SIAM J. NUMER. ANAL.
Vol. 60, No. 3, pp. 1026–1054

STABILITY AND ERROR ANALYSIS OF IMEX SAV SCHEMES FOR THE MAGNETO-HYDRODYNAMIC EQUATIONS*

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Abstract. We construct and analyze first- and second-order implicit-explicit schemes based on the scalar auxiliary variable approach for the magneto-hydrodynamic equations. These schemes are linear, only require solving a sequence of linear differential equations with constant coefficients at each time step, and are unconditionally energy stable. We derive rigorous error estimates for the velocity, pressure, and magnetic field of the first-order scheme in the two-dimensional case without any condition on the time step. Numerical examples are presented to validate the proposed schemes.

Key words. magneto-hydrodynamic equations, implicit-explicit schemes, energy stability, error estimates

AMS subject classifications. 65M12, 65M15, 76E25

DOI. 10.1137/21M1430376

1. Introduction.

We consider in this paper numerical approximation of the following magneto-hydrodynamic (MHD) equations [18]:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p - \alpha(\nabla\times\mathbf{b})\times\mathbf{b} = 0 \quad \text{in } \Omega\times \mathrm{J}, \tag{1.1a}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \eta\nabla\times(\nabla\times\mathbf{b}) + \nabla\times(\mathbf{b}\times\mathbf{u}) = 0 \quad \text{in } \Omega\times \mathrm{J}, \tag{1.1b}$$

$$\nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0 \quad \text{in } \Omega\times \mathrm{J}, \tag{1.1c}$$

with boundary and initial conditions

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{b}\cdot\mathbf{n} = 0, \quad \mathbf{n}\times(\nabla\times\mathbf{b}) = 0 \quad \text{on } \partial\Omega\times \mathrm{J},$$
$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}), \quad \mathbf{b}(\mathbf{x},0) = \mathbf{b}^0(\mathbf{V}) \quad \text{in } \Omega,$$

where Ω is an open bounded domain in $\mathbb{R}^d$ $(d = 2,3)$ with a sufficiently smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal of the domain Ω, $J = (0,T]$, and $(\mathbf{u}, p, \mathbf{b})$ represent, respectively, the unknown velocity, pressure, and magnetic field. The parameters ν and η are kinematic viscosity and magnetic diffusivity, respectively, and $\alpha = 1/(4\pi\mu\rho)$ with μ as the magnetic permeability and ρ as the fluid density.

The MHD system is used to describe the interaction between a viscous, incompressible, electrically conducting fluid and an external magnetic field. When a conducting fluid is placed in an existing magnetic field, the fluid motion produces electric

*Received by the editors June 30, 2021; accepted for publication (in revised form) December 22, 2021; published electronically May 10, 2022.
https://doi.org/10.1137/21M1430376
Funding: The work of the authors was supported by the National Natural Science Foundation of China grants 11901489 and 11971407.
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currents which in turn create forces on the fluid and change the magnetic field itself. It has been widely used in many science and engineering applications, such as liquid metal cooling for nuclear reactors, sustained plasma confinement for controlled thermonuclear fusion, etc. [8, 6]. The mathematical theory of MHD equations can be found in [18].

Numerical approximation of the MHD equations is challenging, as it involves delicate nonlinear coupling between the velocity and magnetic field in addition to the difficulties associated with the Navier–Stokes equations and Maxwell equations. There exists a large literature devoted to constructing compatible spatial discretization for the MHD equations; see [28, 2, 17, 7, 4] and related references. In this paper, we are only concerned with time discretization, which can be coupled with any well developed compatible spatial discretization.

The MHD equations (1.1) are energy dissipative. More precisely, taking the inner products of (1.1a) and (1.1b) with $\mathbf{u}$ and $\alpha\mathbf{b}$, respectively, summing up the results, we find that the nonlinear terms do not contribute to the energy and that the following energy dissipation law holds:

$$\frac{d}{dt}E(\mathbf{u},\mathbf{b}) = -\nu\|\nabla\mathbf{u}\|^2 - \alpha\eta\|\nabla\times\mathbf{b}\|^2 \quad \text{with} \quad E(\mathbf{u},\mathbf{b}) = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{\alpha}{2}\|\mathbf{b}\|^2. \tag{1.2}$$

It is thus desirable to construct numerical schemes which satisfy a discrete energy dissipation law.

Most existing work use fully implicit or semi-implicit treatments for the nonlinear terms so that the effects of nonlinear coupling can cancel each other and a discrete energy dissipation law can be derived. However, one needs to solve a nonlinear system or a coupled linear system with time dependent coefficients at each time step. For examples, Armero and Simo developed in [1] energy dissipative schemes for an abstract evolution equation with applications to the incompressible MHD equations; Tone [25] considered an implicit Euler scheme for the two-dimensional MHD equations and established a uniform H^2 stability; Layton, Tran, and Trenchea constructed in [12] two partitioned methods for uncoupling evolutionary MHD flows; Hiptmair et al. [11] developed a fully divergence-free finite element method for MHD equations with a semi-implicit treatment of the nonlinear terms; Zhang, Yang, and Bi [30] proposed a second-order linear backward difference formula scheme with an extrapolated treatment for the nonlinear terms and proved its unconditionally stability and convergence (cf. also [29]); and most recently, Li, Wang, and Xu [13] proposed a fully discrete linearized H^1 conforming Lagrange finite element method and derived the convergence based on the regularity of the initial conditions and source terms without extra assumptions on the regularity of the solution. To alleviate the cost of solving fully coupled systems at each time step, Badia et al. [3] developed an operator splitting algorithm by a stabilized finite element formulation based on projections; Choi and Shen [5] constructed several efficient splitting schemes based on the standard and rotational pressure-correction schemes with a semi-implicit treatment of the nonlinear terms for the MHD equations.

From a computational point of view, it is desirable for a numerical scheme to treat the nonlinear term explicitly while still being energy dissipative so that one only needs to solve simple linear equations with constant coefficients at each time step. However, with a direct explicit treatment of the nonlinear terms, their energy contribution no longer vanishes, so it becomes very difficult to derive a uniform bound for the numerical solution. Liu and Pego [16] constructed a first-order scheme with fully explicit treatment of the nonlinear terms and showed that its numerical solution is bounded with the time step sufficiently small, but their scheme is not shown to be energy dissipative. The recently proposed scalar auxiliary variable (SAV) approach [21, 20, 22] provides a general approach to construct linear, decoupled unconditionally energy stable schemes for gradient flows. The approach has been extended to Navier–Stokes equations in [15]. However, the scheme in [15] requires solving a nonlinear algebraic equation whose well posedness is not guaranteed. We introduced in [14] a different SAV approach which leads to purely linear and unconditionally stable schemes for the Navier–Stokes equations, and we proved corresponding error estimates.

The aim of this work is to extend the approach proposed in [14] to the MHD equations which are much more complicated with nonlinear couplings between the velocity and magnetic fields. Our main contributions are twofold:

- We construct first- and second-order implicit-explicit (IMEX) SAV schemes for the MHD equations and show that they are unconditionally energy stable. These schemes only require solving a sequence of differential equations with constant coefficients at each time step, so they are very efficient and easy to implement.
- We establish rigorous error estimates for the first-order scheme in the two-dimensional case without any condition on the time step.

Compared to the Navier–Stokes equations or Maxwell's equations, the error analysis for the MHD equations is much more involved due to the nonlinear coupling terms. Our error analysis uses essentially the unconditional and uniform bounds of the numerical solution that we derive for our SAV schemes. To the best of our knowledge, this is the first rigorous error analysis for any scheme with fully explicit treatment of nonlinear terms for the MHD equations.

To simplify the presentation, we mainly concentrate on the time discretization where lies the main novelty of the paper. One can of course couple the time discretization schemes constructed in this paper with any compatible spatial discretization to obtain a fully discrete scheme and obtain corresponding stability results following essentially the same procedure as presented in this paper. Such an example is provided in section 4.

The paper is organized as follows. In section 2, we construct our IMEX SAV schemes and prove their stability. In section 3, we carry out a rigorous error analysis for the first-order IMEX SAV scheme in the two-dimensional case. We present some numerical experiments to validate our schemes in section 4 and conclude with a few remarks in section 5.

2. The SAV schemes and their energy stability. In this section, we construct first- and second-order IMEX schemes based on the SAV approach for the MHD equations and show that they are unconditionally energy stable. We introduce an (SAV),

$$q(t) = \epsilon \exp\left(-\frac{\mathrm{t}}{\mathrm{T}}\right), \tag{2.1}$$

where $\epsilon > 0$ is a parameter, and expand the system (1.1) as follows:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p + \dfrac{q(t)}{q(t)}(\mathbf{u}\cdot\nabla\mathbf{u} - \alpha(\nabla\times\mathbf{b})\times\mathbf{b}) = 0, & (2.2)\\ \dfrac{\partial \mathbf{b}}{\partial t} + \eta\nabla\times(\nabla\times\mathbf{b}) + \dfrac{q(t)}{q(t)}\nabla\times(\mathbf{b}\times\mathbf{u}) = 0, & (2.3)\\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, & (2.4)\\ \dfrac{\mathrm{dq}}{\mathrm{dt}} = -\dfrac{1}{T}q + \dfrac{1}{q(t)}\big((\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{u}) & \\ \qquad -\alpha\left((\nabla\times\mathbf{b})\times\mathbf{b},\mathbf{u}\right) + \alpha\left(\nabla\times(\mathbf{b}\times\mathbf{u}),\mathbf{b}\right)\big). & (2.5) \end{cases}$$

Since the sum of the nonlinear terms in (2.5) is zero, (2.5) is equivalent to the time derivative of (2.1). Hence, with $q(0) = \epsilon$, the exact solution of (2.5) is given by (2.1), so (2.2)–(2.4) is exactly the same as (1.1). Therefore, the above system is equivalent to the original system. Note that we have, in addition to the original energy law (1.2), an additional energy law:

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|^2 + \alpha\|\mathbf{b}\|^2 + |q|^2) = -\nu\|\nabla\mathbf{u}\|^2 - \alpha\eta\|\nabla\times\mathbf{b}\|^2 - \frac{1}{T}|q|^2. \tag{2.6}$$

Since $|q|^2 \le \epsilon^2$, the modified energy is an $O(\epsilon^2)$ approximation of the original energy (1.2). Note that, unlike in the original SAV approach, where the SAV $r(t)$ is related to the nonlinear part of the free energy, here the SAV $q(t)$ is purely artificial, but it allows us to construct unconditionally energy stable schemes, with respect to the modified energy in (2.6), with fully explicit treatment of the nonlinear terms.

2.1. The IMEX SAV schemes. We choose $\epsilon = \Delta t$ and set

$$\Delta t = T/N,\ t^n = n\Delta t,\ d_t g^{n+1} = \frac{g^{n+1} - g^n}{\Delta t} \text{ for } n \le N.$$

Scheme I (first-order): Find $(\mathbf{u}^{n+1}, p^{n+1}, q^{n+1}, \mathbf{b}^{n+1})$ by solving

$$d_t\mathbf{u}^{n+1} - \nu\Delta\mathbf{u}^{n+1} + \nabla p^{n+1} = \frac{q^{n+1}}{q(t^{n+1})}(\alpha(\nabla\times\mathbf{b}^n)\times\mathbf{b}^n - \mathbf{u}^n\cdot\nabla\mathbf{u}^n), \tag{2.7}$$

$$d_t\mathbf{b}^{n+1} + \eta\nabla\times(\nabla\times\mathbf{b}^{n+1}) + \frac{q^{n+1}}{q(t^{n+1})}\nabla\times(\mathbf{b}^n\times\mathbf{u}^n) = 0, \tag{2.8}$$

$$\nabla\cdot\mathbf{u}^{n+1} = 0, \quad \nabla\cdot\mathbf{b}^{n+1} = 0, \tag{2.9}$$

$$\mathbf{u}^{n+1}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{b}^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{n}\times(\nabla\times\mathbf{b}^{n+1})|_{\partial\Omega} = 0, \tag{2.10}$$

$$d_t q^{n+1} = -\frac{1}{T}q^{n+1} + \frac{1}{q(t^{n+1})} \big((\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \mathbf{u}^{n+1}) - \alpha((\nabla\times\mathbf{b}^n)\times\mathbf{b}^n, \mathbf{u}^{n+1}) + \alpha(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n), \mathbf{b}^{n+1})\big). \tag{2.11}$$

The above scheme is linear but coupled. We describe below how to implement it efficiently.

We denote $S^{n+1} = \frac{q^{n+1}}{q(t^{n+1})}$ and set

$$\begin{cases} \mathbf{b}^{n+1} = \mathbf{b}_1^{n+1} + S^{n+1}\mathbf{b}_2^{n+1}, & (2.12)\\ \mathbf{u}^{n+1} = \mathbf{u}_1^{n+1} + S^{n+1}\mathbf{u}_2^{n+1}, & (2.13)\\ p^{n+1} = p_1^{n+1} + S^{n+1}p_2^{n+1}. & (2.14) \end{cases}$$

Plugging (2.12)–(2.14) into the scheme (2.7)–(2.10), we find that $\mathbf{u}_i^{n+1}, p_i^{n+1}$ $(i = 1, 2)$ satisfy

$$\begin{cases} \dfrac{\mathbf{u}_1^{n+1} - \mathbf{u}^n}{\Delta t} = \nu\Delta\mathbf{u}_1^{n+1} - \nabla p_1^{n+1}, & (2.15)\\ \dfrac{\mathbf{u}_2^{n+1}}{\Delta t} + \mathbf{u}^n\cdot\nabla\mathbf{u}^n = \nu\Delta\mathbf{u}_2^{n+1} - \nabla p_2^{n+1} + \alpha(\nabla\times\mathbf{b}^n)\times\mathbf{b}^n, & (2.16)\\ \nabla\cdot\mathbf{u}_i^{n+1} = 0, \quad \mathbf{u}_i^{n+1}|_{\partial\Omega} = \mathbf{0}, \quad i = 1, 2. & (2.17) \end{cases}$$

Next we determine $\mathbf{b}_i^{n+1}$ $(i=1,2)$ from

$$
\begin{cases}
\dfrac{\mathbf{b}_1^{n+1}-\mathbf{b}^n}{\Delta t} + \eta\nabla\times(\nabla\times\mathbf{b}_1^{n+1}) = 0, & (2.18)\\
\dfrac{\mathbf{b}_2^{n+1}}{\Delta t} + \eta\nabla\times(\nabla\times\mathbf{b}_2^{n+1}) + \nabla\times(\mathbf{b}^n\times\mathbf{u}^n) = 0, & (2.19)\\
\nabla\cdot\mathbf{b}_i^{n+1}=0, \quad \mathbf{b}_i^{n+1}\cdot\mathbf{n}|_{\partial\Omega}=0, \quad \mathbf{n}\times(\nabla\times\mathbf{b}_i^{n+1})|_{\partial\Omega}=0, \quad i=1,2. & (2.20)
\end{cases}
$$

Once $\mathbf{u}_i^{n+1}$, p_i^{n+1}, $\mathbf{b}_i^{n+1}$ $(i=1,2)$ are known, we derive from (2.11) that

$$
\left(\frac{T+\Delta t}{T\Delta t}q(t^{n+1}) - \frac{1}{q(t^{n+1})}A_2\right)S^{n+1} = \frac{1}{q(t^{n+1})}A_1 + \frac{1}{\Delta t}q^n, \tag{2.21}
$$

where

$$
A_i = (\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \mathbf{u}_i^{n+1}) - \alpha\left((\nabla\times\mathbf{b}^n)\times\mathbf{b}^n, \mathbf{u}_i^{n+1}\right) + \alpha\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n), \mathbf{b}_i^{n+1}\right),
$$

$i=1,2$. Next, we show that $A_2\le 0$ so that S^{n+1} can be uniquely determined from (2.21). Taking the inner product of (2.16) with $\mathbf{u}_2^{n+1}$ leads to

$$
\frac{\|\mathbf{u}_2^{n+1}\|^2}{\Delta t} + (\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \mathbf{u}_2^{n+1}) = -\nu\|\nabla\mathbf{u}_2^{n+1}\|^2 + \alpha\big((\nabla\times\mathbf{b}^n)\times\mathbf{b}^n, \mathbf{u}_2^{n+1}\big). \tag{2.22}
$$

Taking the inner product of (2.19) with $\alpha\mathbf{b}_2^{n+1}$, we have

$$
\alpha\frac{\|\mathbf{b}_2^{n+1}\|^2}{\Delta t} + \alpha\eta\|\nabla\times\mathbf{b}_2^{n+1}\|^2 + \alpha\big(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n), \mathbf{b}_2^{n+1}\big) = 0. \tag{2.23}
$$

Therefore,

$$
\begin{aligned}
A_2 =& (\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}_2^{n+1}) - \alpha\left((\nabla\times\mathbf{b}^n)\times\mathbf{b}^n, \mathbf{u}_2^{n+1}\right) + \alpha\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n), \mathbf{b}_2^{n+1}\right)\\
=& -\frac{\|\mathbf{u}_2^{n+1}\|^2}{\Delta t} - \nu\|\nabla\mathbf{u}_2^{n+1}\|^2 - \alpha\frac{\|\mathbf{b}_2^{n+1}\|^2}{\Delta t} - \alpha\eta\|\nabla\times\mathbf{b}_2^{n+1}\|^2 \le 0.
\end{aligned} \tag{2.24}
$$

Finally, we can obtain $\mathbf{u}^{n+1}$, p^{n+1}, and $\mathbf{b}^{n+1}$ from (2.12)–(2.14). Hence, the scheme is uniquely solvable.

In summary, at each time step, we only need to solve two generalized Stokes equations in (2.15)–(2.17) and two elliptic equations (2.18)–(2.20) with constant coefficients plus a linear algebraic equation (2.21). Hence, the scheme is very efficient.

Scheme II (second-order): Find $(\mathbf{u}^{n+1}, p^{n+1}, q^{n+1}, \mathbf{b}^{n+1})$ by solving

$$
\begin{aligned}
&\frac{3\mathbf{u}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\Delta t} - \nu\Delta\mathbf{u}^{n+1} + \nabla p^{n+1}\\
&= \frac{q^{n+1}}{q(t^{n+1})}\big(\alpha(\nabla\times\bar{\mathbf{b}}^{n+1})\times\bar{\mathbf{b}}^{n+1} - \bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1}\big),
\end{aligned} \tag{2.25}
$$

$$
\begin{aligned}
&\frac{3\mathbf{b}^{n+1}-4\mathbf{b}^n+\mathbf{b}^{n-1}}{2\Delta t} + \eta\nabla\times(\nabla\times\mathbf{b}^{n+1})\\
&\qquad + \frac{q^{n+1}}{q(t^{n+1})}\nabla\times(\bar{\mathbf{b}}^{n+1}\times\bar{\mathbf{u}}^{n+1}) = 0,
\end{aligned} \tag{2.26}
$$

$$
\nabla\cdot\mathbf{u}^{n+1}=0, \quad \nabla\cdot\mathbf{b}^{n+1}=0, \tag{2.27}
$$

$$
\mathbf{u}^{n+1}|_{\partial\Omega}=\mathbf{0}, \quad \mathbf{b}^{n+1}\cdot\mathbf{n}|_{\partial\Omega}=0, \quad \mathbf{n}\times(\nabla\times\mathbf{b}^{n+1})|_{\partial\Omega}=0, \tag{2.28}
$$

$$\frac{3q^{n+1}-4q^n+q^{n-1}}{2\Delta t} = -\frac{1}{T}q^{n+1} + \frac{1}{q(t^{n+1})}$$
$$\Big[\alpha((\nabla\times(\bar{\mathbf{b}}^{n+1}\times\bar{\mathbf{u}}^{n+1}),\mathbf{b}^{n+1}) - \alpha((\nabla\times\bar{\mathbf{b}}^{n+1})\times\bar{\mathbf{b}}^{n+1},\mathbf{u}^{n+1})$$
$$+(\bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1},\mathbf{u}^{n+1})\Big], \tag{2.29}$$

where $\bar{\mathbf{v}}^{n+1} = 2\mathbf{v}^n - \mathbf{v}^{n-1}$ for any function $\mathbf{v}$. For $n = 0$, we can compute ($\mathbf{u}^1$, p^1, q^1, $\mathbf{b}^1$) by the first-order scheme described above.

The second-order scheme (2.25)–(2.29) can be shown to admit an unique solution and can be implemented the same way as the first-order scheme (2.7)–(2.11).

2.2. Energy stability. We show below that the first- and second-order SAV schemes (2.7)–(2.11) and (2.25)–(2.29) are unconditionally energy stable. We shall use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and inner product in $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ to denote the inner product in $L^2(\partial\Omega)$.

THEOREM 2.1. *The scheme* (2.7)–(2.11) *is uniquely solvable and is unconditionally stable in the sense that*

$$E^{n+1} - E^n \le -\nu\Delta t\|\nabla \boldsymbol{u}^{n+1}\|^2 - \eta\alpha\Delta t\|\nabla\times\boldsymbol{b}^{n+1}\|^2 - \frac{1}{T}\Delta t|q^{n+1}|^2 \quad \forall\Delta t,\ n\ge 0, \tag{2.30}$$

where

$$E^{n+1} = \frac{1}{2}\|\boldsymbol{u}^{n+1}\|^2 + \frac{\alpha}{2}\|\boldsymbol{b}^{n+1}\|^2 + \frac{1}{2}|q^{n+1}|^2.$$

Proof. The unique solvability is already established in the previous subsection, so we only need to prove (2.30).

Taking the inner product of (2.7) with $\Delta t\mathbf{u}^{n+1}$ and using the identity

$$(a-b,a) = \frac{1}{2}(|a|^2 - |b|^2 + |a-b|^2), \tag{2.31}$$

we have

$$\frac{\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2}{2} + \frac{\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2}{2} + \nu\Delta t\|\nabla\mathbf{u}^{n+1}\|^2 + \Delta t(\nabla p^{n+1},\mathbf{u}^{n+1})$$
$$= \Delta t\frac{q^{n+1}}{q(t^{n+1})}\left(\alpha(\nabla\times\mathbf{b}^n)\times\mathbf{b}^n,\mathbf{u}^{n+1}) - \mathbf{u}^n\cdot\nabla\mathbf{u}^n,\mathbf{u}^{n+1})\right). \tag{2.32}$$

Taking the inner product of (2.8) with $\alpha\Delta t\mathbf{b}^{n+1}$, we find

$$\alpha\frac{\|\mathbf{b}^{n+1}\|^2 - \|\mathbf{b}^n\|^2}{2} + \alpha\frac{\|\mathbf{b}^{n+1}-\mathbf{b}^n\|^2}{2} + \eta\alpha\Delta t\|\nabla\times\mathbf{b}^{n+1}\|^2$$
$$+\alpha\Delta t\frac{q^{n+1}}{q(t^{n+1})}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),\mathbf{b}^{n+1}\right) = 0. \tag{2.33}$$

Multiplying (2.11) by $q^{n+1}\Delta t$ leads to

$$\frac{|q^{n+1}|^2 - |q^n|^2}{2} + \frac{1}{2}|q^{n+1}-q^n|^2 + \frac{1}{T}\Delta t|q^{n+1}|^2$$
$$= \Delta t\frac{q^{n+1}}{q(t^{n+1})}\big((\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\mathbf{u}^{n+1}) - \alpha((\nabla\times\mathbf{b}^n)\times\mathbf{b}^n,\mathbf{u}^{n+1})$$
$$+\alpha(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),\mathbf{b}^{n+1})\big). \tag{2.34}$$

Then summing up (2.32) with (2.33)–(2.34) results in

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2-\|\mathbf{u}^n\|^2+\alpha\|\mathbf{b}^{n+1}\|^2-\alpha\|\mathbf{b}^n\|^2+|q^{n+1}|^2-|q^n|^2\\&+|q^{n+1}-q^n|^2+\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2+\|\mathbf{b}^{n+1}-\mathbf{b}^n\|^2\\&\leq -2\nu\Delta t\|\nabla\mathbf{u}^{n+1}\|^2-2\eta\alpha\Delta t\|\nabla\times\mathbf{b}^{n+1}\|^2-\frac{2}{T}\Delta t|q^{n+1}|^2,\end{aligned}$$

which implies the desired result. □

We observe that the discrete energy dissipation law (2.30) is an approximation of the continuous energy dissipation law (2.6) with the modified energy being a second-order approximation of the original energy.

Theorem 2.2. *The scheme* (2.25)–(2.29) *is uniquely solvable and unconditionally stable in the sense that*

$$E^{n+1}-E^n\leq -\Delta t\left(\nu\|\nabla\boldsymbol{u}^{n+1}\|^2+\eta\alpha\|\nabla\times\boldsymbol{b}^{n+1}\|^2+\frac{1}{T}|q^{n+1}|^2\right)\quad\forall\Delta t,\ n\geq 0, \tag{2.35}$$

where

$$\begin{aligned}E^{n+1}=&\frac{1}{4}(\|\boldsymbol{u}^{n+1}\|^2+\alpha\|\boldsymbol{b}^{n+1}\|^2+|q^{n+1}|^2)\\&+\frac{1}{4}(\|2\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\|^2+\alpha\|2\boldsymbol{b}^{n+1}-\boldsymbol{b}^n\|^2+|2q^{n+1}-q^n|^2).\end{aligned}\tag{2.36}$$

Proof. The unique solvability can be established exactly as for the first-order scheme. Next, we prove the energy stability.

Taking the inner product of (2.25) with $4\Delta t\mathbf{u}^{n+1}$ and using the identity

$$2(3a-4b+c,a)=|a|^2+|2a-b|^2-|b|^2-|2b-c|^2+|a-2b+c|^2, \tag{2.37}$$

we have

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2+\|2\mathbf{u}^{n+1}-\mathbf{u}^n\|^2-\|\mathbf{u}^n\|^2-\|2\mathbf{u}^n-\mathbf{u}^{n-1}\|^2+\|\mathbf{u}^{n+1}-2\mathbf{u}^n+\mathbf{u}^{n-1}\|^2\\&\qquad+4\nu\Delta t\|\nabla\mathbf{u}^{n+1}\|^2+4\Delta t(\nabla p^{n+1},\mathbf{u}^{n+1})\\&=4\Delta t\frac{q^{n+1}}{q(t^{n+1})}\left(\alpha((\nabla\times\bar{\mathbf{b}}^{n+1})\times\bar{\mathbf{b}}^{n+1},\mathbf{u}^{n+1})-(\bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1},\mathbf{u}^{n+1})\right).\end{aligned}\tag{2.38}$$

Taking the inner product of (2.26) with $4\alpha\Delta t\mathbf{b}^{n+1}$ leads to

$$\begin{aligned}&\alpha(\|\mathbf{b}^{n+1}\|^2+\|2\mathbf{b}^{n+1}-\mathbf{b}^n\|^2-\|\mathbf{b}^n\|^2-\|2\mathbf{b}^n-\mathbf{b}^{n-1}\|^2+\|\mathbf{b}^{n+1}-2\mathbf{b}^n+\mathbf{b}^{n-1}\|^2)\\&+4\eta\alpha\Delta t\|\nabla\times\mathbf{b}^{n+1}\|^2+4\alpha\Delta t\frac{q^{n+1}}{q(t^{n+1})}\left(\nabla\times(\bar{\mathbf{b}}^{n+1}\times\bar{\mathbf{u}}^{n+1}),\mathbf{b}^{n+1}\right)=0.\end{aligned}\tag{2.39}$$

Multiplying (2.29) by $4\Delta tq^{n+1}$ leads to

$$\begin{aligned}&|q^{n+1}|^2+|2q^{n+1}-q^n|^2-|q^n|^2-|2q^n-q^{n-1}|^2+|q^{n+1}-2q^n+q^{n-1}|^2\\&=-\frac{4\Delta t}{T}|q^{n+1}|^2+4\Delta t\frac{q^{n+1}}{q(t^{n+1})}((\bar{\mathbf{u}}^{n+1}\cdot\nabla)\bar{\mathbf{u}}^{n+1},\mathbf{u}^{n+1})\\&\quad-4\alpha\Delta t\frac{q^{n+1}}{q(t^{n+1})}\left(((\nabla\times\bar{\mathbf{b}}^{n+1})\times\bar{\mathbf{b}}^{n+1},\mathbf{u}^{n+1})-(\nabla\times(\bar{\mathbf{b}}^{n+1}\times\bar{\mathbf{u}}^{n+1}),\mathbf{b}^{n+1})\right).\end{aligned}\tag{2.40}$$

Then summing up (2.38) with (2.39)–(2.40) results in

$$
\begin{aligned}
&\|\mathbf{u}^{n+1}\|^2 + \|2\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 + \alpha\|\mathbf{b}^{n+1}\|^2 + \alpha\|2\mathbf{b}^{n+1} - \mathbf{b}^n\|^2 \\
&+ |q^{n+1}|^2 + |2q^{n+1} - q^n|^2 + \|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2 + \alpha\|\mathbf{b}^{n+1} - 2\mathbf{b}^n + \mathbf{b}^{n-1}\|^2 \\
&+ |q^{n+1} - 2q^n + q^{n-1}|^2 + \frac{4\Delta t}{T}|q^{n+1}|^2 + 4\nu\Delta t\|\nabla\mathbf{u}^{n+1}\|^2 + 4\eta\alpha\Delta t\|\nabla\times\mathbf{b}^{n+1}\|^2 \\
\leq& \|\mathbf{u}^n\|^2 + \|2\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 + \alpha\|\mathbf{b}^n\|^2 + \alpha\|2\mathbf{b}^n - \mathbf{b}^{n-1}\|^2 + |q^n|^2 + |2q^n - q^{n-1}|^2,
\end{aligned}
$$

which implies the desired result. ◻

Note that the discrete energy defined in (2.36) is a second-order approximation of the continuous energy defined in (2.6), and (2.35) is an approximation of the continuous energy dissipation law (2.6).

2.3. A fully discrete scheme and its energy stability. The IMEX SAV schemes can be easily coupled with any compatible spatial discretization. For example, let $\mathbf{X}_h \subset \mathbf{H}_0^1(\Omega)$, $M_h \subset L_0^2(\Omega)$, and $\mathbf{W}_h \subset \mathbf{H}_n^1(\Omega)$ be a set of compatible approximation spaces for the velocity, pressure, and magnetic field; a fully discrete first-order IMEX SAV scheme is as follows: $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \mathbf{b}_h^{n+1})$ in $(\mathbf{X}_h, M_h, \mathbf{W}_h)$ and $q_h^{n+1} \in \mathbb{R}$ such that

$$
\begin{aligned}
&(d_t\mathbf{u}_h^{n+1}, \mathbf{v}_h) + \nu(\nabla\mathbf{u}_h^{n+1}, \mathbf{v}_h) - (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) = \alpha\frac{q_h^{n+1}}{q(t^{n+1})}((\nabla\times\mathbf{b}_h^n)\times\mathbf{b}_h^n, \mathbf{v}_h) \\
&\quad -\frac{q_h^{n+1}}{q(t^{n+1})}(\mathbf{u}_h^n\cdot\nabla\mathbf{u}_h^n, \mathbf{v}_h) \quad \forall\mathbf{v}_h \in \mathbf{X}_h, \qquad (2.41)
\end{aligned}
$$

$$
(\nabla\cdot\mathbf{u}_h^{n+1}, \xi_h) = 0 \quad \forall\xi_h \in M_h, \qquad (2.42)
$$

$$
\begin{aligned}
&(d_t\mathbf{b}_h^{n+1}, \mathbf{w}_h) + \eta(\nabla\times\mathbf{b}_h^{n+1}, \nabla\times\mathbf{w}_h) + \eta(\nabla\cdot\mathbf{b}_h^{n+1}, \nabla\cdot\mathbf{w}_h) \\
&+\frac{q_h^{n+1}}{q(t^{n+1})}(\nabla\times(\mathbf{b}_h^n\times\mathbf{u}_h^n), \mathbf{w}_h) = 0 \quad \forall\mathbf{w}_h \in \mathbf{W}_h, \qquad (2.43)
\end{aligned}
$$

$$
\begin{aligned}
&d_t q_h^{n+1} = -\frac{1}{T}q_h^{n+1} + \frac{1}{q(t^{n+1})} \\
&\big((\mathbf{u}_h^n\cdot\nabla\mathbf{u}_h^n, \mathbf{u}_h^{n+1}) - \alpha((\nabla\times\mathbf{b}_h^n)\times\mathbf{b}_h^n, \mathbf{u}_h^{n+1}) \\
&\qquad +\alpha(\nabla\times(\mathbf{b}_h^n\times\mathbf{u}_h^n), \mathbf{b}_h^{n+1})\big). \qquad (2.44)
\end{aligned}
$$

A second-order fully discrete IMEX SAV scheme can be constructed similarly.

Following the same procedure as in the proof of Theorem 2.1, namely, setting $\mathbf{v}_h = \mathbf{u}_h^{n+1}$, $\xi_h = p_h^{n+1}$, $\mathbf{w}_h = \alpha\mathbf{b}_h^{n+1}$ in (2.41)–(2.43), respectively, and taking the inner product of (2.44) with q_h^{n+1}, we can obtain the following stability result.

THEOREM 2.3. *The scheme* (2.41)–(2.44) *is unconditionally stable in the sense that*

$$
E_h^{n+1} - E_h^n \leq -\nu\Delta t\|\nabla\boldsymbol{u}_h^{n+1}\|^2 - \eta\alpha\Delta t(\|\nabla\times\boldsymbol{b}_h^{n+1}\|^2 + \|\nabla\cdot\boldsymbol{b}_h^{n+1}\|^2) - \frac{1}{T}\Delta t|q_h^{n+1}|^2
$$

for all $\Delta\ t$, $n \geq 0$, *where*

$$
E_h^{n+1} = \frac{1}{2}\|\boldsymbol{u}_h^{n+1}\|^2 + \frac{\alpha}{2}\|\boldsymbol{b}_h^{n+1}\|^2 + \frac{1}{2}|q_h^{n+1}|^2.
$$

3. Error analysis. In this section, we carry out a rigorous error analysis for Scheme I (2.7)–(2.11) in the two-dimensional case. Due to some technical difficulties, we shall only consider $\epsilon = 1$, although numerical results presented in the next section show that essentially the same numerical results are obtained with $\epsilon = 1$ and $\epsilon = \Delta t$. For the sake of clarity and simplicity, we only consider Scheme I, although a similar analysis can be carried out for Scheme II, but the process is much more tedious. We emphasize that while both schemes can be used in the three-dimensional case, the error analysis cannot be easily extended to the three-dimensional case due to some technical issues.

3.1. Preliminaries. We describe below some notations and results which will be frequently used in the analysis. We use C, with or without subscript, to denote a positive constant, which could have different values at different places.

We use the standard notations $L^2(\Omega)$, $H^k(\Omega)$, and $H_0^k(\Omega)$ to denote the usual Sobolev spaces. The norm corresponding to $H^k(\Omega)$ will be denoted simply by $\|\cdot\|_k$. The vector functions and vector spaces will be indicated by boldface type.

We define

$$\begin{aligned}
&L_0^2(\Omega) = \left\{ p \in L^2(\Omega) : \int_\Omega q dx = 0 \right\},\\
&\mathbf{H}^k(\Omega) = (H^k(\Omega))^d, \quad \mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0\},\\
&\mathbf{H}_n^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\},\\
&\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\mathbf{v} = 0\},\\
&\mathbf{H} = \{\mathbf{v} \in (L^2(\Omega))^2 : \nabla\cdot\mathbf{v} = 0,\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}.
\end{aligned}$$

The following formulae are essential and useful for our analysis:

$$(\nabla \times \mathbf{v}) \times \mathbf{v} = (\mathbf{v}\cdot\nabla)\mathbf{v} - \frac{1}{2}\nabla|\mathbf{v}|^2, \tag{3.1}$$

$$\mathbf{v} \times (\mathbf{w} \times \mathbf{z}) = (\mathbf{v}\cdot\mathbf{z})\mathbf{w} - (\mathbf{v}\cdot\mathbf{w})\mathbf{z}, \tag{3.2}$$

$$\nabla \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{w} + (\nabla\cdot\mathbf{w})\mathbf{v} - (\nabla\cdot\mathbf{v})\mathbf{w}, \tag{3.3}$$

$$(\mathbf{v} \times \mathbf{w}) \times \mathbf{z} \cdot \mathbf{q} = (\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{z} \times \mathbf{q}) = -(\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{q} \times \mathbf{z}), \tag{3.4}$$

$$\int_\Omega (\nabla \times \mathbf{v}) \cdot \mathbf{w} d\mathbf{x} = \int_\Omega \mathbf{v} \cdot (\nabla \times \mathbf{w}) d\mathbf{x} + \int_{\partial\Omega} (\mathbf{n} \times \mathbf{v}) \cdot \mathbf{w} ds. \tag{3.5}$$

Define the Stokes operator

$$A\mathbf{u} = -P\Delta\mathbf{u} \quad \forall\ \mathbf{u} \in D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V},$$

where P is the orthogonal projector in $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$ and the Stokes operator A is an unbounded positive self-adjoint closed operator in $\mathbf{H}$ with domain $D(A)$. We then derive from the above and the Poincaré inequality that [24, 10]

$$\|\nabla\mathbf{v}\| \le c_1\|A^{\frac{1}{2}}\mathbf{v}\|, \quad \|\Delta\mathbf{v}\| \le c_1\|A\mathbf{v}\| \quad \forall\ \mathbf{v} \in D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V} \tag{3.6}$$

and

$$\|\mathbf{v}\| \le c_1\|\nabla\mathbf{v}\|\ \forall\ \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad \|\nabla\mathbf{v}\| \le c_1\|A\mathbf{v}\| \quad \forall\ \mathbf{v} \in D(A). \tag{3.7}$$

We recall the following inequalities that will be used in what follows [7, 27]:

$$\|\nabla\times\mathbf{v}\|_0 \le c_1\|\nabla\mathbf{v}\|_0, \quad \|\nabla\cdot\mathbf{v}\|_0 \le c_1\|\nabla\mathbf{v}\|_0 \ \forall\ \mathbf{v}\in\mathbf{H}^1(\Omega), \tag{3.8}$$

$$\|\nabla\times\mathbf{v}\|_0^2 + \|\nabla\cdot\mathbf{v}\|_0^2 \ge c_1\|\mathbf{v}\|_1^2 \ \forall\ \mathbf{v}\in\mathbf{H}_n^1(\Omega), \tag{3.9}$$

and the following well-known inequalities which are valid with $d=2$ [16]:

$$\|\mathbf{v}\|_{L^4} \le c_1\|\mathbf{v}\|_0^{1/2}\|\mathbf{v}\|_1^{1/2} \ \forall\ \mathbf{v}\in\mathbf{H}^1(\Omega), \tag{3.10}$$

$$\|\mathbf{v}\|_{L^\infty} \le c_1\|\mathbf{v}\|_1^{1/2}\|\mathbf{v}\|_2^{1/2} \ \forall\ \mathbf{v}\in\mathbf{H}^2(\Omega), \tag{3.11}$$

where c_1 is a positive constant depending only on Ω.

Next we define the trilinear form $b(\cdot,\cdot,\cdot)$ by

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}d\mathbf{x}.$$

We can easily obtain that the trilinear form $b(\cdot,\cdot,\cdot)$ is a skew-symmetric with respect to its last two arguments, i.e.,

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}) \ \ \forall\ \mathbf{u}\in\mathbf{H}, \quad \mathbf{v},\mathbf{w}\in\mathbf{H}^1(\Omega), \tag{3.12}$$

and

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \ \ \forall\ \mathbf{u}\in\mathbf{H}, \quad \mathbf{v}\in\mathbf{H}^1(\Omega). \tag{3.13}$$

By using a combination of integration by parts, Holder's inequality, and Sobolev inequalities [23, 19, 9], we have that for $d\le 4$,

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \le \begin{cases} c_2\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1, \\ c_2\|\mathbf{u}\|_2\|\mathbf{v}\|\|\mathbf{w}\|_1, \\ c_2\|\mathbf{u}\|_2\|\mathbf{v}\|_1\|\mathbf{w}\|, \\ c_2\|\mathbf{u}\|_1\|\mathbf{v}\|_2\|\mathbf{w}\|, \\ c_2\|\mathbf{u}\|\|\mathbf{v}\|_2\|\mathbf{w}\|_1 \end{cases} \tag{3.14}$$

and that for $d=2$, we have

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \le \begin{cases} c_2\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\|_1^{1/2}\|\mathbf{v}\|^{1/2}\|\mathbf{w}\|_1, \\ c_2\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|^{1/2}\|A\mathbf{v}\|^{1/2}\|\mathbf{v}\|^{1/2}\|\mathbf{w}\|, \\ c_2\|A\mathbf{u}\|^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\|_1\|\mathbf{w}\|, \end{cases} \tag{3.15}$$

where c_2 is a positive constant depending only on Ω.

We will frequently use the following discrete version of the Gronwall lemma.

LEMMA 3.1. *Let a_k, b_k, c_k, d_k, γ_k, Δt_k be nonnegative real numbers such that*

$$a_{k+1} - a_k + b_{k+1}\Delta t_{k+1} + c_{k+1}\Delta t_{k+1} - c_k\Delta t_k \le a_k d_k\Delta t_k + \gamma_{k+1}\Delta t_{k+1} \tag{3.16}$$

for all $0\le k\le m$. Then

$$a_{m+1} + \sum_{k=0}^{m+1} b_k\Delta t_k \le \exp\left(\sum_{k=0}^{m} d_k\Delta t_k\right)\left\{a_0 + (b_0+c_0)\Delta t_0 + \sum_{k=1}^{m+1}\gamma_k\Delta t_k\right\}. \tag{3.17}$$

Finally, we may drop the dependence on $\boldsymbol{x}$ if no confusion can arise. In particular, we set

$$\begin{cases} e_{\mathbf{b}}^{n+1} = \mathbf{b}^{n+1} - \mathbf{b}(t^{n+1}), \quad e_{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}(t^{n+1}), \\ e_p^{n+1} = p^{n+1} - p(t^{n+1}), \quad e_q^{n+1} = q^{n+1} - q(t^{n+1}). \end{cases}$$

3.2. Error estimates for the velocity and magnetic field. In this subsection, we derive the following error estimates for the velocity $\mathbf{u}$ and magnetic field $\mathbf{b}$.

THEOREM 3.2. *Let $d = 2$. Assuming*

$$\boldsymbol{u} \in H^2(0,T;\boldsymbol{H}^{-1}(\Omega)) \bigcap H^1(0,T;\boldsymbol{H}^2(\Omega)) \bigcap L^\infty(0,T;\boldsymbol{H}^2(\Omega))$$

and $\boldsymbol{b} \in H^2(0,T;\boldsymbol{H}^{-1}(\Omega)) \bigcap H^1(0,T;\boldsymbol{H}^2(\Omega)) \bigcap L^\infty(0,T;\boldsymbol{H}^2(\Omega))$, *for the scheme* (2.7)–(2.11), *we have*

$$\begin{aligned}
&\|e_{\boldsymbol{u}}^{m+1}\|^2 + \|e_{\boldsymbol{b}}^{m+1}\|^2 + |e_q^{m+1}|^2 + \nu\Delta t \sum_{n=0}^{m} \|\nabla e_{\boldsymbol{u}}^{n+1}\|^2 \\
&+ \eta\Delta t \sum_{n=0}^{m} \|\nabla e_{\boldsymbol{b}}^{n+1}\|^2 + \Delta t \sum_{n=0}^{m} |e_q^{n+1}|^2 + \sum_{n=0}^{m} \|e_{\boldsymbol{u}}^{n+1} - e_{\boldsymbol{u}}^n\|^2 \\
&+ \sum_{n=0}^{m} \|e_{\boldsymbol{b}}^{n+1} - e_{\boldsymbol{b}}^n\|^2 + \sum_{n=0}^{m} |e_q^{n+1} - e_q^n|^2 \le C(\Delta t)^2 \quad \forall\ 0 \le n \le N-1,
\end{aligned}$$

where C is a positive constant independent of Δt.

The proof of the above theorem will be carried out with a sequence of lemmas below.

We start first with the following uniform bounds which are a direct consequence of the energy stability in Theorem 2.1.

LEMMA 3.3. *Let ($\boldsymbol{u}^{n+1}$, p^{n+1}, q^{n+1}, $\boldsymbol{b}^{n+1}$) be the solution of* (2.7)–(2.11); *then we have*

$$\|\boldsymbol{u}^{m+1}\|^2 + \|\boldsymbol{b}^{m+1}\|^2 + |q^{m+1}|^2 \le k_1 \quad \forall\ 0 \le m \le N-1 \tag{3.18}$$

and

$$\Delta t \sum_{n=0}^{m} \|\boldsymbol{u}^{n+1}\|_1^2 + \Delta t \sum_{n=0}^{m} \|\boldsymbol{b}^{n+1}\|_1^2 \le k_2 \quad \forall\ 0 \le m \le N-1, \tag{3.19}$$

where the constants k_i $(i = 1, 2)$ are independent of Δt.

Next, we derive a first bound for the velocity errors.

LEMMA 3.4. *Under the assumptions of Theorem* 3.2, *we have*

$$\begin{aligned}
&\frac{\|e_{\boldsymbol{u}}^{n+1}\|^2 - \|e_{\boldsymbol{u}}^n\|^2}{2\Delta t} + \frac{\|e_{\boldsymbol{u}}^{n+1} - e_{\boldsymbol{u}}^n\|^2}{2\Delta t} + \frac{\nu}{2}\|\nabla e_{\boldsymbol{u}}^{n+1}\|^2 \\
&\quad \le \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(\alpha((\nabla \times \boldsymbol{b}^n) \times \boldsymbol{b}^n, e_{\boldsymbol{u}}^{n+1}) - (\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n, e_{\boldsymbol{u}}^{n+1})\right) \\
&\qquad + C(\|\boldsymbol{u}(t^n)\|_2^2 + \|\boldsymbol{u}(t^{n+1})\|_2^2 + \|e_{\boldsymbol{u}}^n\|_1^2)\|e_{\boldsymbol{u}}^n\|^2 \\
&\qquad + C(\|e_{\boldsymbol{b}}^n\|_1^2 + \|\boldsymbol{b}(t^{n+1})\|_2^2)\|e_{\boldsymbol{b}}^n\|^2 \\
&\qquad + C\Delta t \int_{t^n}^{t^{n+1}} (\|\boldsymbol{u}_t\|_2^2 + \|\boldsymbol{u}_{tt}\|_{-1}^2 + \|\boldsymbol{b}_t\|_2^2)dt \quad \forall\ 0 \le n \le N-1,
\end{aligned} \tag{3.20}$$

where C is a positive constant independent of Δt.

Proof. Let $\mathbf{R}_{\mathbf{u}}^{n+1}$ be the truncation error defined by

$$\mathbf{R}_{\mathbf{u}}^{n+1} = \frac{\partial \mathbf{u}(t^{n+1})}{\partial t} - \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (t^n - t)\frac{\partial^2 \mathbf{u}}{\partial t^2} dt. \tag{3.21}$$

Subtracting (2.2) at t^{n+1} from (2.7), we obtain

$$\begin{aligned} d_t e_{\mathbf{u}}^{n+1} &- \nu\Delta e_{\mathbf{u}}^{n+1} + \nabla e_p^{n+1} = \mathbf{R}_{\mathbf{u}}^{n+1} \\ &+ \exp\left(\frac{t^{n+1}}{T}\right)(q(t^{n+1})\mathbf{u}(t^{n+1}) \cdot \nabla \mathbf{u}(t^{n+1}) - q^{n+1}\mathbf{u}^n \cdot \nabla \mathbf{u}^n) \\ &+ \alpha \exp\left(\frac{t^{n+1}}{T}\right)(q^{n+1}(\nabla \times \mathbf{b}^n) \times \mathbf{b}^n - q(t^{n+1})(\nabla \times \mathbf{b}(t^{n+1})) \times \mathbf{b}(t^{n+1})). \end{aligned} \tag{3.22}$$

Taking the inner product of (3.22) with $e_{\mathbf{u}}^{n+1}$, we obtain

$$\begin{aligned} &\frac{\|e_{\mathbf{u}}^{n+1}\|^2 - \|e_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\|e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2}{2\Delta t} + \nu\|\nabla e_{\mathbf{u}}^{n+1}\|^2 + (\nabla e_p^{n+1}, e_{\mathbf{u}}^{n+1}) = (\mathbf{R}_{\mathbf{u}}^{n+1}, e_{\mathbf{u}}^{n+1}) \\ &+ \exp\left(\frac{t^{n+1}}{T}\right)\left(q(t^{n+1})\mathbf{u}(t^{n+1}) \cdot \nabla \mathbf{u}(t^{n+1}) - q^{n+1}\mathbf{u}^n \cdot \nabla \mathbf{u}^n, e_{\mathbf{u}}^{n+1}\right) \\ &+ \alpha \exp\left(\frac{t^{n+1}}{T}\right)\left(q^{n+1}(\nabla \times \mathbf{b}^n) \times \mathbf{b}^n - q(t^{n+1})(\nabla \times \mathbf{b}(t^{n+1})) \times \mathbf{b}(t^{n+1}), e_{\mathbf{u}}^{n+1}\right). \end{aligned} \tag{3.23}$$

For the first term on the right-hand side of (3.23), we have

$$(\mathbf{R}_{\mathbf{u}}^{n+1}, e_{\mathbf{u}}^{n+1}) \le \frac{\nu}{16}\|\nabla e_{\mathbf{u}}^{n+1}\|^2 + C\Delta t\int_{t^n}^{t^{n+1}} \|\mathbf{u}_{tt}\|_{-1}^2 dt. \tag{3.24}$$

For the second term on the right-hand side of (3.23), we have

$$\begin{aligned} &\exp\left(\frac{t^{n+1}}{T}\right)\left(q(t^{n+1})\mathbf{u}(t^{n+1}) \cdot \nabla \mathbf{u}(t^{n+1}) - q^{n+1}\mathbf{u}^n \cdot \nabla \mathbf{u}^n, e_{\mathbf{u}}^{n+1}\right) \\ &\quad = \left((\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \cdot \nabla \mathbf{u}(t^{n+1}), e_{\mathbf{u}}^{n+1}\right) + \left(\mathbf{u}^n \cdot \nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), e_{\mathbf{u}}^{n+1}\right) \\ &\quad\quad - \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, e_{\mathbf{u}}^{n+1}\right). \end{aligned} \tag{3.25}$$

Using the Cauchy–Schwarz inequality, Lemma 3.3, and (3.14), the first term on the right-hand side of (3.25) can be bounded by

$$\begin{aligned} &\left((\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \cdot \nabla \mathbf{u}(t^{n+1}), e_{\mathbf{u}}^{n+1}\right) \\ &\quad \le c_2(1 + c_1)\|\mathbf{u}(t^{n+1}) - \mathbf{u}^n\|\|\mathbf{u}(t^{n+1})\|_2\|\nabla e_{\mathbf{u}}^{n+1}\| \\ &\quad \le \frac{\nu}{16}\|\nabla e_{\mathbf{u}}^{n+1}\|^2 + C\|\mathbf{u}(t^{n+1})\|_2^2\|e_{\mathbf{u}}^n\|^2 + C\|\mathbf{u}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}} \|\mathbf{u}_t\|^2 dt. \end{aligned} \tag{3.26}$$

Using (3.14) and (3.15), the second term on the right-hand side of (3.25) can be estimated as follows:

$$
\begin{aligned}
&(\mathbf{u}^n\cdot\nabla(\mathbf{u}(t^{n+1})-\mathbf{u}^n),e_{\mathbf{u}}^{n+1})\\
&\quad=\left(\mathbf{u}^n\cdot\nabla(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)),e_{\mathbf{u}}^{n+1}\right)-\left(e_{\mathbf{u}}^n\cdot\nabla e_{\mathbf{u}}^n,e_{\mathbf{u}}^{n+1}\right)-\left(\mathbf{u}(t^n)\cdot\nabla e_{\mathbf{u}}^n,e_{\mathbf{u}}^{n+1}\right)\\
&\quad\le c_2(1+c_1)\|\nabla e_{\mathbf{u}}^{n+1}\|(\|\mathbf{u}^n\|\|\int_{t^n}^{t^{n+1}}\mathbf{u}_t dt\|_2+\|e_{\mathbf{u}}^n\|\|\mathbf{u}(t^n)\|_2)\\
&\qquad+c_2(1+c_1)\|e_{\mathbf{u}}^n\|^{1/2}\|e_{\mathbf{u}}^n\|_1^{1/2}\|e_{\mathbf{u}}^n\|^{1/2}\|e_{\mathbf{u}}^n\|_1^{1/2}\|\nabla e_{\mathbf{u}}^{n+1}\|\\
&\quad\le\frac{\nu}{16}\|\nabla e_{\mathbf{u}}^{n+1}\|^2+C(\|\mathbf{u}(t^n)\|_2^2+\|e_{\mathbf{u}}^n\|_1^2)\|e_{\mathbf{u}}^n\|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_2^2dt.
\end{aligned}
\tag{3.27}
$$

For the last term on the right-hand side of (3.23), we have

$$
\begin{aligned}
&\exp\left(\frac{t^{n+1}}{T}\right)\left(q^{n+1}(\nabla\times\mathbf{b}^n)\times\mathbf{b}^n-q(t^{n+1})(\nabla\times\mathbf{b}(t^{n+1}))\times\mathbf{b}(t^{n+1}),e_{\mathbf{u}}^{n+1}\right)\\
&\qquad=\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left((\nabla\times\mathbf{b}^n)\times\mathbf{b}^n,e_{\mathbf{u}}^{n+1}\right)\\
&\qquad\quad+\left((\nabla\times(\mathbf{b}^n-\mathbf{b}(t^{n+1})))\times\mathbf{b}^n,e_{\mathbf{u}}^{n+1}\right)\\
&\qquad\quad+\left((\nabla\times\mathbf{b}(t^{n+1}))\times(\mathbf{b}^n-\mathbf{b}(t^{n+1})),e_{\mathbf{u}}^{n+1}\right).
\end{aligned}
\tag{3.28}
$$

The second term on the right-hand side of (3.28) can be transformed into

$$
\begin{aligned}
&((\nabla\times(\mathbf{b}^n-\mathbf{b}(t^{n+1})))\times\mathbf{b}^n,e_{\mathbf{u}}^{n+1})\\
&\qquad=\left((\nabla\times e_{\mathbf{b}}^n)\times e_{\mathbf{b}}^n,e_{\mathbf{u}}^{n+1}\right)+\left((\nabla\times e_{\mathbf{b}}^n)\times\mathbf{b}(t^n),e_{\mathbf{u}}^{n+1}\right)\\
&\qquad\quad+\left((\nabla\times(\mathbf{b}(t^n)-\mathbf{b}(t^{n+1})))\times\mathbf{b}^n,e_{\mathbf{u}}^{n+1}\right).
\end{aligned}
\tag{3.29}
$$

Using the identity (3.1), the first term on the right-hand side of (3.29) can be bounded by

$$
\begin{aligned}
\left((\nabla\times e_{\mathbf{b}}^n)\times e_{\mathbf{b}}^n,e_{\mathbf{u}}^{n+1}\right)&=\left((e_{\mathbf{b}}^n\cdot\nabla)e_{\mathbf{b}}^n,e_{\mathbf{u}}^{n+1}\right)-\frac{1}{2}\left(\nabla|e_{\mathbf{b}}^n|^2,e_{\mathbf{u}}^{n+1}\right)\\
&\le C\|e_{\mathbf{b}}^n\|^{1/2}\|e_{\mathbf{b}}^n\|_1^{1/2}\|e_{\mathbf{b}}^n\|^{1/2}\|e_{\mathbf{b}}^n\|_1^{1/2}\|\nabla e_{\mathbf{u}}^{n+1}\|\\
&\le\frac{\nu}{16}\|\nabla e_{\mathbf{u}}^{n+1}\|^2+C\|e_{\mathbf{b}}^n\|_1^2\|e_{\mathbf{b}}^n\|^2.
\end{aligned}
\tag{3.30}
$$

Using (3.2), (3.4), and integration by parts (3.5), the second term on the right-hand side of (3.29) can be controlled by

$$
\begin{aligned}
\left((\nabla\times e_{\mathbf{b}}^n)\times\mathbf{b}(t^n),e_{\mathbf{u}}^{n+1}\right)&=-\left(e_{\mathbf{u}}^{n+1}\times\mathbf{b}(t^n),\nabla\times e_{\mathbf{b}}^n\right)\\
&=-\left(\nabla\times(e_{\mathbf{u}}^{n+1}\times\mathbf{b}(t^n)),e_{\mathbf{b}}^n\right)-\langle\mathbf{n}\times(e_{\mathbf{u}}^{n+1}\times\mathbf{b}(t^n)),e_{\mathbf{b}}^n\rangle\\
&=\left((e_{\mathbf{u}}^{n+1}\cdot\nabla)\mathbf{b}(t^n),e_{\mathbf{b}}^n\right)-\left((\mathbf{b}(t^n)\cdot\nabla)e_{\mathbf{u}}^{n+1},e_{\mathbf{b}}^n\right)\\
&\le\frac{\nu}{16}\|\nabla e_{\mathbf{u}}^{n+1}\|^2+C\|\mathbf{b}(t^n)\|_2^2\|e_{\mathbf{b}}^n\|^2,
\end{aligned}
\tag{3.31}
$$

where we use the identity

$$
\nabla\times(\mathbf{v}\times\mathbf{w})=(\mathbf{w}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{w}\ \forall\ \mathbf{v},\mathbf{w}\in\mathbf{H}.
$$

Using Lemma 3.3 and (3.14), the last term on the right-hand side of (3.29) can be estimated by

$$\begin{aligned}&\big((\nabla\times(\mathbf{b}(t^n)-\mathbf{b}(t^{n+1})))\times\mathbf{b}^n,e_{\mathbf{u}}^{n+1}\big)\\&\quad\le\frac{\nu}{16}\|\nabla e_{\mathbf{u}}^{n+1}\|^2+C\|\mathbf{b}^n\|^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_t\|_2^2dt.\end{aligned}\tag{3.32}$$

For the last term on the right-hand side of (3.28), we have

$$\begin{aligned}&\big((\nabla\times\mathbf{b}(t^{n+1}))\times(\mathbf{b}^n-\mathbf{b}(t^{n+1})),e_{\mathbf{u}}^{n+1}\big)\\&\quad\le\frac{\nu}{16}\|\nabla e_{\mathbf{u}}^{n+1}\|^2+C\|\mathbf{b}(t^{n+1})\|_2^2\|e_{\mathbf{b}}^n\|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_t\|^2dt.\end{aligned}\tag{3.33}$$

Finally, combining (3.23) with (3.25)–(3.33) leads to the desired result. ∎

We derive below a bound for the errors of the magnetic field.

LEMMA 3.5. *Under the assumptions of Theorem* 3.2*, we have*

$$\begin{aligned}&\frac{\|e_{\boldsymbol{b}}^{n+1}\|^2-\|e_{\boldsymbol{b}}^n\|^2}{2\Delta t}+\frac{\|e_{\boldsymbol{b}}^{n+1}-e_{\boldsymbol{b}}^n\|^2}{2\Delta t}+\frac{\eta}{2}\|\nabla e_{\boldsymbol{b}}^{n+1}\|^2\\&\quad\le-\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\nabla\times(\boldsymbol{b}^n\times\boldsymbol{u}^n),e_{\boldsymbol{b}}^{n+1}\right)\\&\qquad+C(\|\boldsymbol{u}(t^{n+1})\|_2^2+\|e_{\boldsymbol{b}}^n\|_1^2)\|e_{\boldsymbol{b}}^n\|^2\\&\qquad+C(\|e_{\boldsymbol{u}}^n\|_1^2+\|\boldsymbol{b}(t^{n+1})\|_2^2)\|e_{\boldsymbol{u}}^n\|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|\boldsymbol{u}_t\|_2^2dt\\&\qquad+C\Delta t\int_{t^n}^{t^{n+1}}(\|\boldsymbol{b}_t\|^2+\|\boldsymbol{b}_{tt}\|_{-1}^2)dt\quad\forall\ 0\le n\le N-1,\end{aligned}\tag{3.34}$$

where C is a positive constant independent of Δt.

Proof. Let $\mathbf{R}_{\mathbf{b}}^{n+1}$ be the truncation error defined by

$$\mathbf{R}_{\mathbf{b}}^{n+1}=\frac{\partial\mathbf{b}(t^{n+1})}{\partial t}-\frac{\mathbf{b}(t^{n+1})-\mathbf{b}(t^n)}{\Delta t}=\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}(t^n-t)\frac{\partial^2\mathbf{b}}{\partial t^2}dt.\tag{3.35}$$

Subtracting (2.3) at t^{n+1} from (2.8) and using

$$\nabla\times(\nabla\times\mathbf{b}^{n+1})=-\Delta\mathbf{b}^{n+1}+\nabla(\nabla\cdot\mathbf{b}^{n+1}),\tag{3.36}$$

we obtain

$$\begin{aligned}d_te_{\mathbf{b}}^{n+1}-\eta\Delta e_{\mathbf{b}}^{n+1}=&\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\nabla\times(\mathbf{b}(t^{n+1})\times\mathbf{u}(t^{n+1}))\\&-\exp\left(\frac{t^{n+1}}{T}\right)q^{n+1}\nabla\times(\mathbf{b}^n\times\mathbf{u}^n)+\mathbf{R}_{\mathbf{u}}^{n+1}.\end{aligned}\tag{3.37}$$

Taking the inner product of (3.37) with $e_{\mathbf{b}}^{n+1}$, we obtain

$$\begin{aligned}&\frac{\|e_{\mathbf{b}}^{n+1}\|^2-\|e_{\mathbf{b}}^n\|^2}{2\Delta t}+\frac{\|e_{\mathbf{b}}^{n+1}-e_{\mathbf{b}}^n\|^2}{2\Delta t}+\eta\|\nabla e_{\mathbf{b}}^{n+1}\|^2\\&\quad=\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left(\nabla\times(\mathbf{b}(t^{n+1})\times\mathbf{u}(t^{n+1})),e_{\mathbf{b}}^{n+1}\right)\\&\qquad-\exp\left(\frac{t^{n+1}}{T}\right)q^{n+1}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),e_{\mathbf{b}}^{n+1}\right)+(\mathbf{R}_{\mathbf{b}}^{n+1},e_{\mathbf{b}}^{n+1}).\end{aligned}\tag{3.38}$$

The first two terms on the right-hand side of (3.38) can be recast as

$$
\begin{aligned}
&\exp\left(\frac{t^{n+1}}{T}\right)\left(q(t^{n+1})\nabla\times(\mathbf{b}(t^{n+1})\times\mathbf{u}(t^{n+1}))-q^{n+1}\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),e_{\mathbf{b}}^{n+1}\right)\\
&\quad=\left(\nabla\times[(\mathbf{b}(t^{n+1})-\mathbf{b}^n)\times\mathbf{u}(t^{n+1})],e_{\mathbf{b}}^{n+1}\right)\\
&\qquad+\left(\nabla\times[\mathbf{b}^n\times(\mathbf{u}(t^{n+1})-\mathbf{u}^n)],e_{\mathbf{b}}^{n+1}\right)\\
&\qquad-\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),e_{\mathbf{b}}^{n+1}\right).
\end{aligned}\tag{3.39}
$$

By using (3.11), (3.8), and integration by parts (3.5), we have

$$
\begin{aligned}
&\left(\nabla\times[(\mathbf{b}(t^{n+1})-\mathbf{b}^n)\times\mathbf{u}(t^{n+1})],e_{\mathbf{b}}^{n+1}\right)=\left((\mathbf{b}(t^{n+1})-\mathbf{b}^n)\times\mathbf{u}(t^{n+1}),\nabla\times e_{\mathbf{b}}^{n+1}\right)\\
&\quad\le\frac{\eta}{6}\|\nabla e_{\mathbf{b}}^{n+1}\|^2+C\|\mathbf{u}(t^{n+1})\|_2^2 e_{\mathbf{b}}^n\|^2+C\|\mathbf{u}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_t\|^2dt.
\end{aligned}\tag{3.40}
$$

Thanks to (3.10) and (3.8), we have

$$
\begin{aligned}
&\left(\nabla\times[\mathbf{b}^n\times(\mathbf{u}(t^{n+1})-\mathbf{u}^n)],e_{\mathbf{b}}^{n+1}\right)\\
&\quad=\left(\mathbf{b}^n\times(\mathbf{u}(t^{n+1})-\mathbf{u}^n),\nabla\times e_{\mathbf{b}}^{n+1}\right)\\
&\quad=\left(e_{\mathbf{b}}^n\times(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)),\nabla\times e_{\mathbf{b}}^{n+1}\right)-\left(e_{\mathbf{b}}^n\times e_{\mathbf{u}}^n,\nabla\times e_{\mathbf{b}}^{n+1}\right)\\
&\qquad+\left(\mathbf{b}(t^{n+1})\times(\mathbf{u}(t^{n+1})-\mathbf{u}^n),\nabla\times e_{\mathbf{b}}^{n+1}\right)\\
&\quad\le\frac{\eta}{6}\|\nabla e_{\mathbf{b}}^{n+1}\|^2+C\|e_{\mathbf{b}}^n\|_{L^4}^2\|e_{\mathbf{u}}^n\|_{L^4}^2+C\|\mathbf{b}(t^{n+1})\|_2^2\|e_{\mathbf{u}}^n\|^2\\
&\qquad+C\|e_{\mathbf{b}}^n\|^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_2^2dt+C\|\mathbf{b}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|^2dt\\
&\quad\le\frac{\eta}{6}\|\nabla e_{\mathbf{b}}^{n+1}\|^2+C\|e_{\mathbf{b}}^n\|_1^2\|e_{\mathbf{b}}^n\|^2+C(\|e_{\mathbf{u}}^n\|_1^2+\|\mathbf{b}(t^{n+1})\|_2^2)\|e_{\mathbf{u}}^n\|^2\\
&\qquad+C\|e_{\mathbf{b}}^n\|^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_2^2dt+C\|\mathbf{b}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|^2dt.
\end{aligned}\tag{3.41}
$$

For the last term on the right-hand side of (3.38), we have

$$
(\mathbf{R}_{\mathbf{b}}^{n+1},e_{\mathbf{b}}^{n+1})\le\frac{\eta}{6}\|\nabla e_{\mathbf{b}}^{n+1}\|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_{tt}\|_{-1}^2dt.\tag{3.42}
$$

Combining (3.38) with (3.39)–(3.42) leads to the desired result. ∎

In the next lemma, we derive a bound for the errors with respect to q.

LEMMA 3.6. *Under the assumptions of Theorem* 3.2*, we have*

$$
\begin{aligned}
&\frac{|e_q^{n+1}|^2-|e_q^n|^2}{2\Delta t}+\frac{|e_q^{n+1}-e_q^n|^2}{2\Delta t}+\frac{1}{2T}|e_q^{n+1}|^2\\
&\quad\le\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n,e_{\boldsymbol{u}}^{n+1}\right)
\end{aligned}\tag{3.43}
$$

$$
\begin{aligned}
&- \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left((\nabla \times \boldsymbol{b}^n) \times \boldsymbol{b}^n, e_{\boldsymbol{u}}^{n+1}\right) \\
&+ \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(\nabla \times (\boldsymbol{b}^n \times \boldsymbol{u}^n), e_{\boldsymbol{b}}^{n+1}\right) + C\|\boldsymbol{u}^n\|_1^2 \|e_{\boldsymbol{u}}^n\|^2 \\
&+ C(\|e_{\boldsymbol{b}}^n\|_1^2 + \|\boldsymbol{u}^n\|_1^2 + \|\boldsymbol{b}(t^{n+1})\|_1^2)\|e_{\boldsymbol{b}}^n\|^2 + C\Delta t \int_{t^n}^{t^{n+1}} \|q_{tt}\|^2 dt \\
&+ C\Delta t \int_{t^n}^{t^{n+1}} (\|\boldsymbol{u}_t\|_0^2 + \|\boldsymbol{b}_t\|_1^2) dt \quad \forall\ 0 \le n \le N-1,
\end{aligned}
$$

where C is a positive constant independent of Δt.

Proof. Subtracting (2.5) from (2.11) leads to

$$
\begin{aligned}
\frac{e_q^{n+1} - e_q^n}{\Delta t} + \frac{1}{T} e_q^{n+1} &= \mathbf{R}_q^{n+1} \\
&+ \exp\left(\frac{t^{n+1}}{T}\right) \left((\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{u}^{n+1}) - (\mathbf{u}(t^{n+1}) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}))\right) \\
&- \alpha \exp\left(\frac{t^{n+1}}{T}\right) \big(((\nabla \times \mathbf{b}^n) \times \mathbf{b}^n, \mathbf{u}^{n+1}) \\
&- ((\nabla \times \mathbf{b}(t^{n+1})) \times \mathbf{b}(t^{n+1}), \mathbf{u}(t^{n+1}))\big) \\
&+ \alpha \exp\left(\frac{t^{n+1}}{T}\right) \big((\nabla \times (\mathbf{b}^n \times \mathbf{u}^n), \mathbf{b}^{n+1}) \\
&- (\nabla \times (\mathbf{b}(t^{n+1}) \times \mathbf{u}(t^{n+1})), \mathbf{b}(t^{n+1}))\big),
\end{aligned}
\tag{3.44}
$$

where

$$
\mathbf{R}_q^{n+1} = \frac{\mathrm{dq(t^{n+1})}}{\mathrm{dt}} - \frac{q(t^{n+1}) - q(t^n)}{\Delta t} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (t^n - t) \frac{\partial^2 q}{\partial t^2} dt.
\tag{3.45}
$$

Multiplying both sides of (3.44) by e_q^{n+1} yields

$$
\begin{aligned}
&\frac{|e_q^{n+1}|^2 - |e_q^n|^2}{2\Delta t} + \frac{|e_q^{n+1} - e_q^n|^2}{2\Delta t} + \frac{1}{T}|e_q^{n+1}|^2 = \mathbf{R}_q^{n+1} e_q^{n+1} \\
&\quad + \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} ((\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{u}^{n+1}) - (\mathbf{u}(t^{n+1}) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}))) \\
&\quad - \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \big(((\nabla \times \mathbf{b}^n) \times \mathbf{b}^n, \mathbf{u}^{n+1}) \\
&\quad - ((\nabla \times \mathbf{b}(t^{n+1})) \times \mathbf{b}(t^{n+1}), \mathbf{u}(t^{n+1}))\big) \\
&\quad + \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \big((\nabla \times (\mathbf{b}^n \times \mathbf{u}^n), \mathbf{b}^{n+1}) \\
&\quad - (\nabla \times (\mathbf{b}(t^{n+1}) \times \mathbf{u}(t^{n+1})), \mathbf{b}(t^{n+1}))\big).
\end{aligned}
\tag{3.46}
$$

We bound the right-hand side of the above as follows:

$$\mathbf{R}_q^{n+1} e_q^{n+1} \le \frac{1}{12T}|e_q^{n+1}|^2 + C\Delta t \int_{t^n}^{t^{n+1}} \|q_{tt}\|^2 dt. \tag{3.47}$$

The second term on the right-hand side of (3.46) can be estimated as

$$\begin{aligned} &\exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left((\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{u}^{n+1}) - (\mathbf{u}(t^{n+1}) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}))\right) \\ =& \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, e_{\mathbf{u}}^{n+1}\right) \\ &+ \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(\mathbf{u}^n \cdot \nabla (\mathbf{u}^n - \mathbf{u}(t^{n+1})), \mathbf{u}(t^{n+1})\right) \\ &+ \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left((\mathbf{u}^n - \mathbf{u}(t^{n+1})) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1})\right). \end{aligned} \tag{3.48}$$

Thanks to (3.14) and Lemma 3.3, we bound the second term on the right-hand side of (3.48) by

$$\begin{aligned} &\exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(\mathbf{u}^n \cdot \nabla (\mathbf{u}^n - \mathbf{u}(t^{n+1})), \mathbf{u}(t^{n+1})\right) \\ &\le C\|\mathbf{u}^n\|_1 \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n) - e_{\mathbf{u}}^n\|_0 \|\mathbf{u}(t^{n+1})\|_2 |e_q^{n+1}| \\ &\le \frac{1}{12T}|e_q^{n+1}|^2 + C\|\mathbf{u}^n\|_1^2 \|e_{\mathbf{u}}^n\|^2 + C\|\mathbf{u}(t^{n+1})\|_2^2 \Delta t \int_{t^n}^{t^{n+1}} \|\mathbf{u}_t\|_0^2 dt. \end{aligned} \tag{3.49}$$

The third term on the right-hand side of (3.48) can be bounded by

$$\begin{aligned} &\exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left((\mathbf{u}^n - \mathbf{u}(t^{n+1})) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1})\right) \\ &\le C\|\mathbf{u}(t^{n+1}) - \mathbf{u}^n\| \|\mathbf{u}(t^{n+1})\|_1 \|\mathbf{u}(t^{n+1})\|_2 |e_q^{n+1}| \\ &\le \frac{1}{12T}|e_q^{n+1}|^2 + C\|e_{\mathbf{u}}^n\|^2 + C\Delta t \int_{t^n}^{t^{n+1}} \|\mathbf{u}_t\|^2 dt. \end{aligned} \tag{3.50}$$

The second to last term on the right-hand side of (3.46) can be recast as

$$\begin{aligned} &-\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(((\nabla \times \mathbf{b}^n) \times \mathbf{b}^n, \mathbf{u}^{n+1}) - ((\nabla \times \mathbf{b}(t^{n+1})) \times \mathbf{b}(t^{n+1}), \mathbf{u}(t^{n+1}))\right) \\ &= \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(((\nabla \times (\mathbf{b}(t^{n+1}) - \mathbf{b}^n)) \times \mathbf{b}^n, \mathbf{u}(t^{n+1})\right) \\ &\quad + \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(((\nabla \times \mathbf{b}(t^{n+1})) \times (\mathbf{b}(t^{n+1}) - \mathbf{b}^n), \mathbf{u}(t^{n+1})\right) \\ &\quad - \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left((\nabla \times \mathbf{b}^n) \times \mathbf{b}^n, e_{\mathbf{u}}^{n+1}\right). \end{aligned} \tag{3.51}$$

Thanks to (3.14) and (3.15) and using the similar procedure in (3.31), the first term on the right-hand side of (3.51) can be estimated by

$$
\begin{aligned}
&\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\nabla\times(\mathbf{b}(t^{n+1})-\mathbf{b}^n)\times\mathbf{b}^n,\mathbf{u}(t^{n+1})\right)\\
&\quad= -\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\nabla\times(\mathbf{u}(t^{n+1})\times\mathbf{b}^n),\mathbf{b}(t^{n+1})-\mathbf{b}^n\right)\\
&\quad= \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{b}^n,\mathbf{b}(t^{n+1})-\mathbf{b}^n\right)\\
&\qquad -\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\mathbf{b}^n\cdot\nabla)\mathbf{u}(t^{n+1}),\mathbf{b}(t^{n+1})-\mathbf{b}^n\right)\\
&\quad\le \frac{1}{12T}|e_q^{n+1}|^2 + C\|e_{\mathbf{b}}^n\|_1^2\|e_{\mathbf{b}}^n\|^2 + C\|\mathbf{u}(t^{n+1})\|_2^2\|\mathbf{b}^n\|^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_t\|_1^2dt.
\end{aligned}
\tag{3.52}
$$

For the second term on the right-hand side of (3.51), we have

$$
\begin{aligned}
&\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\nabla\times\mathbf{b}(t^{n+1})\times(\mathbf{b}(t^{n+1})-\mathbf{b}^n),\mathbf{u}(t^{n+1})\right)\\
&\le \frac{1}{12T}|e_q^{n+1}|^2 + C\|\mathbf{b}(t^{n+1})\|_1^2\|e_{\mathbf{b}}^n\|^2 + C\|\mathbf{u}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_t\|^2dt.
\end{aligned}
\tag{3.53}
$$

Using (3.10) and (3.8) and integration by parts (3.5), the last term on the right-hand side of (3.46) can be bounded by

$$
\begin{aligned}
&\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),\mathbf{b}^{n+1}) - (\nabla\times(\mathbf{b}(t^{n+1})\times\mathbf{u}(t^{n+1})),\mathbf{b}(t^{n+1}))\right)\\
&\quad\le \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left(\nabla\times((\mathbf{b}^n-\mathbf{b}(t^{n+1}))\times\mathbf{u}^n),\mathbf{b}(t^{n+1})\right)\\
&\qquad +\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left(\nabla\times(\mathbf{b}(t^{n+1})\times(\mathbf{u}^n-\mathbf{u}(t^{n+1}))),\mathbf{b}(t^{n+1})\right)\\
&\qquad +\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),e_{\mathbf{b}}^{n+1}\right)\\
&\quad\le \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),e_{\mathbf{b}}^{n+1}\right) + \frac{1}{12T}|e_q^{n+1}|^2\\
&\qquad + C\|\mathbf{u}^n\|_1^2\|e_{\mathbf{b}}^n\|^2 + C\|e_{\mathbf{u}}^n\|^2 + C\|\mathbf{b}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}(\|\mathbf{b}_t\|_1^2+\|\mathbf{u}_t\|^2)dt.
\end{aligned}
\tag{3.54}
$$

Finally, combining (3.48)–(3.54) in (3.46) leads to the desired result. ∎

Now we are in the position to prove Theorem 3.2 by using Lemmas 3.4–3.6.

Proof of Theorem 3.2. Multiplying both sides of (3.34) by α and summing up this inequality with (3.20) and (3.43) lead to

(3.55)
$$
\begin{aligned}
&\frac{\|e_{\mathbf{u}}^{n+1}\|^2-\|e_{\mathbf{u}}^{n}\|^2}{2\Delta t}+\frac{\|e_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^{n}\|^2}{2\Delta t}+\frac{\nu}{2}\|\nabla e_{\mathbf{u}}^{n+1}\|^2+\alpha\frac{\|e_{\mathbf{b}}^{n+1}\|^2-\|e_{\mathbf{b}}^{n}\|^2}{2\Delta t}\\
&+\alpha\frac{\|e_{\mathbf{b}}^{n+1}-e_{\mathbf{b}}^{n}\|^2}{2\Delta t}+\frac{\alpha\eta}{2}\|\nabla e_{\mathbf{b}}^{n+1}\|^2+\frac{|e_{q}^{n+1}|^2-|e_{q}^{n}|^2}{2\Delta t}+\frac{|e_{q}^{n+1}-e_{q}^{n}|^2}{2\Delta t}+\frac{1}{2T}|e_q^{n+1}|^2\\
\le& C(\|\mathbf{b}(t^{n+1})\|_2^2+\|e_{\mathbf{u}}^{n}\|_1^2)\|e_{\mathbf{u}}^{n}\|^2+C(\|e_{\mathbf{b}}^{n}\|_1^2+\|\mathbf{u}^{n}\|_1^2)\|e_{\mathbf{b}}^{n}\|^2\\
&+C\Delta t\int_{t^n}^{t^{n+1}}(\|\mathbf{u}_t\|_2^2+\|\mathbf{u}_{tt}\|_{-1}^2+\|q_{tt}\|^2)dt\\
&+C\Delta t\int_{t^n}^{t^{n+1}}(\|\mathbf{b}_t\|_2^2+\|\mathbf{b}_{tt}\|_{-1}^2)dt.
\end{aligned}
$$

Multiplying (3.55) by $2\Delta t$ and summing over n, $n=0,1,\dots,m$, thanks to the stability results in (3.19), we can apply the discrete Gronwall lemma (Lemma 3.1) to get

$$
\begin{aligned}
&\|e_{\mathbf{u}}^{m+1}\|^2+\|e_{\mathbf{b}}^{n+1}\|^2+|e_q^{m+1}|^2+\nu\Delta t\sum_{n=0}^{m}\|\nabla e_{\mathbf{u}}^{n+1}\|^2\\
&+\eta\Delta t\sum_{n=0}^{m}\|\nabla e_{\mathbf{b}}^{n+1}\|^2+\Delta t\sum_{n=0}^{m}|e_q^{n+1}|^2+\sum_{n=0}^{m}\|e_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^{n}\|^2\\
&+\sum_{n=0}^{m}\|e_{\mathbf{b}}^{n+1}-e_{\mathbf{b}}^{n}\|^2+\sum_{n=0}^{m}|e_q^{n+1}-e_q^{n}|^2\\
\le& C(\|\mathbf{u}\|^2_{H^1(0,T;H^2(\Omega))}+\|\mathbf{u}\|^2_{H^2(0,T;H^{-1}(\Omega))}+\|\mathbf{u}\|^2_{L^\infty(0,T;\mathbf{H}^2(\Omega))})(\Delta t)^2\\
&+C(\|\mathbf{b}\|^2_{H^1(0,T;H^2(\Omega))}+\|\mathbf{b}\|^2_{H^2(0,T;H^{-1}\Omega))})(\Delta t)^2\\
&+C(\|\mathbf{b}\|^2_{L^\infty(0,T;\mathbf{H}^2(\Omega))}+\|q\|^2_{H^2(0,T)})(\Delta t)^2,
\end{aligned}
\tag{3.56}
$$

which concludes the proof of Theorem 3.2. ∎

3.3. Error estimates for the pressure. With the error estimates for the velocity and magnetic fields established above, we can establish the following error estimate for the pressure.

THEOREM 3.7. *Assuming*

$$\boldsymbol{u}\in H^2(0,T;\boldsymbol{L}^2(\Omega))\bigcap H^1(0,T;\boldsymbol{H}^2(\Omega))\bigcap L^\infty(0,T;\boldsymbol{H}^2(\Omega)),$$

$\boldsymbol{b}\in H^2(0,T;\boldsymbol{L}^2(\Omega))\bigcap H^1(0,T;\boldsymbol{H}^2(\Omega))\bigcap L^\infty(0,T;\boldsymbol{H}^2(\Omega))$, *and* $p\in C((0,T];L_0^2(\Omega))$, *for the first-order scheme* (2.7)–(2.11), *we have*

$$\Delta t\sum_{n=0}^{m}\|e_p^{n+1}\|^2_{L^2(\Omega)/R}\le C(\Delta t)^2\quad\forall\ 0\le m\le N-1 \tag{3.57}$$

where C is a positive constant independent of Δt.

The proof of the above result can be carried out using a rather standard procedure as presented in [14], albeit a very technical one, so we provide it in the appendix.

4. Numerical experiments. In this section we provide some numerical experiments to validate the SAV schemes developed in the previous sections. In our simulation, we use (P_2,P_1,P_2) finite elements to approximate velocity, pressure, and magnetic field, respectively. Note that the (P_2, P_1) finite elements for velocity and pressure satisfy the inf-sup conditions so that one can easily show that the fully discrete scheme (2.41)–(2.44) coupled with (P_2, P_1, P_2) finite elements are well posed and can be solved following the procedure described in section 2.

4.1. Accuracy test for the SAV schemes. In the first example, we set $\Omega = (0,1)\times(0,1)$, $\nu = 0.01$, $\eta = 0.01$, $\alpha = 1$,$T = 1$. The right-hand side of the equations is computed according to the analytic solution given below:

$$\begin{cases} u_1(x,y,t) = \pi k \sin^2(\pi x)\sin(\pi y)\cos(t), \\ u_2(x,y,t) = -\pi k \sin(\pi x)\sin^2(\pi y)\cos(t), \\ p(x,y,t) = k(x-1/2)(y-1/2)\cos(t)/10, \\ b_1(x,y,t) = k\sin(\pi x)\cos(\pi y)\cos(t), \\ b_2(x,y,t) = -k\cos(\pi x)\sin(\pi y)\cos(t), \end{cases}$$

where $k = 0.01$. To test the time accuracy, we choose $h = 0.005$ so that the spatial discretization error is negligible compared to the time discretization error for the time steps used in this experiment.

We first set $\epsilon = \Delta t$ and list the numerical results for this example with first- and second-order schemes in Tables 4.1–4.4. We observe that the results for the first-order scheme (2.7)–(2.11) are consistent with the error estimates in Theorems 3.2 and 3.7, while second-order convergence rates for the velocity, pressure, and magnetic field were observed for the second-order scheme (2.25)–(2.29). We then set $\epsilon = 1$ and observe that the errors for $\mathbf{u}$ and $\mathbf{b}$ are exactly the same (up to the digits shown in the tables) as with $\epsilon = \Delta t$, while there is a slight difference at the second or third digit shown in the tables for the pressure error.

4.2. Driven cavity flow. In this example, we consider the following driven cavity flow problem. We set $\Omega = (-1,1)\times(-1,1)$, $\eta = 1$, $\alpha = 1$, $\Delta t = 0.001$, $h = 1/40$. The boundary conditions are

TABLE 4.1
Errors and convergence rates with the first-order scheme (2.7)–(2.11).

Δt	$\|\mathbf{u}_h - \mathbf{u}\|_1$	Order	$\|\mathbf{u}_h - \mathbf{u}\|_0$	Order	$\|p_h - p\|_0$	Order
1/2	8.26E-3	—	1.34E-3	—	2.52E-5	—
1/4	3.96E-3	1.06	7.16E-4	0.91	1.13E-5	1.16
1/8	1.93E-3	1.04	3.70E-4	0.95	5.34E-6	1.08
1/16	9.52E-4	1.04	1.89E-4	0.97	2.59E-6	1.04
1/32	4.72E-4	1.01	9.51E-5	0.99	1.27E-6	1.03
1/64	2.35E-4	1.01	4.78E-5	0.99	6.32E-7	1.01

TABLE 4.2
Errors and convergence rates with the first-order scheme (2.7)–(2.11).

Δt	$\|\mathbf{b}_h - \mathbf{b}\|_1$	Order	$\|\mathbf{b}_h - \mathbf{b}\|_2$	Order
1/2	4.52E-3	—	1.22E-3	—
1/4	2.10E-3	1.11	6.39E-4	0.94
1/8	1.00E-3	1.07	3.27E-4	0.97
1/16	4.89E-4	1.04	1.65E-4	0.98
1/32	2.41E-4	1.02	8.31E-5	0.99
1/64	1.20E-4	1.01	4.17E-5	1.00

TABLE 4.3
Errors and convergence rates with the second-order scheme (2.25)–(2.29).

Δt	$\|\mathbf{u}_h - \mathbf{u}\|_1$	Order	$\|\mathbf{u}_h - \mathbf{u}\|_2$	Order	$\|p_h - p\|_2$	Order
1/2	6.43E-3	—	8.84E-4	—	1.83E-5	—
1/4	1.99E-3	1.70	2.32E-4	1.93	5.09E-6	1.85
1/8	5.49E-4	1.85	5.35E-5	2.12	1.36E-6	1.90
1/16	1.44E-4	1.93	1.26E-5	2.09	3.51E-7	1.95
1/32	3.70E-5	1.96	3.05E-6	2.04	8.89E-8	1.98
1/64	1.03E-5	1.85	7.52E-7	2.02	2.24E-8	1.99

TABLE 4.4
Errors and convergence rates with the second-order scheme (2.25)–(2.29).

Δt	$\|\mathbf{b}_h - \mathbf{b}\|_1$	Order	$\|\mathbf{b}_h - \mathbf{b}\|_2$	Order
1/2	3.54E-3	—	8.38E-4	—
1/4	1.06E-3	1.74	2.30E-4	1.87
1/8	2.90E-4	1.88	5.57E-5	2.05
1/16	7.54E-5	1.94	1.35E-5	2.04
1/32	1.92E-5	1.97	3.32E-6	2.02
1/64	4.88E-6	1.98	8.23E-7	2.01

$$(4.1) \qquad \begin{cases} \mathbf{u} = (0,0) & \text{on } x = \pm 1 \text{ and } y = -1, \\ \mathbf{u} = (1,0) & \text{on } y = 1, \\ \mathbf{n} \times \mathbf{b} = \mathbf{n} \times \mathbf{b}_0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{b}_0 = (1,0)$. The streamlines of the velocity, the isolines of pressure, and the vectors of the magnetic field at steady state with $\nu = 0.001$ using the second-order SAV scheme (2.25)–(2.29) with $\epsilon = 1$ and $\epsilon = \Delta t$ are shown in Figure 4.1 and are consistent with the numerical results in [26]. We observe that, due to the influence of the magnetic field, steady states of the velocity streamlines for the MHD equations are very different from that of the Navier–Stokes equations with the same viscosity. Note that essentially the same results are obtained with $\epsilon = 1$ and $\epsilon = \Delta t$.

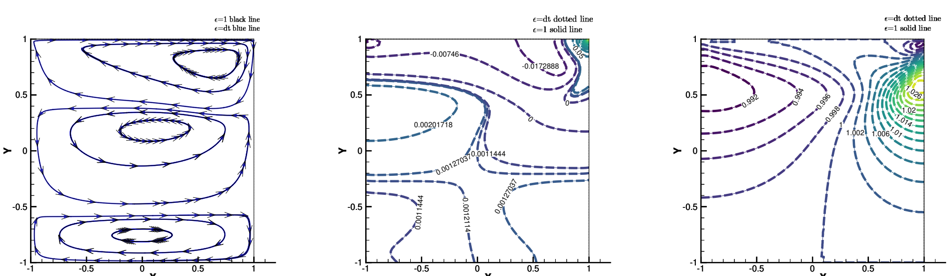

FIG. 4.1. $\mu = 0.001$: (a) *the streamlines of velocity,* (b) *the isolines of pressure,* (c) *the vectors of the magnetic field.*

5. Concluding remarks. We constructed first- and second-order discretization schemes in time based on the SAV approach for the MHD equations. The nonlinear terms are treated explicitly in our schemes, so they only require solving a sequence of linear differential equations with constant coefficients at each time step. Thus, the schemes are efficient and easy to implement.

Despite the fact that the nonlinear terms are treated explicitly, we proved that our schemes are unconditionally energy stable. This is made possible by introducing a purely artificial SAV, $q(t)$, which enables cancellation of the nonlinear contributions to the energy as in the continuous case, leading to the unconditional energy stability.

By using the unconditional energy stability which leads to uniform bounds on the numerical solution, we derived rigorous error estimates for the velocity, pressure, and magnetic field of the first-order scheme in the two-dimensional case without any condition on the time step. This appears to be the first unconditional error estimate of any scheme with fully explicit treatment of the nonlinear terms for the MHD equations. We believe that the error estimates can also be established for the second-order scheme in the two-dimensional case, although the process will surely be much more tedious. However, it appears that the error estimates cannot be easily extended to the three-dimensional case, as our proof uses essentially some inequalities which are only valid in the two-dimensional case.

Appendix A. Proof of Theorem 3.7. In order to prove Theorem 3.7, we need to first establish an estimate on $\|d_t e_{\mathbf{u}}^{n+1}\|$.

Thanks to Theorem 3.2, we have

$$\|e_{\mathbf{u}}^{m+1}\|^2 + \|e_{\mathbf{b}}^{m+1}\|^2 + \Delta t \sum_{n=0}^{m}(\|\nabla e_{\mathbf{u}}^{n+1}\|^2 + \|\nabla e_{\mathbf{b}}^{n+1}\|^2) \le C(\Delta t)^2, \tag{A.1}$$

which implies that

$$\|\mathbf{u}^{n+1}\|_1 \le C\left((\Delta t)^{1/2} + \|\mathbf{u}(t^{n+1})\|_1\right), \quad \|\mathbf{b}^{n+1}\|_1 \le C\left((\Delta t)^{1/2} + \|\mathbf{b}(t^{n+1})\|_1\right). \tag{A.2}$$

Taking the inner product of (3.22) with $Ae_{\mathbf{u}}^{n+1} + d_t e_{\mathbf{u}}^{n+1}$, we obtain

$$\begin{aligned}
&(1+\nu)\frac{\|\nabla e_{\mathbf{u}}^{n+1}\|^2 - \|\nabla e_{\mathbf{u}}^{n}\|^2}{2\Delta t} + \|d_t e_{\mathbf{u}}^{n+1}\|^2 + \nu\|Ae_{\mathbf{u}}^{n+1}\|^2 \\
&\quad = \exp\left(\frac{t^{n+1}}{T}\right)\left(q(t^{n+1})\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}) - q^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^n, Ae_{\mathbf{u}}^{n+1} + d_t e_{\mathbf{u}}^{n+1}\right) \\
&\qquad + \alpha\exp\left(\frac{t^{n+1}}{T}\right)\left(q^{n+1}(\nabla\times\mathbf{b}^n)\times\mathbf{b}^n - q(t^{n+1})(\nabla\times\mathbf{b}(t^{n+1}))\times\mathbf{b}(t^{n+1}),\right. \\
&\qquad \left. Ae_{\mathbf{u}}^{n+1} + d_t e_{\mathbf{u}}^{n+1}\right) + (\mathbf{R}_{\mathbf{u}}^{n+1}, Ae_{\mathbf{u}}^{n+1} + d_t e_{\mathbf{u}}^{n+1}).
\end{aligned} \tag{A.3}$$

For the first term on the right-hand side of (A.3), we have

$$\begin{aligned}
&\exp\left(\frac{t^{n+1}}{T}\right)\left(q(t^{n+1})\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}) - q^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^n, Ae_{\mathbf{u}}^{n+1} + d_t e_{\mathbf{u}}^{n+1}\right) \\
&\quad = -\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, Ae_{\mathbf{u}}^{n+1} + d_t e_{\mathbf{u}}^{n+1}\right) \\
&\qquad + \left((\mathbf{u}(t^{n+1}) - \mathbf{u}^n)\cdot\nabla\mathbf{u}(t^{n+1}), Ae_{\mathbf{u}}^{n+1} + d_t e_{\mathbf{u}}^{n+1}\right) \\
&\qquad + \left(\mathbf{u}^n\cdot\nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), Ae_{\mathbf{u}}^{n+1} + d_t e_{\mathbf{u}}^{n+1}\right).
\end{aligned} \tag{A.4}$$

Thanks to (3.15) and (A.2), the first term on the right-hand side of (A.4) can be bounded by

$$
\begin{aligned}
& -\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad = -\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\mathbf{u}^n\cdot\nabla e_{\mathbf{u}}^n, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\qquad -\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left((\mathbf{u}^n\cdot\nabla\mathbf{u}(t^n), Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad \leq C|e_q^{n+1}|\|\mathbf{u}^n\|^{1/2}\|\nabla\mathbf{u}^n\|^{1/2}\|e_{\mathbf{u}}^n\|^{1/2}\|Ae_{\mathbf{u}}^n\|^{1/2}\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\qquad + C|e_q^{n+1}|\|\mathbf{u}^n\|_1\|\mathbf{u}(t^n)\|_2^2\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\quad \leq \frac{1}{12}\|d_te_{\mathbf{u}}^{n+1}\|^2+\frac{\nu}{24}\|Ae_{\mathbf{u}}^{n+1}\|^2+\frac{\nu}{8}\|Ae_{\mathbf{u}}^n\|^2\\
&\qquad + C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)\|e_{\mathbf{u}}^n\|^2+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)|e_q^{n+1}|^2.
\end{aligned} \tag{A.5}
$$

The second term on the right-hand side of (A.4) can be estimated by

$$
\begin{aligned}
& \left((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla\mathbf{u}(t^{n+1}), Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad \leq C\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|_1\|\mathbf{u}(t^{n+1})\|_2\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\quad \leq \frac{1}{12}\|d_te_{\mathbf{u}}^{n+1}\|^2+\frac{\nu}{24}\|Ae_{\mathbf{u}}^{n+1}\|^2+C\|e_{\mathbf{u}}^n\|_1^2\\
&\qquad + C\|\mathbf{u}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_1^2dt.
\end{aligned} \tag{A.6}
$$

Using (3.15) and (A.2), the last term on the right-hand side of (A.4) can be controlled by

$$
\begin{aligned}
& \left(\mathbf{u}^n\cdot\nabla(\mathbf{u}(t^{n+1})-\mathbf{u}^n), Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad = \left(\mathbf{u}^n\cdot\nabla(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)), Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\qquad - \left(\mathbf{u}^n\cdot\nabla e_{\mathbf{u}}^n, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad \leq C\|\mathbf{u}^n\|_1\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|_2\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\qquad + C\|\mathbf{u}^n\|_1^{1/2}\|\mathbf{u}^n\|_0^{1/2}\|Ae_{\mathbf{u}}^n\|^{1/2}\|e_{\mathbf{u}}^n\|^{1/2}\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\quad \leq \frac{1}{12}\|d_te_{\mathbf{u}}^{n+1}\|^2+\frac{\nu}{24}\|Ae_{\mathbf{u}}^{n+1}\|^2+C(\Delta t+\|\mathbf{u}(t^{n+1})\|_1^2)\|e_{\mathbf{u}}^n\|^2\\
&\qquad + \frac{\nu}{8}\|Ae_{\mathbf{u}}^n\|^2+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_2^2dt.
\end{aligned} \tag{A.7}
$$

For the second term on the right-hand side of (A.3), we have

$$
\begin{aligned}
& \alpha\exp\left(\frac{t^{n+1}}{T}\right)\left(q^{n+1}(\nabla\times\mathbf{b}^n)\times\mathbf{b}^n-q(t^{n+1})(\nabla\times\mathbf{b}(t^{n+1}))\times\mathbf{b}(t^{n+1}),\right.\\
&\qquad\qquad \left. Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad = \alpha\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left((\nabla\times\mathbf{b}^n)\times\mathbf{b}^n, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\qquad + \alpha\left((\nabla\times(\mathbf{b}^n-\mathbf{b}(t^{n+1})))\times\mathbf{b}^n, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\qquad + \alpha\left((\nabla\times\mathbf{b}(t^{n+1}))\times(\mathbf{b}^n-\mathbf{b}(t^{n+1})), Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right).
\end{aligned} \tag{A.8}
$$

Thanks to (3.11) and and (A.2), the first term on the right-hand side of (A.8) can be bounded by

$$
\begin{aligned}
&\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\nabla\times\mathbf{b}^n)\times\mathbf{b}^n, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad=\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\nabla\times\mathbf{b}^n)\times e_{\mathbf{b}}^n, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\qquad+\alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}\left((\nabla\times\mathbf{b}^n)\times\mathbf{b}(t^n), Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad\le C\|\nabla\times\mathbf{b}^n\|\|e_{\mathbf{b}}^n\|_1^{1/2}\|e_{\mathbf{b}}^n\|_2^{1/2}\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\qquad+C|e_q^{n+1}|\|\nabla\times\mathbf{b}^n\|\|\mathbf{b}(t^n)\|_2\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\quad\le\frac{1}{12}\|d_te_{\mathbf{u}}^{n+1}\|^2+\frac{\nu}{24}\|Ae_{\mathbf{u}}^{n+1}\|^2+\frac{\eta}{8}\|\Delta e_{\mathbf{b}}^n\|^2\\
&\qquad+C(\Delta t+\|\mathbf{b}(t^n)\|_1^2)\|e_{\mathbf{b}}^n\|_1^2+C(\Delta t+\|\mathbf{b}(t^n)\|_1^2)|e_q^{n+1}|^2.
\end{aligned}
\tag{A.9}
$$

The last two terms on the right-hand side of (A.8) can be estimated by

$$
\begin{aligned}
&\alpha\big((\nabla\times(\mathbf{b}^n-\mathbf{b}(t^{n+1})))\times\mathbf{b}^n, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\big)\\
&\qquad+\alpha\left((\nabla\times\mathbf{b}(t^{n+1}))\times(\mathbf{b}^n-\mathbf{b}(t^{n+1})), Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\right)\\
&\quad\le C\|e_{\mathbf{b}}^n+\mathbf{b}(t^n)-\mathbf{b}(t^{n+1})\|_1\|e_{\mathbf{b}}^n\|_1^{1/2}\|e_{\mathbf{b}}^n\|_2^{1/2}\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\qquad+C\|e_{\mathbf{b}}^n+\mathbf{b}(t^n)-\mathbf{b}(t^{n+1})\|_1\|\mathbf{b}(t^n)\|_2\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\qquad+C\|\nabla\times\mathbf{b}(t^{n+1})\|_{L^4}\|\mathbf{b}^n-\mathbf{b}(t^{n+1})\|_{L^4}\|Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1}\|\\
&\quad\le\frac{1}{12}\|d_te_{\mathbf{u}}^{n+1}\|^2+\frac{\nu}{24}\|Ae_{\mathbf{u}}^{n+1}\|^2+\frac{\eta}{8}\|\Delta e_{\mathbf{b}}^n\|^2\\
&\qquad+C\|e_{\mathbf{b}}^n\|_1^2+C\|\mathbf{b}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_t\|_1^2dt.
\end{aligned}
\tag{A.10}
$$

For the last term on the right-hand side of (A.3), we have

$$
(\mathbf{R}_{\mathbf{u}}^{n+1}, Ae_{\mathbf{u}}^{n+1}+d_te_{\mathbf{u}}^{n+1})\le\frac{1}{12}\|d_te_{\mathbf{u}}^{n+1}\|^2+\frac{\nu}{24}\|Ae_{\mathbf{u}}^{n+1}\|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_{tt}\|^2dt.
\tag{A.11}
$$

Combining (A.3) with (A.4)–(A.11), we have

$$
\begin{aligned}
&(1+\nu)\frac{\|\nabla e_{\mathbf{u}}^{n+1}\|^2-\|\nabla e_{\mathbf{u}}^n\|^2}{2\Delta t}+\frac{1}{2}\|d_te_{\mathbf{u}}^{n+1}\|^2+\frac{3\nu}{4}\|Ae_{\mathbf{u}}^{n+1}\|^2\\
&\quad\le\frac{\eta}{4}\|\Delta e_{\mathbf{b}}^n\|^2+\frac{\nu}{4}\|Ae_{\mathbf{u}}^n\|^2+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)\|e_{\mathbf{u}}^n\|_1^2+C(\Delta t+\|\mathbf{b}(t^n)\|_1^2)\|e_{\mathbf{b}}^n\|_1^2\\
&\quad+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2+\|\mathbf{b}(t^n)\|_1^2)|e_q^{n+1}|^2\\
&\quad+C\Delta t\int_{t^n}^{t^{n+1}}(\|\mathbf{u}_t\|_2^2+\|\mathbf{u}_{tt}\|^2+\|\mathbf{b}_t\|_1^2)dt.
\end{aligned}
\tag{A.12}
$$

Next we shall balance the first term on the right-hand side of (A.12) by using the error equation (3.37) for magnetic field. We proceed as follows.

Taking the inner product of (3.37) with $-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}$, we obtain

$$\begin{aligned}(1+\eta)&\frac{\|\nabla e_{\mathbf{b}}^{n+1}\|^2-\|\nabla e_{\mathbf{b}}^{n}\|^2}{2\Delta t}+\|d_te_{\mathbf{b}}^{n+1}\|^2+\eta\|\Delta e_{\mathbf{b}}^{n+1}\|^2\\ &=\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left(\nabla\times(\mathbf{b}(t^{n+1})\times\mathbf{u}(t^{n+1})),-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\quad-\exp\left(\frac{t^{n+1}}{T}\right)q^{n+1}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\quad+(\mathbf{R}_{\mathbf{b}}^{n+1},-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}).\end{aligned}\tag{A.13}$$

The first two terms on the right-hand side of (A.13) can be recast as

$$\begin{aligned}\exp&\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left(\nabla\times(\mathbf{b}(t^{n+1})\times\mathbf{u}(t^{n+1})),-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\qquad-\exp\left(\frac{t^{n+1}}{T}\right)q^{n+1}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &=\left(\nabla\times[(\mathbf{b}(t^{n+1})-\mathbf{b}^n)\times\mathbf{u}(t^{n+1})],-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\quad+\left(\nabla\times[\mathbf{b}^n\times(\mathbf{u}(t^{n+1})-\mathbf{u}^n)],-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\quad-\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n),-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right).\end{aligned}\tag{A.14}$$

Noting (3.3) and (3.14), the first term on the right-hand side of (A.14) can be bounded by

$$\begin{aligned}&\left(\nabla\times[(\mathbf{b}(t^{n+1})-\mathbf{b}^n)\times\mathbf{u}(t^{n+1})],-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\quad\leq C\|\mathbf{b}(t^{n+1})-\mathbf{b}^n\|_1\|\mathbf{u}(t^{n+1})\|_2\|d_te_{\mathbf{b}}^{n+1}-\Delta e_{\mathbf{b}}^{n+1}\|\\ &\quad\leq\frac{1}{8}\|d_te_{\mathbf{b}}^{n+1}\|^2+\frac{\eta}{16}\|\Delta e_{\mathbf{b}}^{n+1}\|^2+C\|e_{\mathbf{b}}^{n}\|_1^2\\ &\qquad+C\|\mathbf{u}(t^{n+1})\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_t\|_1^2dt.\end{aligned}\tag{A.15}$$

For the second term on the right-hand side of (A.14), we have

$$\begin{aligned}&\left(\nabla\times[\mathbf{b}^n\times(\mathbf{u}(t^{n+1})-\mathbf{u}^n)],-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\quad=\left(\nabla\times[e_{\mathbf{b}}^n\times(\mathbf{u}(t^{n+1})-\mathbf{u}^n)],-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\qquad+\left(\nabla\times[\mathbf{b}(t^n)\times(\mathbf{u}(t^{n+1})-\mathbf{u}^n)],-\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\ &\quad\leq C\|e_{\mathbf{b}}^n\|_1^{1/2}\|e_{\mathbf{b}}^n\|_2^{1/2}\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|_1\|d_te_{\mathbf{b}}^{n+1}-\Delta e_{\mathbf{b}}^{n+1}\|\\ &\qquad+C\|\mathbf{b}(t^n)\|_2\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|_1\|d_te_{\mathbf{b}}^{n+1}-\Delta e_{\mathbf{b}}^{n+1}\|\\ &\quad\leq\frac{1}{8}\|d_te_{\mathbf{b}}^{n+1}\|^2+\frac{\eta}{16}\|\Delta e_{\mathbf{b}}^{n+1}\|^2+\frac{\eta}{8}\|\Delta e_{\mathbf{b}}^{n}\|^2+C\|e_{\mathbf{b}}^{n}\|_1^2\\ &\qquad+C\|\mathbf{b}(t^n)\|_2^2\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_1^2dt.\end{aligned}\tag{A.16}$$

Thanks to (3.3) and (3.14), the last term on the right-hand side of (A.14) can be

$$
\begin{aligned}
&-\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\nabla\times(\mathbf{b}^n\times\mathbf{u}^n), -\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\
&\qquad=-\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\nabla\times(e_{\mathbf{b}}^n\times\mathbf{u}^n), -\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\
&\qquad\quad-\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left(\nabla\times(\mathbf{b}(t^n)\times\mathbf{u}^n), -\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1}\right)\\
&\qquad\le C|e_q^{n+1}|\|e_{\mathbf{b}}^n\|_1^{1/2}\|e_{\mathbf{b}}^n\|_2^{1/2}\|\mathbf{u}^n\|_1\|d_te_{\mathbf{b}}^{n+1}-\Delta e_{\mathbf{b}}^{n+1}\|\\
&\qquad\quad+C|e_q^{n+1}|\|\mathbf{b}(t^n)\|_2\|\mathbf{u}^n\|_1\|d_te_{\mathbf{b}}^{n+1}-\Delta e_{\mathbf{b}}^{n+1}\|\\
&\qquad\le\frac{1}{8}\|d_te_{\mathbf{b}}^{n+1}\|^2+\frac{\eta}{16}\|\Delta e_{\mathbf{b}}^{n+1}\|^2+\frac{\eta}{8}\|\Delta e_{\mathbf{b}}^n\|^2\\
&\qquad\quad+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)\|e_{\mathbf{b}}^n\|_1^2+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)|e_q^{n+1}|^2.
\end{aligned}\tag{A.17}
$$

For the last term on the right-hand side of (A.13), we have

$$
(\mathbf{R}_{\mathbf{b}}^{n+1}, -\Delta e_{\mathbf{b}}^{n+1}+d_te_{\mathbf{b}}^{n+1})\le\frac{1}{8}\|d_te_{\mathbf{b}}^{n+1}\|^2+\frac{\eta}{16}\|\Delta e_{\mathbf{b}}^{n+1}\|^2+C\Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{b}_{tt}\|^2dt.\tag{A.18}
$$

Combining (A.13) with (A.14)–(A.18), we obtain

$$
\begin{aligned}
&(1+\eta)\frac{\|\nabla e_{\mathbf{b}}^{n+1}\|^2-\|\nabla e_{\mathbf{b}}^n\|^2}{2\Delta t}+\frac{1}{2}\|d_te_{\mathbf{b}}^{n+1}\|^2+\frac{3\eta}{4}\|\Delta e_{\mathbf{b}}^{n+1}\|^2\\
&\quad\le\frac{\eta}{4}\|\Delta e_{\mathbf{b}}^n\|^2+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)\|e_{\mathbf{b}}^n\|_1^2+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)|e_q^{n+1}|^2\\
&\qquad+C\Delta t\int_{t^n}^{t^{n+1}}(\|\mathbf{u}_t\|_1^2+\|\mathbf{b}_t\|_1^2+\|\mathbf{b}_{tt}\|^2)dt.
\end{aligned}\tag{A.19}
$$

Summing up (A.19) with (A.12) leads to

$$
\begin{aligned}
&(1+\nu)\frac{\|\nabla e_{\mathbf{u}}^{n+1}\|^2-\|\nabla e_{\mathbf{u}}^n\|^2}{2\Delta t}+\frac{1}{2}\|d_te_{\mathbf{u}}^{n+1}\|^2+\frac{3\nu}{4}\|Ae_{\mathbf{u}}^{n+1}\|^2\\
&\quad+(1+\eta)\frac{\|\nabla e_{\mathbf{b}}^{n+1}\|^2-\|\nabla e_{\mathbf{b}}^n\|^2}{2\Delta t}+\frac{1}{2}\|d_te_{\mathbf{b}}^{n+1}\|^2+\frac{3\eta}{4}\|\Delta e_{\mathbf{b}}^{n+1}\|^2\\
&\le\frac{\eta}{2}\|\Delta e_{\mathbf{b}}^n\|^2+\frac{\nu}{4}\|Ae_{\mathbf{u}}^n\|^2+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2)\|e_{\mathbf{u}}^n\|_1^2\\
&\quad+C(\Delta t+\|\mathbf{u}(t^n)\|_1^2+\|\mathbf{b}(t^n)\|_1^2)(\|e_{\mathbf{b}}^n\|_1^2+|e_q^{n+1}|^2)\\
&\quad+C\Delta t\int_{t^n}^{t^{n+1}}(\|\mathbf{u}_t\|_2^2+\|\mathbf{u}_{tt}\|^2+\|\mathbf{b}_t\|_1^2+\|\mathbf{b}_{tt}\|^2)dt.
\end{aligned}\tag{A.20}
$$

Multiplying (A.20) by $2\Delta t$ and summing over n, $n=0,2,\dots,m$, and applying the discrete Gronwall lemma (Lemma 3.1), we obtain

$$
\begin{aligned}
&\|\nabla e_{\mathbf{u}}^{m+1}\|^2+\Delta t\sum_{n=0}^{m}\|d_te_{\mathbf{u}}^{n+1}\|^2+\nu\Delta t\sum_{n=0}^{m}\|Ae_{\mathbf{u}}^{n+1}\|^2\\
&\qquad+\|\nabla e_{\mathbf{b}}^{m+1}\|^2+\Delta t\sum_{n=0}^{m}\|d_te_{\mathbf{b}}^{n+1}\|^2+\eta\Delta t\sum_{n=0}^{m}\|\Delta e_{\mathbf{b}}^{n+1}\|^2
\end{aligned}\tag{A.21}
$$

$$
\begin{aligned}
&\le C(\Delta t + \|\mathbf{u}(t^n)\|_1^2 + \|\mathbf{b}(t^n)\|_1^2)\Delta t \sum_{n=0}^{m} (\|e_{\mathbf{u}}^n\|_1^2 + \|e_{\mathbf{b}}^n\|_1^2) \\
&\quad + C\Delta t \sum_{n=0}^{m} |e_q^{n+1}|^2 + C(\Delta t)^2.
\end{aligned}
$$

Combining the above estimate with Theorem 3.2, we obtain

$$
\begin{aligned}
&\Delta t \sum_{n=0}^{m} \|d_t e_{\mathbf{u}}^{n+1}\|^2 + \|\nabla e_{\mathbf{u}}^{m+1}\|^2 + \nu \Delta t \sum_{n=0}^{m} \|A e_{\mathbf{u}}^{n+1}\|^2 + \|\nabla e_{\mathbf{b}}^{m+1}\|^2 \\
&\quad + \Delta t \sum_{n=0}^{m} \|d_t e_{\mathbf{b}}^{n+1}\|^2 + \eta \Delta t \sum_{n=0}^{m} \|\Delta e_{\mathbf{b}}^{n+1}\|^2 \le C(\Delta t)^2.
\end{aligned} \tag{A.22}
$$

We are now in position to prove the pressure estimate. Taking the inner product of (3.22) with $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, we obtain

$$
\begin{aligned}
&(\nabla e_p^{n+1}, \mathbf{v}) = -(d_t e_{\mathbf{u}}^{n+1}, \mathbf{v}) + \nu (\Delta e_{\mathbf{u}}^{n+1}, \mathbf{v}) + (\mathbf{R}_{\mathbf{u}}^{n+1}, \mathbf{v}) \\
&\quad + \exp\left(\frac{t^{n+1}}{T}\right) \left(q(t^{n+1})(\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) - q^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n, \mathbf{v}\right) \\
&\quad + \alpha \exp\left(\frac{t^{n+1}}{T}\right) \left(q^{n+1}(\nabla \times \mathbf{b}^n) \times \mathbf{b}^n - q(t^{n+1})(\nabla \times \mathbf{b}(t^{n+1})) \times \mathbf{b}(t^{n+1}), \mathbf{v}\right).
\end{aligned} \tag{A.23}
$$

We bound below the terms on the right-hand side of the above equation. The first two terms can be handled by (A.23). The third term can be handled similarly as in (3.24). For the fourth term, we derive from (3.25)–(3.27) that, for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$,

$$
\begin{aligned}
&\exp\left(\frac{t^{n+1}}{T}\right) \left(q(t^{n+1})(\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) - q^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n, \mathbf{v}\right) \\
&\qquad = \frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})} \left((\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{v}\right) - \frac{e_q^{n+1}}{\exp(-\frac{t^{n+1}}{T})} \left((\mathbf{u}^n \cdot \nabla)\mathbf{u}^n, \mathbf{v}\right) \\
&\qquad\quad + \frac{q(t^{n+1})}{\exp(-\frac{t^{n+1}}{T})} \left(\mathbf{u}^n \cdot \nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), \mathbf{v}\right) \\
&\qquad \le C(\|e_{\mathbf{u}}^n\|_1 + \|\int_{t^n}^{t^{n+1}} \mathbf{u}_t dt\|_1 + |e_q^{n+1}|)\|\nabla \mathbf{v}\|.
\end{aligned} \tag{A.24}
$$

For the last term on the right-hand side of (A.23), we derive from (3.28)–(3.33) that

$$
\begin{aligned}
&\alpha \exp\left(\frac{t^{n+1}}{T}\right) \left(q^{n+1}(\nabla \times \mathbf{b}^n) \times \mathbf{b}^n - q(t^{n+1})(\nabla \times \mathbf{b}(t^{n+1})) \times \mathbf{b}(t^{n+1}), \mathbf{v}\right) \\
&\qquad = \alpha \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left((\nabla \times \mathbf{b}^n) \times \mathbf{b}^n, e_{\mathbf{u}}^{n+1}\right) \\
&\qquad\quad + \alpha \left(\nabla \times (\mathbf{b}^n - \mathbf{b}(t^{n+1})) \times \mathbf{b}^n, e_{\mathbf{u}}^{n+1}\right) \\
&\qquad\quad + \alpha \left((\nabla \times \mathbf{b}(t^{n+1})) \times (\mathbf{b}^n - \mathbf{b}(t^{n+1})), e_{\mathbf{u}}^{n+1}\right) \\
&\qquad \le C(\|e_{\mathbf{b}}^n\|_1 + \|\mathbf{b}^n\| \|\int_{t^n}^{t^{n+1}} \mathbf{b}_t dt\|_1 + |e_q^{n+1}|)\|\nabla \mathbf{v}\|.
\end{aligned} \tag{A.25}
$$

Finally thanks to Theorem 3.2, (A.22), and (A.24)–(A.25), we derive from (A.23) and

$$\|e_p^{n+1}\|_{L^2(\Omega)/\mathbb{R}} \le \sup_{\mathbf{v}\in \mathbf{H}_0^1(\Omega)} \frac{(\nabla e_p^{n+1}, \mathbf{v})}{\|\nabla \mathbf{v}\|}$$

that

$$\begin{aligned}\Delta t \sum_{n=0}^{m} \|e_p^{n+1}\|^2_{L^2(\Omega)/\mathbb{R}} &\le C\Delta t \sum_{n=0}^{m} \big(\|d_t e_{\mathbf{u}}^{n+1}\|^2 + \|\nabla e_{\mathbf{u}}^{n+1}\|^2 \\ &\quad + \|e_{\mathbf{u}}^n\|_1^2 + \|e_{\mathbf{b}}^n\|_1^2 + |e_q^{n+1}|^2\big) + C(\Delta t)^2 \int_{t^0}^{t^{m+1}} \|\mathbf{b}_t\|_1^2 dt \\ &\quad + C(\Delta t)^2 \int_{t^0}^{t^{m+1}} (\|\mathbf{u}_t\|_1^2 + \|\mathbf{u}_{tt}\|_{-1}^2) dt \le C(\Delta t)^2.\end{aligned}$$

The proof of Theorem 3.7 is complete.

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