}{\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{2\kappa\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{\rho^2 - 1}\right)\right), \tag{3.50}$$

$$S^I_{l,\kappa} = \frac{e^{-2\nu\Psi}}{\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{\kappa\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{\rho^2 - 1}\right)\right), \tag{3.51}$$

where Ψ is defined in (2.19).

We postpone the derivation of the above estimates to Appendix C.

REMARK 3.7 With some careful calculations, one can verify that for $t \in [-1, 1]$,

$$\min\{H^R(t)\} = H^R(t = -1) \approx 0.2493, \quad \max\{H^R(t)\} = H^R(t \approx 0.8004) \approx 1.9291,$$
$$\min\{H^I(t)\} = H^I(t = 1) \approx 0.2479, \quad \max\{H^I(t)\} = H^I(t = 0) = 1.$$

Thus, we roughly have $0.2493 \le H^R(t) \le 1.9291$ and $0.2479 \le H^I(t) \le 1$ for $t \in [-1, 1]$.

A weak form of (3.37) is to find $v_3 \in {}_0H^1(I)$ such that

$$\widehat{\mathbb{B}}^m_l(v_3, w) = G^m_l(w) \quad \forall\, w \in {}_0H^1(I), \tag{3.52}$$

where

$$\widehat{\mathbb{B}}^m_l(v_3, w) = (v_3', w')_\varpi + \beta_l(v_3, w) - k^2(v_3, w)_\varpi - kb^2\big(\mathcal{S}_{l,kb} - (kb)^{-1}\big)\, v_3(b)\bar{w}(b);$$
$$G^m_l(w) := b\, v_2(b)\bar{w}(b) + 2(v_2, w) + b^2 h_Y \bar{w}(b) + (g_3, w)_\varpi \quad \varpi = r^2. \tag{3.53}$$

Alternatively, we can use the equivalent boundary condition (3.38)–(3.39) and modify (3.53) as

$$\widehat{\mathbb{B}}^m_l(v_3, w) = (v_3', w')_\varpi + \beta_l(v_3, w) - k^2(v_3, w)_\varpi;$$
$$G^m_l(w) = b\,\sigma_{l,kb} v_2(b)\bar{w}(b) + 2(v_2, w) + (g_3, w)_\varpi + b^2\frac{h_Y}{k\,\mathcal{S}_{l,kb}}\bar{w}(b). \tag{3.54}$$

THEOREM 3.8 Let θ_0 and $\{\vartheta_i\}_{i=1}^2$ be the same as in (2.6) and (2.8). If $g_2, g_3 \in L^2(I)$ then we have that for all $k \geq k_0 > 0$ (for some fixed constant k_0) and $l \geq 1, 0 \leq |m| \leq l$,

$$\|v_3'\|_{\varpi}^2 + \beta_l\|v_3\|^2 + k^2\|v_3\|_{\varpi}^2 \leq C_{l,k}\left(\frac{1}{\beta_l}\|g_2\|_{\varpi}^2 + \|g_3\|_{\varpi}^2\right) + C\left(1 + \frac{\beta_l^2}{k^4}\right)|h_Y|^2, \tag{3.55}$$

where C is a generic positive constant independent of k, l, m and v_3, and

$$C_{l,k} = C\begin{cases} 1, & \text{if } \rho = \nu/(kb) \in (0, \theta_0] \cup (\vartheta_2, \infty), \\ (kb)^{1-\gamma}, & \text{if } \rho = \nu/(kb) \in (\theta_0, \vartheta_2]. \end{cases} \tag{3.56}$$

Note that for $\rho \in (\theta_0, \vartheta_2]$, we have $\rho = 1 + \xi(kb)^{-\gamma}$ or $\nu = l + 1/2 = kb + \xi(kb)^{\gamma-1}$, for some $\gamma \in [1/3, 1)$, and some constant ξ.

Proof. Taking $w = v_3$ in (3.52), we obtain

$$\begin{aligned} &\|v_3'\|_{\varpi}^2 + \beta_l\|v_3\|^2 - k^2\|v_3\|_{\varpi}^2 - kb^2\,\mathrm{Re}(\mathcal{S}_{l,kb})|v_3(b)|^2 + b|v_3(b)|^2 \\ &\qquad = b\,\mathrm{Re}(v_2(b)\bar{v}_3(b)) + 2\mathrm{Re}(v_2, v_3) + b^2\,\mathrm{Re}(h_Y\bar{v}_3(b)) + \mathrm{Re}(g_3, v_3)_{\varpi}, \end{aligned} \tag{3.57a}$$

$$-kb^2\,\mathrm{Im}(\mathcal{S}_{l,kb})|v_3(b)|^2 = b\,\mathrm{Im}(v_2(b)\bar{v}_3(b)) + 2\mathrm{Im}(v_2, v_3) + b^2\,\mathrm{Im}(h_Y\bar{v}_3(b)) + \mathrm{Im}(g_3, v_3)_{\varpi}. \tag{3.57b}$$

Next taking $w = 2(r-a)v_3'$ in (3.52) and following the derivation of (2.25)–(2.26), we can obtain

$$\begin{aligned} &b^2|I||v_3'(b)|^2 + (\beta_l|I| + b)|v_3(b)|^2 + 2a\|\sqrt{r}v_3'\|^2 + 2k^2\int_a^b \left[1 - \frac{a}{r}\right]|v_3|^2 r^2\,\mathrm{d}r \\ &= \big(k^2b^2|I| + kb^2\,\mathrm{Re}(\mathcal{S}_{l,kb})\big)|v_3(b)|^2 + 2kb^2|I|\mathrm{Re}\big\{(\mathcal{S}_{l,kb} - (kb)^{-1})v_3(v)\bar{v}_3'(b)\big\} \\ &\quad + b\,\mathrm{Re}(v_2(b)\bar{v}_3(b)) + b^2\,\mathrm{Re}(h_Y\bar{v}_3(b)) + 2\mathrm{Re}(v_2, v_3) + \mathrm{Re}(g_3, v_3)_{\varpi} + 2b|I|\,\mathrm{Re}\big(v_2(b)\bar{v}_3'(b)\big) \\ &\quad + 2b^2|I|\,\mathrm{Re}\big(h_Y\bar{v}_3'(b)\big) + 4\,\mathrm{Re}(v_2, (r-a)v_3') + 2\,\mathrm{Re}(g_3, (r-a)v_3')_{\varpi}. \end{aligned} \tag{3.58}$$

Then we can derive the estimate similar to (2.28) (by noting that $\mathcal{S}_{l,kb} - (kb)^{-1}$ should be in place of $\mathcal{T}_{l,kb}$ and the term of the left endpoint $r = a$ is not involved):

$$b^2|I||v_3'(b)|^2 + \mathcal{D}_{l,k}|I||v_3(b)|^2 + a\|\sqrt{r}v_3'\|^2 + k^2\|v_3\|_{\varpi}^2 \leq C\big(\|v_2\|_{\varpi}^2 + |v_2(b)|^2 + \|g_3\|_{\varpi}^2 + |h_Y|^2\big), \tag{3.59}$$

where

$$\mathcal{D}_{l,\kappa} := \beta_l - (1 - \varepsilon_3)|I|^{-1}\,kb^2\,\mathrm{Re}(\mathcal{S}_{l,kb}) - k^2b^2\big(1 + \varepsilon_1^{-1}\big|\mathcal{S}_{l,kb} - (kb)^{-1}\big|^2\big). \tag{3.60}$$

Thus, it remains to bound the term $\mathcal{D}_{l,\kappa}|I||v_3(b)|^2$ (note: it is negative for some range of l) and to estimate the terms of v_2 by using that of u_2 in Theorem 3.4 and its proof. Following the proof of Theorem 3.4, we proceed with four cases.

(i) If $\rho = \frac{\nu}{kb} \in (0, \theta_0)$ for fixed $0 < \theta_0 < 1$, we find from (3.46) that both $kb\mathrm{Re}(\mathcal{S}_{l,kb})$ and $\mathrm{Im}(\mathcal{S}_{l,kb})$ behave like constants. Thus, from (3.57b), we can obtain the bound like (2.31):

$$k^2b^2\,|v_3(b)|^2 \le \varepsilon k^2\|v_3\|_{\varpi}^2 + C\big(\|v_2\|_{\varpi}^2 + |v_2(b)|^2 + \|g_3\|_{\varpi}^2 + |h_Y|^2\big). \tag{3.61}$$

Noting from (3.46) and (3.60) that

$$\mathcal{D}_{l,\kappa} \sim \beta_l - Ck^2b^2, \tag{3.62}$$

we infer from (3.59) that

$$b^2|I||v_3'(b)|^2 + \beta_l|I||v_3(b)|^2 + a\|\sqrt{r}v_3'\|^2 + k^2\|v_3\|_{\varpi}^2 \le C\big(\|v_2\|_{\varpi}^2 + |v_2(b)|^2 + \|g_3\|_{\varpi}^2 + |h_Y|^2\big). \tag{3.63}$$

Recall from (3.35) that $h_2 = -h_Y/(k\,\mathcal{S}_{l,kb}), u_2 = r\beta_l^{-1}v_2$ and $f_2 = r\beta_l^{-1}g_2$. Then by (2.32),

$$\|v_2\|_{\varpi}^2 + |v_2(b)|^2 \le C\left(\frac{1}{k^2}\|g_2\|_{\varpi}^2 + \frac{\beta_l^2}{k^4}|h_Y|^2\right) \le C\left(\frac{1}{\beta_l}\|g_2\|_{\varpi}^2 + \frac{\beta_l^2}{k^4}|h_Y|^2\right). \tag{3.64}$$

Thus, using (3.57a), (3.61), (3.63), (3.64) and the Cauchy–Schwarz inequality, we can obtain (3.55).

(ii) If $\rho = \frac{\nu}{kb} \in [\theta_0, \vartheta_1]$, we start with (2.35) and find from (3.47) that

$$\mathrm{Re}(\mathcal{S}_{l,kb} - (kb)^{-1}) \sim \frac{1}{8\bar{c}_0^2}(kb)^{1-2\gamma_0}, \quad \mathrm{Im}(\mathcal{S}_{l,kb}) \sim \frac{1}{\sqrt{2\bar{c}_0}}(kb)^{(1-\gamma_0)/2}, \tag{3.65}$$

where $1/3 \le \gamma_0 < 1$. Thus, by (3.62)–(3.65), $\mathcal{D}_{l,k} \sim -C(kb)^{3-\gamma_0}$. As with (2.37)–(2.39), we can derive

$$(kb)^{3-\gamma_0}\,|v_3(b)|^2 \le \varepsilon k^2\|v_3\|_{\varpi}^2 + C\left((kb)^{1-\gamma_0}(\|v_2\|_{\varpi}^2 + \|g_3\|_{\varpi}^2) + |v_2(b)|^2 + |h_Y|^2\right). \tag{3.66}$$

Therefore, we have

$$\|\sqrt{r}v_3'\|^2 + k^2\|v_3\|_{\varpi}^2 + (kb)^{3-\gamma_0}|v_3(b)|^2 \le C\left((kb)^{1-\gamma_0}(\|v_2\|_{\varpi}^2 + \|g_3\|_{\varpi}^2) + |v_2(b)|^2 + |h_Y|^2\right). \tag{3.67}$$

Like (3.64), we derive from (2.40) (note: $h_2 = -h_Y/(k\,\mathcal{S}_{l,kb}), u_2 = r\beta_l^{-1}v_2, f_2 = r\beta_l^{-1}g_2$) and (3.65) that

$$k^{1-\gamma_0}\|v_2\|_{\varpi}^2 + |v_2(b)|^2 \le C\left(\frac{1}{k^{1+\gamma_0}}\|g_2\|_{\varpi}^2 + \frac{\beta_l^2}{k^4}|h_Y|^2\right) \le C\left(\frac{k^{1-\gamma_0}}{\beta_l}\|g_2\|_{\varpi}^2 + \frac{\beta_l^2}{k^4}|h_Y|^2\right). \tag{3.68}$$

Thus, as with the previous case, we can obtain the desired estimate.

(iii) If $\rho = \frac{\nu}{kb} \in (\vartheta_1, \vartheta_2)$, we have the range in (2.41). Using (3.48)–(3.49), we can show that in this range, the bound is the same as (2.50) with $\gamma_0 = 1/3$:

$$\|\sqrt{r}v_3'\|^2 + k^2\|v_3\|_{\varpi}^2 + (kb)^{8/3}|v_3(b)|^2 \le C\left((kb)^{2/3}(\|v_2\|_{\varpi}^2 + \|g_3\|_{\varpi}^2) + |v_2(b)|^2 + |h_Y|^2\right). \tag{3.69}$$

Similarly, we can bound the terms involving v_2 by (3.68) with $\gamma_0 = 1/3$.

(iv) If $\rho = \frac{\nu}{kb} \in [\vartheta_2, \infty)$, we find from (3.51) that $\mathrm{Im}(\mathcal{S}_{l,kb})$ decays exponentially with respect to l. However, since $\mathrm{Re}(\mathcal{S}_{l,kb} - (kb)^{-1}) > 0$, we do not have (2.48) to bound the term $\mathcal{D}_{l,k}|I||v_3(b)|^2$ (note: $\mathcal{D}_{l,k} < 0$), as opposite to the estimate of u_2 in Theorem 3.4. For this purpose, we use the equivalent boundary condition (3.38)–(3.39). Correspondingly, we modify the weak form (3.52) as

$$(v_3', w')_\varpi + \beta_l(v_3, w) - k^2(v_3, w)_\varpi = b\sigma_{l,kb} v_2(b)\bar{w}(b) + 2(v_2, w) + (g_3, w)_\varpi + b^2 \frac{h_Y}{k\,\mathcal{S}_{l,kb}} \bar{w}(b), \quad \forall\, w \in {}_0H^1(\varLambda). \tag{3.70}$$

Taking $w = v_3$ in (3.70) leads to

$$\|v_3'\|_\varpi^2 + \beta_l\|v_3\|^2 - k^2\|v_3\|_\varpi^2 = b\,\mathrm{Re}(\sigma_{l,kb}\, v_2(b)\bar{v}_3(b)) + \mathrm{Re}(g_3, v_3)_\varpi + 2\mathrm{Re}(v_2, v_3) + b^2 \mathrm{Re}\left(\frac{h_Y}{k\,\mathcal{S}_{l,kb}} \bar{v}_3(b)\right). \tag{3.71}$$

Next taking $w = 2(r-a)v_3'$ and following the same procedure in deriving (2.25)–(2.26), we have

$$\begin{aligned} &b^2|I||v_3'(b)|^2 + (\beta_l - k^2b^2)|I||v_3(b)|^2 + 2a\|\sqrt{r}v_3'\|^2 + 2k^2\int_a^b \left(1 - \frac{a}{r}\right)|v_3|^2 r^2\,\mathrm{d}r \\ &= 2b|I|\mathrm{Re}\{\sigma_{l,kb}\, v_2(b)\bar{v}_3'(b)\} + b\mathrm{Re}\{\sigma_{l,kb} v_2(b)\bar{v}_3(b)\} + 4\,\mathrm{Re}(v_2, (r-a)v_3') + 2\mathrm{Re}(v_2, v_3) \\ &\quad + 2\mathrm{Re}(g_3, (r-a)v_3')_\varpi + \mathrm{Re}(g_3, v_3)_\varpi + 2b^2|I|\mathrm{Re}\left(\frac{h_Y}{k\,\mathcal{S}_{l,kb}}\bar{v}_3'(b)\right) + b^2\mathrm{Re}\left(\frac{h_Y}{k\,\mathcal{S}_{l,kb}}\bar{v}_3(b)\right). \end{aligned} \tag{3.72}$$

Using the Cauchy–Schwarz inequality, we can derive

$$\begin{aligned} |v_3'(b)|^2 + (\beta_l - k^2b^2)|v_3(b)|^2 + \|\sqrt{r}v_3'\|^2 + k^2\|v_3\|_\varpi^2 \le C\Big\{&|\sigma_{l,kb}|^2\big(1 + (\beta_l - k^2b^2)^{-1}\big)|v_2(b)|^2 \\ &+ \|v_2\|_\varpi^2 + \|g_3\|_\varpi^2 + \frac{1}{(kb)^2\,|\mathcal{S}_{l,kb}|^2}\big(1 + (\beta_l - k^2b^2)^{-1}\big)|h_Y|^2\Big\}. \end{aligned} \tag{3.73}$$

We first consider the range (a) in (2.44), i.e., $\nu \sim kb + \bar{c}_5(kb)^{\gamma_1}$ for $1/3 \le \gamma_1 < 1$ and some constant $\bar{c}_5 > 0$. From (3.39) and (3.50), one verifies

$$\beta_l - k^2b^2 \sim 2\bar{c}_5(kb)^{1+\gamma_1}, \quad |\mathcal{S}_{l,kb}| \sim |\mathrm{Re}(\mathcal{S}_{l,kb})| \sim \frac{1}{\sqrt{2\bar{c}_5(kb)^{\gamma_1-1}}} \quad |\sigma_{l,kb}| \sim 2\bar{c}_5(kb)^{\gamma_1-1}. \tag{3.74}$$

Then we obtain from (3.73)–(3.74) that

$$k^2\|v_3\|_\varpi^2 \le C\left((kb)^{2(\gamma_1-1)}|v_2(b)|^2 + \|v_2\|_\varpi^2 + \|g_3\|_\varpi^2 + (kb)^{-(1+\gamma_1)}|h_Y|^2\right). \tag{3.75}$$

Recalling that $h_2 = -h_Y/(k\,\mathcal{S}_{l,kb})$, $u_2 = r\beta_l^{-1}v_2$ and $f_2 = r\beta_l^{-1}g_2$, we have from (2.50) and (3.74) that

$$\|v_2\|_\varpi^2 + (kb)^{2(\gamma_1-1)}|v_2(b)|^2 \le C\left(\frac{1}{\beta_l}\|g_2\|_\varpi^2 + \frac{\beta_l^2}{k^4}|h_Y|^2\right). \tag{3.76}$$

As $v_3 \in {}_0H^1(I)$, one verifies readily that

$$|v_3(b)| \le \int_a^b |v_3'(r)|\,\mathrm{d}r \le C\|v_3'\|_{\varpi}. \tag{3.77}$$

Thus, using (3.71) and the Cauchy–Schwarz inequality, we can obtain the same upper bound as (3.75) for $\|v_3'\|_{\varpi}^2 + \beta_l\|v_3\|^2$. This leads to the desired estimate for this case.

We then consider the range (b) in (2.44), i.e., $\nu > \eta\, kb$ with $\eta > 1$. Once again, by (3.39) and (3.50),

$$|\mathcal{S}_{l,kb}| \sim |\mathrm{Re}(\mathcal{S}_{l,kb})| \sim \frac{kb}{\nu\sqrt{1-\eta^{-2}}}, \quad |\sigma_{l,kb}| \sim 1-\eta^{-2}. \tag{3.78}$$

It is evident that

$$\beta_l\|v_3\|^2 - k^2\|v_3\|_{\varpi}^2 \ge (\beta_l - k^2b^2)\|v_3\|^2 \ge \beta_l(1-\eta^{-2})\|v_3\|^2. \tag{3.79}$$

Using the Cauchy–Schwarz inequality and (3.77)–(3.79), we have from (3.71) that

$$\|v_3'\|_{\varpi}^2 + \beta_l\|v_3\|^2 \le C\left(|v_2(b)|^2 + \beta_l^{-1}\|v_2\| + \beta_l^{-1}\|g_3\|_{\varpi}^2 + \frac{\beta_l}{k^4}|h_Y|^2\right). \tag{3.80}$$

Then by (2.23), (2.54) and the fact that $h_2 = -h_Y/(k\,\mathcal{S}_{l,kb})$, $u_2 = r\beta_l^{-1}v_2$ and $f_2 = r\beta_l^{-1}g_2$, we obtain

$$|v_2(b)|^2 + \beta_l^{-1}\|v_2\|^2 \le C\left(\frac{1}{\beta_l}\|g_2\|_{\varpi}^2 + \frac{\beta_l^2}{k^4}|h_Y|^2\right). \tag{3.81}$$

Then we can derive the desired estimates. □

REMARK 3.9 It is seen from (3.30) that $\|u_2'\|_{\varpi} = O(1)$, while by (3.55), $\|u_2'\|_{\varpi} = O(k^{-1}\sqrt{C_{l,k}})$ (note: $v_3 = \hat{\partial}_r u_2$).

3.4 *Main result on* a priori *estimates of* $E^{a,b}$

We are in a position to derive *a priori* estimates for the Maxwell equations (1.1) and (1.2). Recall the space $\mathbb{H}(\mathrm{div}^0;\Omega)$ defined in (3.1). We further introduce

$$\begin{aligned}\mathbb{H}(\mathbf{curl};\Omega) &= \big\{\boldsymbol{E} \in (L^2(\Omega))^3 : \nabla\times\boldsymbol{E} \in (L^2(\Omega))^3\big\};\\ \mathbb{H}_0(\mathbf{curl};\Omega) &= \big\{\boldsymbol{E} \in \mathbb{H}(\mathbf{curl};\Omega) : \boldsymbol{E}\times\boldsymbol{e}_r|_{r=a} = \mathbf{0}\big\},\end{aligned} \tag{3.82}$$

which are equipped with the graph norm as defined in Monk (2003).

A weak form of (1.1) and (1.2) is to find $\boldsymbol{E}^{a,b} \in V := \mathbb{H}_0(\mathbf{curl};\Omega)\cap\mathbb{H}(\mathrm{div}^0;\Omega)$ such that

$$\begin{aligned}\mathcal{B}(\boldsymbol{E}^{a,b},\boldsymbol{\Psi}) &:= \big(\nabla\times\boldsymbol{E}^{a,b},\nabla\times\boldsymbol{\Psi}\big)_{\Omega} - k^2\big(\boldsymbol{E}^{a,b},\boldsymbol{\Psi}\big)_{\Omega} - \mathrm{i}kb^2\big\langle\mathscr{T}_b\boldsymbol{E}_S^{a,b},\boldsymbol{\Psi}_S\big\rangle_S\\ &= \big(\boldsymbol{F}^{a,b},\boldsymbol{\Psi}\big)_{\Omega} + b^2\big\langle\boldsymbol{h},\boldsymbol{\Psi}_S\big\rangle_S \quad \forall\,\boldsymbol{\Psi}\in V.\end{aligned} \tag{3.83}$$

Its well posedness can be established using the property: $\mathrm{Re}\langle \mathscr{T}_b \boldsymbol{E}_S^{a,b}, \boldsymbol{E}_S^{a,b} \rangle_S > 0$ (see, e.g., Nédélec, 2001, Chapter 5 and Monk, 2003, Chapter 10).

By Nédélec (2001, (5.3.47)), the surface divergence of $\boldsymbol{h}$ (with the expansion (3.15)) can be expressed as

$$\mathrm{div}_S\, \boldsymbol{h} = -\sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l\, h_{Y,l}^m\, Y_l^m, \quad \text{so} \quad \|\mathrm{div}_S\, \boldsymbol{h}\|_{L^2(S)}^2 = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l^2 \big|h_{Y,l}^m\big|^2. \tag{3.84}$$

THEOREM 3.10 Let $\boldsymbol{E}^{a,b}$ be the solution to (3.83). If $\boldsymbol{F}^{a,b} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{h} \in \boldsymbol{L}_T^2(S)$ and $\mathrm{div}_S\, \boldsymbol{h} \in L^2(S)$ then we have $\boldsymbol{E}^{a,b} \in \mathbb{H}_0(\mathbf{curl};\Omega)$ and

$$\|\nabla \times \boldsymbol{E}^{a,b}\|_\Omega + k\|\boldsymbol{E}^{a,b}\|_\Omega \le C\big(k^{1/3}\|\boldsymbol{F}^{a,b}\|_\Omega + \|\boldsymbol{h}\|_{\boldsymbol{L}_T^2(S)} + k^{-2}\|\mathrm{div}_S \boldsymbol{h}\|_{L^2(S)}\big), \tag{3.85}$$

for all $k \ge k_0 > 0$ (k_0 is some positive constant), where C is independent of $k, \boldsymbol{E}^{a,b}, \boldsymbol{F}^{a,b}$ and $\boldsymbol{h}$.

Proof. With the notation in (3.35), we can rewrite the field $\boldsymbol{E}^{a,b}$ in (3.10) as

$$\boldsymbol{E}^{a,b} = u_0^0\, Y_0^0 \boldsymbol{e}_r + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \big\{u_{1,l}^m\, \boldsymbol{T}_l^m + v_{2,l}^m\, Y_l^m \boldsymbol{e}_r + v_{3,l}^m\, \nabla_S Y_l^m\big\}, \tag{3.86}$$

where we recall (cf. Proposition 3.3): $-k^2 u_0^0 = f_0^0$. Thus, by the orthogonality and (A.1),

$$\|\boldsymbol{E}^{a,b}\|_\Omega^2 = \|u_0^0\|_\varpi^2 + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{\|u_{1,l}^m\|_\varpi^2 + \beta_l^{-1}\|v_{2,l}^m\|_\varpi^2 + \|v_{3,l}^m\|_\varpi^2\right\}. \tag{3.87}$$

Working out $\nabla \times \boldsymbol{E}^{a,b}$ via (3.86) and (A.4)–(A.5), we obtain from (A.1) that

$$\|\nabla \times \boldsymbol{E}^{a,b}\|_\Omega^2 = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{\|\hat{\partial}_r u_{1,l}^m\|_\varpi^2 + \beta_l\|u_{1,l}^m\|^2 + \big\|v_{2,l}^m/r - \hat{\partial}_r v_{3,l}^m\big\|^2\right\}. \tag{3.88}$$

Noting that $\beta_l + 2 \le 2\beta_l$ and $\|\hat{\partial}_r u_{1,l}^m\|_\varpi^2 \le 2\big(\big\|(u_{1,l}^m)'\big\|_\varpi^2 + \|u_{1,l}^m\|^2\big)$, we obtain from (3.87)–(3.88) that

$$\begin{aligned}\|\nabla \times \boldsymbol{E}^{a,b}\|_\Omega^2 + k^2\|\boldsymbol{E}^{a,b}\|_\Omega^2 \le{}& \|u_0^0\|_\varpi^2 + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{2\big(\|(u_{1,l}^m)'\|_\varpi^2 + \beta_l\|u_{1,l}^m\|^2\big) + k^2\|u_{1,l}^m\|_\varpi^2\right\}\\ &+ \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{2\|v_{2,l}^m\|^2 + k^2\beta_l^{-1}\|v_{2,l}^m\|_\varpi^2\right\} + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{4\big(\|(v_{3,l}^m)'\|_\varpi^2 + \|v_{3,l}^m\|^2\big) + k^2\|v_{3,l}^m\|_\varpi^2\right\}.\end{aligned}$$

Similarly, using the orthogonality of VSH, we have

$$\|\boldsymbol{F}^{a,b}\|_{\Omega}^2 = \|f_0^0\|_{\varpi}^2 + \sum_{l=1}^{\infty}\sum_{|m|=0}^{l} \beta_l \left\{\|f_{1,l}^m\|_{\varpi}^2 + \beta_l^{-1}\|g_{2,l}^m\|_{\varpi}^2 + \|g_{3,l}^m\|_{\varpi}^2\right\},$$

$$\|\boldsymbol{h}\|_{\boldsymbol{L}_T^2(S)}^2 = \sum_{l=1}^{\infty}\sum_{|m|=0}^{l} \beta_l\left\{\left|h_{T,l}^m\right|^2 + \left|h_{Y,l}^m\right|^2\right\}. \tag{3.89}$$

Recall from (3.35) that $h_{2,l}^m = -h_{Y,l}^m/(k\,\mathcal{S}_{l,kb})$, $u_{2,l}^m = r\beta_l^{-1}v_{2,l}^m$ and $f_{2,l}^m = r\beta_l^{-1}g_{2,l}^m$. Then by Theorem 3.4,

$$\|v_{2,l}^m\|^2 + k^2\beta_l^{-1}\|v_{2,l}^m\|_{\varpi}^2 \le C\left\{\beta_l^{-1}\|g_{2,l}^m\|_{\varpi}^2 + k^{-4}\beta_l^2|h_{Y,l}^m|^2\right\}, \tag{3.90}$$

where we have used the fact $|\mathcal{S}_{l,kb}|^{-2} \le C\beta_l/k^2$ for all the ranges of l, k in the proof of Theorem 3.8. We further derive from Theorems 3.4 to 3.8 and (3.90) that

$$\|\nabla\times\boldsymbol{E}^{a,b}\|_{\Omega}^2 + k^2\|\boldsymbol{E}^{a,b}\|_{\Omega}^2 \le k^{-2}\|f_0^0\|_{\varpi}^2 + C\sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\beta_l\left\{\|f_{1,l}^m\|_{\varpi}^2 + \left|h_{T,l}^m\right|^2\right\} + C\sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\beta_l\left\{\beta_l^{-1}\|g_{2,l}^m\|_{\varpi}^2\right.$$

$$\left. + k^{-4}\beta_l^2|h_{Y,l}^m|^2\right\} + \sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\beta_l\left\{C_{l,k}\left(\beta_l^{-1}\|g_{2,l}^m\|_{\varpi}^2 + \|g_{3,l}^m\|_{\varpi}^2\right) + C\left(1 + k^{-4}\beta_l^2\right)|h_{Y,l}^m|^2\right\}.$$

Finally, the desired estimate follows from (3.84), (3.89) and the above. □

REMARK 3.11 We point out that the estimate in Theorem 3.10 is suboptimal due to the presence of the factor $k^{1/3}$. In the bound of the 'auxiliary' variable v_3 in Theorem 3.8, we have $C_{l,k} = O(k^{1/3})$, which brings about this, but appears hard to be removed.

4. Spectral-Galerkin approximation and its wavenumber explicit analysis

In this section, we consider the analysis of spectral-Galerkin approximation to (3.83). We look for the approximation of $\boldsymbol{E}^{a,b}$ in the form

$$\boldsymbol{E}_N^L = -k^{-2}f_0^0\,Y_0^0\boldsymbol{e}_r + \sum_{l=1}^{L}\sum_{|m|=0}^{l}\left\{u_{1,l}^{N,m}\,\boldsymbol{T}_l^m + \nabla\times\left(u_{2,l}^{N,m}\,\boldsymbol{T}_l^m\right)\right\}, \tag{4.1}$$

where $u_{1,l}^{N,m} =: u_1^N$ and $u_{2,l}^{N,m} =: u_2^N$ are, respectively, the solutions of the spectral-Galerkin schemes:

(i) Find $u_1^N \in {}_0\mathcal{P}_N := {}_0H^1(I) \cap \mathcal{P}_N$ (where $\mathcal{P}_N$ is the space of polynomials of degree at most N) such that

$$\mathbb{B}_l^m(u_1^N, \varphi) = (f_1, \varphi)_{\varpi} + b^2h_1\bar{\varphi}(b) \quad \forall\,\varphi \in {}_0\mathcal{P}_N. \tag{4.2}$$

(ii) Find $u_2^N \in \mathcal{P}_N$ such that

$$\mathbb{B}_l^m(u_2^N, \psi) - au_2^N(a)\bar{\psi}(a) = (f_2, \psi)_{\varpi} + b^2h_2\bar{\psi}(b) \quad \forall\,\psi \in \mathcal{P}_N. \tag{4.3}$$

Here, the sesquilinear forms $\mathbb{B}_l^m$ is defined in (2.22). It is evident that by Proposition 3.1, the expansion in (4.1) preserves the divergence-free property of the continuous field.

THEOREM 4.1 Theorem 3.4 holds when u_1^N, u_2^N are in place of u_1, u_2 in (3.30), respectively.

REMARK 4.2 The algorithm in the recent work (Ma *et al.*, 2015) was based on the VSH expansion in Nédélec (2001), so the divergence-free condition could only be fulfilled approximately. Moreover, one had to deal three components where two were coupled. In a nutshell, the above algorithm is much more efficient.

4.1 *Error estimates*

As before, we start with the schemes (4.2) and (4.3) in one dimension. To describe the errors more precisely, we introduce the weighted Sobolev space

$$\mathbb{X}^s(I) := \left\{u \in L^2(I) \;:\; [(r-a)(b-r)]^{\frac{l-1}{2}} u^{(l)} \in L^2(I),\ 1 \le l \le s\right\} \quad s \in \mathbb{N} := \{1, 2, \cdots, \},$$

with the norm and seminorm

$$\|u\|_{\mathbb{X}^s(I)} = \left(\|u\|^2 + \sum_{l=1}^{s} \left\|[(r-a)(b-r)]^{\frac{l-1}{2}} u^{(l)}\right\|^2\right)^{1/2}, \quad |u|_{\mathbb{X}^s(I)} = \left\|[(r-a)(b-r)]^{\frac{s-1}{2}} u^{(s)}\right\|.$$

Define $\mathbb{X}^0(I) = L^2(I)$. Following the proof of Shen & Wang (2007, Theorem 4.2) (but using the improved estimate in Theorem 3.4), we have the following error estimate for the scheme (4.2).

LEMMA 4.3 Let u_1 and u_1^N be the solution of (3.28) and (4.2), respectively, and define $e_N^{u_1} = u_1 - u_1^N$. If $u_1 \in {}_0H^1(I) \cap \mathbb{X}^s(I)$ with integer $s \ge 1$ then for all $k \ge k_0$ (where k_0 is a certain constant), we have

$$\left\|(e_N^{u_1})'\right\|_{\varpi} + \sqrt{\beta_l}\|e_N^{u_1}\| + k\|e_N^{u_1}\|_{\varpi} \lesssim \left(\sqrt{\beta_l} + k^2 N^{-1}\right) N^{1-s} |u_1|_{\mathbb{X}^s(I)}, \tag{4.4}$$

where $\beta_l = l(l+1)$ and $\varpi = r^2$ as before.

Now, we turn to (4.3). Consider the orthogonal projection $\pi_N^1 : H^1(I) \to \mathcal{P}_N$ defined by

$$\left((\pi_N^1 v - v)', \phi'\right)_{\varpi} + \left(\pi_N^1 v - v, \phi\right)_{\varpi} = 0, \quad \forall \phi \in \mathcal{P}_N. \tag{4.5}$$

Noting that the weight function ϖ is uniformly bounded below and above, we follow the argument in Shen *et al.* (2011, Chapter 3), and derive the following estimate.

LEMMA 4.4 For any $v \in \mathbb{X}^s(I)$ with $s \in \mathbb{N}$, we have

$$\|(\pi_N^1 v - v)'\|_{\varpi} + N\|\pi_N^1 v - v\|_{\varpi} \lesssim N^{1-s} |v|_{\mathbb{X}^s(I)}. \tag{4.6}$$

LEMMA 4.5 Let u_2 and u_2^N be the solution of (3.29) and (4.3), respectively, and define $e_N^{u_2} = u_2 - u_2^N$. If $u_2 \in \mathbb{X}^s(\Lambda)$ with $s \in \mathbb{N}$ then for all $k \ge k_0$ (where k_0 is a certain constant) the estimate (4.4) holds when u_2 and $e_N^{u_2}$ are in place of u_1 and $e_N^{u_1}$, respectively.

Proof. Let $\hat{e}_N = u_2^N - \pi_N^1 u_2$ and $\tilde{e}_N = u_2 - \pi_N^1 u_2$. Then $e_N^{u_2} = \tilde{e}_N - \hat{e}_N$. By (3.29) and (4.3),

$$\mathbb{B}_l^m(e_N^{u_2}, \psi) - a e_N^{u_2}(a)\bar{\psi}(a) = 0 = \mathbb{B}_l^m(\tilde{e}_N, \psi) - a\tilde{e}_N(a)\bar{\psi}(a) - \mathbb{B}_l^m(\hat{e}_N, \psi) + a\hat{e}_N(a)\bar{\psi}(a)\ \forall\, \psi \in \mathcal{P}_N.$$

Thus, by (4.5),

$$\begin{aligned}&\mathbb{B}_l^m(\hat{e}_N, \psi) - a\hat{e}_N(a)\bar{\psi}(a) = \mathbb{B}_l^m(\tilde{e}_N, \psi) - a\tilde{e}_N(a)\bar{\psi}(a)\\ &= \beta_l(\tilde{e}_N, \psi) - (k^2+1)(\tilde{e}_N, \psi)_\varpi - a\,\tilde{e}_N(a)\bar{\psi}(a) - kb^2\mathcal{T}_{l,kb}\tilde{e}_N(b)\bar{\psi}(b)\ \ \forall\, \psi \in \mathcal{P}_N.\end{aligned} \tag{4.7}$$

Compared with the analysis for (4.2), the only difference is the presence of the extra term '$-a\,\tilde{e}_N(a)\bar{\psi}(a)$', which is akin to the situation in the proof of Theorem 3.4. We omit the details, as one can refer to the proofs of (Shen & Wang, 2007, Theorem 4.2) and Theorem 3.4. □

We now estimate the error between the electric field and its spectral approximation in (4.1)–(4.3). We first introduce suitable functional spaces to characterize the regularity of the electric field. For any $\boldsymbol{E}^{a,b} \in \boldsymbol{L}^2(\Omega)$, we write

$$\boldsymbol{E}^{a,b} = v_{2,0}^0(r)\,Y_0^0\boldsymbol{e}_r + \sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{v_{1,l}^m(r)\,\boldsymbol{T}_l^m + v_{2,l}^m(r)\,Y_l^m\boldsymbol{e}_r + v_{3,l}^m(r)\,\nabla_S Y_l^m\right\}. \tag{4.8}$$

We introduce the anisotropic Sobolev space $\boldsymbol{H}^t(S; H_\varpi^s(I))$ for $t \geq 0$ and integer $s \geq 0$ equipped with the norm:

$$\|\boldsymbol{E}^{a,b}\|_{\boldsymbol{H}^t(S;H_\varpi^s(I))} = \left(\|v_{2,0}^0\|_{H_\varpi^s(I)}^2 + \sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\beta_l^{1+t}\left\{\|v_{1,l}^m\|_{H_\varpi^s(I)}^2 + \beta_l^{-1}\left\|v_{2,l}^m\right\|_{H_\varpi^s(I)}^2 + \|v_{3,l}^m\|_{H_\varpi^s(I)}^2\right\}\right)^{\frac{1}{2}}.$$

Note that $\boldsymbol{H}^0(S; H_\varpi^0(I)) = \boldsymbol{L}^2(\Omega)$. Here, we are interested in the divergence-free fields. In this case, like Proposition 3.1, we can rewrite $\boldsymbol{E}^{a,b} \in \mathbb{H}_0(\mathbf{curl}; \Omega)$ in the divergence-free form:

$$\boldsymbol{E}^{a,b} = \frac{c}{r^2}Y_0^0\boldsymbol{e}_r + \sum_{l=1}^{\infty}\sum_{|m|=0}^{l}\left\{u_{1,l}^m(r)\,\boldsymbol{T}_l^m + \nabla\times\left(u_{2,l}^m(r)\,\boldsymbol{T}_l^m\right)\right\}, \tag{4.9}$$

where c is an arbitrary constant, and for $l \geq 1$,

$$v_{1,l}^m(r) = u_{1,l}^m(r), \quad v_{2,l}^m(r) = \frac{\beta_l}{r}u_{2,l}^m(r), \quad v_{3,l}^m(r) = \left(\frac{d}{dr} + \frac{1}{r}\right)u_{2,l}^m(r). \tag{4.10}$$

Note that we can substitute (4.10) into (4.1) to express the norm in (4.1) in terms of $\{u_{1,l}^m, u_{2,l}^m\}$.

THEOREM 4.6 If $\boldsymbol{E}^{a,b} \in \mathbb{H}_0(\mathbf{curl};\Omega) \cap \boldsymbol{L}^2(S;H^s_\varpi(I)) \cap \boldsymbol{H}^s(S;L^2_\varpi(I))$ with $s \in \mathbb{N}$, then

$$\|\boldsymbol{E}^{a,b} - \boldsymbol{E}_N^L\|_\Omega \lesssim (1 + k^{-1}N)(L + k^2N^{-1})N^{-s}\big\|\boldsymbol{E}^{a,b}\big\|_{\boldsymbol{L}^2(S;H^s_\varpi(I))} + L^{-s}\big\|\boldsymbol{E}^{a,b}\big\|_{\boldsymbol{H}^s(S;L^2_\varpi(I))}, \tag{4.11}$$

for all $k \ge k_0$ with k_0 being a positive constant.

Proof. By (3.5) and (4.1),

$$\begin{aligned}\boldsymbol{E}^{a,b} - \boldsymbol{E}_N^L &= \sum_{l=1}^{L}\sum_{|m|=0}^{l}\big\{(u_{1,l}^m - u_{1,l}^{N,m})\,\boldsymbol{T}_l^m + \nabla\times\big((u_{2,l}^m - u_{2,l}^{N,m})\,\boldsymbol{T}_l^m\big)\big\}\\ &\quad + \sum_{l=L+1}^{\infty}\sum_{|m|=0}^{l}\big\{u_{1,l}^m\,\boldsymbol{T}_l^m + \nabla\times\big(u_{2,l}^m\,\boldsymbol{T}_l^m\big)\big\} = \boldsymbol{S}_1 + \boldsymbol{S}_2,\end{aligned} \tag{4.12}$$

where $\boldsymbol{S}_2$ counts the error from truncating the VSH series. It is clear that by the orthogonality of VSH, (4.1) and (4.10),

$$\|\boldsymbol{S}_2\|_\Omega^2 = \sum_{l=L+1}^{\infty}\sum_{|m|=0}^{l}\beta_l\big\{\|u_{1,l}^m\|_\varpi^2 + \big\|\hat{\partial}_r u_{2,l}^m\big\|_\varpi^2 + \beta_l\|u_{2,l}^m\|^2\big\} \le L^{-2s}\big\|\boldsymbol{E}^{a,b}\big\|^2_{\boldsymbol{H}^s(S;L^2_\varpi(I))}. \tag{4.13}$$

Next, by (3.87), Lemma 4.3, Lemma 4.5 and (4.10),

$$\begin{aligned}\|\boldsymbol{S}_1\|_\Omega^2 &\lesssim \sum_{l=1}^{L}\sum_{|m|=0}^{l}\beta_l\Big\{\|u_{1,l}^m - u_{1,l}^{N,m}\|_\varpi^2 + \big\|(u_{2,l}^m - u_{2,l}^{N,m})'\big\|_\varpi^2 + \beta_l\big\|u_{2,l}^m - u_{2,l}^{N,m}\big\|^2\Big\}\\ &\lesssim \sum_{l=1}^{L}\sum_{|m|=0}^{l}\beta_l\big(\sqrt{\beta_l} + k^2N^{-1}\big)^2k^{-2}N^{2-2s}|u_{1,l}^m|^2_{\mathbb{X}^s(I)}\\ &\quad + \sum_{l=1}^{L}\sum_{|m|=0}^{l}\beta_l\big(\sqrt{\beta_l} + k^2N^{-1}\big)^2N^{-2s}|u_{2,l}^m|^2_{\mathbb{X}^{s+1}(I)}.\end{aligned} \tag{4.14}$$

By (4.10) and a direct calculation,

$$\begin{aligned}|u_{2,l}^m|^2_{\mathbb{X}^{s+1}(I)} &\lesssim \|\partial_r^{s+1}u_{2,l}^m\|^2_{L^2(I)} = \|\partial_r^s(\hat{\partial}_r u_{2,l}^m) - \partial_r^s(u_{2,l}^m/r)\|^2_{L^2(I)}\\ &\lesssim \|\partial_r^s(\hat{\partial}_r u_{2,l}^m)\|^2_{L^2(I)} + \|\partial_r^s(u_{2,l}^m/r)\|^2_{L^2(I)} = \|\partial_r^s v_{3,l}^m\|^2_{L^2(I)} + \beta_l^{-2}\|\partial_r^s v_{2,l}^m\|^2_{L^2(I)}.\end{aligned} \tag{4.15}$$

As the weight ϖ is uniformly bounded below and above for $r \in (a,b)$, we derive from (4.1), (4.10) and (4.14)–(4.15) that

$$\|\boldsymbol{S}_1\|_\Omega \lesssim (1 + k^{-1}N)(L + k^2N^{-1})N^{-s}\big\|\boldsymbol{E}^{a,b}\big\|_{\boldsymbol{L}^2(S;H^s_\varpi(I))}. \tag{4.16}$$

A combination of (4.13) and (4.16) leads to the desired estimate. □

Note that the estimate in (4.11) is in the L^2-norm not in the usual energy norm. For the continuous problem, we were able to obtain the bound for the energy norm through a further estimate of $\hat{\partial}_r u_{2,l}^m$ in subsection 3.3. However, this approach does not carry over to the discrete problem, as the second test function does not belong to the finite-dimensional space for the spectral-Galerkin approximation of (3.52). We shall derive below a sub-optimal error estimate in the energy norm through a different approach.

THEOREM 4.7 If $\boldsymbol{E}^{a,b} \in \boldsymbol{L}^2(S; H^s_\varpi(I)) \cap \boldsymbol{H}^{s-1}(S; H^1_\varpi(I)) \cap \boldsymbol{H}^s(S; L^2_\varpi(I))$ with $s \geq 3$, then

$$\begin{aligned}\left\| \nabla \times (\boldsymbol{E}^{a,b} - \boldsymbol{E}_N^L) \right\|_{w,\Omega} \lesssim & \big(N + (1 + kN^{-1})(L + k^2 N^{-1})\big) N^{1-s} \left\| \boldsymbol{E}^{a,b} \right\|_{\boldsymbol{L}^2(S; H^s_\varpi(I))} \\ & + L^{1-s} \big\{ \|\boldsymbol{E}^{a,b}\|_{\boldsymbol{H}^{s-1}(S; H^1_\varpi(I))} + \|\boldsymbol{E}^{a,b}\|_{\boldsymbol{H}^s(S; L^2_\varpi(I))} \big\},\end{aligned} \tag{4.17}$$

for all $k \geq k_0$ with k_0 being a positive constant, where $w = (b - r)(r - a)$.

Proof. For notational convenience, let $e_{lm}^{u_i} = u_{i,l}^m - u_{i,l}^{N,m} (i = 1, 2)$. By (4.12), (A.1) and (A.5)–(A.4),

$$\begin{aligned}\left\| \nabla \times (\boldsymbol{E}^{a,b} - \boldsymbol{E}_N^L) \right\|^2_{w,\Omega} \lesssim & \sum_{l=1}^{L} \sum_{|m|=0}^{l} \beta_l \big\{ \|r \hat{\partial}_r e_{lm}^{u_1}\|_w^2 + \beta_l \|e_{lm}^{u_1}\|_w^2 + \|r \mathcal{L}_l(e_{lm}^{u_2})\|_w^2 \big\} \\ & + \sum_{l=L+1}^{\infty} \sum_{|m|=0}^{l} \beta_l \big\{ \|r \hat{\partial}_r u_{1,l}^m\|_w^2 + \beta_l \|u_{1,l}^m\|_w^2 + \|r \mathcal{L}_l(u_{2,l}^m)\|_w^2 \big\} = T_1 + T_2.\end{aligned} \tag{4.18}$$

We first estimate T_2. It is clear that by (4.1) and (4.10),

$$\begin{aligned}& \|r \hat{\partial}_r u_{1,l}^m\|_w^2 + \beta_l \|u_{1,l}^m\|_w^2 \lesssim \|v_{1,l}^m\|^2_{H^1_\varpi(I)} + \beta_l \|v_{1,l}^m\|^2_{L^2_\varpi(I)}, \\ & \|r \mathcal{L}_l(u_{2,l}^m)\|_w^2 = \|r \hat{\partial}_r^2 u_{2,l}^m - \beta_l r^{-1} u_{2,l}^m\|_w^2 = \|r \hat{\partial}_r^2 u_{2,l}^m - \beta_l r^{-1} u_{2,l}^m\|_w^2 = \|r \hat{\partial}_r v_{3,l}^m - v_{2,l}^m\|_w^2 \\ & \qquad \lesssim \|v_{3,l}^m\|^2_{H^1_\varpi(I)} + \|v_{2,l}^m\|^2_{L^2_\varpi(I)},\end{aligned} \tag{4.19}$$

so we have

$$\begin{aligned}T_2 \leq & \sum_{l=L+1}^{\infty} \sum_{|m|=0}^{l} \beta_l \big\{ \|v_{1,l}^m\|^2_{H^1_\varpi(I)} + \|v_{2,l}^m\|^2_{L^2_\varpi(I)} + \|v_{3,l}^m\|^2_{H^1_\varpi(I)} \big\} \\ & + \sum_{l=L+1}^{\infty} \sum_{|m|=0}^{l} \beta_l^2 \|v_{1,l}^m\|^2_{L^2_\varpi(I)} \lesssim \beta_{L+1}^{1-s} \big\{ \|\boldsymbol{E}^{a,b}\|^2_{\boldsymbol{H}^{s-1}(S; H^1_\varpi(I))} + \|\boldsymbol{E}^{a,b}\|^2_{\boldsymbol{H}^s(S; L^2_\varpi(I))} \big\}.\end{aligned} \tag{4.20}$$

We next turn to estimating T_1. We see that it is necessary to obtain H^2-estimate of $e_{lm}^{u_2}$. To simplify the notation, we will drop l, m from the notations if no confusion may arise. Taking $\psi = w\hat{e}_N'' (\in \mathcal{P}_N)$ with $w(r) = (r - a)(b - r)$ in (4.7), and using integration by parts, we obtain

$$\begin{aligned}\mathbb{B}_l^m(\hat{e}_N, w\hat{e}_N'') &= -((r^2 \hat{e}_N')', w\hat{e}_N'') + \beta_l(\hat{e}_N, w\hat{e}_N'') - k^2(r^2 \hat{e}_N, w\hat{e}_N'') \\ &= \beta_l(\tilde{e}_N, w\hat{e}_N'') - (k^2 + 1)(r^2 \tilde{e}_N, w\hat{e}_N'').\end{aligned} \tag{4.21}$$

Using integration by parts again, we derive from a direct calculation that

$$-\mathrm{Re}\big((r^2\hat{e}_N')', w\hat{e}_N''\big) = -\|r\hat{e}_N''\|_w^2 - 2\mathrm{Re}\big(r\hat{e}_N', w\hat{e}_N''\big) = -\|r\hat{e}_N''\|_w^2 + \int_a^b |\hat{e}_N'|^2 (rw)'\,\mathrm{d}r;$$

$$\begin{aligned}\mathrm{Re}(\hat{e}_N, w\hat{e}_N'') &= -\|\hat{e}_N'\|_w^2 - \mathrm{Re}\int_a^b \hat{e}_N\overline{\hat{e}_N'}w'\,\mathrm{d}r = -\|\hat{e}_N'\|_w^2 - \frac{1}{2}|\hat{e}_N|^2 w'\big|_a^b + \frac{1}{2}\int_a^b |\hat{e}_N|^2 w''\,\mathrm{d}r\\ &= -\|\hat{e}_N'\|_w^2 + \frac{b-a}{2}\big(|\hat{e}_N(a)|^2 + |\hat{e}_N(b)|^2\big) - \|\hat{e}_N\|^2;\\ -\mathrm{Re}(r^2\hat{e}_N, w\hat{e}_N'') &= \|r\hat{e}_N'\|_w^2 + \frac{1}{2}|\hat{e}_N|^2(r^2w)'\big|_a^b - \frac{1}{2}\int_a^b |\hat{e}_N|^2 (r^2w)''\,\mathrm{d}r\\ &= \|r\hat{e}_N'\|_w^2 - \frac{b-a}{2}\big(a^2|\hat{e}_N(a)|^2 + b^2|\hat{e}_N(b)|^2\big) - \frac{1}{2}\int_a^b |\hat{e}_N|^2(r^2w)''\,\mathrm{d}r,\end{aligned}$$

and further by the Cauchy–Schwartz inequality,

$$\begin{aligned}|(\tilde{e}_N, w\hat{e}_N'')| &\le \int_a^b |(w\tilde{e}_N)'||\hat{e}_N'|\,\mathrm{d}r \le \frac{1}{2}\|\hat{e}_N'\|^2 + \frac{1}{2}\|(w\tilde{e}_N)'\|^2 \le \frac{1}{2}\|\hat{e}_N'\|^2 + c\big(\|\tilde{e}_N\|^2 + \|\tilde{e}_N'\|^2\big);\\ |(r^2\tilde{e}_N, w\hat{e}_N'')| &\le \int_a^b |(r^2w\tilde{e}_N)'||\hat{e}_N'|\,\mathrm{d}r \le \frac{1}{2}\|\hat{e}_N'\|^2 + \frac{1}{2}\|(r^2w\tilde{e}_N)'\|^2 \le \frac{1}{2}\|\hat{e}_N'\|^2 + c\big(\|\tilde{e}_N\|^2 + \|\tilde{e}_N'\|^2\big).\end{aligned}$$

Thus, we obtain from (4.21) and the above estimates that

$$\|r\hat{e}_N''\|_w^2 \lesssim (\beta_l + k^2)\big(\|\hat{e}_N\|_{H^1(I)}^2 + \|\tilde{e}_N\|_{H^1(I)}^2\big). \tag{4.22}$$

Recall that $\hat{e}_N = u_2^N - \pi_N^1 u_2$, $\tilde{e}_N = u_2 - \pi_N^1 u_2$ and $e_N^{u_2} = \tilde{e}_N - \hat{e}_N$, so we derive from Lemma 4.3 and Lemma 4.5 that

$$\begin{aligned}\|r(e_N^{u_2})''\|_w^2 &\lesssim \|r(\tilde{e}_N)''\|_w^2 + (\beta_l + k^2)\big(\|e_N^{u_2}\|_{H^1(I)}^2 + \|\tilde{e}_N\|_{H^1(I)}^2\big)\\ &\lesssim \|(u_2 - \pi_N^1 u_2)''\|^2 + (\beta_l + k^2)(\sqrt{\beta_l} + k^2N^{-1})^2 N^{-2s}|u_2|_{\mathbb{X}^{s+1}(I)}^2.\end{aligned} \tag{4.23}$$

To estimate $\|(u_2 - \pi_N^1 u_2)''\|^2$, we need to use the orthogonal projection $\pi_N^2 : H^2(I) \to \mathcal{P}_N$, and recall its approximation result (cf. Shen *et al.*, 2011, Chapter 4): for any $v \in \mathbb{X}^s(I)$,

$$\|\pi_N^2 v - v\|_{H^\mu(I)} \lesssim N^{\mu-s}|v|_{\mathbb{X}^s(I)} \quad \mu = 0, 1, 2, \quad s \ge 2. \tag{4.24}$$

Applying the inverse inequality (cf. Shen *et al.*, 2011, Theorem 3.33) and the above approximation result, we obtain

$$\|(\pi_N^1 v - \pi_N^2 v)''\| \lesssim N^2\|(\pi_N^1 v - \pi_N^2 v)'\| \lesssim N^{3-s}|v|_{\mathbb{X}^s(I)} \quad s \ge 2.$$

Therefore, we have

$$\|(\pi_N^1 v - v)''\| \le \|(\pi_N^1 v - \pi_N^2 v)''\| + \|(v - \pi_N^2 v)''\| \lesssim N^{3-s}|v|_{\mathbb{X}^s(I)}. \tag{4.25}$$

From (4.23) and (4.25), we have

$$\|(e_N^{u_2})''\|_w^2 \lesssim \big\{N^4 + (\beta_l + k^2)(\sqrt{\beta_l} + k^2N^{-1})^2\big\}N^{-2s}|u_2|^2_{\mathbb{X}^{s+1}(I)}. \tag{4.26}$$

Now, we are ready to estimate T_1 in (4.18). Using Lemma 4.5, we obtain

$$\|r\mathcal{L}_l(e_{lm}^{u_2})\|_w^2 \lesssim \|(e_{lm}^{u_2})''\|_w^2 + \beta_l^2\|e_{lm}^{u_2}\|^2 \lesssim \big\{N^4 + (\beta_l + k^2)(\sqrt{\beta_l} + k^2N^{-1})^2\big\}N^{-2s}|u_{2,l}^m|^2_{\mathbb{X}^{s+1}(I)}. \tag{4.27}$$

Therefore, we derive from Lemma 4.3, (4.27) and (4.15),

$$\begin{aligned} T_1 &\lesssim \sum_{l=1}^{L}\sum_{|m|=0}^{l} \beta_l\big(\sqrt{\beta_l} + k^2N^{-1}\big)^2 N^{2-2s}|u_{1,l}^m|^2_{\mathbb{X}^s(I)} \\ &\quad + \sum_{l=1}^{L}\sum_{|m|=0}^{l} \big\{N^4 + (\beta_l + k^2)(\sqrt{\beta_l} + k^2N^{-1})^2\big\}N^{-2s}|u_{2,l}^m|^2_{\mathbb{X}^{s+1}(I)} \\ &\lesssim \big\{N^2 + (1 + k^2N^{-2})(L + k^2N^{-1})^2\big\}N^{2-2s}\big\|\boldsymbol{E}^{a,b}\big\|^2_{\boldsymbol{L}^2(S;H^s_{\varpi}(I))}. \end{aligned} \tag{4.28}$$

A combination of (4.18), (4.20) and (4.28) leads to the desired estimate. □

5. Perturbed scatterers through TFE

We consider a perturbed scatterer enclosed by

$$D = \big\{(r,\theta,\phi)\ :\ 0 < r < a + g(\theta,\phi),\ \theta \in [0,\pi],\ \phi \in [0,2\pi)\big\}$$

for some $a > 0$ and given g. Let us choose the radius b of the artificial spherical boundary such that $b > \max_{\theta,\phi}\{a + g(\theta,\phi)\}$ and consider the Maxwell equations (1.1) and (1.2) in the domain $\hat{\Omega} = \{a + g(\theta,\phi) < r < b\}$. An effective approach to deal with scattering problems in general domains with moderately large wave numbers is the so-called TFE (David & Fernando, 2004). It has been successfully applied to various situations, including in particular acoustic scattering problems in two dimensions (Nicholls & Shen, 2006) and three dimensions (Fang *et al.*, 2007).

In our recent work (Ma *et al.*, 2015), we applied the TFE approach to the Maxwell equations (1.1) and (1.2) in $\hat{\Omega}$. We outline below the essential steps of this approach and refer to Ma *et al.* (2015) for more details.

- The first step is to transform the general domain $\hat{\Omega} = \{a + g < r < b\}$ to the spherical shell $\Omega = \{a < r' < b\}$ in (1.1) with the change of variables:

$$r' = \frac{(b-a)r - b\,g(\theta,\phi)}{b - a - g(\theta,\phi)} \quad \theta' = \theta, \quad \phi' = \phi. \tag{5.1}$$

  With this change of variable, the Maxwell equations (1.1) and (1.2) in $\hat{\Omega}$ is transformed to a Maxwell equation in Ω, which can still be written in the form (1.1) and (1.2) with the understanding that all new terms (induced by the transform) are included in $\boldsymbol{F}^{a,b}$ and $\boldsymbol{h}$ (cf. Ma *et al.*, 2015, (3.6)). With a slight abuse of notation, we shall still use r to denote r' and the same notations to denote the transformed functions.

- The second step is to assume $g(\theta,\phi) = \epsilon f(\theta,\phi)$ and for clarity, we denote the electric field and the data by $\boldsymbol{E}^{\epsilon f}, \boldsymbol{F}^{\epsilon f}$ and $\boldsymbol{h}^{\epsilon f}$, respectively. We expand them in ϵ-power series:

$$\boldsymbol{E}^{\epsilon f}(r,\theta,\phi) = \sum_{n=0}^{\infty} \boldsymbol{E}_n^{a,b}(r,\theta,\phi)\epsilon^n, \quad \boldsymbol{F}^{\epsilon f}(r,\theta,\phi) = \sum_{n=0}^{\infty} \boldsymbol{F}_n^{a,b}(r,\theta,\phi)\epsilon^n,$$

$$\boldsymbol{h}^{\epsilon f}(\theta,\phi) = \sum_{n=0}^{\infty} \boldsymbol{h}_n(\theta,\phi)\epsilon^n. \tag{5.2}$$

One can then derive a recursion formula for $\boldsymbol{E}_n^{a,b}$ (for $n \geq 0$):

$$\nabla\times\nabla\times\boldsymbol{E}_n^{a,b} - k^2\boldsymbol{E}_n^{a,b} = \boldsymbol{F}_n^{a,b} + \boldsymbol{G}_n^{a,b}, \quad \text{in } \ \Omega; \tag{5.3}$$

$$\boldsymbol{E}_n^{a,b}\times\boldsymbol{e}_r = \boldsymbol{0} \quad \text{at } \ r = a; \tag{5.4}$$

$$(\nabla\times\boldsymbol{E}_n^{a,b})\times\boldsymbol{e}_r - \mathrm{i}k\mathscr{T}_b\big[(\boldsymbol{E}_n^{a,b})_S\big] = \boldsymbol{h}_n + \boldsymbol{g}_n \quad \text{at } \ r = b, \tag{5.5}$$

where $\boldsymbol{G}_n^{a,b}$ and $\boldsymbol{g}_n$ are given by explicit recurrence formulae in Ma *et al.* (2015, Appendix B).

- The third step is to obtain the approximation $\boldsymbol{E}_{n,N}^L$ (in the form of (4.1)) to $\boldsymbol{E}_n^{a,b}$ (for $0 \leq n \leq M$) by solving the above Maxwell equations (5.3)–(5.5) in the spherical shell Ω using the decoupled method presented in Section 4. Then, we define our approximation to $\boldsymbol{E}^{\epsilon f}$ by

$$\boldsymbol{E}_N^{L,M}(r,\theta,\phi) = \sum_{n=0}^{M} \boldsymbol{E}_{n,N}^L(r,\theta,\phi)\,\epsilon^n. \tag{5.6}$$

Next, we shall use the general convergence theory developed in Nicholls & Shen (2009) to give an error estimate for $\boldsymbol{E}^{\epsilon f} - \boldsymbol{E}_N^{L,M}$. Using essentially the same argument as in the proof of Nicholls & Shen (2009, Theorem 5.5) for the Helmholtz equation, we can prove the following bounds.

Proposition 5.1 Let $\boldsymbol{F}_n^{a,b} \in (H^{s-2}(\Omega))^3, f \in H^s(S)$ and $\boldsymbol{h}_n \in (H^{s-3/2}(S))^2$ for an integer $s \geq 2$. Then, the expansion (5.2) converges strongly, i.e., there exists $C_1, C_2 > 0$ such that

$$\|\boldsymbol{E}_n^{a,b}\|_{(H^s(\Omega))^3} \leq C_1\big(\|\boldsymbol{F}_n^{a,b}\|_{(H^{s-2}(\Omega))^3} + \|\boldsymbol{h}_n\|_{(H^{s-3/2}(S))^2}\big)B^n, \quad \text{for some } \ B > C_2\|f\|_{H^s(S)}. \tag{5.7}$$

On the other hand, it can be shown that the space with the norm in (4.1) satisfies $\boldsymbol{H}^t(S; H_{\varpi}^s(I)) \subseteq (H^{s+t}(\Omega))^3$. Therefore, with the above result and Theorems 4.6–4.7 at our disposal, we can then apply Theorem 2.1 in Nicholls & Shen (2009) to obtain the following estimates.

Theorem 5.2 Let $\boldsymbol{E}^{\epsilon f}$ be the solution of the Maxwell equations in $\hat{\Omega}$ and $\boldsymbol{E}_N^{L,M}$ be its approximation defined in (5.6). Then, under the condition of Proposition 5.1 and Theorems 4.6–4.7, we have

$$\|\boldsymbol{E}^{\epsilon f} - \boldsymbol{E}_N^{L,M}\|_{\hat{\Omega}} \lesssim (B\epsilon)^{M+1} + \big\{(1+k^{-1}N)(L+k^2N^{-1})N^{-s} + L^{-s}\big\}(\|\boldsymbol{F}^{\epsilon f}\|_{(H^{s-2}(\hat{\Omega}))^3} + \|\boldsymbol{h}^{\epsilon f}\|_{(H^s(S))^2}),$$

and

$$\|\nabla \times (\boldsymbol{E}^{\epsilon f} - \boldsymbol{E}_N^{L,M})\|_{w,\hat{\Omega}} \lesssim (B\epsilon)^{M+1} + \big\{\big(N + (1 + kN^{-1})(L + k^2N^{-1})\big)N^{1-s} + L^{1-s}\big\}\big(\|\boldsymbol{F}^{\epsilon f}\|_{(H^{s-2}(\hat{\Omega}))^3} + \|\boldsymbol{h}^{\epsilon f}\|_{(H^s(S))^2}\big),$$

for any $B > C_2\|f\|_{H^s(S)}$, where C_2 is the constant in Proposition 5.1.

6. Concluding remarks

We summarize below the major contributions of this article.

Firstly, we considered the Maxwell equations in a spherical shell.

- We reduced the Maxwell system into two sequences of decoupled one-dimensional problems by using divergence-free VSH. This reduction not only led to a more efficient spectral-Galerkin algorithm, but also greatly simplified its analysis.
- We derived wavenumber explicit bounds for the (continuous) Maxwell system with (exact) TBCs, and wavenumber explicit error estimates for its spectral-Galerkin approximation.
- We derived optimal wavenumber explicit *a priori* bounds and error estimates for the Helmholtz equation, which improved the results in Shen & Wang (2007).

Then, we applied the TFE approach (David & Fernando, 2004) to deal with general scatterers. By using the general framework developed in Nicholls & Shen (2009), we derived rigorous wavenumber explicit error estimates for the complete algorithm for the ϵ-perturbed variant. To the best of our knowledge, these are the first estimates for time-harmonic Maxwell equations with *exact TBCs*.

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Appendix A. Properties of VSH

We adopt the notation and normalization of SPH in Nédélec (2001). Let (r,θ,ϕ) (with $\theta \in [0,\pi]$ and $\phi \in [0,2\pi)$) be the spherical coordinates. Then the (right-handed) orthonormal coordinate basis consists of $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\}$. Denote by ∇_S and Δ_S the tangent gradient operator and the Laplace–Beltrami operator on S (the unit spherical surface). We denote by $\{Y_l^m(\theta,\phi)\}$ the (scalar) SPH that are eigenfunctions of Δ_S, and form an orthonormal basis of $L^2(S)$.

We use the family of VSH: $\left\{Y_l^m \boldsymbol{e}_r, \nabla_S Y_l^m, \boldsymbol{T}_l^m = \nabla_S Y_l^m \times \boldsymbol{e}_r\right\}$ in Swarztrauber & Spotz (2000) (also see Morse & Feshbach, 1953). They are mutually orthogonal in $\boldsymbol{L}^2(S)$ (for vector fields) and normalized such that

$$\left\langle \boldsymbol{T}_l^m, \boldsymbol{T}_l^m \right\rangle_S = l(l+1), \quad \left\langle \nabla_S Y_l^m, \nabla_S Y_l^m \right\rangle_S = l(l+1), \quad \left\langle Y_l^m \boldsymbol{e}_r, Y_l^m \boldsymbol{e}_r \right\rangle_S = 1. \tag{A.1}$$

We have

$$\boldsymbol{T}_l^m \times \boldsymbol{e}_r = -\nabla_S Y_l^m, \quad \nabla_S Y_l^m \times \boldsymbol{e}_r = \boldsymbol{T}_l^m, \quad Y_l^m \boldsymbol{e}_r \times \boldsymbol{e}_r = \boldsymbol{0}. \tag{A.2}$$

Define the differential operators:

$$\mathrm{d}_l^{\pm} = \frac{d}{dr} \pm \frac{l}{r}, \quad \mathcal{L}_l = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}, \quad \hat{\partial}_r = \frac{d}{dr} + \frac{1}{r}. \tag{A.3}$$

Let f be a scalar function of r. The following properties can be derived from Hill (1954):

$$\operatorname{div}\big(f\boldsymbol{T}_l^m\big) = 0, \quad \Delta\big(f\boldsymbol{T}_l^m\big) = \mathcal{L}_l(f)\boldsymbol{T}_l^m, \quad \nabla \times \big(f\boldsymbol{T}_l^m\big) = \hat{\partial}_r f\, \nabla_S Y_l^m + l(l+1)\frac{f}{r} Y_l^m \boldsymbol{e}_r, \tag{A.4}$$

$$\nabla \times \big(f\nabla_S Y_l^m\big) = -\hat{\partial}_r f\, \boldsymbol{T}_l^m, \quad \nabla \times \big(fY_l^m \boldsymbol{e}_r\big) = \frac{f}{r}\boldsymbol{T}_l^m. \tag{A.5}$$

Moreover, we have

$$\operatorname{div}\big(f\nabla_S Y_l^m\big) = \frac{l(l+1)}{2l+1}\big(d_{l-1}^- - d_{l+2}^+\big)f\, Y_l^m = -l(l+1)\frac{f}{r}\, Y_l^m, \tag{A.6}$$

$$\operatorname{div}\big(fY_l^m \boldsymbol{e}_r\big) = \frac{1}{2l+1}\big(ld_{l-1}^- + (l+1)d_{l+2}^+\big)f\, Y_l^m = \left(\frac{d}{dr} + \frac{2}{r}\right) f\, Y_l^m. \tag{A.7}$$

Appendix B. Proof of Theorem 2.2

Case (i) $\rho = \nu/\kappa \in (0, \theta_0)$. Set $\sec\beta = \kappa/\nu = \rho^{-1}$, i.e., $\cos\beta = \rho$ with $0 < \beta < \pi/2$. One verifies

$$\sin\beta = \sqrt{1-\rho^2}, \quad \tan\beta = \frac{\sqrt{1-\rho^2}}{\rho}, \quad \cot\beta = \frac{\rho}{\sqrt{1-\rho^2}}, \quad 0 < \rho < \theta_0 < 1. \tag{B.1}$$

Recall the formulas (cf. Abramowitz & Stegun, 1964, (9.3.15–9.3.20))

$$J_\nu(\nu\sec\beta) = \sqrt{\frac{2}{\pi\nu\tan\beta}}\big(L_1\cos\psi + M_1\sin\psi\big), \quad Y_\nu(\nu\sec\beta) = \sqrt{\frac{2}{\pi\nu\tan\beta}}\big(L_1\sin\psi - M_1\cos\psi\big),$$

$$J_\nu'(\nu\sec\beta) = -\sqrt{\frac{\sin 2\beta}{\pi\nu}}\big(L_2\sin\psi + M_2\cos\psi\big), \quad Y_\nu'(\nu\sec\beta) = \sqrt{\frac{\sin 2\beta}{\pi\nu}}\big(L_2\cos\psi - M_2\sin\psi\big),$$

where $\psi = \nu(\tan\beta - \beta) - 1/4$, and $L_i = L_i(\nu,\beta), M_i = M_i(\nu,\beta), i = 1,2$ are given in Abramowitz & Stegun (1964, pp. 366–367). Inserting them into (2.3) leads to

$$\operatorname{Re}(\mathcal{T}_{l,\kappa}) = -\frac{1}{2\kappa} - \sin\beta\,\frac{L_1M_2 + L_2M_1}{L_1^2 + M_1^2}, \quad \operatorname{Im}(\mathcal{T}_{l,\kappa}) = \frac{\rho\tan\beta}{L_1^2 + M_1^2}. \tag{B.2}$$

We find it suffices to take the leading term of $L_i, M_i, i = 1, 2$ in Abramowitz & Stegun (1964, pp. 366–367), that is,

$$L_1 \sim 1, \quad L_2 \sim 1, \quad M_1 \sim \frac{3\cot\beta + 5\cot^3\beta}{24\nu}, \quad M_2 \sim \frac{9\cot\beta + 7\cot^3\beta}{24\nu}. \tag{B.3}$$

By a direct calculation and using (B.1), we obtain

$$\sin\beta\left(L_1M_2 + L_2M_1\right) \sim \sin\beta\,\frac{\cot\beta + \cot^3\beta}{2\nu} = \frac{1}{2\kappa}\frac{1}{1-\rho^2}, \tag{B.4}$$

and

$$M_1^2 \sim \frac{3+5\rho^2}{192(1-\rho^2)^2}\frac{1}{\kappa^2}, \qquad \frac{1}{L_1^2 + M_1^2} \sim 1 - M_1^2 = 1 + O(\kappa^{-2}). \tag{B.5}$$

Then we obtain (2.14) from (B.2) and the above.

Cases (ii)–(iii) $\rho = \nu/\kappa \in [\theta_0, \vartheta_1] \cup (\vartheta_1, \vartheta_2)$. We adopt the asymptotic formulas (Abramowitz & Stegun, 1964, (9.3.23–9.3.28)):

$$\begin{aligned} J_\nu(\nu + z\sqrt[3]{\nu}) &\sim \left(\frac{2}{\nu}\right)^{\frac{1}{3}} \mathrm{Ai}(-\sqrt[3]{2}z) + O(\nu^{-1}), \quad Y_\nu(\nu + z\sqrt[3]{\nu}) \sim -\left(\frac{2}{\nu}\right)^{\frac{1}{3}} \mathrm{Bi}(-\sqrt[3]{2}z) + O(\nu^{-1}), \\ J_\nu'(\nu + z\sqrt[3]{\nu}) &\sim -\left(\frac{2}{\nu}\right)^{\frac{2}{3}} \mathrm{Ai}'(-\sqrt[3]{2}z) + O(\nu^{-\frac{4}{3}}), \quad Y_\nu'(\nu + z\sqrt[3]{\nu}) \sim \left(\frac{2}{\nu}\right)^{\frac{2}{3}} \mathrm{Bi}'(-\sqrt[3]{2}z) + O(\nu^{-\frac{4}{3}}), \end{aligned} \tag{B.6}$$

where $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$ are Airy functions of the first and second kinds, respectively. Set

$$t = -\sqrt[3]{2}\,z, \quad \kappa = \nu + z\sqrt[3]{\nu} \ \ (\text{i.e.,} \ \ z = (\kappa - \nu)/\sqrt[3]{\nu}). \tag{B.7}$$

We obtain from (B.6) and (2.3) that

$$\mathrm{Re}(\mathcal{T}_{l,\kappa}) \sim -\frac{1}{2\kappa} - \left(\frac{2}{\nu}\right)^{1/3} T_R(t), \quad \mathrm{Im}(\mathcal{T}_{l,\kappa}) \sim \frac{2}{\pi\kappa}\left(\frac{\nu}{2}\right)^{2/3} T_I(t), \tag{B.8}$$

where

$$T_R(t) = \frac{\mathrm{Ai}(t)\mathrm{Ai}'(t) + \mathrm{Bi}(t)\mathrm{Bi}'(t)}{\mathrm{Ai}^2(t) + \mathrm{Bi}^2(t)}, \quad T_I(t) = \frac{1}{\mathrm{Ai}^2(t) + \mathrm{Bi}^2(t)}. \tag{B.9}$$

Note that the Airy functions have different asymptotic behaviours for $t \le -1$ and $-1 < t < 1$ (see, e.g., Abramowitz & Stegun, 1964; Zhang & Jin, 1996). We therefore solve the equations: $t = -\sqrt[3]{2}\,z = -\sqrt[3]{2}(\kappa - \nu)/\sqrt[3]{\nu} = \mp 1$, that is,

$$\nu + 2^{-\frac{1}{3}}\nu^{\frac{1}{3}} - \kappa = 0, \quad \nu - 2^{-\frac{1}{3}}\nu^{\frac{1}{3}} - \kappa = 0. \tag{B.10}$$

Both are cubic equations in $\nu^{\frac{1}{3}}$ with only one real root each. We find the real root of the first equation is $\kappa\,\vartheta_1$, whereas that of the second one is $\kappa\,\vartheta_2$, where ϑ_1 and ϑ_2 are given in (2.8).

(a) For $\rho \in [\theta_0, \vartheta_1]$ (note: $t = -\sqrt[3]{2}z \le -1$), we recall the asymptotic formulas (see Abramowitz & Stegun, 1964, (10.4.60))

$$\begin{aligned}
\mathrm{Ai}(t) &\sim \frac{1}{\sqrt[4]{-\pi^2 t}}\left(\sin\xi - \frac{5}{72\eta}\cos\xi\right), \quad \mathrm{Ai}'(t) \sim -\sqrt[4]{-\frac{t}{\pi^2}}\left(\cos\xi - \frac{7}{72\eta}\sin\xi\right),\\
\mathrm{Bi}(t) &\sim \frac{1}{\sqrt[4]{-\pi^2 t}}\left(\cos\xi + \frac{5}{72\eta}\sin\xi\right), \quad \mathrm{Bi}'(t) \sim \sqrt[4]{-\frac{t}{\pi^2}}\left(\sin\xi + \frac{7}{72\eta}\cos\xi\right),
\end{aligned} \tag{B.11}$$

where

$$\xi = \eta + \frac{\pi}{4}, \quad \eta = \frac{2}{3}(-t)^{3/2} = \frac{2}{3}\left(\sqrt[3]{2}z\right)^{3/2}.$$

Thus, a direct calculation leads to

$$\begin{aligned}
&\mathrm{Ai}(t)\mathrm{Ai}'(t) + \mathrm{Bi}(t)\mathrm{Bi}'(t) \sim \frac{1}{6\pi\eta} = \frac{1}{4\pi}(-t)^{-3/2},\\
&\mathrm{Ai}^2(t) + \mathrm{Bi}^2(t) \sim \frac{1}{\pi\sqrt{-t}}\left(1 + \left(\frac{5}{72\eta}\right)^2\right) = \frac{1}{\pi\sqrt{-t}} + O((-t)^{-7/2}).
\end{aligned} \tag{B.12}$$

Inserting them into (B.9), we obtain

$$T_R(t) \sim -\frac{1}{4t} = \frac{1}{4\sqrt[3]{2}}\frac{\sqrt[3]{\nu}}{\kappa - \nu}, \quad T_I(t) \sim \frac{\pi\sqrt{-t}}{1 + O((-t)^{-3})} \sim 2^{1/6}\pi\left(\frac{\kappa - \nu}{\nu^{1/3}}\right)^{1/2}. \tag{B.13}$$

We derive from (B.8) that

$$\mathrm{Re}(\mathcal{T}_{l,\kappa}) \sim -\frac{1}{2\kappa} - \frac{1}{4(\kappa - \nu)}, \quad \mathrm{Im}(\mathcal{T}_{l,\kappa}) \sim \frac{\nu}{\kappa}\sqrt{2\left(\frac{\kappa}{\nu} - 1\right)}. \tag{B.14}$$

This yields (2.15).

(b) For $\rho \in (\vartheta_1, \vartheta_2)$ (note: $|t| = \sqrt[3]{2}|z| < 1$), we approximate $T_R(t)$ and $T_I(t)$ in (B.9) by their Taylor expansions at $t = 0$, which requires to evaluate $\mathrm{Ai}^{(m)}(0)$ and $\mathrm{Bi}^{(m)}(0)$ for $m \ge 1$. Recall that the Airy functions satisfy the Airy equation: $w''(t) - tw(t) = 0, \quad t \in \mathbb{R}$, and some special values are

$$\mathrm{Ai}(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}, \quad \mathrm{Ai}'(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})}, \quad \mathrm{Bi}(0) = \frac{1}{3^{\frac{1}{6}}\Gamma(\frac{2}{3})}, \quad \mathrm{Bi}'(0) = \frac{3^{\frac{1}{6}}}{\Gamma(\frac{1}{3})}. \tag{B.15}$$

With these and some tedious calculation, we can obtain

$$T_R(t) = T_R(0) + T_R'(0)t + \frac{T_R''(0)}{2}t^2 + O(t^3), \; T_I(t) = T_I(0) + T_I'(0)t + \frac{T_I''(0)}{2}t^2 + O(t^3),$$

with

$$c_1 = T_R(0) = \frac{3^{\frac{1}{3}}}{2}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \approx 0.3645, \quad T_R'(0) = 2c_1^2, \quad T_R''(0) = 1 - 16c_1^3,$$

$$T_I(0) = \frac{3^{\frac{4}{3}}}{4}\left(\Gamma\left(\frac{2}{3}\right)\right)^2 = \sqrt{3}\pi c_1, \quad T_I'(0) = -2\sqrt{3}\pi c_1^2, \quad T_I''(0) = 0.$$

Noting that $t = -\sqrt[3]{2}(\kappa - \nu)/\sqrt[3]{\nu}$, Thus, we derive from (B.8)–(B.9) that

$$\mathrm{Re}(\mathcal{T}_{l,\kappa}) \sim -\sqrt[3]{\frac{2}{\nu}}\left(c_1 + 2c_1^2 t + \frac{1}{2}\big(1 - 16c_1^3\big)t^2\right) - \frac{1}{2\kappa}, \tag{B.16}$$

$$\mathrm{Im}(\mathcal{T}_{l,\kappa}) \sim 2^{\frac{1}{3}}\sqrt{3}c_1\frac{\nu^{2/3}}{\kappa}\big(1 - 2c_1 t\big), \quad \text{where } \ t = -\sqrt[3]{2}\,\frac{\kappa - \nu}{\sqrt[3]{\nu}}. \tag{B.17}$$

Hence, we obtain the desired estimates for this case.

Case (iv) $\rho = \nu/\kappa \in [\vartheta_2, \infty)$. Set $\operatorname{sech}\alpha = \rho^{-1}$, i.e., $\cosh\alpha = \rho$ with $\alpha > 0$. One verifies

$$\sinh\alpha = \sqrt{\rho^2 - 1}, \quad \tanh\alpha = \frac{\sqrt{\rho^2 - 1}}{\rho}, \quad \Psi = \alpha - \tanh\alpha > 0. \tag{B.18}$$

Recall the asymptotic formulas (Abramowitz & Stegun, 1964, (9.3.7–9.3.8)):

$$J_\nu(\nu\operatorname{sech}\alpha) \sim \frac{e^{-\nu\Psi}}{\sqrt{2\pi\nu\tanh\alpha}}\big\{1 + O(\nu^{-1})\big\}; \quad Y_\nu(\nu\operatorname{sech}\alpha) \sim -\frac{e^{\nu\Psi}}{\sqrt{\pi/2\ \nu\tanh\alpha}}\big\{1 + O(\nu^{-1})\big\}. \tag{B.19}$$

Note that by (B.18),

$$\Psi(\rho) = \operatorname{arccosh}\rho - \sqrt{1 - \rho^{-2}} = \ln(\rho + \sqrt{\rho^2 - 1}) - \frac{\sqrt{\rho^2 - 1}}{\rho} \quad \rho > 1, \tag{B.20}$$

which is monotonically increasing with respect to ρ. By (2.10), we have

$$\begin{aligned}\Psi(\vartheta_2) &\sim \ln(1 + \tau + \sqrt{2\tau + \tau^2}) - \frac{\sqrt{2\tau + \tau^2}}{1 + \tau} \sim \tau + \sqrt{2\tau + \tau^2} - \frac{\sqrt{2\tau + \tau^2}}{1 + \tau}\\ &= \tau + \tau\frac{\sqrt{2\tau + \tau^2}}{1 + \tau} \sim \tau, \quad \text{where } \ \tau = \frac{1}{\sqrt[3]{2}\,\kappa^{2/3}}.\end{aligned} \tag{B.21}$$

Thus, we observe from (B.19) that in the range of interest, J_ν, J_ν' decay exponentially, whereas Y_ν, Y_ν' grow exponentially. By (2.3) and (B.19),

$$\mathrm{Im}(\mathcal{T}_{l,\kappa}) = \frac{2}{\pi\kappa}\frac{1}{J_\nu^2(\kappa) + Y_\nu^2(\kappa)} \sim \frac{4\nu}{\kappa}\tanh\alpha\,\frac{e^{-2\nu\Psi}}{4 + e^{-4\nu\Psi}} \sim \sqrt{\rho^2 - 1}\ e^{-2\nu\Psi}, \tag{B.22}$$

which leads to the estimate of the imaginary part in (2.18). As $\mathrm{Im}(\mathcal{T}_{l,\kappa})$ decays exponentially with respect to l. We derive from (2.4) that

$$\mathrm{Re}(\mathcal{T}_{l,\kappa}) = \frac{l}{\kappa} - \frac{Y_{\nu+1}(\kappa)}{Y_\nu(\kappa)} - \mathrm{Im}(\mathcal{T}_{l,\kappa})\frac{J_\nu(\kappa)}{Y_\nu(\kappa)} \sim \frac{l}{\kappa} - \frac{Y_{\nu+1}(\kappa)}{Y_\nu(\kappa)}. \tag{B.23}$$

To obtain better estimate, we resort to the asymptotic approximation of the ratio (cf. Kiefer & Weiss, 1972):

$$\frac{Y_{\nu+1}(\kappa)}{Y_\nu(\kappa)} = \frac{1+\sqrt{1-\rho^{-2}}}{\rho^{-1}}\left\{1 - \frac{1-\sqrt{1-\rho^{-2}}}{2(1-\rho^{-2})}\frac{1}{\nu} + O\left(\frac{1}{\nu^2}\right)\right\}, \tag{B.24}$$

which is valid for $\nu > \kappa$ and $\kappa \sim \nu$. In fact, as shown in Kiefer & Weiss (1972), it is derived from the formula (B.19) with more terms. Inserting (B.24) into (B.23) leads to the estimate of the real part in (2.18).

Appendix C. Proof of Theorem 3.6

Case (i) $\rho = \nu/\kappa \in (0, \theta_0)$. By (3.43) and (2.14),

$$\mathrm{Re}(\mathcal{S}_{l,\kappa}) \sim \frac{\rho^2}{2\kappa}\frac{1-\rho^2}{(1-\rho^2)^3 + \kappa^{-2}\rho^4} \sim \frac{\rho^2}{2\kappa}\frac{1}{(1-\rho^2)^2}, \quad \mathrm{Im}(\mathcal{S}_{l,\kappa}) \sim \frac{(1-\rho^2)^2\sqrt{1-\rho^2}}{(1-\rho^2)^3 + 4\kappa^{-2}\rho^4} \sim \frac{1}{\sqrt{1-\rho^2}}.$$

This leads to (3.46).

Case (ii) $\rho = \nu/\kappa \in [\theta_0, \vartheta_1]$. By (3.43) and (2.15),

$$\begin{aligned}\mathrm{Re}(\mathcal{S}_{l,\kappa}) &\sim \frac{1}{2\kappa}\left(1 + \frac{1}{2(1-\rho)}\right)\left(\frac{1}{4\kappa^2}\left(1+\frac{1}{2(1-\rho)}\right)^2 + 2\rho(1-\rho)\right)^{-1} \\ &\sim \frac{1}{4\rho(1-\rho)\kappa}\left(1+\frac{1}{2(1-\rho)}\right)\end{aligned}$$

and

$$\mathrm{Im}(\mathcal{S}_{l,\kappa}) \sim \frac{\sqrt{2\rho(1-\rho)}}{\frac{1}{4\kappa^2}(1+\frac{1}{2(1-\rho)})^2 + 2\rho(1-\rho)} \sim \frac{1}{\sqrt{2\rho(1-\rho)}},$$

so (3.47) follows.

Case (iii) $\rho = \nu/\kappa \in (\vartheta_1, \vartheta_2)$. By (3.43) and (2.16),

$$\mathrm{Re}(\mathcal{S}_{l,\kappa}) \sim \frac{\sqrt[3]{2/\nu}\big(c_1 + 2c_1^2 t + \frac{1}{2}\big(1 - 16c_1^3\big)t^2\big) - 1/(2\kappa)}{\big(\sqrt[3]{2/\nu}\big(c_1 + 2c_1^2 t + \frac{1}{2}\big(1 - 16c_1^3\big)t^2\big) - 1/(2\kappa)\big)^2 + \big(\sqrt[3]{2/\nu}\sqrt{3}c_1\rho(1 - 2c_1 t)\big)^2}$$

$$\sim \frac{1}{c_1}\left(\frac{\nu}{2}\right)^{1/3} \frac{1 + 2c_1 t + c_2 t^2}{(1 + 2c_1 t + c_2 t^2)^2 + 3\rho^2(1 - 2c_1 t)^2} \sim \frac{1}{c_1}\left(\frac{\nu}{2}\right)^{1/3} \frac{1 + 2c_1 t + c_2 t^2}{(1 + 2c_1 t + c_2 t^2)^2 + 3(1 - 2c_1 t)^2}$$

$$= \frac{1}{4c_1}\left(\frac{\nu}{2}\right)^{1/3} \frac{1 + 2c_1 t + c_2 t^2}{1 - 2c_1 t + (4c_1^2 + c_2/2)t^2 + c_1 c_2 t^3 + c_2^2 t^4/4},$$

where $c_2 = (1 - 16c_1^3)/(2c_1) \approx 0.3088$. In the above, we dropped the term $-1/(2\kappa)$, and used $\rho \approx 1$.

Similarly, we can derive

$$\mathrm{Im}(\mathcal{S}_{l,\kappa}) \sim \frac{\sqrt{3}}{4c_1}\left(\frac{\nu}{2}\right)^{1/3} \frac{1 - 2c_1 t}{1 - 2c_1 t + (4c_1^2 + c_2/2)t^2 + c_1 c_2 t^3 + c_2^2 t^4/4}.$$

Thus, we obtain (3.48).

Case (iv) $\rho = \nu/\kappa \in [\vartheta_2, \infty)$. Noticing from (2.18) that $\mathrm{Im}(\mathcal{T}_{l,\kappa})$ is exponentially small in this range, we obtain from (3.43) and (2.18) that

$$\mathrm{Re}(\mathcal{S}_{l,\kappa}) \sim \left(\sqrt{\rho^2 - 1} - \frac{1}{2\kappa}\left(1 + \frac{1}{\rho^2 - 1}\right)\right)^{-1} = \frac{1}{\sqrt{\rho^2 - 1}}\left(1 - \frac{1}{2\kappa\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{\rho^2 - 1}\right)\right)^{-1}$$

$$\sim \frac{1}{\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{2\kappa\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{\rho^2 - 1}\right)\right) \quad \text{and } \mathrm{Im}(\mathcal{S}_{l,\kappa})$$

$$\sim \frac{e^{-2\nu\Psi}}{\sqrt{\rho^2 - 1}}\left(1 - \frac{1}{2\kappa\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{\rho^2 - 1}\right)\right)^{-2}$$

$$\sim \frac{e^{-2\nu\Psi}}{\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{\kappa\sqrt{\rho^2 - 1}}\left(1 + \frac{1}{\rho^2 - 1}\right)\right),$$

where we used $(1 - y)^{-1} \sim 1 + y$, $(1 - y)^{-2} \sim 1 + 2y$ for $y \sim 0$. This ends the proof.