Journal of Computational and Applied Mathematics 421 (2023) 114892

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Error analysis of a fully discrete consistent splitting MAC scheme for time dependent Stokes equations[☆]

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ARTICLE INFO

*Article history:*
Received 24 April 2022
Received in revised form 16 September 2022

*MSC:*
76D07
65M06
65M12
65M15

*Keywords:*
Stokes equations
Consistent splitting
Finite-difference
Staggered grids
Error estimates

ABSTRACT

We construct and analyze a numerical scheme based on the truly consistent splitting approach in time and the MAC discretization in space for the time dependent Stokes equations. The scheme only requires solving several Poisson type equations for the velocity and pressure at each time step. We establish the equivalence between two different formulations of the fully discrete consistent splitting schemes, prove unconditional stability, and establish first-order in time and second-order in space error estimates for velocity and pressure in different discrete norms. We also provide numerical experiments to verify our theoretical results.



## 1. Introduction

How to efficiently and accurately solve the incompressible Navier–Stokes equations have attracted continued attention (e.g., [1–6] and the references therein). The two main difficulties associated with the incompressible Navier–Stokes equations are (i) the nonlinearity and (ii) the coupling of velocity and pressure by the incompressibility constraint. In this paper, we shall concentrate on the latter and consider the time-dependent incompressible Stokes problem (Extension to the incompressible Navier–Stokes equations will be discussed at the end of the paper):

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad in\ \Omega \times J, \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad in\ \Omega \times J, \tag{1.1b}$$

$$\mathbf{u} = \mathbf{0} \quad on\ \partial\Omega \times J, \tag{1.1c}$$

with the initial condition $\mathbf{u}|_{t=0} = \mathbf{u}^0$ in $\Omega$, where $\nu > 0$ is the viscosity, $\Omega$ is a bounded domain and $J = (0, T]$.

Numerical methods to deal with the incompressibility constraint can be classified into two categories: coupled approach with a mixed formulation and decoupled approach through a projection type method. At each time step,

☆ The work of X. Li is supported by the National Natural Science Foundation of China under Grant Number 12271302, 11971407. The work of J. Shen is supported in part by National Science Foundation grants DMS-1720442 and AFOSR, USA grant FA9550-16-1-0102.

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https://doi.org/10.1016/j.cam.2022.114892
0377-0427/

the coupled approach needs to solve a saddle point problem, where the computational cost can be expensive although many efficient solution techniques are now available [1,7–10]. Since Chorin [11] and Temam [12] proposed the original projection method in the later sixties, an enormous amount of work has been devoted to the decouple approach which only requires solving a sequence of Poisson type equations at each time step. These schemes generally fall into three categories [13]: the pressure-correction method [14–19], the velocity correction method [20–23] and the consistent splitting method [24–26] (see also the gauge method [27,28]).

The basic idea of the consistent splitting scheme is to first compute the velocity with an explicit pressure treatment, and then update the pressure by a consistent pressure Poisson equation. Compared with other projection-type algorithms, the main advantage of the consistent splitting scheme is that it is free of the operator splitting error so it can achieve the full accuracy of the time discretization. At the space continuous level, there are two equivalent forms of the consistent splitting [24]. The first version of the first-order consistent splitting scheme for (1.1) is

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}-\nu\Delta\mathbf{u}^{n+1}+\nabla p^{n}=\mathbf{f}^{n+1},\ \ \mathbf{u}^{n+1}|_{\partial\Omega}=0, \tag{1.2a}$$

$$(\nabla p^{n+1},\nabla q)=(\mathbf{f}^{n+1}-\nu\nabla\times\nabla\times\mathbf{u}^{n+1},\nabla q),\ \ \forall q\in H^{1}(\Omega); \tag{1.2b}$$

while the second version is

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}-\nu\Delta\mathbf{u}^{n+1}+\nabla p^{n}=\mathbf{f}^{n+1},\ \ \mathbf{u}^{n+1}|_{\partial\Omega}=0, \tag{1.3a}$$

$$(\nabla\Psi^{n+1},\nabla q)=(\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t},\nabla q),\ \ \forall q\in H^{1}(\Omega), \tag{1.3b}$$

$$p^{n+1}=\Psi^{n+1}+p^{n}-\nu\nabla\cdot\mathbf{u}^{n+1}. \tag{1.3c}$$

However, the fully discrete versions of the above schemes are usually not equivalent, and their stability and error analysis require very different approaches.

While the consistent splitting schemes have been frequently used in practice as evidenced by the large number of papers citing [24,25], only very limited works on their stability and error analysis are available. Guermond & Shen established stability for the semi-discrete first-order consistent splitting scheme in [24]; Johnston and Liu [25] carried out a normal mode analysis for a second-order consistent splitting scheme in the periodic channel case; Shen and Yang analyzed a $C^0$ finite element approximation for the second version of the consistent splitting scheme [26]; a series of work were carried out by Liu, Liu and Pego [29,30], including an error analysis for a $C^1$ finite element approximation for the first version of the consistent splitting scheme. In particular, there is no analysis available for any $C^0$ finite element approximation for the first version of the consistent splitting scheme.

The main purpose in this paper is to construct a fully discrete consistent splitting scheme for the Stokes equations with the marker and cell (MAC) method [31,32] for the spatial discretization. The MAC method has been extensively used in scientific and engineering applications due to its simplicity while satisfying the discrete incompressibility constraint, as well as locally conserving the mass, momentum and kinetic energy [33,34]. There are many works to study the error estimates and stability for the MAC scheme [7,35–37]. Most of the error analysis only achieve first-order accuracy for both the velocity and pressure, although Nicolaides [38] pointed out that numerical results suggest that the velocity is second order convergent. Inspired by the analysis techniques in [39,40] for Darcy–Forchheimer and Maxwell's equations, Rui and Li [7,41] established the discrete LBB condition and obtained the second-order accuracy for both the velocity and pressure in discrete $L^2$ norms for the Stokes and Brinkman equations respectively. With regards to the Navier–Stokes equations, they derived the second-order accuracy for the velocity and pressure by using the characteristics technique in [8]. Very recently, Li and Shen [42] constructed an efficient numerical scheme based on the scalar auxiliary variable (SAV) and MAC scheme for the Navier–Stokes equations.

This work is unique in the following aspects: (i) we construct a numerical scheme based on the truly consistent splitting approach in time and the MAC discretization in space for the Stokes equations, where one only needs to solve several Poisson type equations for the velocity and pressure; (ii) we obtain the equivalence of the two consistent splitting versions by constructing an interpolation operator and performing appropriate approximation for the discrete operators on the staggered grids; (iii) we derive optimal error estimates in space and time for both velocity and pressure.

The paper is organized as follows. In Section 2 we describe some notations and present some preliminary results on the MAC method. In Section 3 we present the fully discrete consistent splitting scheme based on the MAC discretization and establish the equivalence of the two discrete pressure Poisson equations. Stability of the constructed scheme is derived in Section 4. In Section 5 we carry out error estimates for the fully discrete scheme for the Stokes equation. In Section 6, we demonstrate some numerical experiments to verify the accuracy of the proposed numerical schemes.

## 2. Preliminaries

We describe in this section the various operators, discrete functional spaces and inf–sup conditions associated with the MAC discretization.

### 2.1. MAC discretization

Set

$$\Delta t = T/N,\ t^n = n\Delta t,\ \text{for } n \le N,\ [d_t f]^n = \frac{f^n - f^{n-1}}{\Delta t}.$$

To fix the idea, we consider $\Omega = (L_{lx}, L_{rx}) \times (L_{ly}, L_{ry})$. Three dimensional rectangular domains can be dealt with similarly. We partition $\Omega$ by $\Omega_x \times \Omega_y$, where

$$\Omega_x : L_{lx} = x_0 < x_1 < \cdots < x_{N_x-1} < x_{N_x} = L_{rx},$$
$$\Omega_y : L_{ly} = y_0 < y_1 < \cdots < y_{N_y-1} < y_{N_y} = L_{ry}.$$

For simplicity we also use the following notations:

$$\begin{cases} x_{-1/2} = x_0 = L_{lx}, \quad x_{N_x+1/2} = x_{N_x} = L_{rx}, \\ y_{-1/2} = y_0 = L_{ly}, \quad y_{N_y+1/2} = y_{N_y} = L_{ry}. \end{cases} \tag{2.1}$$

For $i, j$, $0 \le i \le N_x,\ 0 \le j \le N_y$, define

$$x_{i+1/2} = \frac{x_i + x_{i+1}}{2}, \quad h_{i+1/2} = x_{i+1} - x_i, \quad h = \max_i h_{i+1/2},$$
$$h_i = x_{i+1/2} - x_{i-1/2} = \frac{h_{i+1/2} + h_{i-1/2}}{2},$$
$$y_{j+1/2} = \frac{y_j + y_{j+1}}{2}, \quad k_{j+1/2} = y_{j+1} - y_j, \quad k = \max_j k_{j+1/2},$$
$$k_j = y_{j+1/2} - y_{j-1/2} = \frac{k_{j+1/2} + k_{j-1/2}}{2},$$
$$\Omega_{i+1/2,j+1/2} = (x_i, x_{i+1}) \times (y_j, y_{j+1}).$$

It is clear that

$$h_0 = \frac{h_{1/2}}{2},\ h_{N_x} = \frac{h_{N_x-1/2}}{2},\ \ k_0 = \frac{k_{1/2}}{2},\ k_{N_y} = \frac{k_{N_y-1/2}}{2}.$$

For a function $f(x, y)$, let $f_{l,m}$ denote $f(x_l, y_m)$ where $l$ may take values $i,\ i + 1/2$ for integer $i$, and $m$ may take values $j,\ j + 1/2$ for integer $j$. For discrete functions with values at proper nodal-points, define

$$\begin{cases} [d_x f]_{i+1/2,m} = \dfrac{f_{i+1,m} - f_{i,m}}{h_{i+1/2}}, \qquad [D_y f]_{l,j+1} = \dfrac{f_{l,j+3/2} - f_{l,j+1/2}}{k_{j+1}}, \\ [D_x f]_{i+1,m} = \dfrac{f_{i+3/2,m} - f_{i+1/2,m}}{h_{i+1}}, \quad [d_y f]_{l,j+1/2} = \dfrac{f_{l,j+1} - f_{l,j}}{k_{j+1/2}}. \end{cases} \tag{2.2}$$

For functions $f$ and $g$, we define some discrete $l^2$ inner products and norms as follows.

$$(f, g)_{l^2,M} \equiv \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} h_{i+1/2} k_{j+1/2} f_{i+1/2,j+1/2} g_{i+1/2,j+1/2}, \tag{2.3}$$

$$(f, g)_{l^2,T_x} \equiv \sum_{i=0}^{N_x} \sum_{j=1}^{N_y-1} h_i k_j f_{i,j} g_{i,j}, \tag{2.4}$$

$$(f, g)_{l^2,T_y} \equiv \sum_{i=1}^{N_x-1} \sum_{j=0}^{N_y} h_i k_j f_{i,j} g_{i,j}, \tag{2.5}$$

$$\|f\|^2_{l^2,\xi} \equiv (f, f)_{l^2,\xi}, \qquad \xi = M,\ T_x,\ T_y; \tag{2.6}$$

and

$$(f, g)_{l^2,T,M} \equiv \sum_{i=1}^{N_x-1} \sum_{j=0}^{N_y-1} h_i k_{j+1/2} f_{i,j+1/2} g_{i,j+1/2}, \tag{2.7}$$

$$(f, g)_{l^2,M,T} \equiv \sum_{i=0}^{N_x-1} \sum_{j=1}^{N_y-1} h_{i+1/2} k_j f_{i+1/2,j} g_{i+1/2,j}, \tag{2.8}$$

$$\|f\|^2_{l^2,T,M} \equiv (f, f)_{l^2,T,M}, \quad \|f\|^2_{l^2,M,T} \equiv (f, f)_{l^2,M,T}. \tag{2.9}$$

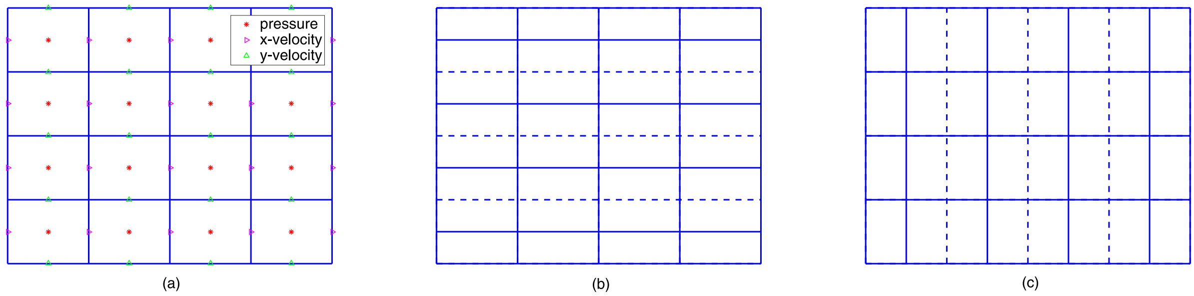


**Fig. 1.** Partitions: (a) $\mathcal{T}_h$, (b) $\mathcal{T}_h^1$, (c) $\mathcal{T}_h^2$.

For vector-valued functions $\mathbf{u} = (u_1, u_2)$, it is clear that

$$\|d_x u_1\|^2_{l^2,M} \equiv \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} h_{i+1/2} k_{j+1/2} |d_x u_{1,i+1/2,j+1/2}|^2, \tag{2.10}$$

$$\|D_y u_1\|^2_{l^2,T_y} \equiv \sum_{i=1}^{N_x-1} \sum_{j=0}^{N_y} h_i k_j |D_y u_{1,i,j}|^2, \tag{2.11}$$

and $\|d_y u_2\|_{l^2,M}$, $\|D_x u_2\|_{l^2,T_x}$ can be represented similarly. Finally we define the discrete $H^1$-norm and discrete $l^2$-norm of a vectored-valued function $\mathbf{u}$ as follows.

$$\|D\mathbf{u}\|^2 \equiv \|d_x u_1\|^2_{l^2,M} + \|D_y u_1\|^2_{l^2,T_y} + \|D_x u_2\|^2_{l^2,T_x} + \|d_y u_2\|^2_{l^2,M}. \tag{2.12}$$

$$\|\mathbf{u}\|^2_{l^2} \equiv \|u_1\|^2_{l^2,T,M} + \|u_2\|^2_{l^2,M,T}. \tag{2.13}$$

For simplicity we only consider the case that for all $h_{i+1/2} = h$, $k_{j+1/2} = k$, i.e. uniform meshes are used both in $x$ and $y$-directions.

### 2.2. The discrete inf–sup condition

In order to carry out stability and error analysis, we need the discrete inf–sup condition. Below, we use the same notation and results as Rui and Li [7, Lemma3.3]. Let

$$b(\mathbf{v}, q) = -\int_\Omega q \nabla \cdot \mathbf{v} dx, \ \ \mathbf{v} \in \mathbf{V}, \ \ q \in W,$$

where

$$\mathbf{V} = H_0^1(\Omega) \times H_0^1(\Omega), \quad W = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}.$$

Then we construct the finite-dimensional subspaces of $W$ and $\mathbf{V}$ by introducing three different partitions $\mathcal{T}_h, \mathcal{T}_h^1, \mathcal{T}_h^2$ of $\Omega$. The original partition $\delta_x \times \delta_y$ is denoted by $\mathcal{T}_h$ (see Fig. 1(a)). The partition $\mathcal{T}_h^1$ is generated by connecting all the midpoints of the vertical sides of $\Omega_{i+1/2,j+1/2}$ and extending the resulting mesh to the boundary $\Gamma$ (see Fig. 1(b)). Similarly, for all $\Omega_{i+1/2,j+1/2} \in \mathcal{T}_h$ we connect all the midpoints of the horizontal sides of $\Omega_{i+1/2,j+1/2}$ and extend the resulting mesh to the boundary $\Gamma$, then the third partition is obtained which is denoted by $\mathcal{T}_h^2$ (see Fig. 1(c)).

Let $Q_k$ denote the space of all polynomials of degree $\leq k$ with respect to each of the two variables $x$ and $y$. Corresponding to the quadrangulation $\mathcal{T}_h$, we define $W_h \subset W$ by

$$W_h = \left\{ q_h : \ q_h|_T \in Q_0, \ \forall T \in \mathcal{T}_h \ and \int_\Omega q dx = 0 \right\}.$$

We also define $\mathbf{V}_h = S_h^1 \times S_h^2 \subset \mathbf{V}$ with

$$S_h^l = \left\{ g \in C^{(0)}(\overline{\Omega}) : \ g|_{T^l} \in Q_1(T^l), \ \forall T^l \in \mathcal{T}_h^l, \ and \ g|_\Gamma = 0 \right\}, \ \ l = 1, 2.$$

Next we introduce the bilinear forms

$$b_h(\mathbf{v}_h, q_h) = -\sum_{\Omega_{i+1/2,j+1/2} \in \mathcal{T}_h} \int_{\Omega_{i+1/2,j+1/2}} q_h \Pi_h(\nabla \cdot \mathbf{v}_h) dx, \ \ \mathbf{v}_h \in \mathbf{V}_h, \ \ q_h \in W_h,$$

where $\Pi_h : C^{(0)}(\overline{\Omega}_{i+1/2,j+1/2}) \to Q_0(\Omega_{i+1/2,j+1/2})$ such that

$$(\Pi_h \varphi)_{i+1/2,j+1/2} = \varphi_{i+1/2,j+1/2}, \quad \forall\ \Omega_{i+1/2,j+1/2} \in \mathcal{T}_h. \tag{2.14}$$

Then, we have the following result [7]:

**Lemma 2.1.** *There exists $\beta > 0$, independent of $h$ and $k$, such that*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b_h(\mathbf{v}_h, q_h)}{\|D\mathbf{v}_h\|} \geq \beta \|q_h\|_{l^2,M} \quad \forall q_h \in W_h. \tag{2.15}$$

It is easy to verify that the following discrete integration-by-part formulae hold.

**Lemma 2.2** ([42–44]). *Let $\{V_{1,i,j+1/2}\}$, $\{V_{2,i+1/2,j}\}$ and $\{q_{1,i+1/2,j+1/2}\}$, $\{q_{2,i+1/2,j+1/2}\}$ be discrete functions with $V_{1,0,j+1/2} = V_{1,N_x,j+1/2} = V_{2,i+1/2,0} = V_{2,i+1/2,N_y} = 0$, with proper integers $i$ and $j$. Then there holds*

$$\begin{cases} (D_x q_1, V_1)_{l^2,T,M} &= -(q_1, d_x V_1)_{l^2,M}, \\ (D_y q_2, V_2)_{l^2,M,T} &= -(q_2, d_y V_2)_{l^2,M}. \end{cases} \tag{2.16}$$

### *2.3. Additional notations*

Let $L^m(\Omega)$ be the standard Banach space with norm

$$\|v\|_{L^m(\Omega)} = \left( \int_\Omega |v|^m d\Omega \right)^{1/m}.$$

For simplicity, let

$$(f, g) = (f, g)_{L^2(\Omega)} = \int_\Omega fg d\Omega$$

denote the $L^2(\Omega)$ inner product, $\|v\|_\infty = \|v\|_{L^\infty(\Omega)}$. And $W_p^k(\Omega)$ be the standard Sobolev space

$$W_p^k(\Omega) = \{g : \ \|g\|_{W_p^k(\Omega)} < \infty\},$$

where

$$\|g\|_{W_p^k(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha g\|^p_{L^p(\Omega)} \right)^{1/p}. \tag{2.17}$$

Throughout the paper we use $C$, with or without subscript, to denote a positive constant, independent of discretization parameters, which could have different values at different places.

## 3. The MAC discretizations of the consistent splitting scheme

Denote by $\{\mathbf{U}^k, P^k\}_{k=1}^N$, the approximations to $\{\mathbf{u}^k, p^k\}_{k=1}^N$ respectively. Starting from the initial conditions

$$\begin{cases} U^0_{1,i,j+1/2} = u^0_{1,i,j+1/2}, & 0 \leq i \leq N_x, 0 \leq j \leq N_y, \\ U^0_{2,i+1/2,j} = u^0_{2,i+1/2,j}, & 0 \leq i \leq N_x, 0 \leq j \leq N_y, \end{cases} \tag{3.1}$$

the fully discrete consistent splitting scheme based on (1.2) and the MAC discretization is as follows:

**Step 1.** Find $(U_1^{n+1}, U_2^{n+1}) \in \mathbf{V}_h$ such that

$$\begin{aligned} &[d_t U_1]^{n+1}_{i,j+1/2} - \nu D_x (d_x U_1)^{n+1}_{i,j+1/2} - \nu d_y (D_y U_1)^{n+1}_{i,j+1/2} + [D_x P]^n_{i,j+1/2} \\ &\qquad = [f_1]^{n+1}_{i,j+1/2}, \quad 1 \leq i \leq N_x - 1, \ 0 \leq j \leq N_y - 1, \end{aligned} \tag{3.2a}$$

$$\begin{aligned} &[d_t U_2]^{n+1}_{i+1/2,j} - \nu D_y (d_y U_2)^{n+1}_{i+1/2,j} - \nu d_x (D_x U_2)^{n+1}_{i+1/2,j} + [D_y P]^n_{i+1/2,j} \\ &\qquad = [f_2]^{n+1}_{i+1/2,j}, \quad 0 \leq i \leq N_x - 1, \ 1 \leq j \leq N_y - 1, \end{aligned} \tag{3.2b}$$

with the homogeneous Dirichlet boundary conditions for $\mathbf{U}^{n+1}$:

$$\begin{cases} U^{n+1}_{1,0,j+1/2} = U^{n+1}_{1,N_x,j+1/2} = 0, & 0 \leq j \leq N_y - 1, \\ U^{n+1}_{1,i,0} = U^{n+1}_{1,i,N_y} = 0, & 0 \leq i \leq N_x, \\ U^{n+1}_{2,0,j} = U^{n+1}_{2,N_x,j} = 0, & 0 \leq j \leq N_y, \\ U^{n+1}_{2,i+1/2,0} = U^{n+1}_{2,i+1/2,N_y} = 0, & 0 \leq i \leq N_x - 1. \end{cases} \tag{3.3}$$

**Step 2.** Find $P^{n+1} \in W_h$ such that

$$[D_xP]^{n+1}_{i,j+1/2} = \nu d_y(D_yU_1)^{n+1}_{i,j+1/2} - \nu d_y(D_xU_2)^{n+1}_{i,j+1/2} + [f_1]^{n+1}_{i,j+1/2}, \; 0 \le i \le N_x, \; 0 \le j \le N_y - 1, \tag{3.4a}$$

$$[D_yP]^{n+1}_{i+1/2,j} = \nu d_x(D_xU_2)^{n+1}_{i+1/2,j} - \nu d_x(D_yU_1)^{n+1}_{i+1/2,j} + [f_2]^{n+1}_{i+1/2,j}, \; 0 \le i \le N_x - 1, \; 0 \le j \le N_y. \tag{3.4b}$$

By applying the divergence operator to (3.4a) and (3.4b) and recalling (3.3), we can obtain the discrete Poisson equation for the pressure with the following nonhomogeneous Neumann boundary condition:

$$[D_xP]^{n+1}_{0,j+1/2} = [f_1]^{n+1}_{0,j+1/2} - \nu \frac{U^{n+1}_{2,1/2,j+1} - U^{n+1}_{2,1/2,j}}{h_0 k_{j+1/2}}, \tag{3.5a}$$

$$[D_xP]^{n+1}_{N_x,j+1/2} = [f_1]^{n+1}_{N_x,j+1/2} + \nu \frac{U^{n+1}_{2,N_x-1/2,j+1} - U^{n+1}_{2,N_x-1/2,j}}{h_{N_x} k_{j+1/2}}, \tag{3.5b}$$

$$[D_yP]^{n+1}_{i+1/2,0} = [f_2]^{n+1}_{i+1/2,0} - \nu \frac{U^{n+1}_{1,i+1,1/2} - U^{n+1}_{1,i,1/2}}{h_{i+1/2} k_0}, \tag{3.5c}$$

$$[D_yP]^{n+1}_{i+1/2,N_y} = [f_2]^{n+1}_{i+1/2,N_y} + \nu \frac{U^{n+1}_{1,i+1,N_y-1/2} - U^{n+1}_{1,i,N_y-1/2}}{h_{i+1/2} k_{N_y}}. \tag{3.5d}$$

It is difficult to conduct stability and error analysis for the above scheme due to the second-order derivatives in (3.4). Hence, we shall first derive an equivalent form of (3.4).

**Theorem 3.1.** *The system (3.4) is equivalent to*

$$d_x(D_x\Psi)^{n+1}_{i+1/2,j+1/2} + d_y(D_y\Psi)^{n+1}_{i+1/2,j+1/2} = d_x(d_tU_1)^{n+1}_{i+1/2,j+1/2} + d_y(d_tU_2)^{n+1}_{i+1/2,j+1/2}, \quad 0 \le i \le N_x - 1, \; 0 \le j \le N_y - 1, \tag{3.6}$$

*with the homogeneous Neumann boundary condition:*

$$[D_x\Psi]^{n+1}_{0,j+1/2} = 0, \; [D_x\Psi]^{n+1}_{N_x,j+1/2} = 0, \tag{3.7a}$$

$$[D_y\Psi]^{n+1}_{i+1/2,0} = 0, \; [D_y\Psi]^{n+1}_{i+1/2,N_y} = 0; \tag{3.7b}$$

*and*

$$P^{n+1}_{i+1/2,j+1/2} = \Psi^{n+1}_{i+1/2,j+1/2} + P^{n}_{i+1/2,j+1/2} - \nu \Pi_h[d_xU_1 + d_yU_2]^{n+1}_{i+1/2,j+1/2}, \quad 0 \le i \le N_x - 1, \; 0 \le j \le N_y - 1, \tag{3.8}$$

*where $\Pi_h$ is defined by (2.14).*

**Proof.** We first prove that the discrete form of the identity $-\nabla \times \nabla u = -\Delta u + \nabla\nabla \cdot u$ holds for the interior points. Since for $1 \le i \le N_x - 1, \; 0 \le j \le N_y - 1$, we have

$$\begin{aligned} d_y(D_xU_2)_{i,j+1/2} =& \frac{1}{k_{j+1/2}} \left(D_xU_{2,i,j+1} - D_xU_{2,i,j}\right) \\ =& \frac{1}{k_{j+1/2}} \left( \frac{U_{2,i+1/2,j+1} - U_{2,i-1/2,j+1}}{h_i} - \frac{U_{2,i+1/2,j} - U_{2,i-1/2,j}}{h_i} \right) \\ =& \frac{1}{h_i} \left( \frac{U_{2,i+1/2,j+1} - U_{2,i+1/2,j}}{k_{j+1/2}} - \frac{U_{2,i-1/2,j+1} - U_{2,i-1/2,j}}{k_{j+1/2}} \right) \\ =& D_x(d_yU_2)_{i,j+1/2}. \end{aligned} \tag{3.9}$$

Similarly for $0 \le i \le N_x - 1, \; 1 \le j \le N_y - 1$, we have

$$d_x(D_yU_1)_{i+1/2,j} = D_y(d_xU_1)_{i+1/2,j}. \tag{3.10}$$

Thus we can easily obtain that

$$\begin{cases} d_y(D_xU_2) - d_y(D_yU_1) = -D_x(d_xU_1) - d_y(D_yU_1) + D_x(d_xU_1) \\ +D_x(d_yU_2), \text{ in } (x_i, y_{j+1/2}), \quad 1 \le i \le N_x - 1, \; 0 \le j \le N_y - 1, \\ d_x(D_yU_1) - d_x(D_xU_2) = -D_y(d_yU_2) - d_x(D_xU_2) + D_y(d_yU_2) \\ +D_y(d_xU_1), \text{ in } (x_{i+1/2}, y_j), \; 0 \le i \le N_x - 1, \; 1 \le j \le N_y - 1, \end{cases} \tag{3.11}$$

which is the discrete form of the identity $-\nabla\times\nabla u = -\Delta u + \nabla\nabla\cdot u$ at the interior points. Furthermore, we observe that $\Pi_h(d_xU_1 + d_yU_2)$ is included in the pressure space $W_h$. Thus we can obtain that (3.4) is equivalent to

$$[D_x\Psi]^{n+1}_{i,j+1/2} = [d_tU_1]^{n+1}_{i,j+1/2}, \quad 1\le i\le N_x-1,\ 0\le j\le N_y-1, \tag{3.12a}$$

$$[D_y\Psi]^{n+1}_{i+1/2,j} = [d_tU_2]^{n+1}_{i+1/2,j}, \quad 0\le i\le N_x-1,\ 1\le j\le N_y-1. \tag{3.12b}$$

Applying the discrete divergence operator, we obtain the discrete pressure Poisson Eq. (3.6) at the interior points.

Next we shall prove that the boundary condition for (3.4) is equivalent to (3.7). Without loss of generality, one only need to prove that $D_x\Psi^{n+1}_{0,j+1/2} = 0$ for the case that $i = 0$ in (3.4a).

Since for $i = 0$ in (3.4a), we have

$$D_xP^{n+1}_{0,j+1/2} = \nu d_y(D_yU_1)^{n+1}_{0,j+1/2} - \nu d_y(D_xU_2)^{n+1}_{0,j+1/2} + [f_1]^{n+1}_{0,j+1/2}. \tag{3.13}$$

Recalling (3.8) and (3.2a), we have

$$\begin{aligned}
D_x\Psi^{n+1}_{0,j+1/2} =& D_xP^{n+1}_{0,j+1/2} - D_xP^{n}_{0,j+1/2} + \nu D_x\Pi_h(d_xU_1 + d_yU_2)^{n+1}_{0,j+1/2}\\
=&\nu d_y(D_yU_1)^{n+1}_{0,j+1/2} - \nu d_y(D_xU_2)^{n+1}_{0,j+1/2} + [f_1]^{n+1}_{0,j+1/2}\\
&- D_xP^{n}_{0,j+1/2} + \nu D_x\Pi_h(d_xU_1 + d_yU_2)^{n+1}_{0,j+1/2}\\
=&\nu d_y(D_yU_1)^{n+1}_{0,j+1/2} - \nu d_y(D_xU_2)^{n+1}_{0,j+1/2} + [d_tU_1]^{n+1}_{0,j+1/2}\\
&- \nu D_x(d_xU_1)^{n+1}_{0,j+1/2} - \nu d_y(D_yU_1)^{n+1}_{0,j+1/2}\\
&+ \nu D_x\Pi_h(d_xU_1 + d_yU_2)^{n+1}_{0,j+1/2}\\
=&[d_tU_1]^{n+1}_{0,j+1/2} = 0,
\end{aligned} \tag{3.14}$$

which implies that the system (3.4) is equivalent to (3.6)–(3.8). □

## 4. A stability result

**Theorem 4.1.** *The fully discrete scheme (3.2)–(3.3)* and *(3.6)–(3.8) is unconditionally stable in the sense that*

$$\begin{aligned}
&\Delta t\|d_t\mathbf{U}^{m+1}\|^2_{l^2} + 2\nu\|d_xU_1^{m+1} + d_yU_2^{m+1}\|^2_{l^2,M} + \sum_{n=0}^{m+1}\Delta t\|\nabla_h\Psi^n\|^2_{l^2}\\
&\quad\le C\|d_xU_1^0 + d_yU_2^0\|^2_{l^2,M} + C\Delta t\|d_t\mathbf{U}^0\|^2_{l^2} + C\sum_{n=0}^{m}(\Delta t)^2\|d_t\mathbf{f}^{n+1}\|^2_{l^2}.
\end{aligned}$$

**Proof.** Taking the time difference of two consecutive steps in (3.2a), we have

$$\begin{aligned}
&[d_{tt}U_1]^{n+1}_{i,j+1/2} - \nu D_x(d_xd_tU_1)^{n+1}_{i,j+1/2} - \nu d_y(D_yd_tU_1)^{n+1}_{i,j+1/2}\\
&\quad+ [D_xd_tP]^{n}_{i,j+1/2} = [d_tf_1]^{n+1}_{i,j+1/2}, \quad 1\le i\le N_x-1,\ 0\le j\le N_y-1,
\end{aligned} \tag{4.1}$$

where $d_{tt}U_1^{n+1} = \dfrac{d_tU_1^{n+1} - d_tU_1^n}{\Delta t}$.

Taking notice of (3.8), we can obtain for $1\le i\le N_x-1,\ 0\le j\le N_y-1$,

$$[D_xd_tP]^n_{i,j+1/2} = (\Delta t)^{-1}[D_x\Psi]^n_{i,j+1/2} - \nu(\Delta t)^{-1}D_x(\Pi_h[d_xU_1 + d_yU_2])^n_{i,j+1/2}. \tag{4.2}$$

Substituting (4.2) into (4.1) yields

$$\begin{aligned}
&[d_{tt}U_1]^{n+1}_{i,j+1/2} - \nu D_x(d_xd_tU_1)^{n+1}_{i,j+1/2} - \nu d_y(D_yd_tU_1)^{n+1}_{i,j+1/2}\\
&\quad- \nu(\Delta t)^{-1}D_x(\Pi_h[d_xU_1 + d_yU_2])^n_{i,j+1/2} + (\Delta t)^{-1}[D_x\Psi]^n_{i,j+1/2} = [d_tf_1]^{n+1}_{i,j+1/2}.
\end{aligned} \tag{4.3}$$

Multiplying (4.3) by $2\Delta t[d_tU_1]^{n+1}_{i,j+1/2}$, making summation on $i, j$ for $1\le i\le N_x-1,\ 0\le j\le N_y-1$, and taking notice of Lemma 2.2, we arrive at

$$\begin{aligned}
&\|d_tU_1^{n+1}\|^2_{l^2,T,M} - \|d_tU_1^n\|^2_{l^2,T,M} + (\Delta t)^2\|d_{tt}U_1^{n+1}\|^2_{l^2,T,M} + 2\nu\Delta t\|d_xd_tU_1^{n+1}\|^2_{l^2,M}\\
&+ 2\nu\Delta t\|D_yd_tU_1^{n+1}\|^2_{l^2,T_y} + 2\nu(\Pi_h[d_xU_1 + d_yU_2]^n, d_xd_tU_1^{n+1})_{l^2,M}\\
&+ 2(D_x\Psi^n, d_tU_1^{n+1})_{l^2,T,M} = 2\Delta t(d_tf_1^{n+1}, d_tU_1^{n+1})_{l^2,T,M}.
\end{aligned} \tag{4.4}$$

Similarly in the $y$ direction, we have

$$\begin{aligned}&\|d_tU_2^{n+1}\|^2_{l^2,M,T}-\|d_tU_2^n\|^2_{l^2,M,T}+(\Delta t)^2\|d_{tt}U_2^{n+1}\|^2_{l^2,M,T}+2\nu\Delta t\|d_yd_tU_2^{n+1}\|^2_{l^2,M}\\&+2\nu\Delta t\|D_xd_tU_2^{n+1}\|^2_{l^2,T_x}+2\nu(\Pi_h[d_xU_1+d_yU_2]^n,d_yd_tU_2^{n+1})_{l^2,M}\\&+2(D_y\Psi^n,d_tU_2^{n+1})_{l^2,M,T}=2\Delta t(d_tf_2^{n+1},d_tU_2^{n+1})_{l^2,M,T}.\end{aligned}\tag{4.5}$$

Combining (4.4) with (4.5) leads to

$$\begin{aligned}&\|d_t\mathbf{U}^{n+1}\|^2_{l^2}-\|d_t\mathbf{U}^n\|^2_{l^2}+(\Delta t)^2\|d_{tt}\mathbf{U}^{n+1}\|^2_{l^2}+2\nu\Delta t\|Dd_t\mathbf{U}\|^2\\&\quad+2\nu(\Delta t)^{-1}\|d_xU_1^{n+1}+d_yU_2^{n+1}\|^2_{l^2,M}-2\nu(\Delta t)^{-1}\|d_xU_1^n+d_yU_2^n\|^2_{l^2,M}\\&\quad+2(\nabla_h\Psi^n,d_t\mathbf{U}^{n+1})_{l^2}\\&=2\nu\Delta t\|d_xd_tU_1^{n+1}+d_yd_tU_2^{n+1}\|^2_{l^2,M}+2\Delta t(d_t\mathbf{f}^{n+1},d_t\mathbf{U}^{n+1})_{l^2},\end{aligned}\tag{4.6}$$

where $\nabla_h=(D_x,D_y)^T$.

Multiplying (3.12a) by $2D_x\Psi^{n+1}_{i,j+1/2}$, making summation on $i,j$ for $1\le i\le N_x-1,\ 0\le j\le N_y-1$ and replacing $n+1$ by $n$, we have

$$2\|D_x\Psi^n\|^2_{l^2,T,M}=2(d_tU_1^n,D_x\Psi^n)_{l^2,T,M}.\tag{4.7}$$

Taking the time difference of two consecutive steps to (3.12a) yields

$$[D_xd_t\Psi]^{n+1}_{i,j+1/2}=[d_{tt}U_1]^{n+1}_{i,j+1/2},\quad 1\le i\le N_x-1,\ 0\le j\le N_y-1.\tag{4.8}$$

Multiplying (4.8) by $2\Delta tD_x\Psi^n_{i,j+1/2}$ and making summation on $i,j$ for $1\le i\le N_x-1,\ 0\le j\le N_y-1$ arrives at

$$\begin{aligned}&\|D_x\Psi^{n+1}\|^2_{l^2,T,M}-\|D_x\Psi^n\|^2_{l^2,T,M}-(\Delta t)^2\|D_xd_t\Psi^{n+1}\|^2_{l^2,T,M}\\&\quad=2\Delta t(d_{tt}U_1^{n+1},D_x\Psi^n)_{l^2,T,M}.\end{aligned}\tag{4.9}$$

Combining (4.9) with (4.7) leads to

$$\begin{aligned}&\|D_x\Psi^{n+1}\|^2_{l^2,T,M}+\|D_x\Psi^n\|^2_{l^2,T,M}-(\Delta t)^2\|D_xd_t\Psi^{n+1}\|^2_{l^2,T,M}\\&\quad=2(d_tU_1^{n+1},D_x\Psi^n)_{l^2,T,M}.\end{aligned}\tag{4.10}$$

Similarly in the $y$ direction, we have

$$\begin{aligned}&\|D_y\Psi^{n+1}\|^2_{l^2,M,T}+\|D_y\Psi^n\|^2_{l^2,M,T}-(\Delta t)^2\|D_yd_t\Psi^{n+1}\|^2_{l^2,M,T}\\&\quad=2(d_tU_2^{n+1},D_y\Psi^n)_{l^2,M,T}.\end{aligned}\tag{4.11}$$

Taking notice of

$$\|\nabla_hd_t\Psi^{n+1}\|^2_{l^2}=\|d_{tt}\mathbf{U}^{n+1}\|^2_{l^2}$$

and combining (4.6) with (4.10) and (4.11), we have

$$\begin{aligned}&\|d_t\mathbf{U}^{n+1}\|^2_{l^2}-\|d_t\mathbf{U}^n\|^2_{l^2}+(\Delta t)^2\|d_{tt}\mathbf{U}^{n+1}\|^2_{l^2}+2\nu\Delta t\|Dd_t\mathbf{U}^{n+1}\|^2\\&\quad+2\nu(\Delta t)^{-1}\|d_xU_1^{n+1}+d_yU_2^{n+1}\|^2_{l^2,M}-2\nu(\Delta t)^{-1}\|d_xU_1^n+d_yU_2^n\|^2_{l^2,M}\\&\quad+\|\nabla_h\Psi^{n+1}\|^2_{l^2}+\|\nabla_h\Psi^n\|^2_{l^2}\\&=2\nu\Delta t\|d_xd_tU_1^{n+1}+d_yd_tU_2^{n+1}\|^2_{l^2,M}+(\Delta t)^2\|\nabla_hd_t\Psi^{n+1}\|^2_{l^2}\\&\quad+2\Delta t(d_t\mathbf{f}^{n+1},d_t\mathbf{U}^{n+1})_{l^2}\\&\le2\nu\Delta t\|Dd_t\mathbf{U}^{n+1}\|^2+(\Delta t)^2\|d_{tt}\mathbf{U}^{n+1}\|^2_{l^2}+\Delta t\|d_t\mathbf{f}^{n+1}\|^2_{l^2}+\Delta t\|d_t\mathbf{U}^{n+1}\|^2_{l^2}.\end{aligned}\tag{4.12}$$

Multiplying (4.12) by $\Delta t$, summing over $n$ from 0 to $m$ and using discrete Gronwall inequality give that

$$\begin{aligned}&\Delta t\|d_t\mathbf{U}^{m+1}\|^2_{l^2}+2\nu\|d_xU_1^{m+1}+d_yU_2^{m+1}\|^2_{l^2,M}+\sum_{n=0}^{m+1}\Delta t\|\nabla_h\Psi^n\|^2_{l^2}\\&\quad\le C\|d_xU_1^0+d_yU_2^0\|^2_{l^2,M}+C\Delta t\|d_t\mathbf{U}^0\|^2_{l^2}+C\sum_{n=0}^{m}(\Delta t)^2\|d_t\mathbf{f}^{n+1}\|^2_{l^2},\end{aligned}\tag{4.13}$$

which implies the desired result. □

## 5. Error analysis

In this section we carry out an error analysis for the fully discrete scheme (3.2)–(3.3) and (3.6)–(3.8). Before we embark on the error estimates, we first construct an auxiliary problem.

### 5.1. An auxiliary problem

We consider first an auxiliary problem which will be used in the sequel.

Let $(\mathbf{u}, p)$ be the solution of Stokes system, and set $\mathbf{g} = \mathbf{f} - \frac{\partial \mathbf{u}}{\partial t}$. For each time step $n+1$, we rewrite (1.1a)–(1.1c) as

$$-\nu \Delta \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{g}^{n+1} \quad \text{in } \Omega \times J, \tag{5.1a}$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0 \quad \text{in } \Omega \times J, \tag{5.1b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times J \tag{5.1c}$$

and consider its approximation by the MAC scheme: For each $n = 1, \ldots, N$, let $\{\widehat{U}^{n+1}_{1,i,j+1/2}\}$, $\{\widehat{U}^{n+1}_{2,i+1/2,j}\}$ and $\{\widehat{P}^{n+1}_{i+1/2,j+1/2}\}$ such that

$$-\nu D_x(d_x\widehat{U}_1)^{n+1}_{i,j+1/2} - \nu d_y(D_y\widehat{U}_1)^{n+1}_{i,j+1/2} + [D_x\widehat{P}]^{n+1}_{i,j+1/2} = g^{n+1}_{1,i,j+1/2}, \quad 1 \le i \le N_x - 1, 0 \le j \le N_y - 1, \tag{5.2}$$

$$-\nu D_y(d_y\widehat{U}_2)^{n+1}_{i+1/2,j} - \nu d_x(D_x\widehat{U}_2)^{n+1}_{i+1/2,j} + [D_y\widehat{P}]^{n+1}_{i+1/2,j} = g^{n+1/2}_{2,i+1/2,j}, \quad 0 \le i \le N_x - 1, 1 \le j \le N_y - 1, \tag{5.3}$$

$$d_x\widehat{U}^{n+1}_{1,i+1/2,j+1/2} + d_y\widehat{U}^{n+1}_{2,i+1/2,j+1/2} = 0, \quad 0 \le i \le N_x - 1, 0 \le j \le N_y - 1, \tag{5.4}$$

where the boundary and initial approximations are same as system (1.1a)–(1.1c).

Inspired by [45], we extend the work in Rui and Li [7] to the above approximation. By following closely the same arguments as in [7,26] and taking the time difference of two consecutive steps in (5.2)–(5.4), we can establish the following results:

**Lemma 5.1.** *Assuming that* $\mathbf{u} \in W^3_\infty(J; W^4_\infty(\Omega))^2$, $p \in W^3_\infty(J; W^3_\infty(\Omega))$, *we have the following results:*

$$\begin{aligned} &\|\widehat{\mathbf{U}}^{n+1} - \widehat{\mathbf{u}}^{n+1}\|^2_{l^2} + \|d_t(\widehat{\mathbf{U}}^{n+1} - \widehat{\mathbf{u}}^{n+1})\|^2_{l^2} + \sum_{l=2}^{n} \Delta t \|d_{tt}(\widehat{\mathbf{U}}^l - \widehat{\mathbf{u}}^l)\|^2_{l^2} \\ &\qquad + \sum_{l=1}^{n} \Delta t \|(\widehat{Z} - p)^l\|^2_{l^2,M} \le O(\Delta t^2 + h^4 + k^4), \end{aligned} \tag{5.5}$$

$$\|d_x(\widehat{U}^{n+1}_1 - u^{n+1}_1)\|^2_{l^2,M} + \|d_y(\widehat{U}^{n+1}_2 - u^{n+1}_2)\|^2_{l^2,M} \le O(\Delta t^2 + h^4 + k^4), \tag{5.6}$$

$$\|D_y(\widehat{U}^{n+1}_1 - u^{n+1}_1)\|^2_{l^2,T_y} \le O(\Delta t^2 + h^4 + k^3), \tag{5.7}$$

$$\|D_x(\widehat{U}^{n+1}_2 - u^{n+1}_2)\|^2_{l^2,T_x} \le O(\Delta t^2 + h^3 + k^4), \tag{5.8}$$

$$\begin{aligned} &\sum_{l=1}^{n} \Delta t \|\nabla_h \widehat{P}^l\|^2_{l^2,M} + \sum_{l=1}^{n} \Delta t \|d_t \nabla_h \widehat{P}^l\|^2_{l^2,M} + \sum_{l=2}^{n} \Delta t \|d_{tt} \nabla_h \widehat{P}^l\|^2_{l^2,M} \\ &\qquad \le C(\|\mathbf{u}\|^2_{W^3_\infty(J;W^4_\infty(\Omega))^2} + \|p\|^2_{W^3_\infty(J;W^3_\infty(\Omega))}). \end{aligned} \tag{5.9}$$

**Lemma 5.2.** *For the discrete function* $\mathbf{U}$ *satisfying the boundary condition* (3.3), *we have*

$$\|D\mathbf{U}\|^2 = \|d_xU_1 + d_yU_2\|^2_{l^2,M} + \|D_xU_2 - D_yU_1\|^2_{T_{xy}}, \tag{5.10}$$

*where we define*

$$\|D_xU_2 - D_yU_1\|^2_{T_{xy}} \equiv \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} h_i k_j |D_xU_{2,i,j} - D_yU_{1,i,j}|^2. \tag{5.11}$$

**Proof.** Taking notice of (2.12), we have

$$\begin{aligned}\|D\mathbf{U}\|^2 =&\|d_xU_1\|^2_{l^2,M}+\|D_yU_1\|^2_{l^2,T_y}+\|D_xU_2\|^2_{l^2,T_x}+\|d_yU_2\|^2_{l^2,M}\\ =&\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}h_{i+1/2}k_{j+1/2}(\frac{U_{1,i+1,j+1/2}-U_{1,i,j+1/2}}{h_{i+1/2}})^2\\ &+\sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y}h_ik_j(\frac{U_{1,i,j+1/2}-U_{1,i,j-1/2}}{k_j})^2\\ &+\sum_{i=0}^{N_x}\sum_{j=1}^{N_y-1}h_ik_j(\frac{U_{2,i+1/2,j}-U_{2,i-1/2,j}}{h_i})^2\\ &+\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}h_{i+1/2}k_{j+1/2}(\frac{U_{2,i+1/2,j+1}-U_{2,i+1/2,j}}{k_{j+1/2}})^2.\end{aligned} \tag{5.12}$$

Using the definitions (2.2) and (2.10) leads to

$$\begin{aligned}&\|d_xU_1+d_yU_2\|^2_{l^2,M}\\ =&\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}h_{i+1/2}k_{j+1/2}(\frac{U_{1,i+1,j+1/2}-U_{1,i,j+1/2}}{h_{i+1/2}}+\frac{U_{2,i+1/2,j+1}-U_{2,i+1/2,j}}{k_{j+1/2}})^2\\ =&\|d_xU_1\|^2_{l^2,M}+\|d_yU_2\|^2_{l^2,M}+2\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}(U_{1,i+1,j+1/2}-U_{1,i,j+1/2})(U_{2,i+1/2,j+1}-U_{2,i+1/2,j}).\end{aligned} \tag{5.13}$$

For the discrete function $\mathbf{U}$ satisfying the boundary condition (3.3), we have

$$\begin{aligned}&\|D_xU_2-D_yU_1\|^2_{T_{xy}}\\ =&\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}h_ik_j(\frac{U_{2,i+1/2,j}-U_{2,i-1/2,j}}{h_i}-\frac{U_{1,i,j+1/2}-U_{1,i,j-1/2}}{k_j})^2\\ =&\|D_yU_1\|^2_{l^2,T_y}+\|D_xU_2\|^2_{l^2,T_x}-2\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}(U_{1,i,j+1/2}-U_{1,i,j-1/2})(U_{2,i+1/2,j}-U_{2,i-1/2,j}).\end{aligned} \tag{5.14}$$

Combining (5.13) with (5.14) yields

$$\begin{aligned}&\|d_xU_1+d_yU_2\|^2_{l^2,M}+\|D_xU_2-D_yU_1\|^2_{T_{xy}}\\ =&\|D\mathbf{U}\|^2+2\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}(U_{1,i+1,j+1/2}-U_{1,i,j+1/2})(U_{2,i+1/2,j+1}-U_{2,i+1/2,j})\\ &-2\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}(U_{1,i,j+1/2}-U_{1,i,j-1/2})(U_{2,i+1/2,j}-U_{2,i-1/2,j}).\end{aligned} \tag{5.15}$$

Since

$$\begin{aligned}&\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}U_{1,i,j-1/2}U_{2,i-1/2,j}=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}U_{1,i,j-1/2}U_{2,i-1/2,j}\\ =&\sum_{i=0}^{N_x-1}\sum_{j=1}^{N_y}U_{1,i+1,j-1/2}U_{2,i+1/2,j}=\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}U_{1,i+1,j+1/2}U_{2,i+1/2,j+1},\end{aligned} \tag{5.16}$$

we can easily obtain that

$$
\begin{aligned}
&2\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}(U_{1,i+1,j+1/2}-U_{1,i,j+1/2})(U_{2,i+1/2,j+1}-U_{2,i+1/2,j}) \\
&\qquad =2\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}(U_{1,i,j+1/2}-U_{1,i,j-1/2})(U_{2,i+1/2,j}-U_{2,i-1/2,j}),
\end{aligned} \tag{5.17}
$$

which implies the desired result (5.10). □

### 5.2. Sub-optimal velocity estimate

In this subsection we derive the error estimates of the velocity. Denote

$$
\begin{aligned}
&e_{\mathbf{u}}^{n+1}=\mathbf{U}^{n+1}-\widehat{\mathbf{U}}^{n+1}+\widehat{\mathbf{U}}^{n+1}-\mathbf{u}^{n+1}=\widehat{e}_{\mathbf{u}}^{n+1}+\widetilde{e}_{\mathbf{u}}^{n+1},\\
&e_{p}^{n+1}=P^{n+1}-\widehat{P}^{n+1}+\widehat{P}^{n+1}-p^{n+1}=\widehat{e}_{p}^{n+1}+\widetilde{e}_{p}^{n+1},\\
&e_{\Psi}^{n+1}=\widehat{e}_{p}^{n+1}-\widehat{e}_{p}^{n}+\nu\Pi_h(d_x\widehat{e}_{\mathbf{u},1}^{n+1}+d_y\widehat{e}_{\mathbf{u},2}^{n+1})-\Delta t d_t\widehat{P}^{n+1}.
\end{aligned} \tag{5.18}
$$

**Lemma 5.3.** *Suppose that $\mathbf{u}\in W_\infty^3(J;W_\infty^4(\Omega))^2$, $p\in W_\infty^3(J;W_\infty^3(\Omega))$, then we have*

$$
\|d_t\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}^2+\|d_{tt}\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}^2\le C(\Delta t+h^4+k^4),\ \ 0\le m\le N-1, \tag{5.19}
$$

$$
\begin{aligned}
&\|d_x\widehat{e}_{\mathbf{u},1}^{m+1}+d_y\widehat{e}_{\mathbf{u},2}^{m+1}\|_{l^2,M}^2+\|d_td_x\widehat{e}_{\mathbf{u},1}^{m+1}+d_td_y\widehat{e}_{\mathbf{u},2}^{m+1}\|_{l^2,M}^2\\
&\qquad\le C\Delta t^2+C\Delta t(h^4+k^4),\ \ 0\le m\le N-1,
\end{aligned} \tag{5.20}
$$

*and*

$$
\|\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}^2\le C(\Delta t+h^4+k^4),\ \ 0\le m\le N-1, \tag{5.21}
$$

*where $C$ is a positive constant independent of $h$, $k$ and $\Delta t$.*

**Proof.** Subtracting (5.2) from (3.2a), we can obtain

$$
\begin{aligned}
&d_t\widehat{e}_{\mathbf{u},1,i,j+1/2}^{n+1}-\nu D_x(d_x\widehat{e}_{\mathbf{u},1})_{i,j+1/2}^{n+1}-\nu d_y(D_y\widehat{e}_{\mathbf{u},1})_{i,j+1/2}^{n+1}+[D_x\widehat{e}_p]_{i,j+1/2}^{n}\\
&\qquad=\frac{\partial u_1}{\partial t}|_{i,j+1/2}^{n+1}-[d_t\widehat{U}_1]_{i,j+1/2}^{n+1}+\Delta t[D_xd_t\widehat{P}]_{i,j+1/2}^{n+1},\quad 1\le i\le N_x-1,\ 0\le j\le N_y-1.
\end{aligned} \tag{5.22}
$$

Taking the time difference of two consecutive steps in (5.22), we have

$$
\begin{aligned}
&d_{tt}\widehat{e}_{\mathbf{u},1,i,j+1/2}^{n+1}-\nu D_x(d_xd_t\widehat{e}_{\mathbf{u},1})_{i,j+1/2}^{n+1}-\nu d_y(D_yd_t\widehat{e}_{\mathbf{u},1})_{i,j+1/2}^{n+1}+[D_xd_t\widehat{e}_p]_{i,j+1/2}^{n}\\
&\qquad=d_t(\frac{\partial u_1}{\partial t})|_{i,j+1/2}^{n+1}-[d_{tt}\widehat{U}_1]_{i,j+1/2}^{n+1}+\Delta t[D_xd_{tt}\widehat{P}]_{i,j+1/2}^{n+1},\quad 1\le i\le N_x-1,\ 0\le j\le N_y-1.
\end{aligned} \tag{5.23}
$$

Recalling (3.12a), (3.8) and (5.4) yields

$$
[D_xe_\Psi]_{i,j+1/2}^{n+1}=[d_tU_1]_{i,j+1/2}^{n+1},\quad 1\le i\le N_x-1,\ 0\le j\le N_y-1. \tag{5.24}
$$

Taking the time difference of two consecutive steps in (5.24) leads to

$$
[D_xd_te_\Psi]_{i,j+1/2}^{n+1}=[d_{tt}U_1]_{i,j+1/2}^{n+1},\quad 1\le i\le N_x-1,\ 0\le j\le N_y-1. \tag{5.25}
$$

Multiplying (5.23) by $2\Delta td_t\widehat{e}_{\mathbf{u},1,i,j+1/2}^{n+1}hk$ and making summation for $i,j$ with $i=1,\dots,N_x-1,\ j=0,\dots,N_y-1$, and recalling Lemma 2.2 lead to

$$
\begin{aligned}
&\|d_t\widehat{e}_{\mathbf{u},1}^{n+1}\|_{l^2,T,M}^2-\|d_t\widehat{e}_{\mathbf{u},1}^{n}\|_{l^2,T,M}^2+\Delta t^2\|d_{tt}\widehat{e}_{\mathbf{u},1}^{n+1}\|_{l^2,T,M}^2+2\nu\Delta t\|d_xd_t\widehat{e}_{\mathbf{u},1}^{n+1}\|_{l^2,M}^2\\
&+2\nu\Delta t\|D_yd_t\widehat{e}_{\mathbf{u},1}^{n+1}\|_{l^2,T_y}^2+2\Delta t(D_xd_t\widehat{e}_p^{n},d_t\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M}\\
=&2\Delta t\left(d_t(\frac{\partial u_1}{\partial t})^{n+1}-d_{tt}\widehat{U}_1^{n+1},d_t\widehat{e}_{\mathbf{u},1}^{n+1}\right)_{l^2,T,M}\\
&+2\Delta t^2(D_xd_{tt}\widehat{P}^{n+1},d_t\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M}.
\end{aligned} \tag{5.26}
$$

Taking notice of the last equation in (5.18) and recalling Lemma 2.2, the last term on the left hand side of (5.26) can be estimated by

$$\begin{aligned}2\Delta t(D_x d_t\widehat{e}_p^n, d_t\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M} = & 2(D_x e_\Psi^n, d_t\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M} \\ & + 2\nu(\Pi_h(d_x\widehat{e}_{\mathbf{u},1}^n + d_y\widehat{e}_{\mathbf{u},2}^n), d_x d_t\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,M} \\ & + 2\Delta t(D_x d_t\widehat{P}^n, d_t\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M}.\end{aligned} \tag{5.27}$$

We can establish the similar results as (5.26) and (5.27) in the $y$ direction, thus it can be easily obtained

$$\begin{aligned} & \|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2 - \|d_t\widehat{e}_{\mathbf{u}}^n\|_{l^2}^2 + \Delta t^2\|d_{tt}\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2 + 2\nu\Delta t\|Dd_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2,M}^2 \\ & + 2(\nabla_h e_\Psi^n, d_t\widehat{e}_{\mathbf{u}}^{n+1})_{l^2} + 2\nu(\Delta t)^{-1}\|d_x\widehat{e}_{\mathbf{u},1}^{n+1} + d_y\widehat{e}_{\mathbf{u},2}^{n+1}\|_{l^2,M}^2 \\ & - 2\nu(\Delta t)^{-1}\|d_x\widehat{e}_{\mathbf{u},1}^n + d_y\widehat{e}_{\mathbf{u},2}^n\|_{l^2,M}^2 \\ = & 2\Delta t\left(d_t(\frac{\partial\mathbf{u}}{\partial t})^{n+1} - d_{tt}\widehat{\mathbf{U}}^{n+1}, d_t\widehat{e}_{\mathbf{u}}^{n+1}\right)_{l^2} + 2\Delta t^2(\nabla_h d_{tt}\widehat{P}^{n+1}, d_t\widehat{e}_{\mathbf{u}}^{n+1})_{l^2} \\ & + 2\nu\Delta t\|d_t(d_x\widehat{e}_{\mathbf{u},1}^{n+1} + d_y\widehat{e}_{\mathbf{u},2}^{n+1})\|_{l^2,M}^2 + 2\Delta t(\nabla_h d_t\widehat{P}^n, d_t\widehat{e}_{\mathbf{u}}^{n+1})_{l^2}.\end{aligned} \tag{5.28}$$

Multiplying (5.24) by $2[D_x e_\Psi]_{i,j+1/2}^{n+1}hk$ and making summation for $i, j$ with $i = 1, \ldots, N_x - 1,\ j = 0, \ldots, N_y - 1$ leads to

$$2\|D_x e_\Psi^{n+1}\|_{l^2,T,M}^2 = 2(d_t U_1^{n+1}, D_x e_\Psi^{n+1})_{l^2,T,M}. \tag{5.29}$$

Replacing $n+1$ by $n$ in (5.29) yields

$$2\|D_x e_\Psi^n\|_{l^2,T,M}^2 = 2(d_t U_1^n, D_x e_\Psi^n)_{l^2,T,M}. \tag{5.30}$$

Multiplying (5.25) by $2\Delta t[D_x e_\Psi]_{i,j+1/2}^n hk$ and making summation for $i, j$ with $i = 1, \ldots, N_x - 1,\ j = 0, \ldots, N_y - 1$ leads to

$$\|D_x e_\Psi^{n+1}\|_{l^2,T,M}^2 - \|D_x e_\Psi^n\|_{l^2,T,M}^2 - \Delta t^2\|D_x d_t e_\Psi^{n+1}\|_{l^2,T,M}^2 = 2\Delta t(d_{tt}U_1^{n+1}, D_x e_\Psi^n)_{l^2,T,M}. \tag{5.31}$$

Combining (5.31) with (5.30), we have

$$\begin{aligned} & \|D_x e_\Psi^{n+1}\|_{l^2,T,M}^2 + \|D_x e_\Psi^n\|_{l^2,T,M}^2 \\ & \quad = \Delta t^2\|D_x d_t e_\Psi^{n+1}\|_{l^2,T,M}^2 + 2(d_t\widehat{e}_{\mathbf{u},1}^{n+1}, D_x e_\Psi^n)_{l^2,T,M} - 2(d_x d_t\widehat{U}_1^{n+1}, e_\Psi^n)_{l^2,M}.\end{aligned} \tag{5.32}$$

Similarly in the $y$ direction, we have

$$\begin{aligned} & \|D_y e_\Psi^{n+1}\|_{l^2,M,T}^2 + \|D_y e_\Psi^n\|_{l^2,M,T}^2 \\ & \quad = \Delta t^2\|D_y d_t e_\Psi^{n+1}\|_{l^2,M,T}^2 + 2(d_t\widehat{e}_{\mathbf{u},2}^{n+1}, D_y e_\Psi^n)_{l^2,M,T} - 2(d_y d_t\widehat{U}_2^{n+1}, e_\Psi^n)_{l^2,M}.\end{aligned} \tag{5.33}$$

Combining (5.32) with (5.33) and recalling (5.4) yields

$$\begin{aligned} & \|\nabla_h e_\Psi^{n+1}\|_{l^2}^2 + \|\nabla_h e_\Psi^n\|_{l^2}^2 \\ & \quad = \Delta t^2\|\nabla_h d_t e_\Psi^{n+1}\|_{l^2}^2 + 2(d_t\widehat{e}_{\mathbf{u}}^{n+1}, \nabla_h e_\Psi^n)_{l^2}.\end{aligned} \tag{5.34}$$

Next we focus on the first term on the right hand side of (5.34). Multiplying (5.25) by $[D_x d_t e_\Psi]_{i,j+1/2}^{n+1}hk$ and making summation for $i, j$ with $i = 1, \ldots, N_x - 1,\ j = 0, \ldots, N_y - 1$, we have

$$\|D_x d_t e_\Psi^{n+1}\|_{l^2,T,M}^2 = (d_{tt}\widehat{e}_{\mathbf{u},1}^{n+1}, D_x d_t e_\Psi^{n+1})_{l^2,T,M} - (d_x d_{tt}\widehat{U}_1^{n+1}, d_t e_\Psi^{n+1})_{l^2,M}. \tag{5.35}$$

By following the similar procedure as (5.34) and using (5.4) and (5.35), we get

$$\|\nabla_h d_t e_\Psi^{n+1}\|_{l^2}^2 = (d_{tt}\widehat{e}_{\mathbf{u}}^{n+1}, \nabla_h d_t e_\Psi^{n+1})_{l^2}. \tag{5.36}$$

Hence we can easily derive from above

$$\|\nabla_h d_t e_\Psi^{n+1}\|_{l^2}^2 \le \|d_{tt}\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2. \tag{5.37}$$

Combining (5.28) with (5.34) and (5.37) arrives at

$$\begin{aligned}
&\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2-\|d_t\widehat{e}_{\mathbf{u}}^{n}\|_{l^2}^2+\Delta t^2\|d_{tt}\widetilde{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2+2\nu\Delta t\|Dd_t\widetilde{e}_{\mathbf{u}}^{n+1}\|_{l^2,M}^2\\
&+2\nu(\Delta t)^{-1}\|d_x\widehat{e}_{\mathbf{u},1}^{n+1}+d_y\widehat{e}_{\mathbf{u},2}^{n+1}\|_{l^2,M}^2-2\nu(\Delta t)^{-1}\|d_x\widehat{e}_{\mathbf{u},1}^{n}+d_y\widehat{e}_{\mathbf{u},2}^{n}\|_{l^2,M}^2\\
&+\|\nabla_h e_{\Psi}^{n+1}\|_{l^2}^2+\|\nabla_h e_{\Psi}^{n}\|_{l^2}^2\\
=&2\Delta t\left(d_t(\frac{\partial\mathbf{u}}{\partial t})^{n+1}-d_{tt}\widehat{\mathbf{U}}^{n+1},d_t\widetilde{e}_{\mathbf{u}}^{n+1}\right)_{l^2}+2\Delta t^2(\nabla_h d_{tt}\widehat{P}^{n+1},d_t\widetilde{e}_{\mathbf{u}}^{n+1})_{l^2}\\
&+\Delta t^2\|\nabla_h d_t e_{\Psi}^{n+1}\|_{l^2}^2+2\Delta t(\nabla_h d_t\widehat{P}^{n},\nabla_h e_{\Psi}^{n})_{l^2}\\
&+2\nu\Delta t\|d_t(d_x\widehat{e}_{\mathbf{u},1}^{n+1}+d_y\widehat{e}_{\mathbf{u},2}^{n+1})\|_{l^2,M}^2\\
\le&\Delta t^2\|d_{tt}\widetilde{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2+C\Delta t\|d_t\widetilde{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2+C\|\mathbf{u}\|_{W^{2,\infty}(L^2(\Omega))}^2\Delta t^3\\
&+C\Delta t\|d_{tt}\widetilde{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2+C\Delta t^3\|\nabla_h d_{tt}\widehat{P}^{n+1}\|_{l^2}^2+C\Delta t^2\|\nabla_h d_t\widehat{P}^{n}\|_{l^2}^2\\
&+\|\nabla_h e_{\Psi}^{n}\|_{l^2}^2+2\nu\Delta t\|d_t(d_x\widehat{e}_{\mathbf{u},1}^{n+1}+d_y\widehat{e}_{\mathbf{u},2}^{n+1})\|_{l^2,M}^2.
\end{aligned}\tag{5.38}$$

Recalling Lemma 5.2, summing (5.38) over $n$ from 1 to $m$ and applying Gronwall's inequality result in

$$\begin{aligned}
&\|d_t\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}^2+2\nu\sum_{n=1}^{m}\Delta t\|d_t(D_x\widetilde{e}_{\mathbf{u},2}^{n+1}-D_y\widetilde{e}_{\mathbf{u},1}^{n+1})\|_{T_{xy}}^2\\
&+2\nu(\Delta t)^{-1}\|d_x\widehat{e}_{\mathbf{u},1}^{m+1}+d_y\widehat{e}_{\mathbf{u},2}^{m+1}\|_{l^2,M}^2+\sum_{n=1}^{m}\|\nabla_h e_{\Psi}^{n+1}\|_{l^2}^2\\
\le&\|d_t\widehat{e}_{\mathbf{u}}^{1}\|_{l^2}^2+2\nu(\Delta t)^{-1}\|d_x\widehat{e}_{\mathbf{u},1}^{1}+d_y\widehat{e}_{\mathbf{u},2}^{1}\|_{l^2,M}^2\\
&+C\sum_{n=1}^{m}\Delta t^2\|d_{tt}\widetilde{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2+C\sum_{n=1}^{m}\Delta t^3\|\nabla_h d_{tt}\widehat{P}^{n+1}\|_{l^2}^2\\
&+C\sum_{n=1}^{m}\Delta t^2\|\nabla_h d_t\widehat{P}^{n}\|_{l^2}^2+C\sum_{n=1}^{m}\Delta t^3\|\mathbf{u}\|_{W^{2,\infty}(L^2(\Omega))}^2.
\end{aligned}\tag{5.39}$$

Next we focus our concentration on the error analysis for the first two terms on the right hand side of (5.39) at $n=0$. Multiplying (5.22) by $2\widetilde{e}_{\mathbf{u},1,i,j+1/2}^{1}hk$ and making summation for $i,j$ with $i=1,\dots,N_x-1,\ j=0,\dots,N_y-1$, and recalling Lemma 2.2 lead to

$$\begin{aligned}
&\frac{\|\widehat{e}_{\mathbf{u},1}^{1}\|_{l^2,T,M}^2-\|\widehat{e}_{\mathbf{u},1}^{0}\|_{l^2,T,M}^2}{\Delta t}+\Delta t\|d_t\widetilde{e}_{\mathbf{u},1}^{1}\|_{l^2,T,M}^2+2\nu\|d_x\widetilde{e}_{\mathbf{u},1}^{1}\|_{l^2,M}^2\\
&+2\nu\|D_y\widetilde{e}_{\mathbf{u},1}^{1}\|_{l^2,T_y}^2+2(D_x\widehat{e}_{p}^{0},\widetilde{e}_{\mathbf{u},1}^{1})_{l^2,T,M}\\
=&2(\frac{\partial u_1^1}{\partial t}-d_t\widehat{U}_1^1,\widetilde{e}_{\mathbf{u},1}^{1})_{l^2,T,M}+2\Delta t(D_xd_t\widehat{P}^1,\widetilde{e}_{\mathbf{u},1}^{1})_{l^2,T,M}.
\end{aligned}\tag{5.40}$$

We can establish the similar results as (5.40) in the $y$ direction, thus it can be easily obtained

$$\|\widehat{e}_{\mathbf{u}}^{1}\|_{l^2}^2+\Delta t^2\|d_t\widetilde{e}_{\mathbf{u}}^{1}\|_{l^2}^2+2\nu\Delta t\|D\widetilde{e}_{\mathbf{u}}^{1}\|^2\le C(\Delta t)^4+(\Delta t)^2(h^4+k^4),\tag{5.41}$$

which implies that

$$\|d_t\widehat{e}_{\mathbf{u}}^{1}\|_{l^2}^2+2\nu(\Delta t)^{-1}\|d_x\widehat{e}_{\mathbf{u},1}^{1}+d_y\widehat{e}_{\mathbf{u},2}^{1}\|_{l^2,M}^2\le C(\Delta t)^2+C(h^4+k^4).\tag{5.42}$$

Thus combining (5.39) with (5.42) and recalling Lemma 5.1, we have

$$\begin{aligned}
&\|d_t\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}^2+2\nu\sum_{n=1}^{m}\Delta t\|d_t(D_x\widetilde{e}_{\mathbf{u},2}^{n+1}-D_y\widetilde{e}_{\mathbf{u},1}^{n+1})\|_{T_{xy}}^2\\
&\qquad+2\nu(\Delta t)^{-1}\|d_x\widehat{e}_{\mathbf{u},1}^{m+1}+d_y\widehat{e}_{\mathbf{u},2}^{m+1}\|_{l^2,M}^2+\sum_{n=1}^{m}\|\nabla_h e_{\Psi}^{n+1}\|_{l^2}^2\\
&\qquad\le C(\Delta t+h^4+k^4).
\end{aligned}\tag{5.43}$$

Using (5.43), taking the time difference of two consecutive steps again and giving the identical estimations as above, we can obtain the desired results (5.19) and (5.20).

Note that

$$\|\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}^2 \le 2\|\widehat{e}_{\mathbf{u}}^{0}\|_{l^2}^2 + 2T\sum_{n=0}^{m}\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2, \tag{5.44}$$

which implies the desired results (5.21). □

### 5.3. Improved velocity estimate

Our aim in this subsection is to improve the velocity results in Lemma 5.3 by constructing an auxiliary discrete inverse Stokes problem, similarly as in [14,26].

Given $\mathbf{U} \in V_h$, $\mathbf{V} = I_h\mathbf{U}$ is the solution of the following discrete Stokes equations:

$$-\nu D_x(d_xV_1)_{i,j+1/2}^{n+1} - \nu d_y(D_yV_1)_{i,j+1/2}^{n+1} + [D_xQ]_{i,j+1/2}^{n+1} = U_{1,i,j+1/2}^{n+1}, \quad 1 \le i \le N_x - 1, 0 \le j \le N_y - 1, \tag{5.45}$$

$$-\nu D_y(d_yV_2)_{i+1/2,j}^{n+1} - \nu d_x(D_xV_2)_{i+1/2,j}^{n+1} + [D_yQ]_{i+1/2,j}^{n+1} = U_{2,i+1/2,j}^{n+1/2}, \quad 0 \le i \le N_x - 1, 1 \le j \le N_y - 1, \tag{5.46}$$

$$d_xV_{1,i+1/2,j+1/2}^{n+1} + d_yV_{2,i+1/2,j+1/2}^{n+1} = 0, \quad 0 \le i \le N_x - 1, 0 \le j \le N_y - 1, \tag{5.47}$$

with the boundary conditions

$$\begin{cases} V_{1,0,j+1/2}^n = V_{1,N_x,j+1/2}^n = 0, & 0 \le j \le N_y - 1, \\ V_{1,i,0}^n = V_{1,i,N_y}^n = 0, & 0 \le i \le N_x, \\ V_{2,0,j}^n = V_{2,N_x,j}^n = 0, & 0 \le j \le N_y, \\ V_{2,i+1/2,0}^n = V_{2,i+1/2,N_y}^n = 0, & 0 \le i \le N_x - 1, \end{cases} \tag{5.48}$$

where $I_h$ is the discrete inverse Stokes operator.

We can easily obtain the following stability results:

$$\|D\mathbf{V}^{m+1}\|^2 + \|Q^{m+1}\|_{l^2,M}^2 \le C_1\|\mathbf{U}^{m+1}\|_{l^2}^2. \tag{5.49}$$

Define a discrete norm on $V_h$ by $\|\mathbf{U}\|_{I_h} = (I_h\mathbf{U}, \mathbf{U})_{l^2}^{1/2}$, which is identical to [26]. In addition, we have

$$\|DI_h\mathbf{U}\|^2 \le C_2\|\mathbf{U}\|_{I_h}^2. \tag{5.50}$$

**Lemma 5.4.** *Suppose that $\mathbf{u} \in W_\infty^3(J; W_\infty^4(\Omega))^2$, $p \in W_\infty^3(J; W_\infty^3(\Omega))$, then we have*

$$\sum_{n=0}^{m}\Delta t\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2 \le C(\Delta t^2 + h^4 + k^4), \ \ 0 \le m \le N - 1, \tag{5.51}$$

*and*

$$\sum_{n=0}^{m}\Delta t\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2 + \|\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}^2 \le C(\Delta t^2 + h^4 + k^4), \ \ 0 \le m \le N - 1, \tag{5.52}$$

*where $C$ is a positive constant independent of $h$, $k$ and $\Delta t$.*

**Proof.** Multiplying (5.22) by $2\Delta t I_h\widehat{e}_{\mathbf{u},1,i,j+1/2}^{n+1}hk$ and making summation for $i, j$ with $i = 1, \dots, N_x - 1, \ j = 0, \dots, N_y - 1$, and recalling Lemma 2.2 lead to

$$\begin{aligned} &2\Delta t(d_t\widehat{e}_{\mathbf{u},1}^{n+1}, I_h\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M} - 2\nu\Delta t(D_xd_x\widehat{e}_{\mathbf{u},1}^{n+1}, I_h\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M} \\ &- 2\nu\Delta t(d_yD_y\widehat{e}_{\mathbf{u},1}^{n+1}, I_h\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M} + 2\Delta t(D_x\widehat{e}_{p,i,j+1/2}^{n}, I_h\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M} \\ &\qquad = 2\Delta t(\frac{\partial u_1^{n+1}}{\partial t} - d_t\widehat{U}_1^{n+1}, I_h\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M} \\ &\qquad\quad + 2\Delta t^2(D_xd_t\widehat{P}^{n+1}, I_h\widehat{e}_{\mathbf{u},1}^{n+1})_{l^2,T,M}. \end{aligned} \tag{5.53}$$

Similar results as (5.53) can be established in the $y$ direction, thus using (5.45)–(5.47), we have

$$\begin{aligned} &\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{I_h}^2 - \|\widehat{e}_{\mathbf{u}}^{n}\|_{I_h}^2 + \Delta t^2\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{I_h}^2 \\ &+ 2\nu\Delta t(\widehat{e}_{\mathbf{u}}^{n+1} - \nabla_hQ^{n+1}, \widehat{e}_{\mathbf{u}}^{n+1})_{l^2} \\ =&2\Delta t(\frac{\partial \mathbf{u}^{n+1}}{\partial t} - d_t\widehat{\mathbf{U}}^{n+1}, I_h\widehat{e}_{\mathbf{u}}^{n+1})_{l^2}, \end{aligned} \tag{5.54}$$

which leads to

$$\begin{aligned}
&\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{l_h}^2 - \|\widehat{e}_{\mathbf{u}}^{n}\|_{l_h}^2 + \Delta t^2\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l_h}^2 + 2\nu\Delta t\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2 \\
\leq&\frac{\nu}{C_1}\Delta t\|Q^{n+1}\|_{l^2,M}^2 + C\Delta t\|d_x\widehat{e}_{\mathbf{u},1}^{m+1} + d_y\widehat{e}_{\mathbf{u},2}^{m+1}\|_{l^2,M}^2 \\
&+ 2\Delta t\|\frac{\partial \mathbf{u}^{n+1}}{\partial t} - d_t\widehat{\mathbf{U}}^{n+1}\|_{l^2}^2 + C\Delta t\|DI_h\widehat{e}_{\mathbf{u}}^{n+1}\|^2 \\
\leq&\nu\Delta t\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2 + C\Delta t\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{l_h}^2 + C\Delta t\|d_x\widehat{e}_{\mathbf{u},1}^{m+1} + d_y\widehat{e}_{\mathbf{u},2}^{m+1}\|_{l^2,M}^2 \\
&+ 2\Delta t\|\frac{\partial \mathbf{u}^{n+1}}{\partial t} - d_t\widehat{\mathbf{U}}^{n+1}\|_{l^2}^2.
\end{aligned} \tag{5.55}$$

Recalling (5.5) and (5.20), summing over $n$ from 0 to $m$ and applying Gronwall's inequality result in

$$\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{l_h}^2 + \sum_{n=0}^{m}\Delta t^2\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l_h}^2 + 2\nu\sum_{n=0}^{m}\Delta t\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}^2 \leq C(\Delta t^2 + h^4 + k^4), \tag{5.56}$$

which leads to the desired result (5.51). Similarly we can establish the error estimate (5.52) by taking the time difference of two consecutive steps again, giving the identical estimations as above and using (5.44). □

### 5.4. Error estimate for the pressure

In this subsection, we derive the optimal error estimate for the pressure.

**Theorem 5.5.** *Suppose that $\mathbf{u} \in W_\infty^3(J; W_\infty^4(\Omega))^2$, $p \in W_\infty^3(J; W_\infty^3(\Omega))$, then we have*

$$\sum_{n=0}^{m}\Delta t\|\widehat{e}_p^n\|^2 \leq C(\Delta t^2 + h^4 + k^4), \ \ 0 \leq m \leq N-1, \tag{5.57}$$

*where the positive constant $C$ is independent of $h$, $k$ and $\Delta t$.*

**Proof.** For a discrete function $\{v_{1,i,j+1/2}^{n+1}\}$ such that $v_{1,i,j+1/2}^{n+1}|_{\partial\Omega} = 0$, multiplying (5.22) by $v_{1,i,j+1/2}^{n+1}hk$ and make summation for $i, j$ with $i = 1, \ldots, N_x - 1,\ j = 0, \ldots, N_y - 1$, and recalling Lemma 2.2 lead to

$$\begin{aligned}
&(d_t\widehat{e}_{\mathbf{u},1}^{n+1}, v_1^{n+1})_{l^2,T,M} + \nu(d_x\widehat{e}_{\mathbf{u},1}^{n+1}, d_xv_1^{n+1})_{l^2,M} \\
&+ \nu(D_y\widehat{e}_{\mathbf{u},1}^{n+1}, D_yv_1^{n+1})_{l^2,T_y} - (\widehat{e}_p^n, d_xv_1^{n+1})_{l^2,M} \\
=&(\frac{\partial u_1^{n+1}}{\partial t} - d_t\widehat{U}_1^{n+1}, v_1^{n+1})_{l^2,T,M} + \Delta t(D_xd_t\widehat{P}^{n+1}, v_1^{n+1})_{l^2,T,M}.
\end{aligned} \tag{5.58}$$

Performing similar procedure in the $y$ direction yields

$$\begin{aligned}
&(d_t\widehat{e}_{\mathbf{u},2}^{n+1}, v_2^{n+1})_{l^2,M,T} + \nu(d_y\widehat{e}_{\mathbf{u},2}^{n+1}, d_yv_2^{n+1})_{l^2,M} \\
&+ \nu(D_x\widehat{e}_{\mathbf{u},2}^{n+1}, D_xv_2^{n+1})_{l^2,T_x} - (\widehat{e}_p^n, d_yv_2^{n+1})_{l^2,M} \\
=&(\frac{\partial u_2^{n+1}}{\partial t} - d_t\widehat{U}_2^{n+1}, v_2^{n+1})_{l^2,M,T} + \Delta t(D_yd_t\widehat{P}^{n+1}, v_2^{n+1})_{l^2,M,T}.
\end{aligned} \tag{5.59}$$

Hence we have

$$\begin{aligned}
&(d_t\widehat{e}_{\mathbf{u}}^{n+1}, \mathbf{v}^{n+1})_{l^2} + \nu(d_x\widehat{e}_{\mathbf{u},1}^{n+1}, d_xv_1^{n+1})_{l^2,M} \\
&+ \nu(D_y\widehat{e}_{\mathbf{u},1}^{n+1}, D_yv_1^{n+1})_{l^2,T_y} + \nu(d_y\widehat{e}_{\mathbf{u},2}^{n+1}, d_yv_2^{n+1})_{l^2,M} \\
&+ \nu(D_x\widehat{e}_{\mathbf{u},2}^{n+1}, D_xv_2^{n+1})_{l^2,T_x} - (\widehat{e}_p^n, d_xv_1^{n+1} + d_yv_2^{n+1})_{l^2,M} \\
=&(\frac{\partial \mathbf{u}^{n+1}}{\partial t} - d_t\widehat{\mathbf{U}}^{n+1}, \mathbf{v}^{n+1})_{l^2} + \Delta t(\nabla_hd_t\widehat{P}^{n+1}, \mathbf{v}^{n+1})_{l^2}.
\end{aligned} \tag{5.60}$$

Using Lemmas 2.1 and 5.1 and applying the discrete Poincaré inequality, we can obtain

$$\begin{aligned}
\|\widehat{e}_p^n\|_{l^2,M} \leq&\frac{1}{\beta}\sup_{\mathbf{v}\in\mathbf{V}_h}\frac{(\widehat{e}_p^n, d_xv_1^{n+1} + d_yv_2^{n+1})_{l^2,M}}{\|D\mathbf{v}\|} \\
\leq&C_3(\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2} + \|d_x\widehat{e}_{\mathbf{u},1}^{n+1}\|_{l^2,M} + \|D_y\widehat{e}_{\mathbf{u},1}^{n+1}\|_{l^2,T_y} \\
&+ \|d_y\widehat{e}_{\mathbf{u},2}^{n+1}\|_{l^2,M} + \|D_x\widehat{e}_{\mathbf{u},2}^{n+1}\|_{l^2,T_x}) \\
&+ C\Delta t\|\nabla_hd_t\widehat{P}^{n+1}\|_{l^2} + O(\Delta t + h^2 + k^2).
\end{aligned} \tag{5.61}$$

Setting $\mathbf{v}^{n+1}=\widehat{e}_{\mathbf{u}}^{n+1}$ in (5.60), multiplying it by $2\Delta t$ and using (5.20), we have

$$\begin{aligned}&\|\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}-\|\widehat{e}_{\mathbf{u}}^{n}\|_{l^2}+\Delta t^2\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}+2\nu\Delta t\|D\widehat{e}_{\mathbf{u}}^{n+1}\|^2\\ \le&\frac{\nu}{2(C_3)^2}\Delta t\|\widehat{e}_{p}^{n}\|_{l^2,M}^2+\frac{\nu}{2}\Delta t\|D\widehat{e}_{\mathbf{u}}^{n+1}\|^2+C\Delta t\|d_x\widehat{e}_{\mathbf{u},1}^{n+1}+d_y\widehat{e}_{\mathbf{u},2}^{n+1}\|_{l^2,M}^2\\ &+C\Delta t^2\|\nabla_h d_t\widehat{P}^{n+1}\|_{l^2}^2+C\Delta t(\Delta t^2+h^4+k^4)\\ \le&\nu\Delta t\|D\widehat{e}_{\mathbf{u}}^{n+1}\|^2+C\Delta t\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}\\ &+C\Delta t^3\|\nabla_h d_t\widehat{P}^{n+1}\|_{l^2}^2+C\Delta t(\Delta t^2+h^4+k^4).\end{aligned}\tag{5.62}$$

Recalling Lemmas 5.1 and 5.4 and summing (5.62) over $n$ from 0 to $m$ leads to

$$\|\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}+\nu\sum_{n=0}^{m}\Delta t\|D\widehat{e}_{\mathbf{u}}^{n+1}\|^2+\sum_{n=0}^{m}\Delta t^2\|d_t\widehat{e}_{\mathbf{u}}^{n+1}\|_{l^2}\le C(\Delta t^2+h^4+k^4),\tag{5.63}$$

which implies the desired result (5.57) by combining (5.61) with (5.63) . □

Combining the above results together, we finally obtain our main results:

**Theorem 5.6.** *Suppose that* $\mathbf{u}\in W_\infty^3(J;W_\infty^4(\Omega))^2$, $p\in W_\infty^3(J;W_\infty^3(\Omega))$, *then we have the following error estimates for the fully discrete scheme* (3.2)–(3.3) *and* (3.6)–(3.8)*:*

$$\begin{aligned}&\|\mathbf{U}^{m+1}-\mathbf{u}^{m+1}\|_{l^2}^2+\sum_{n=0}^{m}\Delta t\|d_t(\mathbf{U}^{n+1}-\mathbf{u}^{n+1})\|_{l^2}^2+\sum_{n=0}^{m}\Delta t\|P^n-p^n\|_{l^2,M}^2\\ &\qquad\le C(\Delta t^2+h^4+k^4),\ \ 0\le m\le N-1,\end{aligned}\tag{5.64}$$

$$\begin{aligned}&\sum_{n=0}^{m}\Delta t\|d_x(U_1^{n+1}-u_1^{n+1})\|_{l^2,M}^2+\sum_{n=0}^{m}\Delta t\|d_y(U_2^{n+1}-u_2^{n+1})\|_{l^2,M}^2\\ &\qquad\le O(\Delta t^2+h^4+k^4),\ \ 0\le m\le N-1,\end{aligned}\tag{5.65}$$

$$\sum_{n=0}^{m}\Delta t\|D_y(U_1^{n+1}-u_1^{n+1})\|_{l^2,T_y}^2\le O(\Delta t^2+h^4+k^3),\ \ 0\le m\le N-1,\tag{5.66}$$

$$\sum_{n=0}^{m}\Delta t\|D_x(U_2^{n+1}-u_2^{n+1})\|_{l^2,T_x}^2\le O(\Delta t^2+h^3+k^4),\ \ 0\le m\le N-1,\tag{5.67}$$

*where* $C$ *is a positive constant independent of* $h$, $k$ *and* $\Delta t$.

**Remark 5.1.** We observe from the above that the results in (5.64) and (5.65) are both optimal in time (first-order) and in space (second-order). However, the result in (5.66) (resp. (5.67)) is optimal in time and in the $x-$direction (resp. $y-$direction), but sub-optimal in the $y-$direction (resp. $x-$direction). Note that the sub-optimal results can be improved to optimal if the exact solution satisfies

$$\frac{\partial^2 u_1}{\partial y^2}=0\ \text{ at }\ y=0,1;\qquad \frac{\partial^2 u_2}{\partial x^2}=0\ \text{ at }\ x=0,1.\tag{5.68}$$

These error estimates are confirmed in our numerical experiments below.

## 6. Numerical experiments

In this section we perform some numerical experiments to validate the fully discrete consistent splitting scheme (3.2)–(3.3) and (3.6)–(3.8). Here we take $\Omega=(0,1)\times(0,1)$, $T=1$, $\nu=1$ and set $\Delta t=h^2=k^2$.

We denote

$$\begin{cases}\|f-g\|_{\infty,2}=\max_{0\le n\le m}\{\|f^n-g^n\|_X\},\\ \|f-g\|_{2,2}=\left(\sum_{n=0}^{m}\Delta t\,\|f^n-g^n\|_X^2\right)^{1/2},\end{cases}$$

where $X$ is the corresponding discrete $L^2$ norm.

**Example 1.** Consider the following exact solution for (1.1):

$$\begin{cases}p(x,y,t)=\exp(t)(x^3-1/4),\\ u_1(x,y,t)=-\exp(t)x^2(x-1)^2y(y-1)(2y-1),\\ u_2(x,y,t)=\exp(t)x(x-1)(2x-1)y^2(y-1)^2.\end{cases}$$

**Table 1**
Convergence rates of the velocity and pressure in discrete $L^2$ norm for Example 1.

| $N_x \times N_y$ | $\|U-\mathbf{u}\|_{\infty,2}$ | Rate | $\|P-p\|_{2,2}$ | Rate |
|---|---|---|---|---|
| 10 × 10 | 2.21E−3 | – | 9.02E−3 | – |
| 20 × 20 | 5.73E−4 | 1.95 | 2.52E−3 | 1.84 |
| 40 × 40 | 1.45E−4 | 1.99 | 6.58E−4 | 1.94 |
| 80 × 80 | 3.62E−5 | 2.00 | 1.67E−4 | 1.98 |
| 160 × 160 | 9.06E−6 | 2.00 | 4.20E−5 | 1.99 |

**Table 2**
Convergence rates of the velocity in discrete $H_1$ semi-norm for Example 1.

| $N_x \times N_y$ | $\|d_xU_1-d_xu_1\|_{2,2}$ | Rate | $\|D_yU_1-D_yu_1\|_{2,2}$ | Rate |
|---|---|---|---|---|
| 10 × 10 | 4.66E−3 | – | 5.55E−3 | – |
| 20 × 20 | 1.22E−3 | 1.94 | 1.66E−3 | 1.74 |
| 40 × 40 | 3.09E−4 | 1.98 | 5.15E−4 | 1.69 |
| 80 × 80 | 7.74E−5 | 1.99 | 1.67E−4 | 1.63 |
| 160 × 160 | 1.94E−5 | 2.00 | 5.59E−5 | 1.57 |

**Table 3**
Convergence rates of the velocity and pressure in discrete $L^2$ norm for Example 2.

| $N_x \times N_y$ | $\|U-\mathbf{u}\|_{\infty,2}$ | Rate | $\|P-p\|_{2,2}$ | Rate |
|---|---|---|---|---|
| 10 × 10 | 2.41E−3 | – | 5.93E−3 | – |
| 20 × 20 | 5.15E−4 | 2.22 | 1.85E−3 | 1.68 |
| 40 × 40 | 1.24E−4 | 2.05 | 5.09E−4 | 1.87 |
| 80 × 80 | 3.08E−5 | 2.01 | 1.32E−4 | 1.95 |
| 160 × 160 | 7.68E−6 | 2.00 | 3.34E−5 | 1.98 |

**Table 4**
Convergence rates of the velocity in discrete $H_1$ semi-norm for Example 2.

| $N_x \times N_y$ | $\|d_xU_1-d_xu_1\|_{2,2}$ | Rate | $\|D_yU_1-D_yu_1\|_{2,2}$ | Rate |
|---|---|---|---|---|
| 10 × 10 | 3.55E−2 | – | 6.15E−2 | – |
| 20 × 20 | 8.88E−3 | 2.00 | 1.54E−2 | 2.00 |
| 40 × 40 | 2.22E−3 | 2.00 | 3.84E−3 | 2.00 |
| 80 × 80 | 5.55E−4 | 2.00 | 9.60E−4 | 2.00 |
| 160 × 160 | 1.39E−4 | 2.00 | 2.40E−4 | 2.00 |

The numerical results for Example 1 are shown in Tables 1–2. For brevity, we only present the numerical results for $u_1$, the results for $u_1$ are similar. We observe that the results are consistent with the error estimates in Theorem 5.6, i.e., all errors converge at second-order except that the convergence rate for $D_yu_1$ (and $D_x(u_2)$ which is not shown in the table for brevity) is 1.5.

**Example 2.** Consider the following exact solution for (1.1):

$$\begin{cases} p(x,y,t) = \sin(\pi t)(\sin(\pi y) - 2/\pi), \\ u_1(x,y,t) = \sin(\pi t)\sin^2(\pi x)\sin(2\pi y), \\ u_2(x,y,t) = -\sin(\pi t)\sin(2\pi x)\sin^2(\pi y). \end{cases}$$

The numerical results for Example 2 are shown in Tables 3–4. Since the exact solution satisfies (5.68) in Remark 5.1, we observe uniform second-order convergence for all quantities, consistent with Remark 5.1.

## 7. Concluding remarks

We developed a fully discrete consistent splitting scheme for the time dependent Stokes equations based on the MAC discretization. By constructing an interpolation operator and performing appropriate approximation for the discrete operators on the staggered grids, we established the equivalence of the two discrete pressure Poisson problems (3.4) and (3.12), proved the unconditional stability, and derived first-order accuracy in time and second-order error estimates in space for velocity and pressure in different discrete norms.

Based on the presentation in this paper, one can easily construct fully discrete consistent splitting schemes for the time dependent Navier–Stokes equations either with a skew-symmetric semi-implicit treatment for the nonlinear term as in [44] so that the corresponding scheme is still linear, but with variable coefficients, and unconditionally stable, or with an explicit treatment for the nonlinear term coupled with a SAV approach as in [19] so that the corresponding scheme is still unconditionally stable and can be efficiently solved as the scheme for the Stokes equations presented in this paper.

We only considered the first-order in time discretization in this paper. While second-order consistent splitting schemes are constructed and have numerically shown to be effective in [24,25], but their stability and error analysis are still elusive.

## Data availability

Data will be made available on request.

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