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Error estimate of a consistent splitting GSAV scheme for the Navier-Stokes equations ☆

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ABSTRACT

We carry out a rigorous error analysis of the first-order semi-discrete (in time) consistent splitting scheme coupled with a generalized scalar auxiliary variable (GSAV) approach for the Navier-Stokes equations with no-slip boundary conditions. The scheme is linear, unconditionally stable, and only requires solving a sequence of Poisson type equations at each time step. By using the build-in unconditional stability of the GSAV approach, we derive optimal global (resp. local) in time error estimates in the two (resp. three) dimensional case for the velocity and pressure approximations.

1. Introduction

We consider in this paper numerical approximation of the following time-dependent incompressible Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad in\ \Omega\times J, \tag{1.1a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad in\ \Omega\times J, \tag{1.1b}$$

$$\mathbf{u} = \mathbf{0} \quad on\ \partial\Omega\times J, \tag{1.1c}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a sufficiently smooth boundary $\partial\Omega$, $J = (0, T]$, $(\mathbf{u}, p)$ represent the unknown velocity and pressure, $\mathbf{f}$ is an external body force, ν is the viscosity coefficient, and $\mathbf{n}$ is the unit outward normal of the domain Ω. The first equation represents the momentum conservation while the second equation represents mass conservation or incompressibility of the fluid.

There exists a large number of work devoted to the numerical approximations of the Navier-Stokes equations, see, for instance, [5,33,6,11]. and the references therein. As we all know, the nonlinearity and the coupling of velocity and pressure have long been the main source of difficulties in both numerical analysis and simulations for the Navier-Stokes equations.

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In view of numerical computation, it is desirable to treat the nonlinear term explicitly so that one only needs to solve simple linear equations with constant coefficients at each time step. However, a simple explicitly treatment usually leads to a severe stability constraint on the time step.

The recently developed schemes [19,17,13,18,35] based on the scalar auxiliary variable (SAV) approach [28] with explicit treatment of nonlinear term can be unconditionally energy diminishing, and ample numerical results presented in the above work demonstrate that the above (implicit-explicit) IMEX type schemes (i.e. the nonlinear term is treated explicitly) are very efficient and robust.

On the other hand, there are in general two classes of numerical approaches to deal with the incompressible constraint: the coupled approach and the decoupled approach. The coupled approach is computationally expensive since it requires solving a saddle point problem at each time step [5,1,4]. The decoupled approach, originated from the so called projection method [2,30], can be more effective for the reason that one only needs to solve a sequence of Poisson type equations to solve at each time step. There have been extensive efforts in constructing various projection type schemes which can be roughly classified into three categories [9]: the pressure-correction method [26,34,8,27,18], the velocity-correction method [10,24] and the consistent splitting method [7,15,29] (see also the gauge method [3,23]). Among these, the consistent splitting scheme has outstanding advantages in the following two aspects: (i) It is free of the operator splitting error so it can achieve the full accuracy of the time discretization; (ii) The inf-sup condition between the velocity and the pressure approximation spaces is not mandatory from a computational point of view.

However, there are only very limited works on the stability and error analysis of the consistent splitting methods. Liu et al. derived a key inequality in [20] for the commutator between the Laplacian and Leray projection operators, and established local error estimates in [21] for the first-order consistent splitting schemes in two- and three-dimensional cases. Recently, based on the generalized SAV approach [14], Huang et al. constructed high-order consistent splitting schemes for the Navier-Stokes equations with periodic boundary conditions in [13] and no-slip boundary conditions in [35]. Furthermore, in the case of periodic boundary conditions, they established in [13] optimal error estimates (up to first-order) which are globally (resp. locally) in time in the two (resp. three) dimensional case. However, it is non trivial to extend the corresponding error analysis to the case with non-periodic boundary conditions: (i) additional difficulty due to the pressure Poisson equation; and (ii) weaker stability result compared with the case of periodic boundary conditions.

The main purpose of this paper is to carry out a rigorous error analysis for the first-order consistent splitting SAV scheme for the Navier-Stokes equations with no-slip boundary conditions. Our main contributions are:

- Global in time error estimates in $L^\infty(0,T;H^1(\Omega))\bigcap L^2(0,T;H^2(\Omega))$ for the velocity and $L^\infty(0,T;L^2(\Omega))\bigcap L^2(0,T;H^1(\Omega))$ for the pressure are established in the two-dimensional case. To the best of our knowledge, this appears to be the first global-in-time error estimate for a consisting splitting scheme with no-slip boundary conditions.
- Local in time error estimates in $L^\infty(0,T_*;H^1(\Omega))\bigcap L^2(0,T_*;H^2(\Omega))$ $(T_*\le T)$ for the velocity and $L^\infty(0,T_*;L^2(\Omega))\bigcap L^2(0,T_*;H^1(\Omega))$ for the pressure are established in the three-dimensional case.

The paper is organized as follows. In Section 2, we provide some preliminaries. In Section 3, we construct the first-order consistent-splitting scheme based on the SAV approach. In Section 4, we carry out a rigorous error estimates for the first-order GSAV consistent splitting scheme. Some numerical experiments are presented in Section 5 to validate our theoretical results.

2. Preliminaries

We describe below some notations and results which will be frequently used in this paper.

Throughout the paper, we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances.

Let Ω be an open bounded domain in $\mathbb{R}^d$ $(d=2,3)$, we will use the standard notations $L^2(\Omega)$, $H^k(\Omega)$ and $H^k_0(\Omega)$ to denote the usual Sobolev spaces over Ω. The norm corresponding to $H^k(\Omega)$ will be denoted simply by $\|\cdot\|_k$. In particular, we use $\|\cdot\|$ to denote the norm in $L^2(\Omega)$. Besides, $(\cdot,\cdot)$ is used to denote the inner product in $L^2(\Omega)$, and boldface letters are used to denote vector functions and vector spaces.

We define

$$\mathbf{H}=\{\mathbf{v}\in\mathbf{L}^2(\Omega):\mathrm{div}\,\mathbf{v}=0,\mathbf{v}\cdot\mathbf{n}|_\Gamma=0\},\quad \mathbf{V}=\{\mathbf{v}\in\mathbf{H}^1_0(\Omega):\mathrm{div}\,\mathbf{v}=0\}.$$

Next the trilinear form $b(\cdot,\cdot,\cdot)$ is defined by

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}d\mathbf{x}.$$

We can easily obtain that the trilinear form $b(\cdot,\cdot,\cdot)$ is a skew-symmetric with respect to its last two arguments, i.e.,

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=-b(\mathbf{u},\mathbf{w},\mathbf{v}),\quad\forall\,\mathbf{u}\in\mathbf{H},\quad\mathbf{v},\mathbf{w}\in\mathbf{H}^1_0(\Omega),\tag{2.1}$$

and

$$b(\mathbf{u},\mathbf{v},\mathbf{v})=0,\ \ \forall\,\mathbf{u}\in\mathbf{H},\ \ \mathbf{v}\in\mathbf{H}_0^1(\Omega). \tag{2.2}$$

By applying a combination of integration by parts, Holder's inequality, and Sobolev inequalities [32,26], we have that for $d\le 4$,

$$b(\mathbf{u},\mathbf{v},\mathbf{w})\le\begin{cases}c_2\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1, & \forall\,\mathbf{u},\mathbf{v}\in\mathbf{H},\mathbf{w}\in\mathbf{H}_0^1(\Omega),\\ c_2\|\mathbf{u}\|_2\|\mathbf{v}\|\|\mathbf{w}\|_1, & \forall\,\mathbf{u}\in\mathbf{H}^2(\Omega)\cap\mathbf{H},\ \mathbf{v}\in\mathbf{H},\mathbf{w}\in\mathbf{H}_0^1(\Omega),\\ c_2\|\mathbf{u}\|_2\|\mathbf{v}\|_1\|\mathbf{w}\|, & \forall\,\mathbf{u}\in\mathbf{H}^2(\Omega)\cap\mathbf{H},\ \mathbf{v}\in\mathbf{H},\mathbf{w}\in\mathbf{H}_0^1(\Omega),\\ c_2\|\mathbf{u}\|_1\|\mathbf{v}\|_2\|\mathbf{w}\|, & \forall\,\mathbf{v}\in\mathbf{H}^2(\Omega)\cap\mathbf{H},\ \mathbf{u}\in\mathbf{H},\mathbf{w}\in\mathbf{H}_0^1(\Omega),\\ c_2\|\mathbf{u}\|\|\mathbf{v}\|_2\|\mathbf{w}\|_1, & \forall\,\mathbf{v}\in\mathbf{H}^2(\Omega)\cap\mathbf{H},\ \mathbf{u}\in\mathbf{H},\mathbf{w}\in\mathbf{H}_0^1(\Omega).\end{cases} \tag{2.3}$$

In addition, we have the following more precise inequalities [31,13]:

$$b(\mathbf{u},\mathbf{v},\mathbf{w})\le\begin{cases}c_2\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\|_1^{1/2}\|\mathbf{v}\|_2^{1/2}\|\mathbf{w}\|, & d\le 2,\\ c_2\|\mathbf{u}\|_1\|\nabla\mathbf{v}\|_{1/2}\|\mathbf{w}\|, & d\le 3,\end{cases} \tag{2.4}$$

where c_2 is a positive constant depending only on Ω.

We will frequently use the following discrete version of the Gronwall lemma [25,12]:

Lemma 2.1. *Let a_k, b_k, c_k, d_k, γ_k, Δt_k be nonnegative real numbers such that*

$$a_{k+1}-a_k+b_{k+1}\Delta t_{k+1}+c_{k+1}\Delta t_{k+1}-c_k\Delta t_k\le a_k d_k\Delta t_k+\gamma_{k+1}\Delta t_{k+1} \tag{2.5}$$

for all $0\le k\le m$. Then

$$a_{m+1}+\sum_{k=0}^{m+1}b_k\Delta t_k\le\exp\left(\sum_{k=0}^{m}d_k\Delta t_k\right)\{a_0+(b_0+c_0)\Delta t_0+\sum_{k=1}^{m+1}\gamma_k\Delta t_k\}. \tag{2.6}$$

To obtain the local error estimates in the three-dimensional case, we recall the following lemma [20,22]:

Lemma 2.2. *Suppose that $F:(0,\infty)\to(0,\infty)$ is continuous and increasing, and let T_* satisfy that $0<T_*<\int_M^\infty dx/F(x)$ with $M>0$. Suppose that quantities $x_n,\ \omega_n\ge 0$ satisfy*

$$x_n+\sum_{k=0}^{n-1}\Delta t\omega(k)\le M+\sum_{k=0}^{n-1}\Delta tF(x_k),\ \forall n\le n_*,$$

with $n_*\Delta t\le T_*$. Then we have $M+\sum_{k=0}^{n_*-1}\Delta tF(x_k)\le C_*$, where C_* is independent of Δt.

3. The first-order consistent-splitting scheme based on the SAV approach

In this section, we construct the first-order consistent-splitting scheme based on the SAV approach for the Navier-Stokes equation [35].

Set

$$\Delta t=T/N,\ t^n=n\Delta t,\ d_tg^{n+1}=\frac{g^{n+1}-g^n}{\Delta t},\ \text{for}\ n\le N,$$

and introduce a SAV

$$R(t)=E(\mathbf{u})+K_0,\ E(\mathbf{u})=\frac{1}{2}\|\mathbf{u}\|^2, \tag{3.1}$$

with some $K_0>0$, and recast the governing system as the following equivalent form:

$$\begin{cases}\dfrac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla p=\mathbf{f}, & \text{(a)}\\ \dfrac{dR}{dt}=\dfrac{R}{E(\mathbf{u})+K_0}\Big(-\nu\|\nabla\mathbf{u}\|^2+(\mathbf{f},\mathbf{u})\Big), & \text{(b)}\\ \nabla\cdot\mathbf{u}=0. & \text{(c)}\end{cases} \tag{3.2}$$

It is clear that the above system is equivalent to the original system. Motivated by the SAV approach and the consistent splitting scheme, we construct the following first-order linear and decoupled scheme, which is almost the same to the first-order scheme in [35]: Find $(\tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, \xi^{n+1}, R^{n+1})$ by solving

$$\frac{\tilde{\mathbf{u}}^{n+1}-\tilde{\mathbf{u}}^{n}}{\Delta t} - \nu\Delta\tilde{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1} - (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n - \nabla p^n, \quad \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = 0; \tag{3.3}$$

$$\frac{R^{n+1}-R^n}{\Delta t} = \frac{R^{n+1}}{E(\tilde{\mathbf{u}}^{n+1})+K_0}\left(-\nu\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 + (\mathbf{f}^{n+1}, \tilde{\mathbf{u}}^{n+1})\right); \tag{3.4}$$

$$\xi^{n+1} = \frac{R^{n+1}}{E(\tilde{\mathbf{u}}^{n+1})+K_0}, \quad \eta^{n+1} = 1-(1-\xi^{n+1})^2, \quad \mathbf{u}^{n+1} = \eta^{n+1}\tilde{\mathbf{u}}^{n+1}; \tag{3.5}$$

$$(\nabla p^{n+1}, \nabla q) = \left(\mathbf{f}^{n+1} - (\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1} - \nu\nabla\times\nabla\times\tilde{\mathbf{u}}^{n+1}, \nabla q)\right), \quad \forall q\in H^1(\Omega). \tag{3.6}$$

By using similar procedure in [35], we can easily obtain the following unconditional energy stability:

Theorem 3.1. *Let $\|\boldsymbol{f}(\cdot,t)\| \le M_{\boldsymbol{f}}$, $\forall t\in[0,T]$, and $K_0 \ge \max\{2M_{\boldsymbol{f}}, 1\}$. Then given $R^n > 0$, we have $\xi^{n+1} > 0$ and*

$$0 < R^{n+1} < R^n, \quad \forall n \le T/\Delta t. \tag{3.7}$$

In addition, there exists a constant C_T only depends on T such that

$$\|\boldsymbol{u}^n\| + \nu\sum_{k=0}^{n}\Delta t\xi^{k+1}\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 \le C_T, \quad \forall n\le T/\Delta t, \tag{3.8}$$

where $\boldsymbol{u}^n$ is the solution of scheme (3.3)-(3.6).

4. Error analysis

In this section, we carry out a rigorous error analysis for the first-order semi-discrete scheme (3.3)-(3.6) in two- and three-dimensional cases.

We set

$$\begin{cases}\tilde{e}_{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}^{n+1} - \mathbf{u}(t^{n+1}), & e_{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}(t^{n+1}),\\ e_p^{n+1} = p^{n+1} - p(t^{n+1}), & e_R^{n+1} = R^{n+1} - R(t^{n+1}).\end{cases}$$

Next we give some preliminaries to estimate the part of pressure. Similar to [20], we let $\mathcal{P}$ denote the Leray-Helmholtz projection operator onto divergence-free fields, defined as follows. Given any $\mathbf{b}\in L^2(\Omega,\mathbb{R}^d)$, there is a unique $q\in H^1(\Omega)$ with $\int_\Omega q = 0$ such that $\mathcal{P}\mathbf{b} = \mathbf{b}+\nabla q$ satisfies

$$(\mathbf{b}+\nabla q, \nabla\phi) = (\mathcal{P}\mathbf{b}, \nabla\phi) = 0, \quad \forall\phi\in H^1(\Omega). \tag{4.1}$$

Then for $\mathbf{u}\in L^2(\Omega,\mathbb{R}^d)$, we have [20]

$$\Delta\mathcal{P}\mathbf{u} = \Delta\mathbf{u} - \nabla\nabla\cdot\mathbf{u} = -\nabla\times\nabla\times\mathbf{u}. \tag{4.2}$$

Next we recall the estimate for commutator of the Laplacian and Leray-Helmholtz projection operators.

Lemma 4.1. *[20] Let $\Omega\subset\mathbb{R}^d$ be a connected bounded domain with C^3 boundary. Then for any $\epsilon > 0$, there exists a positive constant $C \ge 0$ such that for all vector fields $\boldsymbol{u}\in H^2\cap H_0^1(\Omega,\mathbb{R}^d)$,*

$$\int_\Omega |(\Delta\mathcal{P}-\mathcal{P}\Delta)\boldsymbol{u}|^2 \le (\frac{1}{2}+\epsilon)\int_\Omega |\Delta\boldsymbol{u}|^2 + C\int_\Omega |\nabla\boldsymbol{u}|^2. \tag{4.3}$$

We define the Stokes pressure $p_s(\mathbf{u})$ by

$$\nabla p_s(\mathbf{u}) = (\Delta\mathcal{P}-\mathcal{P}\Delta)\mathbf{u},$$

where the Stokes pressure is generated by the tangential part of vorticity at the boundary in two and three dimensions by [20,21]

$$\int_{\Omega} \nabla p_s(\mathbf{u}) \cdot \nabla \phi = \int_{\Gamma} (\nabla \times \mathbf{u}) \cdot (\mathbf{n} \times \nabla \phi), \ \forall \phi \in H^1(\Omega). \tag{4.4}$$

Then by using (4.2), we have

$$\nabla p_s(\mathbf{u}) = (\Delta \mathcal{P} - \mathcal{P}\Delta)\mathbf{u} = (I - \mathcal{P})\Delta \mathbf{u} - \nabla\nabla \cdot \mathbf{u} = (I - \mathcal{P})(\Delta \mathbf{u} - \nabla\nabla \cdot \mathbf{u}). \tag{4.5}$$

Recalling (4.1), we have

$$\int_{\Omega} \nabla p_s(\mathbf{u}) \cdot \nabla \phi = \int_{\Omega} (\Delta \mathbf{u} - \nabla\nabla \cdot \mathbf{u}) \cdot \nabla \phi, \ \forall \phi \in H^1(\Omega). \tag{4.6}$$

The main result of this section is stated in the following theorem.

Theorem 4.2. *Assume $\boldsymbol{u} \in H^2(0,T;\boldsymbol{L}^2(\Omega)) \bigcap H^1(0,T;\boldsymbol{H}^2(\Omega)) \bigcap L^\infty(0,T;\boldsymbol{H}^2(\Omega))$, $p \in H^1(0,T;L^2(\Omega))$ and*

$$\Delta t \sum_{k=0}^{n+1} \|\boldsymbol{f}^k\|^2 + \Delta t \|\Delta \boldsymbol{u}^0\|^2 + \|\nabla \boldsymbol{u}^0\|^2 \le C^*$$

with $C^* > 0$, then for the first-order scheme (3.3)-(3.6) with $\Delta t \le \frac{1}{1+C_0^2}$, we have

$$\|\boldsymbol{u}^{n+1} - \boldsymbol{u}(t^{n+1})\|^2 + \|\nabla(\boldsymbol{u}^{n+1} - \boldsymbol{u}(t^{n+1}))\|^2 + \Delta t \sum_{k=0}^{n} \|\Delta(\boldsymbol{u}^{k+1} - \boldsymbol{u}(t^{k+1}))\|^2 \le \begin{cases} C(\Delta t)^2, & d=2, \ \forall n \le T/\Delta t, \\ C(\Delta t)^2, & d=3, \ \forall n \le T_*/\Delta t, \end{cases} \tag{4.7}$$

and

$$\|p^{n+1} - p(t^{n+1})\|^2 + \Delta t \sum_{k=0}^{n} \|\nabla(p^{k+1} - p(t^{k+1}))\|^2 \le \begin{cases} C(\Delta t)^2, & d=2, \ \forall n \le T/\Delta t, \\ C(\Delta t)^2, & d=3, \ \forall n \le T_*/\Delta t, \end{cases} \tag{4.8}$$

where T_* is defined in (4.23) and the constants C_0 and C are some positive constants independent of Δt.

Proof. First we shall make the hypothesis that there exists a positive constant C_0 such that

$$|1 - \xi^k| \le C_0 \Delta t, \ \forall k \le T/\Delta t, \tag{4.9}$$

which will be proved in the induction process below by using a bootstrap argument.

We can easily obtain that (4.9) holds for $k = 0$. Now we suppose

$$|1 - \xi^k| \le C_0 \Delta t, \ \forall k \le n, \tag{4.10}$$

and we shall prove that $|1 - \xi^{n+1}| \le C_0 \Delta t$ holds true.

Step 1: H^2 bounds for $\tilde{\mathbf{u}}^k$ and $\mathbf{u}^k$ with $k \le n$ in two- and three-dimensional cases. First using exactly the same procedure in [13], we can easily obtain that

$$\frac{1}{2} \le |\xi^k|, \ |\eta^k| \le 2, \tag{4.11}$$

under the condition $\Delta t \le \min\{\frac{1}{4C_0}, 1\}$. Recalling Theorem 3.1, we have

$$\|\tilde{\mathbf{u}}^k\| + \nu \sum_{l=0}^{n} \Delta t \|\nabla \tilde{\mathbf{u}}^{k+1}\|^2 + \nu \sum_{l=0}^{n} \Delta t \|\nabla \mathbf{u}^{k+1}\|^2 \le 8C_T, \ \ C_0 \ge 1, \tag{4.12}$$

where C_T is independent of C_0.

Noting

$$\Delta \mathbf{u} - \nabla\nabla \cdot \mathbf{u} = -\nabla \times \nabla \times \mathbf{u}, \tag{4.13}$$

and taking $q = p^{k+1}$ in (3.6) lead to

$$\|\nabla p^{k+1}\| \leq \|\mathbf{f}^{k+1} - (\mathbf{u}^{k+1} \cdot \nabla)\mathbf{u}^{k+1}\| + \nu\|\nabla p_s^{k+1}(\tilde{\mathbf{u}})\|. \tag{4.14}$$

Recalling (4.6) and Lemma 4.1, we have

$$\nu\|\nabla p_s^{k+1}(\tilde{\mathbf{u}})\|^2 \leq \nu\alpha\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 + \nu C_\alpha\|\nabla\tilde{\mathbf{u}}^{k+1}\|^2, \tag{4.15}$$

where the positive constant $\frac{1}{2} < \alpha < 1$.

Taking the inner product of (3.3) with $-2\Delta t\Delta\tilde{\mathbf{u}}^{k+1}$, we obtain

$$\begin{aligned}
&(\|\nabla\tilde{\mathbf{u}}^{k+1}\|^2 - \|\nabla\tilde{\mathbf{u}}^k\|^2 + \|\nabla\tilde{\mathbf{u}}^{k+1} - \nabla\tilde{\mathbf{u}}^k\|^2) + 2\nu\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 \\
\leq& 2\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|\|\mathbf{f}^{k+1} - (\mathbf{u}^k \cdot \nabla)\mathbf{u}^k\| + 2\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|\|\nabla p^k\| \\
\leq& 2\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|(\|\mathbf{f}^{k+1} - (\mathbf{u}^k \cdot \nabla)\mathbf{u}^k\| + \|\mathbf{f}^k - (\mathbf{u}^k \cdot \nabla)\mathbf{u}^k\| + \nu\|\nabla p_s^k(\tilde{\mathbf{u}})\|) \\
\leq& \nu\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 + \nu\alpha\Delta t\|\Delta\tilde{\mathbf{u}}^k\|^2 + \nu\Delta t C_\alpha\|\nabla\tilde{\mathbf{u}}^k\|^2 + \frac{1-\alpha}{4}\nu\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 \\
&+ \frac{8}{(1-\alpha)\nu}\Delta t(\|\mathbf{f}^k\|^2 + \|\mathbf{f}^{k+1}\|^2) + \frac{16}{(1-\alpha)\nu}\Delta t\|(\mathbf{u}^k \cdot \nabla)\mathbf{u}^k\|^2.
\end{aligned} \tag{4.16}$$

Next we shall estimate the nonlinear term. Taking notice of the fact that by using Ladyzhenskaya's inequalities and Sobolev embedding theorems [16,20] and (2.4), we have

$$\|(\mathbf{u}^k \cdot \nabla)\mathbf{u}^k\|^2 \leq \begin{cases} \|\mathbf{u}^k\|_{L^4}^2\|\nabla\mathbf{u}^k\|_{L^4}^2 \leq C\|\mathbf{u}^k\|_{L^2}\|\nabla\mathbf{u}^k\|_{L^2}^2\|\nabla\mathbf{u}^k\|_{H^1}, & d=2, \\ \|\mathbf{u}^k\|_{L^6}^2\|\nabla\mathbf{u}^k\|_{L^3}^2 \leq C\|\nabla\mathbf{u}^k\|_{L^2}^3\|\nabla\mathbf{u}^k\|_{H^1}, & d=3. \end{cases}$$

In addition, by using the elliptic regularity estimate $\|\mathbf{u}^k\|_{H^2} \leq C\|\Delta\mathbf{u}^k\|$ and recalling $\mathbf{u}^k = \eta^k\tilde{\mathbf{u}}^k$, we have

$$\|(\mathbf{u}^k \cdot \nabla)\mathbf{u}^k\|^2 \leq \begin{cases} \frac{(1-\alpha)^2\nu^2}{64}\|\Delta\tilde{\mathbf{u}}^k\|^2 + C\|\mathbf{u}^k\|^2\|\nabla\tilde{\mathbf{u}}^k\|^4, & d=2, \\ \frac{(1-\alpha)^2\nu^2}{64}\|\Delta\tilde{\mathbf{u}}^k\|^2 + C\|\nabla\tilde{\mathbf{u}}^k\|^6, & d=3. \end{cases} \tag{4.17}$$

Thus for $d=2$, we can recast (4.16) as follows:

$$\begin{aligned}
&(\|\nabla\tilde{\mathbf{u}}^{k+1}\|^2 - \|\nabla\tilde{\mathbf{u}}^k\|^2 + \|\nabla\tilde{\mathbf{u}}^{k+1} - \nabla\tilde{\mathbf{u}}^k\|^2) + \nu\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 \\
\leq& \frac{1-\alpha}{4}\nu\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 + \frac{1-\alpha}{4}\nu\Delta t\|\Delta\tilde{\mathbf{u}}^k\|^2 + \nu\alpha\Delta t\|\Delta\tilde{\mathbf{u}}^k\|^2 + \nu\Delta t C_\alpha\|\nabla\tilde{\mathbf{u}}^k\|^2 \\
&+ C\Delta t(\|\mathbf{f}^k\|^2 + \|\mathbf{f}^{k+1}\|^2) + C\Delta t\|\mathbf{u}^k\|^2\|\nabla\tilde{\mathbf{u}}^k\|^2\|\nabla\tilde{\mathbf{u}}^k\|^2 \\
\leq& \frac{1-\alpha}{4}\nu\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 + (\nu C_\alpha + C\|\mathbf{u}^k\|^2\|\nabla\tilde{\mathbf{u}}^k\|^2)\Delta t\|\nabla\tilde{\mathbf{u}}^k\|^2 \\
&+ (\frac{1-\alpha}{4}\nu + \alpha\nu)\Delta t\|\Delta\tilde{\mathbf{u}}^k\|^2 + C\Delta t(\|\mathbf{f}^k\|^2 + \|\mathbf{f}^{k+1}\|^2).
\end{aligned} \tag{4.18}$$

Summing (4.18) over k, $k = 0, 2, \ldots, n$, using (4.12) and Lemma 2.1, we can arrive at

$$\begin{aligned}
&\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 + \Delta t\sum_{k=0}^{n}\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 \\
&\quad \leq C\Delta t\sum_{k=0}^{n+1}\|\mathbf{f}^k\|^2 + C\Delta t\|\Delta\tilde{\mathbf{u}}^0\|^2 + C\|\nabla\tilde{\mathbf{u}}^0\|^2 \leq C^*, \quad d=2,
\end{aligned} \tag{4.19}$$

where C^* is independent of Δt and C_0. Recalling (4.11), we have for $d=2$,

$$\|\nabla\mathbf{u}^n\|^2 + \Delta t\sum_{k=0}^{n}\|\Delta\mathbf{u}^k\|^2 \leq C, \tag{4.20}$$

where C is independent of Δt and C_0.

Next we consider the case with $d=3$. Using (4.17), we can transform (4.16) into the following:

$$\begin{aligned}
&(\|\nabla\tilde{\mathbf{u}}^{k+1}\|^2 - \|\nabla\tilde{\mathbf{u}}^k\|^2 + \|\nabla\tilde{\mathbf{u}}^{k+1} - \nabla\tilde{\mathbf{u}}^k\|^2) + \nu\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 \\
\leq& \frac{1-\alpha}{4}\nu\Delta t\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2 + \frac{1-\alpha}{4}\nu\Delta t\|\Delta\tilde{\mathbf{u}}^k\|^2 + \nu\alpha\Delta t\|\Delta\tilde{\mathbf{u}}^k\|^2 + \nu\Delta t C_\alpha\|\nabla\tilde{\mathbf{u}}^k\|^2 \\
&+ C\Delta t(\|\mathbf{f}^k\|^2 + \|\mathbf{f}^{k+1}\|^2) + C\Delta t\|\nabla\tilde{\mathbf{u}}^k\|^6.
\end{aligned} \tag{4.21}$$

Summing up (4.21) over k from 0 to n leads to

$$\begin{aligned}&\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2+\Delta t\sum_{k=0}^{n}\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2\\&\quad\le C\Delta t\sum_{k=0}^{n}\|\nabla\tilde{\mathbf{u}}^{k}\|^6+C\Delta t\sum_{k=0}^{n+1}\|\mathbf{f}^k\|^2+C\Delta t\|\Delta\tilde{\mathbf{u}}^0\|^2+C\|\nabla\tilde{\mathbf{u}}^0\|^2,\ \ d=3.\end{aligned}\tag{4.22}$$

Recalling Lemma 2.2, we let $M_0>0$ and $F(x)=x^6$, and choose T^* satisfy that $0<T^*<\int_{M_0}^{\infty}dx/F(x)$, then we can estimate (4.22) as follows:

$$\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2+\|\nabla\mathbf{u}^{n+1}\|^2+\Delta t\sum_{k=0}^{n}\|\Delta\tilde{\mathbf{u}}^{k+1}\|^2+\Delta t\sum_{k=0}^{n}\|\Delta\mathbf{u}^{k+1}\|^2\le C_*,\ \ d=3,\ \forall n\le T_*/\Delta t,\tag{4.23}$$

where $T_*=\min\{T^*,T\}$ and C_* is independent of Δt and C_0.

Step 2: Estimates for H^2 bounds for $\tilde{e}_{\mathbf{u}}^{n+1}$ in two- and three-dimensional cases.

We shall first start by establishing an error equation corresponding to (3.3). Let $\mathbf{S}_{\mathbf{u}}^{k+1}$ be the truncation error defined by

$$\mathbf{S}_{\mathbf{u}}^{k+1}=\frac{\partial\mathbf{u}(t^{k+1})}{\partial t}-\frac{\mathbf{u}(t^{k+1})-\mathbf{u}(t^k)}{\Delta t}=\frac{1}{\Delta t}\int_{t^k}^{t^{k+1}}(t^k-t)\frac{\partial^2\mathbf{u}}{\partial t^2}dt.\tag{4.24}$$

Subtracting (3.2a) at t^{k+1} from (3.3), we obtain

$$\begin{aligned}&\frac{\tilde{e}_{\mathbf{u}}^{k+1}-\tilde{e}_{\mathbf{u}}^{k}}{\Delta t}-\nu\Delta\tilde{e}_{\mathbf{u}}^{k+1}=(\mathbf{u}(t^{k+1})\cdot\nabla)\mathbf{u}(t^{k+1})\\&\qquad-(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k-\nabla(p^k-p(t^{k+1}))+\mathbf{S}_{\mathbf{u}}^{k+1}.\end{aligned}\tag{4.25}$$

Next we establish an error equation for pressure corresponding to (3.6) by

$$\begin{aligned}(\nabla e_p^{k+1},\nabla q)=&\Big((\mathbf{u}(t^{k+1})\cdot\nabla)\mathbf{u}(t^{k+1})-(\mathbf{u}^{k+1}\cdot\nabla)\mathbf{u}^{k+1},\nabla q)\Big)\\&-(\nu\nabla\times\nabla\times\tilde{e}_{\mathbf{u}}^{k+1},\nabla q),\ \forall q\in H^1(\Omega).\end{aligned}\tag{4.26}$$

Taking $q=e_p^{k+1}$ in (4.26) leads to

$$\begin{aligned}\|\nabla e_p^{k+1}\|\le&\|(\mathbf{u}(t^{k+1})\cdot\nabla)\mathbf{u}(t^{k+1})-(\mathbf{u}^{k+1}\cdot\nabla)\mathbf{u}^{k+1}\|+\nu\|\nabla e_{ps}^{k+1}(\tilde{e}_{\mathbf{u}})\|\\\le&\nu\|\nabla e_{ps}^{k+1}(\tilde{e}_{\mathbf{u}})\|+\|(e_{\mathbf{u}}^{k+1}\cdot\nabla)\mathbf{u}(t^{k+1})\|\\&+\|(\mathbf{u}^{k+1}\cdot\nabla)e_{\mathbf{u}}^{k+1}\|\end{aligned}\tag{4.27}$$

Recalling (4.6) and Lemma 4.1, we have

$$\nu\|\nabla e_{ps}^{k+1}(\tilde{e}_{\mathbf{u}})\|^2\le\nu\alpha\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2+\nu C_\alpha\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2,\tag{4.28}$$

where the positive constant $\frac{1}{2}<\alpha<1$. Taking the inner product of (4.25) with $-2\Delta t\Delta\tilde{e}_{\mathbf{u}}^{k+1}$, we obtain

$$\begin{aligned}&(\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2-\|\nabla\tilde{e}_{\mathbf{u}}^{k}\|^2+\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}-\nabla\tilde{e}_{\mathbf{u}}^{k}\|^2)+2\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2\\\le&2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|\|(\mathbf{u}(t^{k+1})\cdot\nabla)\mathbf{u}(t^{k+1})-(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\|\\&+2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|\|\nabla e_p^k\|+2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|(\|p(t^n)-p(t^{k+1})\|+\|\mathbf{S}_{\mathbf{u}}^{k+1}\|)\\\le&2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|\|\nabla e_p^k\|+2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|(\|(e_{\mathbf{u}}^k\cdot\nabla)\mathbf{u}(t^k)\|+\|(\mathbf{u}^k\cdot\nabla)e_{\mathbf{u}}^k\|)\\&+2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|(\|p(t^n)-p(t^{k+1})\|+\|\mathbf{S}_{\mathbf{u}}^{k+1}\|+\|(\mathbf{u}(t^{k+1})\cdot\nabla)\mathbf{u}(t^{k+1})-(\mathbf{u}(t^k)\cdot\nabla)\mathbf{u}(t^k)\|)\\\le&2\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|\|\nabla e_{ps}^k(\tilde{e}_{\mathbf{u}})\|+3\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|(\|(e_{\mathbf{u}}^k\cdot\nabla)\mathbf{u}(t^k)\|+\|(\mathbf{u}^k\cdot\nabla)e_{\mathbf{u}}^k\|)\\&+2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|(\|p(t^n)-p(t^{k+1})\|+\|\mathbf{S}_{\mathbf{u}}^{k+1}\|+\|(\mathbf{u}(t^{k+1})\cdot\nabla)\mathbf{u}(t^{k+1})-(\mathbf{u}(t^k)\cdot\nabla)\mathbf{u}(t^k)\|).\end{aligned}\tag{4.29}$$

Using Cauchy-Schwarz inequality, the first term on the right hand side of (4.29) can be estimated by

$$2\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|\|\nabla e_{ps}^k(\tilde{e}_{\mathbf{u}})\|\le\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2+\nu\alpha\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k}\|^2+\nu C_\alpha\Delta t\|\nabla\tilde{e}_{\mathbf{u}}^{k}\|^2.\tag{4.30}$$

Recalling the Sobolev embedding theorems and Ladyzhenskaya's inequalities, we have

$$\int_\Omega |(\mathbf{f}\cdot\nabla)\mathbf{g}|^2 \le \left(\int_\Omega |\mathbf{f}|^6\right)^{1/3} \left(\int_\Omega |\nabla\mathbf{g}|^3\right)^{2/3} \le C\|\nabla\mathbf{f}\|^2\|\nabla\mathbf{g}\|\|\nabla\mathbf{g}\|_{H^1}. \tag{4.31}$$

Thanks to (4.31) and the H^2 boundedness for $\tilde{\mathbf{u}}^k$ in (4.19) and (4.23), the second term on the right hand side of (4.29) can be estimated by

$$\begin{aligned}
&3\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|(\|(e_{\mathbf{u}}^k\cdot\nabla)\mathbf{u}(t^k)\| + \|(\mathbf{u}^k\cdot\nabla)e_{\mathbf{u}}^k\|)\\
\le&\frac{(1-\alpha)\nu}{6}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + C\Delta t\|(e_{\mathbf{u}}^k\cdot\nabla)\mathbf{u}(t^k)\|^2 + C\Delta t\|(\mathbf{u}^k\cdot\nabla)e_{\mathbf{u}}^k\|^2\\
\le&\frac{(1-\alpha)\nu}{6}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + C\Delta t\|(e_{\mathbf{u}}^k\cdot\nabla)\mathbf{u}(t^k)\|^2 + C\Delta t\|(\mathbf{u}^k\cdot\nabla)e_{\mathbf{u}}^k\|^2\\
\le&\frac{(1-\alpha)\nu}{6}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + C\Delta t\|\nabla e_{\mathbf{u}}^k\|^2\|\nabla\mathbf{u}(t^k)\|\|\nabla\mathbf{u}(t^k)\|_{H^1} + C\Delta t\|\nabla\mathbf{u}^k\|^2\|\nabla e_{\mathbf{u}}^k\|\|\nabla e_{\mathbf{u}}^k\|_{H^1}\\
\le&\frac{(1-\alpha)\nu}{6}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + \frac{(1-\alpha)\nu}{12}\Delta t\|\Delta e_{\mathbf{u}}^k\|^2 + C\Delta t\|\nabla e_{\mathbf{u}}^k\|^2.
\end{aligned} \tag{4.32}$$

From (3.5) and (4.10), we have

$$\begin{aligned}
\|\Delta e_{\mathbf{u}}^k\|^2 \le& 2\|\Delta\tilde{e}_{\mathbf{u}}^k\|^2 + 2|1-\eta^k|^2\|\Delta\tilde{\mathbf{u}}^k\|^2\\
\le& 2\|\Delta\tilde{e}_{\mathbf{u}}^k\|^2 + 2\|\Delta\tilde{\mathbf{u}}^k\|^2 C_0^4(\Delta t)^4,
\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}
\|\nabla e_{\mathbf{u}}^k\|^2 \le& 2\|\nabla\tilde{e}_{\mathbf{u}}^k\|^2 + 2|1-\eta^k|^2\|\nabla\tilde{\mathbf{u}}^k\|^2\\
\le& 2\|\nabla\tilde{e}_{\mathbf{u}}^k\|^2 + 2\|\nabla\tilde{\mathbf{u}}^k\|^2 C_0^4(\Delta t)^4.
\end{aligned} \tag{4.34}$$

Thus we can recast (4.32) as

$$\begin{aligned}
&3\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|(\|(e_{\mathbf{u}}^k\cdot\nabla)\mathbf{u}(t^k)\| + \|(\mathbf{u}^k\cdot\nabla)e_{\mathbf{u}}^k\|)\\
\le&\frac{(1-\alpha)\nu}{6}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + \frac{(1-\alpha)\nu}{6}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^k\|^2 + C\Delta t\|\nabla\tilde{e}_{\mathbf{u}}^k\|^2\\
&+ C(\|\Delta\tilde{\mathbf{u}}^k\|^2 + \|\nabla\tilde{\mathbf{u}}^k\|^2)C_0^4(\Delta t)^5.
\end{aligned} \tag{4.35}$$

Using Cauchy-Schwarz inequality, the last term on the right hand side of (4.29) can be bounded by

$$\begin{aligned}
&2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|(\|p(t^k)-p(t^{k+1})\| + \|\mathbf{S}_{\mathbf{u}}^{k+1}\| + \|(\mathbf{u}(t^{k+1})\cdot\nabla)\mathbf{u}(t^{k+1}) - (\mathbf{u}(t^k)\cdot\nabla)\mathbf{u}(t^k)\|)\\
\le&\frac{(1-\alpha)\nu}{6}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + C(\Delta t)^2\left(\int_{t^n}^{t^{k+1}}\|p_t\|^2dt + \int_{t^n}^{t^{k+1}}\|\mathbf{u}_{tt}\|^2dt\right)\\
&+ C(\Delta t)^2\left(\int_{t^n}^{t^{k+1}}\|\nabla\mathbf{u}_t\|^2dt\|\mathbf{u}(t^{k+1})\|_{H^2}^2 + \|\mathbf{u}(t^k)\|_{H^1}^2\int_{t^n}^{t^{k+1}}\|\mathbf{u}_t\|_{H^2}^2dt\right).
\end{aligned} \tag{4.36}$$

Finally, combining (4.29) with (4.30)-(4.36), we obtain

$$\begin{aligned}
&(\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2 - \|\nabla\tilde{e}_{\mathbf{u}}^k\|^2 + \|\nabla\tilde{e}_{\mathbf{u}}^{k+1} - \nabla\tilde{e}_{\mathbf{u}}^k\|^2) + (1-\frac{1-\alpha}{3})\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2\\
\le&(\alpha+\frac{1-\alpha}{6})\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^k\|^2 + \nu C_\alpha\Delta t\|\nabla\tilde{e}_{\mathbf{u}}^k\|^2 + C\|\nabla\tilde{e}_{\mathbf{u}}^k\|^2\\
&+ C(\|\Delta\tilde{\mathbf{u}}^k\|^2 + \|\nabla\tilde{\mathbf{u}}^k\|^2)C_0^4(\Delta t)^5 + C(\Delta t)^2\left(\int_{t^n}^{t^{k+1}}\|p_t\|^2dt + \int_{t^n}^{t^{k+1}}\|\mathbf{u}_{tt}\|^2dt\right)\\
&+ C(\Delta t)^2\left(\int_{t^n}^{t^{k+1}}\|\nabla\mathbf{u}_t\|^2dt\|\mathbf{u}(t^{k+1})\|_{H^2}^2 + \|\mathbf{u}(t^k)\|_{H^1}^2\int_{t^n}^{t^{k+1}}\|\mathbf{u}_t\|_{H^2}^2dt\right).
\end{aligned} \tag{4.37}$$

Summing up (4.37) over k from 0 to n, using (4.19), (4.23) and Lemma 2.1, we can arrive at

$$\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+\Delta t\sum_{k=0}^{n}\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2\le\begin{cases}C_1\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, & d=2,\ \forall n\le T/\Delta t,\\ C_1\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, & d=3,\ \forall n\le T_*/\Delta t,\end{cases}\tag{4.38}$$

where C_1 is independent of C_0 and Δt.

Next we estimate $\|\tilde{e}_{\mathbf{u}}^{n+1}\|$. Taking the inner product of (4.25) with $2\Delta t\tilde{e}_{\mathbf{u}}^{k+1}$ and using the similar procedure as above, we can easily obtain that

$$\begin{aligned}\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2+\Delta t\sum_{k=0}^{n}\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2\le&C_2\Delta t\sum_{k=0}^{n}\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2+C\Delta t\sum_{k=0}^{n}\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2\\&+C(\|\Delta\tilde{\mathbf{u}}^k\|^2+\|\nabla\tilde{\mathbf{u}}^k\|^2)C_0^4(\Delta t)^4+C(\Delta t)^2,\end{aligned}\tag{4.39}$$

where C_2 is independent of C_0 and Δt. Choosing $\Delta t\le\frac{1}{2C_2}$, using (4.38) and discrete Gronwall inequality, we have

$$\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2+\Delta t\sum_{k=0}^{n}\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2\le\begin{cases}C_3\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, & d=2,\ \forall n\le T/\Delta t,\\ C_3\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, & d=3,\ \forall n\le T_*/\Delta t,\end{cases}\tag{4.40}$$

where C_3 is independent of C_0 and Δt.

Step 3: Estimates for $|1-\xi^{n+1}|$. We shall first start by establishing an error equation corresponding to (3.4). Let $\mathbf{S}_R^{k+1}$ be the truncation error defined by

$$\mathbf{S}_R^{k+1}=\frac{\partial R(t^{k+1})}{\partial t}-\frac{R(t^{k+1})-R(t^k)}{\Delta t}=\frac{1}{\Delta t}\int_{t^k}^{t^{k+1}}(t^k-t)\frac{\partial^2 R}{\partial t^2}dt.\tag{4.41}$$

Subtracting (3.2b) at t^{k+1} from (3.4), we obtain

$$\begin{aligned}\frac{e_R^{k+1}-e_R^k}{2\Delta t}=&\frac{R^{k+1}}{E(\tilde{\mathbf{u}}^{k+1})+K_0}\left(-\nu\|\nabla\tilde{\mathbf{u}}^{k+1}\|^2+(\mathbf{f}^{k+1},\tilde{\mathbf{u}}^{k+1})\right)\\&-\frac{R(t^{k+1})}{E(\mathbf{u}(t^{k+1}))+K_0}\left(-\nu\|\nabla\mathbf{u}(t^{k+1})\|^2+(\mathbf{f}^{k+1},\mathbf{u}(t^{k+1}))\right)+\mathbf{S}_R^{k+1}.\end{aligned}\tag{4.42}$$

Using (3.7), the first two terms on the right hand side of (4.42) can be estimated by

$$\begin{aligned}&\frac{R^{k+1}}{E(\tilde{\mathbf{u}}^{k+1})+K_0}\left(-\nu\|\nabla\tilde{\mathbf{u}}^{k+1}\|^2+(\mathbf{f}^{k+1},\tilde{\mathbf{u}}^{k+1})\right)\\&-\frac{R(t^{k+1})}{E(\mathbf{u}(t^{k+1}))+K_0}\left(-\nu\|\nabla\mathbf{u}(t^{k+1})\|^2+(\mathbf{f}^{k+1},\mathbf{u}(t^{k+1}))\right)\\=&\frac{R^{k+1}}{E(\tilde{\mathbf{u}}^{k+1})+K_0}\left(\nu\|\nabla\mathbf{u}(t^{k+1})\|^2-\nu\|\nabla\tilde{\mathbf{u}}^{k+1}\|^2+(\mathbf{f}^{k+1},\tilde{e}_{\mathbf{u}}^{k+1})\right)\\&+\left(\frac{R^{k+1}}{E(\tilde{\mathbf{u}}^{k+1})+K_0}-\frac{R(t^{k+1})}{E(\mathbf{u}(t^{k+1}))+K_0}\right)\left(-\nu\|\nabla\mathbf{u}(t^{k+1})\|^2+(\mathbf{f}^{k+1},\mathbf{u}(t^{k+1}))\right)\\\le&C\|\nabla\tilde{\mathbf{u}}^{k+1}\|\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|+C\|\tilde{e}_{\mathbf{u}}^{k+1}\|+C|E(\mathbf{u}(t^{k+1}))-E(\tilde{\mathbf{u}}^{k+1})|+C|e_R^{k+1}|\\\le&C\|\nabla\tilde{\mathbf{u}}^{k+1}\|\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|+C\|\tilde{e}_{\mathbf{u}}^{k+1}\|+C\|\tilde{\mathbf{u}}^{k+1}\|\|\tilde{e}_{\mathbf{u}}^{k+1}\|+C|e_R^{k+1}|.\end{aligned}\tag{4.43}$$

Then taking the inner product of (4.42) with $2\Delta te_R^{k+1}$ leads to

$$\begin{aligned}&(|e_R^{k+1}|^2-|e_R^k|^2+|e_R^{k+1}-e_R^k|^2)\\\le&C\Delta t|e_R^{k+1}|^2+C\Delta t\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2+C\Delta t\|\tilde{e}_{\mathbf{u}}^{k+1}\|^2\end{aligned}\tag{4.44}$$

Summing up (4.44) over k from 0 to n, and using (4.38) and (4.40) lead to

$$
\begin{aligned}
|e_R^{n+1}|^2 \leq & C_4\Delta t\sum_{k=0}^{n}|e_R^{k+1}|^2 + C\Delta t\sum_{k=0}^{n}\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + C\Delta t\sum_{k=0}^{n}\|\tilde{e}_{\mathbf{u}}^{k+1}\|^2 \\
\leq & \begin{cases} C_4\Delta t\sum\limits_{k=0}^{n}|e_R^{k+1}|^2 + C\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, & d=2,\ \forall n\leq T/\Delta t, \\ C_4\Delta t\sum\limits_{k=0}^{n}|e_R^{k+1}|^2 + C\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, & d=3,\ \forall n\leq T_*/\Delta t, \end{cases}
\end{aligned}
$$

where C_4 and C are independent of C_0 and Δt. Thus choosing $\Delta t \leq \frac{1}{2C_4}$ and using discrete Gronwall inequality, we have

$$
|e_R^{n+1}|^2 \leq \begin{cases} C_5\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, & d=2,\ \forall n\leq T/\Delta t, \\ C_5\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, & d=3,\ \forall n\leq T_*/\Delta t, \end{cases} \tag{4.45}
$$

where C_5 is independent of C_0 and Δt.

Next we finish the induction process as follows. Recalling (3.5), we have

$$
\begin{aligned}
|1-\xi^{n+1}| = & \left|\frac{R(t^{k+1})}{E(\mathbf{u}(t^{k+1}))+K_0} - \frac{R^{n+1}}{E(\tilde{\mathbf{u}}^{n+1})+K_0}\right| \\
\leq & C(|e_R^{n+1}| + \|\tilde{e}_{\mathbf{u}}^{n+1}\|) \\
\leq & \begin{cases} C_6\Delta t\sqrt{1+C_0^4(\Delta t)^2}, & d=2,\ \forall n\leq T/\Delta t, \\ C_6\Delta t\sqrt{1+C_0^4(\Delta t)^2}, & d=3,\ \forall n\leq T_*/\Delta t, \end{cases}
\end{aligned} \tag{4.46}
$$

where C_6 is independent of C_0 and Δt.

Let $C_0 = \max\{2C_6, \sqrt{2C_4}, \sqrt{2C_2}, 4\}$ and $\Delta t \leq \frac{1}{1+C_0^2}$, we can obtain

$$
C_6\sqrt{1+C_0^4(\Delta t)^2} \leq C_6(1+C_0^2\Delta t) \leq C_0. \tag{4.47}
$$

Then combining (4.46) with (4.47) results in

$$
|1-\xi^{n+1}| \leq \begin{cases} C_0\Delta t, & d=2,\ \forall n\leq T/\Delta t, \\ C_0\Delta t, & d=3,\ \forall n\leq T_*/\Delta t, \end{cases} \tag{4.48}
$$

which completes the induction process (4.9).

Now combining (4.38) with (4.40), we have

$$
\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \Delta t\sum_{k=0}^{n}\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 \leq \begin{cases} C(\Delta t)^2, & d=2,\ \forall n\leq T/\Delta t, \\ C(\Delta t)^2, & d=3,\ \forall n\leq T_*/\Delta t. \end{cases} \tag{4.49}
$$

Noting (3.5) and (4.48), and using the stability results (4.23) and (4.20), we have

$$
\begin{aligned}
& \|e_{\mathbf{u}}^{n+1}\|^2 + \|\nabla e_{\mathbf{u}}^{n+1}\|^2 + \Delta t\sum_{k=0}^{n}\|\Delta e_{\mathbf{u}}^{k+1}\|^2 \\
\leq & 2(\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + |\xi^{n+1}-1|^4\|\tilde{\mathbf{u}}^{n+1}\|^2) + 2(\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + |\xi^{n+1}-1|^4\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2) \\
& + 2\Delta t\sum_{k=0}^{n}(\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + |\xi^{n+1}-1|^4\|\Delta\tilde{\mathbf{u}}^{n+1}\|^2) \\
\leq & 2(\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + |\xi^{n+1}-1|^4\|\tilde{\mathbf{u}}^{n+1}\|^2) + 2(\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + |\xi^{n+1}-1|^4\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2) \\
& + 2\Delta t\sum_{k=0}^{n}(\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + |\xi^{n+1}-1|^4\|\Delta\tilde{\mathbf{u}}^{n+1}\|^2) \\
\leq & \begin{cases} C(\Delta t)^2, & d=2,\ \forall n\leq T/\Delta t, \\ C(\Delta t)^2, & d=3,\ \forall n\leq T_*/\Delta t, \end{cases}
\end{aligned} \tag{4.50}
$$

which leads to the desired results (4.7).

Table 5.1
Errors and convergence rates for Example 1 with $\nu = 1$.

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|\nabla e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^\infty}$	Rate	$\|\nabla e_p\|_{l^2}$	Rate
1/10	8.45E-3	–	4.30E-2	–	5.27E-2	–	2.85E-1	–
1/20	4.43E-3	0.93	2.28E-2	0.92	2.87E-2	0.88	1.92E-1	0.57
1/40	2.24E-3	0.98	1.15E-2	0.98	1.46E-2	0.97	1.12E-1	0.78
1/80	1.12E-3	1.00	5.78E-3	1.00	7.32E-3	1.00	5.98E-2	0.90

It remains to estimate the pressure error. Recalling (4.28), we can transform (4.27) into the following:

$$
\begin{aligned}
\Delta t\sum_{k=0}^{n}\|\nabla e_p^{k+1}\|^2 \leq & C\Delta t\sum_{k=0}^{n}\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + C\Delta t\sum_{k=0}^{n}\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2 \\
& + C\Delta t\sum_{k=0}^{n}\|\nabla e_{\mathbf{u}}^{k+1}\|^2\|\nabla\mathbf{u}(t^{k+1})\|\|\nabla\mathbf{u}(t^{k+1})\|_{H^1} \\
& + C\Delta t\sum_{k=0}^{n}\Delta t\|\nabla\mathbf{u}^{k+1}\|^2\|\nabla e_{\mathbf{u}}^{k+1}\|\|\nabla e_{\mathbf{u}}^{k+1}\|_{H^1} \\
\leq & \begin{cases} C(\Delta t)^2, & d=2,\ \forall n\leq T/\Delta t, \\ C(\Delta t)^2, & d=3,\ \forall n\leq T_*/\Delta t. \end{cases}
\end{aligned}
$$

Taking $q = \Delta^{-1}e_p^{k+1}$ in (4.26) and using (2.3) and (4.49), we can obtain

$$
\begin{aligned}
\|e_p^{k+1}\|^2 = & \Big((\mathbf{u}(t^{k+1})\cdot\nabla)\mathbf{u}(t^{k+1}) - (\mathbf{u}^{k+1}\cdot\nabla)\mathbf{u}^{k+1}, \Delta^{-1/2}e_p^{k+1}\Big) \\
& - (\nu\nabla\times\nabla\times\tilde{e}_{\mathbf{u}}^{k+1}, \Delta^{-1/2}e_p^{k+1}) \\
\leq & C\|e_{\mathbf{u}}^{k+1}\|_1^2\|\nabla\mathbf{u}(t^{k+1})\|_1^2 + C\|\mathbf{u}^{k+1}\|_1^2\|e_{\mathbf{u}}^{k+1}\|_1^2 + C\|\nabla e_{\mathbf{u}}^{k+1}\|^2 \\
& + \frac{1}{2}\|e_p^{k+1}\|^2 + \nu C\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2 + \nu C\|\tilde{e}_{\mathbf{u}}^{k+1}\|^2 \\
\leq & \begin{cases} C(\Delta t)^2, & d=2,\ \forall n\leq T/\Delta t, \\ C(\Delta t)^2, & d=3,\ \forall n\leq T_*/\Delta t, \end{cases}
\end{aligned}
$$

which leads to the desired results (4.8). □

5. Numerical experiments and concluding remarks

We present in this section some numerical experiments followed by some concluding remarks.

5.1. Numerical results

We first present some numerical tests to verify the accuracy of the first-order GSAV scheme with consistent splitting method (3.3)-(3.6) for the Navier-Stokes equations. In all examples below, we take $\Omega = (0,1)\times(0,1)$. We set $T = 1$, $K_0 = 1$ and the spatial discretization is based on the MAC scheme on the staggered grid with $N_x = N_y = 250$ so that the spatial discretization error is negligible compared to the time discretization error for the time steps used in the experiments.

Example 1. The right hand side of the equations are computed according to the analytic solution given by:

$$
\begin{cases} p(x,y,t) = t(x^3 - 0.25), \\ u_1(x,y,t) = -tx^2(x-1)^2y(y-1)(2y-1), \\ u_2(x,y,t) = ty^2(y-1)^2x(x-1)(2x-1). \end{cases}
$$

Example 2. The right hand side of the equations are computed according to the analytic solution given by:

$$
\begin{cases} p(x,y,t) = \sin(t)(\sin(\pi y) - 2/\pi), \\ u_1(x,y,t) = \sin(t)\sin^2(\pi x)\sin(2\pi y), \\ u_2(x,y,t) = -\sin(t)\sin(2\pi x)\sin^2(\pi y). \end{cases}
$$

Table 5.2
Errors and convergence rates for Example 1 with $\nu = 0.1$.

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|\nabla e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^\infty}$	Rate	$\|\nabla e_p\|_{l^2}$	Rate
1/10	5.10E-2	–	2.74E-1	–	3.36E-2	–	1.86E-1	–
1/20	2.71E-2	0.91	1.47E-1	0.90	1.85E-2	0.86	1.19E-1	0.64
1/40	1.40E-2	0.95	7.62E-2	0.95	9.70E-3	0.93	6.88E-2	0.80
1/80	7.10E-3	0.98	3.88E-2	0.97	4.95E-3	0.97	3.69E-2	0.90

Table 5.3
Errors and convergence rates for Example 1 with $\nu = 0.01$.

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|\nabla e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^\infty}$	Rate	$\|\nabla e_p\|_{l^2}$	Rate
1/10	1.00E-1	–	7.66E-1	–	1.45E-2	–	8.36E-2	–
1/20	5.11E-2	0.97	3.97E-1	0.95	6.41E-3	1.18	4.76E-2	0.81
1/40	2.58E-2	0.98	2.03E-1	0.97	3.01E-3	1.09	2.63E-2	0.86
1/80	1.30E-2	0.99	1.02E-1	0.99	1.46E-3	1.04	1.39E-2	0.92

Table 5.4
Errors and convergence rates for Example 2 with $\nu = 1$.

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|\nabla e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^\infty}$	Rate	$\|\nabla e_p\|_{l^2}$	Rate
1/10	6.59E-3	–	5.04E-2	–	4.17E-2	–	4.06E-1	–
1/20	3.24E-3	1.02	2.50E-2	1.01	2.15E-2	0.96	2.62E-1	0.63
1/40	1.59E-3	1.03	1.22E-2	1.03	1.04E-2	1.04	1.48E-1	0.82
1/80	7.86E-4	1.02	6.02E-3	1.02	5.05E-3	1.04	7.79E-2	0.93

Table 5.5
Errors and convergence rates for Example 2 with $\nu = 0.1$.

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|\nabla e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^\infty}$	Rate	$\|\nabla e_p\|_{l^2}$	Rate
1/10	6.48E-2	–	4.59E-1	–	4.10E-2	–	3.45E-1	–
1/20	3.34E-2	0.95	2.58E-1	0.94	2.26E-2	0.86	2.05E-1	0.66
1/40	1.69E-2	0.98	1.32E-1	0.97	1.17E-2	0.95	1.17E-1	0.81
1/80	8.52E-3	0.99	6.62E-2	0.99	5.90E-3	0.98	6.21E-2	0.91

Table 5.6
Errors and convergence rates for Example 2 with $\nu = 0.01$.

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|\nabla e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|e_p\|_{l^\infty}$	Rate	$\|\nabla e_p\|_{l^2}$	Rate
1/10	2.54E-1	–	2.50E-0	–	1.36E-1	–	5.96E-1	–
1/20	1.37E-1	0.89	1.36E-0	0.88	7.57E-2	0.85	3.01E-1	0.98
1/40	7.07E-2	0.95	7.00E-1	0.96	3.89E-2	0.96	1.48E-1	1.02
1/80	3.59E-2	0.98	3.54E-1	0.98	1.96E-2	0.99	7.32E-2	1.02

We demonstrate numerical results for Examples 1 and 2 with different viscosity coefficients $\nu = 1, 0.1, 0.01$ in Tables 5.1-5.6. It can be easily observed that the numerical results for the velocity and pressure in different norms are all consistent with the error estimates in Theorem 4.2.

5.2. Concluding remarks

We carried out a rigorous error analysis of the first-order semi-discrete (in time) consistent splitting GSAV scheme for the Navier-Stokes equations with no-slip boundary conditions. The scheme is linear, unconditionally stable, and only requires solving a sequence of Poisson type equations at each time step. Thanks to its unconditional stability, we were able to derive optimal global (resp. local) in time error estimates in the two (resp. three) dimensional case for the velocity and pressure approximations. To the best of our knowledge, this is the first global in time error estimate for a consistent splitting scheme for the Navier-Stokes equations with no-slip boundary conditions.

Although we only considered semi-discrete (in time) case in this paper, the analysis can be extended, albeit tedious, to fully discrete approximations with C^1 subspaces for the velocity and C^0 subspaces for the pressure similarly as in [21]. The consistent splitting GSAV scheme can also be easily extended to higher-order [35]. However, it is a non-trivial matter to extend the current error analysis to high-order, which will be a subject of future study.

References

[1] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, vol. 15, Springer Science & Business Media, 2012.

[2] A.J. Chorin, Numerical solution of the Navier-Stokes equations, Math. Comput. 22 (1968) 745–762.
[3] W. E, J.-G. Liu, Gauge method for viscous incompressible flows, Commun. Math. Sci. 1 (2003) 317–332.
[4] H.C. Elman, D.J. Silvester, A.J. Wathen, Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics, Oxford University Press, USA, 2014.
[5] V. Girault, P.-A. Raviart, Finite Element Approximation of the Navier-Stokes Equations, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1979, p. 749.
[6] R. Glowinski, Finite element methods for incompressible viscous flow, Handb. Numer. Anal. 9 (2003) 3–1176.
[7] J. Guermond, J. Shen, A new class of truly consistent splitting schemes for incompressible flows, J. Comput. Phys. 192 (2003) 262–276.
[8] J. Guermond, J. Shen, On the error estimates for the rotational pressure-correction projection methods, Math. Comput. 73 (2004) 1719–1737.
[9] J.L. Guermond, P. Minev, J. Shen, An overview of projection methods for incompressible flows, Comput. Methods Appl. Mech. Eng. 195 (2006) 6011–6045.
[10] J.-L. Guermond, J. Shen, Velocity-correction projection methods for incompressible flows, SIAM J. Numer. Anal. 41 (2003) 112–134.
[11] M.D. Gunzburger, Finite Element Methods for Viscous Incompressible Flows: a Guide to Theory, Practice, and Algorithms, Elsevier, 2012.
[12] Y. He, W. Sun, Stability and convergence of the Crank-Nicolson/Adams-Bashforth scheme for the time-dependent Navier-Stokes equations, SIAM J. Numer. Anal. 45 (2007) 837–869.
[13] F. Huang, J. Shen, Stability and error analysis of a class of high-order IMEX schemes for Navier–Stokes equations with periodic boundary conditions, SIAM J. Numer. Anal. 59 (2021) 2926–2954.
[14] F. Huang, J. Shen, A new class of implicit-explicit BDF*k* SAV schemes for general dissipative systems and their error analysis, Comput. Methods Appl. Mech. Eng. 392 (2022) 114718.
[15] H. Johnston, J.-G. Liu, Accurate, stable and efficient Navier–Stokes solvers based on explicit treatment of the pressure term, J. Comput. Phys. 199 (2004) 221–259.
[16] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon & Breach, 1969.
[17] X. Li, J. Shen, Error analysis of the SAV-MAC scheme for the Navier-Stokes equations, SIAM J. Numer. Anal. 58 (2020) 2465–2491.
[18] X. Li, J. Shen, Z. Liu, New SAV-pressure correction methods for the Navier-Stokes equations: stability and error analysis, Math. Comput. 91 (2022) 141–167.
[19] L. Lin, Z. Yang, S. Dong, Numerical approximation of incompressible Navier-Stokes equations based on an auxiliary energy variable, J. Comput. Phys. 388 (2019) 1–22.
[20] J.-G. Liu, J. Liu, R.L. Pego, Stability and convergence of efficient Navier-Stokes solvers via a commutator estimate, Commun. Pure Appl. Math. 60 (2007) 1443–1487, A Journal Issued by the Courant Institute of Mathematical Sciences.
[21] J.-G. Liu, J. Liu, R.L. Pego, Error estimates for finite-element Navier-Stokes solvers without standard inf-sup conditions, Chin. Ann. Math., Ser. B 30 (2009) 743–768.
[22] J.-G. Liu, R. Pego, Stable discretization of magnetohydrodynamics in bounded domains, Commun. Math. Sci. 8 (2010) 235–251.
[23] R. Nochetto, J.-H. Pyo, Error estimates for semi-discrete gauge methods for the Navier-Stokes equations, Math. Comput. 74 (2005) 521–542.
[24] D. Serson, J. Meneghini, S.J. Sherwin, Velocity-correction schemes for the incompressible Navier–Stokes equations in general coordinate systems, J. Comput. Phys. 316 (2016) 243–254.
[25] J. Shen, Long time stability and convergence for fully discrete nonlinear Galerkin methods, Appl. Anal. 38 (1990) 201–229.
[26] J. Shen, On error estimates of projection methods for Navier-Stokes equations: first-order schemes, SIAM J. Numer. Anal. 29 (1992) 57–77.
[27] J. Shen, Modeling and numerical approximation of two-phase incompressible flows by a phase-field approach, in: Multiscale Modeling and Analysis for Materials Simulation, 2012, pp. 147–195.
[28] J. Shen, J. Xu, J. Yang, A new class of efficient and robust energy stable schemes for gradient flows, SIAM Rev. 61 (2019) 474–506.
[29] J. Shen, X. Yang, Error estimates for finite element approximations of consistent splitting schemes for incompressible flows, Discrete Contin. Dyn. Syst., Ser. B 8 (2007) 663–675.
[30] R. Temam, Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires (ii), Arch. Ration. Mech. Anal. 33 (1969) 377–385.
[31] R. Témam, Nonlinear Functional Analysis and Navier-Stokes Equations, SIAM, Philadelphia, 1983.
[32] R. Temam, Navier-Stokes Equations and Nonlinear Functional Analysis, second ed., CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 66, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.
[33] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, vol. 343, American Mathematical Soc., 2001.
[34] E. Weinan, J.-G. Liu, Projection method I: convergence and numerical boundary layers, SIAM J. Numer. Anal. (1995) 1017–1057.
[35] K. Wu, F. Huang, J. Shen, A new class of higher-order decoupled schemes for the incompressible Navier-Stokes equations and applications to rotating dynamics, J. Comput. Phys. 458 (2022) 111097.