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On a SAV-MAC scheme for the Cahn–Hilliard–Navier–Stokes phase-field model and its error analysis for the corresponding Cahn–Hilliard–Stokes case

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We construct a numerical scheme based on the scalar auxiliary variable (SAV) approach in time and the MAC discretization in space for the Cahn–Hilliard–Navier–Stokes phase-field model, prove its energy stability, and carry out error analysis for the corresponding Cahn–Hilliard–Stokes model only. The scheme is linear, second-order, unconditionally energy stable and can be implemented very efficiently. We establish second-order error estimates both in time and space for phase-field variable, chemical potential, velocity and pressure in different discrete norms for the Cahn–Hilliard–Stokes phase-field model. We also provide numerical experiments to verify our theoretical results and demonstrate the robustness and accuracy of our scheme.

Keywords: Cahn–Hilliard–Navier–Stokes; scalar auxiliary variable; finite-difference; staggered grids; energy stability; error estimates.

AMS Subject Classification: 35G25, 65M06, 65M12, 65M15, 65Z05, 76D07

1. Introduction

Interfacial dynamics in the mixture of different fluids, solids or gas has been one of the fundamental issues in many fields of science and engineering, particularly in materials science and fluid dynamics.[1,2,18,28] In recent years, the phase-field (i.e.

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diffuse interface) methods, have been successfully used to approximate a variety of interfacial dynamics. The basic idea for the phase-field methods is that the interface is represented as a thin transition layer between two phases.[3,23]

The phase-field model can be derived from an energy variational approach. Thus, a crucial goal in algorithm design is to preserve the energy law at the discrete level. A large number of numerical schemes that have been developed for phase-field models. Among them, the convex splitting approach[13,17,24] and stabilized linearly implicit approach[15,20,26,30] are two popular ways to construct unconditionally energy stable schemes. Unfortunately, the convex splitting approach usually leads to nonlinear schemes, and the stabilized linearly implicit approach results in additional accuracy issues and may not be easy to obtain second-order unconditionally energy stable schemes. Recently, a novel numerical method of invariant energy quadratization (IEQ) has been proposed.[4,10,27,29] This method is a generalization of the method of Lagrange multipliers or of auxiliary variable. The IEQ approach is remarkable as it permits us to construct linear and second-order unconditionally energy stable schemes for a large class of gradient flows. However, it leads to coupled systems with time-dependent variable coefficients. The scalar auxiliary variable (SAV) approach[18,19] inherits advantages of the IEQ approach but leads to decoupled systems with constant coefficients so it is both accurate and very efficient.

As for the Cahn–Hilliard–Navier–Stokes phase-field models, Shen and Yang[21,22] constructed several efficient time discretization schemes for two-phase incompressible flows with different densities and viscosities, established discrete energy laws but no error estimates were derived. Second order in time numerical scheme based on the convex-splitting for the Cahn–Hilliard equation and pressure-projection for the Navier–Stokes equation has been constructed by Han and Wang.[12] With regards to the numerical analysis, Feng *et al.,*[9] proposed and analyzed some semi-discrete and fully discrete finite element schemes with the abstract convergence by making use of the discrete energy law. Grün[11] proved an abstract convergence result of a fully discrete scheme for a diffuse interface models for two-phase incompressible fluids. Diegel *et al.*,[7] developed a fully discrete mixed finite element convex-splitting scheme for the Cahn–Hilliard–Darcy–Stokes system. The time discretization used is a first-order implicit Euler. They proved unconditional energy stability and error estimates for the phase-field variable, chemical potential and velocity. No convergence rate for pressure was demonstrated in their work.

The work presented in this paper for the Cahn–Hilliard–Navier–Stokes phase-field model is unique in the following aspects. First, we construct fully discrete linear, second-order (in space and time), unconditionally energy stable scheme for the Cahn–Hilliard–Navier–Stokes phase-field model. Furthermore, the scheme can be very efficiently implemented. Second, we carry out a rigorous error analysis to derive second-order error estimates both in time and space for phase-field variable, chemical potential, velocity and pressure in different discrete norms for the Cahn–Hilliard–Stokes phase-field model. We believe that this is the first such result for any fully discrete linear schemes for Cahn–Hilliard–Stokes or Cahn–Hilliard–Navier–Stokes models without assuming a uniform Lipschitz condition on the nonlinear potential.

The paper is organized as follows. In Sec. 2, we describe the problem and present some notations. In Sec. 3, we present the fully discrete SAV-MAC schemes and prove their stability. In Sec. 4, we carry out error estimates for the fully discrete SAV-MAC scheme for the Cahn–Hilliard–Stokes system. In Sec. 5, we present some numerical experiments to verify the accuracy of the proposed numerical schemes.

2. The Problem Description and Notations

We consider the following incompressible Cahn–Hilliard–Navier–Stokes phase-field model[3,7,9]:

$$\frac{\partial \phi}{\partial t} = M\Delta\mu - \mathbf{u}\cdot\nabla\phi \qquad \text{in } \Omega\times J, \tag{2.1a}$$

$$\mu = -\lambda\Delta\phi + \lambda F'(\phi) \qquad \text{in } \Omega\times J, \tag{2.1b}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \gamma\mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mu\nabla\phi \qquad \text{in } \Omega\times J, \tag{2.1c}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega\times J, \tag{2.1d}$$

$$\frac{\partial \phi}{\partial \mathbf{n}} = \frac{\partial \mu}{\partial \mathbf{n}} = 0,\ \mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega\times J, \tag{2.1e}$$

where $F(\phi) = \dfrac{1}{4\epsilon^2}(1-\phi^2)^2$, $M > 0$ is the mobility constant, $\nu > 0$ is the fluid viscosity. $\lambda > 0$ is the mixing coefficient, Ω is a bounded domain and $J = (0, T]$. The unknowns are the velocity $\mathbf{u}$, the pressure p, the phase function ϕ and the chemical potential μ. It models the dynamics of the mixture of two-incompressible fluids with the same density, which is set to be $\rho_0 = 1$ for simplicity. γ is an additional parameter that we added to distinguish the Cahn–Hilliard–Navier–Stokes model ($\gamma = 1$) and the Cahn–Hilliard–Stokes model ($\gamma = 0$). When the viscosity ν is not sufficient small, the Cahn–Hilliard–Stokes model can be used as a good approximation to the Cahn–Hilliard–Navier–Stokes model.

Taking the inner products of (2.1a) with μ, (2.1b) with $\frac{\partial\phi}{\partial t}$, (2.1c) with $\mathbf{u}$ respectively, we obtain the following energy dissipation law:

$$\frac{dE(\phi,\mathbf{u})}{dt} = -M\|\nabla\mu\|^2 - \nu\|\nabla\mathbf{u}\|^2, \tag{2.2}$$

where $E(\phi,\mathbf{u}) = \int_\Omega\{\frac{1}{2}|\mathbf{u}|^2 + \lambda(\frac{1}{2}|\nabla\phi|^2 + F(\phi))\}$ is the total energy.

We now introduce some standard notations. Let $L^m(\Omega)$ be the standard Banach space with norm

$$\|v\|_{L^m(\Omega)} = \left(\int_\Omega |v|^m d\Omega\right)^{1/m}.$$

For simplicity, let

$$(f,g) = (f,g)_{L^2(\Omega)} = \int_\Omega fg d\Omega$$

denote the $L^2(\Omega)$ inner product, $\|v\|_\infty = \|v\|_{L^\infty(\Omega)}$. And $W_p^k(\Omega)$ be the standard Sobolev space

$$W_p^k(\Omega) = \{g : \|g\|_{W_p^k(\Omega)} < \infty\},$$

where

$$\|g\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha g\|^p_{L^p(\Omega)}\right)^{1/p}. \tag{2.3}$$

Throughout the paper, we use C, with or without subscript, to denote a positive constant, independent of discretization parameters, which could have different values at different places.

3. The SAV Schemes and Their Stability

In this section, we first reformulate the phase-field system into an equivalent system with an additional scalar auxiliary variable (SAV). Then, we construct semi-discrete and fully discrete SAV schemes, and prove that they are unconditionally energy stable.

3.1. *The SAV reformulation*

We introduce a scalar auxiliary variable $r(t) = \sqrt{E_1(\phi)+\delta}$ with any $\delta > 0$, and reformulate the system (2.1) as:

$$\frac{\partial \phi}{\partial t} = M\Delta\mu - \mathbf{u}\cdot\nabla\phi \qquad \text{in } \Omega\times J, \tag{3.1a}$$

$$\mu = -\lambda\Delta\phi + \lambda\frac{r}{\sqrt{E_1(\phi)+\delta}}F'(\phi) \qquad \text{in } \Omega\times J, \tag{3.1b}$$

$$r_t = \frac{1}{2\sqrt{E_1(\phi)+\delta}}\int_\Omega F'(\phi)\phi_t d\mathbf{x} \qquad \text{in } \Omega\times J, \tag{3.1c}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \gamma\mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mu\nabla\phi \qquad \text{in } \Omega\times J, \tag{3.1d}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega\times J, \tag{3.1e}$$

where $E_1(\phi) = \int_\Omega F(\phi)d\mathbf{x}$. It is clear that with $r(0) = \sqrt{E_1(\phi|_{t=0})+\delta}$, the above system is equivalent to (2.1). Taking the inner products of (3.1a) with μ, (3.1b) with $\frac{\partial\phi}{\partial t}$, (3.1c) with $2\lambda r$ and (3.1d) with **u,** respectively, we obtain the following energy dissipation law:

$$\frac{d\tilde{E}(\phi,\mathbf{u},r)}{dt} = -M\|\nabla\mu\|^2 - \nu\|\nabla\mathbf{u}\|^2, \tag{3.2}$$

where $\tilde{E}(\phi,\mathbf{u},r) = \int_\Omega \frac{1}{2}\{|\mathbf{u}|^2 + \lambda|\nabla\phi|^2\}d\mathbf{x} + \lambda r^2$ is the total energy.

3.2. *The semi-discrete SAV/CN scheme*

Set $\Delta t = T/N,\ t^n = n\Delta t,\ for\ n \le N$, and define

$$[d_t f]^n = \frac{f^n - f^{n-1}}{\Delta t}, \quad f^{n+1/2} = \frac{f^n + f^{n+1}}{2}.$$

Then, a second-order SAV scheme based on Crank–Nicolson is:

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = M\Delta\mu^{n+1/2} - \mathbf{u}^{n+1/2}\cdot\nabla\tilde{\phi}^{n+1/2}, \tag{3.3a}$$

$$\mu^{n+1/2} = -\lambda\Delta\phi^{n+1/2} + \lambda\frac{r^{n+1/2}}{\sqrt{E_1(\tilde{\phi}^{n+1/2}) + \delta}}F'(\tilde{\phi}^{n+1/2}), \tag{3.3b}$$

$$\frac{r^{n+1} - r^n}{\Delta t} = \frac{1}{2\sqrt{E_1(\tilde{\phi}^{n+1/2}) + \delta}}\int_\Omega F'(\tilde{\phi}^{n+1/2})\frac{\phi^{n+1} - \phi^n}{\Delta t}d\mathbf{x}, \tag{3.3c}$$

$$\begin{aligned}&\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \gamma\tilde{\mathbf{u}}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2} - \nu\Delta\mathbf{u}^{n+1/2}\\ &\quad + \nabla p^{n+1/2} = \mu^{n+1/2}\nabla\tilde{\phi}^{n+1/2},\end{aligned} \tag{3.3d}$$

$$\nabla\cdot\mathbf{u}^{n+1/2} = 0, \tag{3.3e}$$

where $\tilde{\mathbf{u}}^{n+1/2} = (3\mathbf{u}^n - \mathbf{u}^{n-1})/2$ and $\tilde{\phi}^{n+1/2} = (3\phi^n - \phi^{n-1})/2$. We also set $\mathbf{u}^{-1} = \mathbf{u}^0$.

Theorem 3.1. *The scheme* (3.3) *is unconditionally energy stable in the sense that*

$$\tilde{E}^{n+1}(\phi, \boldsymbol{u}, r) - \tilde{E}^n(\phi, \boldsymbol{u}, r) = -M\Delta t\|\nabla\mu^{n+1/2}\|^2 - \nu\Delta t\|\nabla\boldsymbol{u}^{n+1/2}\|^2,$$

where

$$\tilde{E}^{n+1}(\phi, \boldsymbol{u}, r) = \int_\Omega \frac{1}{2}\{|\boldsymbol{u}^{n+1}|^2 + \lambda|\nabla\phi^{n+1}|^2\}d\boldsymbol{x} + \lambda|r^{n+1}|^2.$$

Proof. The proof is quite straightforward. Taking the inner products of (3.3a) with $\mu^{n+\frac{1}{2}}$, (3.3b) with $\frac{\phi^{n+1}-\phi^n}{\Delta t}$, (3.3c) with $2\lambda r^{n+1/2}$ and (3.3d) with $\mathbf{u}^{n+1/2}$ respectively, we obtain immediately the desired result. □

Remark 3.1.

- The above scheme is second order in time and linear, but it is weakly coupled. The above stability result indicates that this weakly coupled system is positive definite.
- If $\mathbf{u}^{n+1/2}$ in (3.3a) is replaced by an explicit second-order extrapolation, $(\phi^{n+1}, \mu^{n+1}, r^{n+1})$ can be obtained from (3.3a)–(3.3c) efficiently by solving decoupled elliptic systems with constant coefficients (Ref. 18). Once μ^{n+1} is known,

we can solve $(\mathbf{u}^{n+1}, p^{n+1})$ from (3.3d)–(3.3e) which is essentially a generalized Stokes problem that can be solved efficiently with a MAC scheme (see below).

- We can use the decoupled scheme with explicit treatment of $\mathbf{u}^{n+1/2}$ in (3.3a) as a preconditioner for the weakly coupled scheme.

3.3. *Spacial discretization by finite differences*

To fix the idea, we consider $\Omega = (L_{lx}, L_{rx}) \times (L_{ly}, L_{ry})$. Three-dimensional rectangular domains can be dealt similarly.

The two-dimensional domain Ω is partitioned by $\Omega_x \times \Omega_y$, where

$$\Omega_x : L_{lx} = x_0 < x_1 < \cdots < x_{N_x-1} < x_{N_x} = L_{rx},$$

$$\Omega_y : L_{ly} = y_0 < y_1 < \cdots < y_{N_y-1} < y_{N_y} = L_{ry}.$$

For simplicity, we also use the following notations:

$$\begin{cases} x_{-1/2} = x_0 = L_{lx}, x_{N_x+1/2} = x_{N_x} = L_{rx}, \\ y_{-1/2} = y_0 = L_{ly}, y_{N_y+1/2} = y_{N_y} = L_{ry}. \end{cases} \tag{3.4}$$

For possible integers i, j, $0 \le i \le N_x,\ 0 \le j \le N_y$, define

$$x_{i+1/2} = \frac{x_i + x_{i+1}}{2}, \quad h_{i+1/2} = x_{i+1} - x_i, \quad h = \max_i h_{i+1/2},$$

$$h_i = x_{i+1/2} - x_{i-1/2} = \frac{h_{i+1/2} + h_{i-1/2}}{2},$$

$$y_{j+1/2} = \frac{y_j + y_{j+1}}{2}, \quad k_{j+1/2} = y_{j+1} - y_j, \quad k = \max_j k_{j+1/2},$$

$$k_j = y_{j+1/2} - y_{j-1/2} = \frac{k_{j+1/2} + k_{j-1/2}}{2},$$

$$\Omega_{i+1/2,j+1/2} = (x_i, x_{i+1}) \times (y_j, y_{j+1}).$$

It is clear that

$$h_0 = \frac{h_{1/2}}{2}, \quad h_{N_x} = \frac{h_{N_x-1/2}}{2}, \quad k_0 = \frac{k_{1/2}}{2}, \quad k_{N_y} = \frac{k_{N_y-1/2}}{2}.$$

For a function $f(x, y)$, let $f_{l,m}$ denote $f(x_l, y_m)$ where l may take values $i,\ i + 1/2$ for integer i, and m may take values $j,\ j + 1/2$ for integer j. For discrete functions with values at proper nodal-points, define

$$\begin{cases} [d_x f]_{i+1/2,m} = \dfrac{f_{i+1,m} - f_{i,m}}{h_{i+1/2}}, & [D_y f]_{l,j+1} = \dfrac{f_{l,j+3/2} - f_{l,j+1/2}}{k_{j+1}}, \\ [D_x f]_{i+1,m} = \dfrac{f_{i+3/2,m} - f_{i+1/2,m}}{h_{i+1}}, & [d_y f]_{l,j+1/2} = \dfrac{f_{l,j+1} - f_{l,j}}{k_{j+1/2}}. \end{cases} \tag{3.5}$$

For functions f and g, define some discrete l^2 inner products and norms as follows:

$$(f,g)_{l^2,M} \equiv \sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1} h_{i+1/2}k_{j+1/2}f_{i+1/2,j+1/2}g_{i+1/2,j+1/2}, \tag{3.6}$$

$$(f,g)_{l^2,T_x} \equiv \sum_{i=0}^{N_x}\sum_{j=1}^{N_y-1} h_i k_j f_{i,j}g_{i,j}, \tag{3.7}$$

$$(f,g)_{l^2,T_y} \equiv \sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y} h_i k_j f_{i,j}g_{i,j}, \tag{3.8}$$

$$\|f\|^2_{l^2,\xi} \equiv (f,f)_{l^2,\xi}, \quad \xi = M,\ T_x,\ T_y. \tag{3.9}$$

Further define discrete l^2 inner products and norms as follows:

$$(f,g)_{l^2,T,M} \equiv \sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y-1} h_i k_{j+1/2}f_{i,j+1/2}g_{i,j+1/2}, \tag{3.10}$$

$$(f,g)_{l^2,M,T} \equiv \sum_{i=0}^{N_x-1}\sum_{j=1}^{N_y-1} h_{i+1/2}k_j f_{i+1/2,j}g_{i+1/2,j}, \tag{3.11}$$

$$\|f\|^2_{l^2,T,M} \equiv (f,f)_{l^2,T,M}, \quad \|f\|^2_{l^2,M,T} \equiv (f,f)_{l^2,M,T}. \tag{3.12}$$

For vector-valued functions $\mathbf{u} = (u_1, u_2)$, it is clear that

$$\|d_x u_1\|^2_{l^2,M} \equiv \sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1} h_{i+1/2}k_{j+1/2}|d_x u_{1,i+1/2,j+1/2}|^2, \tag{3.13}$$

$$\|D_y u_1\|^2_{l^2,T_y} \equiv \sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y} h_i k_j |D_y u_{1,i,j}|^2, \tag{3.14}$$

and $\|d_y u_2\|_{l^2,M}$, $\|D_x u_2\|_{l^2,T_x}$ can be represented similarly. Finally, define the discrete H^1-norm and discrete l^2-norm of a vectored-valued function $\mathbf{u}$,

$$\|D\mathbf{u}\|^2 \equiv \|d_x u_1\|^2_{l^2,M} + \|D_y u_1\|^2_{l^2,T_y} + \|D_x u_2\|^2_{l^2,T_x} + \|d_y u_2\|^2_{l^2,M}. \tag{3.15}$$

$$\|\mathbf{u}\|^2_{l^2} \equiv \|u_1\|^2_{l^2,T,M} + \|u_2\|^2_{l^2,M,T}. \tag{3.16}$$

For simplicity, we only consider the case that for all $h_{i+1/2} = h$, $k_{j+1/2} = k$, i.e. uniform meshes are used both in x and y-directions.

Denote by $\{Z^n, W^n, R^n, \mathbf{U}^n, P^n\}_{n=1}^N$, the approximations to $\{\phi^n, \mu^n, r^n, \mathbf{u}^n, p^n\}_{n=1}^N$, respectively, with the boundary conditions

$$\begin{cases} [D_xZ]^n_{0,j+1/2} = [D_xZ]^n_{N_x,j+1/2} = 0, & 0 \le j \le N_y - 1, \\ [D_yZ]^n_{i+1/2,0} = [D_yZ]^n_{i+1/2,N_y} = 0, & 0 \le i \le N_x - 1, \\ [D_xW]^n_{0,j+1/2} = [D_xW]^n_{N_x,j+1/2} = 0, & 0 \le j \le N_y - 1, \\ [D_yW]^n_{i+1/2,0} = [D_yW]^n_{i+1/2,N_y} = 0, & 0 \le i \le N_x - 1, \\ U^n_{1,0,j+1/2} = U^n_{1,N_x,j+1/2} = 0, & 0 \le j \le N_y - 1, \\ U^n_{1,i,0} = U^n_{1,i,N_y} = 0, & 0 \le i \le N_x, \\ U^n_{2,0,j} = U^n_{2,N_x,j} = 0, & 0 \le j \le N_y, \\ U^n_{2,i+1/2,0} = U^n_{2,i+1/2,N_y} = 0, & 0 \le i \le N_x - 1, \end{cases} \tag{3.17}$$

and initial conditions

$$\begin{cases} Z^0_{i+1/2,j+1/2} = \phi^0_{i+1/2,j+1/2}, & 0 \le i \le N_x - 1, \quad 0 \le j \le N_y - 1, \\ U^0_{1,i,j+1/2} = u^0_{1,i,j+1/2}, & 0 \le i \le N_x, \quad 0 \le j \le N_y, \\ U^0_{2,i+1/2,j} = u^0_{2,i+1/2,j}, & 0 \le i \le N_x, \quad 0 \le j \le N_y, \end{cases} \tag{3.18}$$

where ϕ^0, $\mathbf{u}^0$ are given initial conditions, respectively.

Then, the fully discrete SAV/CN scheme based on the MAC discretization is as follows:

$$[d_tZ]^{n+1} = M[d_xD_xW + d_yD_yW]^{n+1/2} - \mathcal{P}^y_h\mathcal{P}^x_h[U_1D_x\tilde{Z} + U_2D_y\tilde{Z}]^{n+1/2}, \tag{3.19a}$$

$$W^{n+1/2} = -\lambda[d_xD_xZ + d_yD_yZ]^{n+1/2} + \lambda\frac{R^{n+1/2}}{\sqrt{E^h_1(\tilde{Z}^{n+1/2}) + \delta}}F'(\tilde{Z}^{n+1/2}), \tag{3.19b}$$

$$[d_tR]^{n+1} = \frac{1}{2\sqrt{E^h_1(\tilde{Z}^{n+1/2}) + \delta}}(F'(\tilde{Z}^{n+1/2}), d_tZ^{n+1})_{l^2,M}, \tag{3.19c}$$

$$\begin{aligned} &[d_tU_1]^{n+1} + \frac{\gamma}{2}[\tilde{U}_1D_x(\mathcal{P}^x_hU_1) + \mathcal{P}^x_hd_x(U_1\tilde{U}_1) + \mathcal{P}^y_h(\mathcal{P}^x_h\tilde{U}_2D_yU_1) \\ &\quad + d_y(\mathcal{P}^y_hU_1\mathcal{P}^x_h\tilde{U}_2)]^{n+1/2} - \nu D_x(d_xU_1)^{n+1/2} - \nu d_y(D_yU_1)^{n+1/2} \\ &\quad + [D_xP]^{n+1/2} = \mathcal{P}^x_hW^{n+1/2}[D_x\tilde{Z}]^{n+1/2}, \end{aligned} \tag{3.19d}$$

$$[d_tU_2]^{n+1} + \frac{\gamma}{2}[\mathcal{P}_h^x(\mathcal{P}_h^y\tilde{U}_1 D_x U_2) + d_x(\mathcal{P}_h^y\tilde{U}_1\mathcal{P}_h^x U_2) + \tilde{U}_2 D_y(\mathcal{P}_h^y U_2)$$
$$+ \mathcal{P}_h^y(d_y(U_2\tilde{U}_2))]^{n+1/2} - \nu D_y(d_yU_2)^{n+1/2} - \nu d_x(D_xU_2)^{n+1/2}$$
$$+ [D_yP]^{n+1/2} = \mathcal{P}_h^y W^{n+1/2}[D_y\tilde{Z}]^{n+1/2}, \tag{3.19e}$$
$$[d_xU_1]^{n+1/2} + [d_yU_2]^{n+1/2} = 0, \tag{3.19f}$$

where $\mathcal{P}_h^x$ and $\mathcal{P}_h^y$ are linear interpolation operators in the x and y directions, respectively, and $\tilde{H}^{n+1/2} = \frac{3}{2}H^n - \frac{1}{2}H^{n-1}$ for any sequence $\{H^k\}$.

It is easy to verify that the following discrete integration-by-part formulae hold.

Lemma 3.1. ([Ref. 25]) *Let* $\{V_{1,i,j+1/2}\}, \{V_{2,i+1/2,j}\}$ *and* $\{q_{1,i+1/2,j+1/2}\}$, $\{q_{2,i+1/2,j+1/2}\}$ *be discrete functions with* $V_{1,0,j+1/2} = V_{1,N_x,j+1/2} = V_{2,i+1/2,0} = V_{2,i+1/2,N_y} = 0$, *with proper integers* i *and* j. *Then, there holds*

$$\begin{cases} (D_xq_1, V_1)_{l^2,T,M} = -(q_1, d_xV_1)_{l^2,M}, \\ (D_yq_2, V_2)_{l^2,M,T} = -(q_2, d_yV_2)_{l^2,M}. \end{cases} \tag{3.20}$$

Theorem 3.2. *The scheme* (3.19a)–(3.19f) *is unconditionally energy stable in the sense that*

$$\tilde{E}^{n+1}(Z, \boldsymbol{U}, R) - \tilde{E}^n(Z, \boldsymbol{U}, R) = -M\Delta t\|DW^{n+1/2}\|_{l^2}^2 - \nu\Delta t\|D\,\boldsymbol{U}^{n+1/2}\|_{l^2}^2,$$

where $\boldsymbol{D}H = (D_xH, D_yH)$ *for any discrete scalar or vector function* H, *and*

$$\tilde{E}^{n+1}(Z, \boldsymbol{U}, R) = \frac{1}{2}\|\boldsymbol{U}\|_{l^2}^2 + \lambda\left(\frac{1}{2}\|DZ^{n+1}\|_{l^2}^2 + (R^{n+1})^2\right).$$

Proof. Multiplying (3.19a) by $W_{i+1/2,j+1/2}^{n+1/2}hk$, and making summation on i, j for $0 \le i \le N_x - 1,\ 0 \le j \le N_y - 1$, we have

$$(d_tZ^{n+1}, W^{n+1/2})_{l^2,M} = M(d_xD_xW^{n+1/2} + d_yD_yW^{n+1/2}, W^{n+1/2})_{l^2,M}$$
$$- (\mathcal{P}_h^y\mathcal{P}_h^x[U_1D_x\tilde{Z} + U_2D_y\tilde{Z}]^{n+1/2}, W^{n+1/2})_{l^2,M}. \tag{3.21}$$

Taking note of Lemma 3.1, the first term on the right-hand side of (3.21) can be transformed into the following

$$M(d_xD_xW^{n+1/2} + d_yD_yW^{n+1/2}, W^{n+1/2})_{l^2,M}$$
$$= -M\|D_xW^{n+1/2}\|_{l^2,T,M}^2 - M\|D_yW^{n+1/2}\|_{l^2,M,T}^2$$
$$= -M\|\mathbf{D}W^{n+1/2}\|_{l^2}. \tag{3.22}$$

Multiplying (3.19b) by $d_tZ_{i+1/2,j+1/2}^{n+1}hk$, and making summation on i, j for $0 \le i \le N_x - 1,\ 0 \le j \le N_y - 1$, we have

$$(d_tZ^{n+1}, W^{n+1/2})_{l^2,M} = -\lambda(d_xD_xZ^{n+1/2} + d_yD_yZ^{n+1/2}, d_tZ^{n+1})_{l^2,M}$$
$$+ \lambda\frac{R^{n+1/2}}{\sqrt{E_1^h(\tilde{Z}^{n+1/2}) + \delta}}(F'(\tilde{Z}^{n+1/2}), d_tZ^{n+1})_{l^2,M}. \tag{3.23}$$

Recalling Lemma 3.1, the first term on the right-hand side of (3.23) can be estimated by:

$$\begin{aligned}
&-\lambda(d_xD_xZ^{n+1/2}+d_yD_yZ^{n+1/2},d_tZ^{n+1})_{l^2,M}\\
&\quad=\lambda(D_xZ^{n+1/2},d_tD_xZ^{n+1})_{l^2,T,M}+\lambda(D_yZ^{n+1/2},d_tD_yZ^{n+1})_{l^2,M,T}\\
&\quad=\lambda\frac{\|\mathbf{D}Z^{n+1}\|_{l^2}^2-\|\mathbf{D}Z^n\|_{l^2}^2}{2\Delta t}. \qquad (3.24)
\end{aligned}$$

Multiplying Eq. (3.19c) by $(R^{n+1}+R^n)$ leads to

$$\frac{(R^{n+1})^2-(R^n)^2}{\Delta t}=\frac{R^{n+1/2}}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}(F'(\tilde{Z}^{n+1/2}),d_tZ^{n+1})_{l^2,M}. \qquad (3.25)$$

Combining (3.25) with (3.21)–(3.24) gives that

$$\begin{aligned}
&\lambda\frac{(R^{n+1})^2-(R^n)^2}{\Delta t}+\lambda\frac{\|\mathbf{D}Z^{n+1}\|_{l^2}^2-\|\mathbf{D}Z^n\|_{l^2}^2}{2\Delta t}\\
&\quad=-M\|\mathbf{D}W^{n+1/2}\|_{l^2}^2-(\mathcal{P}_h^y\mathcal{P}_h^x[U_1D_x\tilde{Z}+U_2D_y\tilde{Z}]^{n+1/2},W^{n+1/2})_{l^2,M}.
\end{aligned} \qquad (3.26)$$

Multiplying (3.19d) by $U_{1,i,j+1/2}^{n+1/2}hk$, and making summation on i,j for $1\le i\le N_x-1,\ 0\le j\le N_y-1$, we have

$$\begin{aligned}
&(d_tU_1^{n+1},U_1^{n+1/2})_{l^2,T,M}+\frac{\gamma}{2}\Big((\tilde{U}_1^{n+1/2}D_x(\mathcal{P}_h^xU_1^{n+1/2}),U_1^{n+1/2})_{l^2,T,M}\\
&\qquad+(\mathcal{P}_h^xd_x(U_1^{n+1/2}\tilde{U}_1^{n+1/2}),U_1^{n+1/2})_{l^2,T,M}\\
&\qquad+(\mathcal{P}_h^y(\mathcal{P}_h^x\tilde{U}_2^{n+1/2}D_yU_1^{n+1/2}),U_1^{n+1/2})_{l^2,T,M}\\
&\qquad+(d_y(\mathcal{P}_h^yU_1^{n+1/2}\mathcal{P}_h^x\tilde{U}_2^{n+1/2}),U_1^{n+1/2})_{l^2,T,M}\Big)+\nu\|d_xU_1^{n+1/2}\|_{l^2,M}^2\\
&\qquad+\nu\|D_yU_1^{n+1/2}\|_{l^2,T_y}^2-(P^{n+1/2},d_xU_1^{n+1/2})_{l^2,M}\\
&\quad=(\mathcal{P}_h^xW^{n+1/2}D_x\tilde{Z}^{n+1/2},U_1^{n+1/2})_{l^2,T,M}. \qquad (3.27)
\end{aligned}$$

Thanks to Lemma 3.1, we have

$$\begin{aligned}
&(\tilde{U}_1^{n+1/2}D_x(\mathcal{P}_h^xU_1^{n+1/2}),U_1^{n+1/2})_{l^2,T,M}\\
&\quad=-(\mathcal{P}_h^xU_1^{n+1/2},d_x(\tilde{U}_1^{n+1/2}U_1^{n+1/2}))_{l^2,M}\\
&\quad=-(\mathcal{P}_h^xd_x(\tilde{U}_1^{n+1/2}U_1^{n+1/2}),U_1^{n+1/2})_{l^2,T,M}. \qquad (3.28)
\end{aligned}$$

The fifth term on the left-hand side of (3.27) can be estimated as follows:

$$\begin{aligned}
&(d_y(\mathcal{P}_h^yU_1^{n+1/2}\mathcal{P}_h^x\tilde{U}_2^{n+1/2}),U_1^{n+1/2})_{l^2,T,M}\\
&\quad=-(\mathcal{P}_h^yU_1^{n+1/2}\mathcal{P}_h^x\tilde{U}_2^{n+1/2},D_yU_1^{n+1/2})_{l^2,M}\\
&\quad=-(\mathcal{P}_h^y(\mathcal{P}_h^x\tilde{U}_2^{n+1/2}D_yU_1^{n+1/2}),U_1^{n+1/2})_{l^2,T,M}. \qquad (3.29)
\end{aligned}$$

Multiplying (3.19e) by $U_{2,i+1/2,j}^{n+1/2}hk$, and making summation on i,j for $0 \le i \le N_x - 1,\ 1 \le j \le N_y - 1$, we can obtain

$$\begin{aligned}
&(d_t U_2^{n+1}, U_2^{n+1/2})_{l^2,M,T} + \frac{\gamma}{2}\Big((\mathcal{P}_h^x(\mathcal{P}_h^y \tilde{U}_1^{n+1/2} D_x U_2^{n+1/2}), U_2^{n+1/2})_{l^2,M,T} \\
&\quad + (d_x(\mathcal{P}_h^y \tilde{U}_1^{n+1/2} \mathcal{P}_h^x U_2^{n+1/2}), U_2^{n+1/2})_{l^2,M,T} \\
&\quad + (\tilde{U}_2^{n+1/2} D_y(\mathcal{P}_h^y U_2^{n+1/2}), U_2^{n+1/2})_{l^2,M,T} \\
&\quad + (\mathcal{P}_h^y(d_y(U_2^{n+1/2}\tilde{U}_2^{n+1/2})), U_2^{n+1/2})_{l^2,M,T}\Big) + \nu\|d_y U_2^{n+1/2}\|_{l^2,M}^2 \\
&\quad + \nu\|D_x U_2^{n+1/2}\|_{l^2,T_x}^2 - (P^{n+1/2}, d_y U_2^{n+1/2})_{l^2,M} \\
&= (\mathcal{P}_h^y W^{n+1/2} D_y \tilde{Z}^{n+1/2}, U_2^{n+1/2})_{l^2,M,T}.
\end{aligned} \tag{3.30}$$

Similar to the estimates of (3.28) and (3.29), we have

$$\begin{aligned}
&(\mathcal{P}_h^x(\mathcal{P}_h^y \tilde{U}_1^{n+1/2} D_x U_2^{n+1/2}), U_2^{n+1/2})_{l^2,M,T} \\
&\quad + (d_x(\mathcal{P}_h^y \tilde{U}_1^{n+1/2} \mathcal{P}_h^x U_2^{n+1/2}), U_2^{n+1/2})_{l^2,M,T} = 0,
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
&(\tilde{U}_2^{n+1/2} D_y(\mathcal{P}_h^y U_2^{n+1/2}), U_2^{n+1/2})_{l^2,M,T} \\
&\quad + (\mathcal{P}_h^y(d_y(U_2^{n+1/2}\tilde{U}_2^{n+1/2})), U_2^{n+1/2})_{l^2,M,T} = 0.
\end{aligned} \tag{3.32}$$

Combining (3.27)–(3.32) and recalling (3.19f) lead to

$$\begin{aligned}
\frac{\|\mathbf{U}^{n+1}\|_{l^2}^2 - \|\mathbf{U}^n\|_{l^2}^2}{2\Delta t} + \nu\|D\mathbf{U}\|^2 &= (\mathcal{P}_h W^{n+1/2} D_x \tilde{Z}^{n+1/2}, U_1^{n+1/2})_{l^2,T,M} \\
&\quad + (\mathcal{P}_h W^{n+1/2} D_y \tilde{Z}^{n+1/2}, U_2^{n+1/2})_{l^2,M,T}.
\end{aligned} \tag{3.33}$$

Taking note of (3.26), we have

$$\begin{aligned}
&\lambda\frac{(R^{n+1})^2 - (R^n)^2}{\Delta t} + \lambda\frac{\|\mathbf{D}Z^{n+1}\|_{l^2}^2 - \|\mathbf{D}Z^n\|_{l^2}^2}{2\Delta t} \\
&\quad + \frac{\|\mathbf{U}^{n+1}\|_{l^2}^2 - \|\mathbf{U}^n\|_{l^2}^2}{2\Delta t} + \nu\|D\mathbf{U}\|^2 = -M\|\mathbf{D}W^{n+1/2}\|_{l^2}^2 \le 0,
\end{aligned} \tag{3.34}$$

which implies the desired result. □

4. Error Estimates

In this section, we carry out an error analysis for the full discrete scheme (3.19a)–(3.19f) with $\gamma = 0$, i.e. for the Cahn–Hilliard–Stokes system. The analysis for the case of $\gamma = 1$, i.e. for the Cahn–Hilliard–Navier–Stokes system, will be extremely technical as it requires a high order upwind method to deal with the nonlinear convection term.

4.1. *An auxiliary problem*

We consider first an auxiliary problem which will be used in the sequel.

Let $(\phi, \mu, \mathbf{u}, p)$ be the solution of Cahn–Hilliard–Stokes system, and set $\mathbf{g} = \mu\nabla\phi - \frac{\partial \mathbf{u}}{\partial t}$. For each time step n, we rewrite (2.1c)–(2.1d) with $\gamma = 0$ as

$$-\nu\Delta\mathbf{u}^n + \nabla p^n = \mathbf{g}^n \qquad \text{in } \Omega \times J, \tag{4.1a}$$

$$\nabla \cdot \mathbf{u}^n = 0 \qquad \text{in } \Omega \times J, \tag{4.1b}$$

and consider its approximation by the MAC scheme: For each $n = 1, \dots, N$, let $\{\widehat{U}^n_{1,i,j+1/2}\}, \{\widehat{U}^n_{2,i+1/2,j}\}$ and $\{\widehat{P}^n_{i+1/2,j+1/2}\}$ such that

$$\begin{aligned}
&-\nu\frac{d_x\widehat{U}^{n+1/2}_{1,i+1/2,j+1/2} - d_x\widehat{U}^{n+1/2}_{1,i-1/2,j+1/2}}{h_i} - \nu\frac{D_y\widehat{U}^{n+1/2}_{1,i,j+1} - D_y\widehat{U}^{n+1/2}_{1,i,j}}{k_{j+1/2}} \\
&\qquad + D_x\widehat{P}^{n+1/2}_{i,j+1/2} = g^{n+1/2}_{1,i,j+1/2}, \quad 1 \le i \le N_x - 1, 0 \le j \le N_y - 1,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&-\nu\frac{D_x\widehat{U}^{n+1/2}_{1,i+1,j} - D_x\widehat{U}^{n+1/2}_{1,i,j}}{h_{i+1/2}} - \nu\frac{d_y\widehat{U}^{n+1/2}_{2,i+1/2,j+1/2} - d_y\widehat{U}^{n+1/2}_{2,i+1/2,j-1/2}}{k_j} \\
&\qquad + D_y\widehat{P}^{n+1/2}_{i+1/2,j} = g^{n+1/2}_{2,i+1/2,j}, \ 0 \le i \le N_x - 1, 1 \le j \le N_y - 1,
\end{aligned} \tag{4.3}$$

$$d_x\widehat{U}^{n+1/2}_{1,i+1/2,j+1/2} + d_y\widehat{U}^{n+1/2}_{2,i+1/2,j+1/2} = 0, \ 0 \le i \le N_x - 1, 0 \le j \le N_y - 1, \tag{4.4}$$

where the boundary and initial approximations are same as Eqs. (3.17) and (3.18).

Inspired by Ref. 6, we extend the work in Rui and Li[16] to the above approximation. By following closely the same arguments as in Ref. 16, we can prove the following

Lemma 4.1. *Assuming that* $\boldsymbol{u} \in W^3_\infty(J; W^4_\infty(\Omega))^2, \ p \in W^3_\infty(J; W^3_\infty(\Omega))$, *we have the following results*:

$$\|d_x(\widehat{U}^{n+1}_1 - u^{n+1}_1)\|_{l^2,M} + \|d_y(\widehat{U}^{n+1}_2 - u^{n+1}_2)\|_{l^2,M} \le O(\Delta t^2 + h^2 + k^2), \tag{4.5}$$

$$\|d_t(\widehat{U}^{n+1}_1 - u^{n+1}_1)\|_{l^2,T,M} + \|d_t(\widehat{U}^{n+1}_2 - u^{n+1}_2)\|_{l^2,M,T} \le O(\Delta t^2 + h^2 + k^2), \tag{4.6}$$

$$\|\widehat{U}^{n+1}_1 - u^{n+1}_1\|_{l^2,T,M} + \|\widehat{U}^{n+1}_2 - u^{n+1}_2\|_{l^2,M,T} \le O(\Delta t^2 + h^2 + k^2), \tag{4.7}$$

$$\|D_y(\widehat{U}^{n+1}_1 - u^{n+1}_1)\|_{l^2,T_y} \le O(\Delta t^2 + h^2 + k^{3/2}), \tag{4.8}$$

$$\|D_x(\widehat{U}^{n+1}_2 - u^{n+1}_2)\|_{l^2,T_x} \le O(\Delta t^2 + h^{3/2} + k^2), \tag{4.9}$$

$$\left(\sum_{l=1}^{N} \Delta t \|(\widehat{Z} - p)^{l-1/2}\|^2_{l^2,M}\right)^{1/2} \le O(\Delta t^2 + h^2 + k^2). \tag{4.10}$$

4.2. *Discrete LBB condition*

In order to carry out error analysis, we need the discrete LBB condition.

Here we use the same notation and results as Rui and Li. [16, Lemma 3.3] Let

$$b(\mathbf{v}, q) = -\int_\Omega q div \mathbf{v} dx, \quad \mathbf{v} \in \mathbf{V}, \quad q \in W,$$

where

$$\mathbf{V} = H_0^1(\Omega) \times H_0^1(\Omega), \quad W = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}.$$

Then, we construct the finite-dimensional subspaces of W and $\mathbf{V}$ by introducing three different partitions $\mathcal{T}_h, \mathcal{T}_h^1, \mathcal{T}_h^2$ of Ω. The original partition $\delta_x \times \delta_y$ is denoted by $\mathcal{T}_h$ (see Fig 1). The partition $\mathcal{T}_h^1$ is generated by connecting all the midpoints of the vertical sides of $\Omega_{i+1/2,j+1/2}$ and extending the resulting mesh to the boundary Γ. Similarly, for all $\Omega_{i+1/2,j+1/2} \in \mathcal{T}_h$, we connect all the midpoints of the horizontal sides of $\Omega_{i+1/2,j+1/2}$ and extend the resulting mesh to the boundary Γ, then the third partition is obtained which is denoted by $\mathcal{T}_h^2$.

Corresponding to the quadrangulation $\mathcal{T}_h$, define W_h, a subspace of W,

$$W_h = \left\{ q_h : \ q_h|_T = constant, \ \forall T \in \mathcal{T}_h \ and \int_\Omega q dx = 0 \right\}.$$

Furthermore, let $\mathbf{V}_h$ be a subspace of $\mathbf{V}$ such that $\mathbf{V}_h = S_h^1 \times S_h^2$, where

$$S_h^l = \left\{ g \in C^{(0)}(\overline{\Omega}) : g|_{T^l} \in Q_1(T^l), \ \forall T^l \in \mathcal{T}_h^l, \ and \ g|_\Gamma = 0 \right\}, \ l = 1, 2,$$

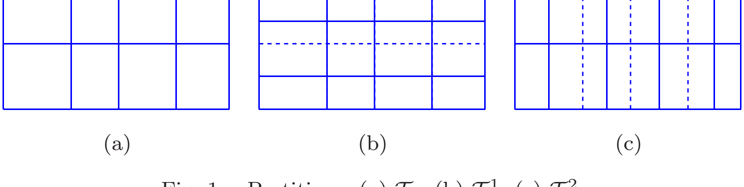
and Q_1 denotes the space of all polynomials of degree ≤ 1 with respect to each of the two variables x and y.

Then, we introduce the bilinear forms

$$b_h(\mathbf{v}_h, q_h) = -\sum_{\Omega_{i+1/2,j+1/2} \in \mathcal{T}_h} \int_{\Omega_{i+1/2,j+1/2}} q_h \Pi_h (div \mathbf{v}_h) dx, \ \mathbf{v}_h \in \mathbf{V}_h, \ q_h \in W_h,$$

where

$$\begin{aligned} &\Pi_h : \ C^{(0)}(\overline{\Omega}_{i+1/2,j+1/2}) \to Q_0(\Omega_{i+1/2,j+1/2}), \ such \ that \\ &\qquad (\Pi_h \varphi)_{i+1/2,j+1/2} = \varphi_{i+1/2,j+1/2}, \quad \forall \, \Omega_{i+1/2,j+1/2} \in \mathcal{T}_h. \end{aligned}$$



(a) (b) (c)

Fig. 1. Partitions: (a) $\mathcal{T}_h$, (b) $\mathcal{T}_h^1$, (c) $\mathcal{T}_h^2$.

Then, we have the following result:

Lemma 4.2. *There is a constant $\beta > 0$, independent of h and k such that*

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b_h(\boldsymbol{v}_h, q_h)}{\|D\boldsymbol{v}_h\|} \geq \beta \|q_h\|_{l^2,M} \quad \forall\, q_h \in W_h. \tag{4.11}$$

4.3. *A first error estimate with a L^∞ bound assumption*

We shall first derive an error estimate assuming that there exists two positive constant C_* and C^* such that

$$\|Z^n\|_\infty \leq C_*, \tag{4.12a}$$

$$\|\mathbf{D}Z^n\|_\infty \leq C^*. \tag{4.12b}$$

Later we shall verify this assumption using an induction process.

We define the operator $\mathbf{I}_h : \mathbf{V} \to \mathbf{V}_h$, such that

$$(\nabla \cdot \mathbf{I}_h \mathbf{v}, w) = (\nabla \cdot \mathbf{v}, w)\ \forall\, w \in W_h, \tag{4.13}$$

with approximation properties[6]

$$\|\mathbf{v} - \mathbf{I}_h \mathbf{v}\| \leq C \|\mathbf{v}\|_{W_2^1(\Omega)} \hat{h}, \tag{4.14}$$

$$\|\nabla \cdot (\mathbf{v} - \mathbf{I}_h \mathbf{v})\| \leq C \|\nabla \cdot \mathbf{v}\|_{W_2^1(\Omega)} \hat{h}, \tag{4.15}$$

where $\hat{h} = \max\{h, k\}$.

Besides, by the definition of $\mathbf{I}_h \mathbf{v}$ and the midpoint rule of integration, the L^∞ norm of the projection is obtained by

$$\|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_\infty \leq C \|\mathbf{v}\|_{W_\infty^2(\Omega)} \hat{h}. \tag{4.16}$$

Furthermore from Durán,[8] we have the following estimates which is necessary for the derivative and analysis of our numerical scheme:

$$\|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{l^2} \leq C \hat{h}^2. \tag{4.17}$$

For simplicity, we set

$$\begin{aligned}
&e_\phi^n = Z^n - \phi^n, \quad e_\mu^n = W^n - \mu^n, \quad e_r^n = R^n - r^n,\\
&e_{\mathbf{u}}^n = \mathbf{U}^n - \widehat{\mathbf{U}}^n + \widehat{\mathbf{U}}^n - \mathbf{u}^n = \widehat{e}_{\mathbf{u}}^n + \widetilde{e}_{\mathbf{u}}^n,\\
&e_p^n = P^n - \widehat{P}^n + \widehat{P}^n - p^n = \widehat{e}_p^n + \widetilde{e}_p^n.
\end{aligned}$$

Lemma 4.3. *Suppose that the hypotheses* (4.12) *hold, and* $\phi \in W_\infty^3(J; W_\infty^4(\Omega))$, $\mu \in L^\infty(J; W_\infty^4(\Omega))$, $\boldsymbol{u} \in W_\infty^3(J; W_\infty^4(\Omega))^2$, $p \in W_\infty^3(J; W_\infty^3(\Omega))$, *then the* *approximate errors of discrete phase function and chemical potential satisfy*

$$\begin{aligned}
&\|e_\phi^{m+1}\|_{l^2,M}^2 + \frac{M}{2}\sum_{n=0}^{m}\Delta t\|e_\mu^{n+1/2}\|_{l^2,M}^2 + \lambda(e_r^{m+1})^2 \\
&\quad + \frac{\lambda}{2}\|\boldsymbol{D}e_\phi^{m+1}\|_{l^2}^2 + \frac{M}{4}\sum_{n=0}^{m}\Delta t\|\boldsymbol{D}e_\mu^{n+1/2}\|_{l^2}^2 \\
&\le C\sum_{n=0}^{m+1}\Delta t\|\boldsymbol{D}e_\phi^n\|_{l^2}^2 + C\sum_{n=0}^{m}\Delta t\|\widehat{e}_{\boldsymbol{u}}^{n+1/2}\|_{l^2}^2 \\
&\quad + C\sum_{n=0}^{m+1}\Delta t\|e_\phi^n\|_{l^2,M}^2 + C\sum_{n=0}^{m+1}\Delta t(e_r^n)^2 \\
&\quad + C(\Delta t^4 + h^4 + k^4), \qquad m \le N,
\end{aligned} \tag{4.18}$$

where the positive constant C is independent of h, k and Δt.

Proof. Denote

$$\begin{aligned}
\delta_x(\phi) &= D_x\phi - \frac{\partial\phi}{\partial x}, \quad \delta_y(\phi) = D_y\phi - \frac{\partial\phi}{\partial y}, \\
\delta_x(\mu) &= D_x\mu - \frac{\partial\mu}{\partial x}, \quad \delta_y(\mu) = D_y\mu - \frac{\partial\mu}{\partial y}.
\end{aligned}$$

Subtracting (3.1a) from (3.19a), we obtain

$$\begin{aligned}
[d_t e_\phi]_{i+1/2,j+1/2}^{n+1} &= M[d_x(D_x e_\mu + \delta_x(\mu)) + d_y(D_y e_\mu + \delta_y(\mu))]_{i+1/2,j+1/2}^{n+1/2} \\
&\quad - \mathcal{P}_h^y\mathcal{P}_h^x[U_1 D_x\tilde{Z} + U_2 D_y\tilde{Z}]_{i+1/2,j+1/2}^{n+1/2} \\
&\quad + \mathbf{u}_{i+1/2,j+1/2}^{n+1/2}\cdot\nabla\phi_{i+1/2,j+1/2}^{n+1/2} \\
&\quad + T_{1,i+1/2,j+1/2}^{n+1/2} + T_{2,i+1/2,j+1/2}^{n+1/2},
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
T_{1,i+1/2,j+1/2}^{n+1/2} &= \frac{\partial\phi}{\partial t}\Big|_{i+1/2,j+1/2}^{n+1/2} - [d_t\phi]_{i+1/2,j+1/2}^{n+1} \\
&\le C\|\phi\|_{W_\infty^3(J;L^\infty(\Omega))}\Delta t^2,
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
T_{2,i+1/2,j+1/2}^{n+1/2} &= M\left[d_x\frac{\partial\mu}{\partial x} + d_y\frac{\partial\mu}{\partial y}\right]_{i+1/2,j+1/2}^{n+1/2} - M\Delta\mu_{i+1/2,j+1/2}^{n+1/2} \\
&\le CM(h^2+k^2)\|\mu\|_{L^\infty(J;W_\infty^4(\Omega))}.
\end{aligned} \tag{4.21}$$

Subtracting (3.1b) from (3.19b) leads to

$$\begin{aligned} e_{\mu,i+1/2,j+1/2}^{n+1/2} &= -\lambda[d_x(D_x e_\phi + \delta_x(\phi)) + d_y(D_y e_\phi + \delta_y(\phi))]_{i+1/2,j+1/2}^{n+1/2} \\ &\quad + \lambda \frac{R^{n+1/2}}{\sqrt{E_1^h(\tilde{Z}^{n+1/2}) + \delta}} F'(\tilde{Z}_{i+1/2,j+1/2}^{n+1/2}) \\ &\quad - \lambda \frac{r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2}) + \delta}} F'(\phi_{i+1/2,j+1/2}^{n+1/2}) \\ &\quad + \lambda T_{3,i+1/2,j+1/2}^{n+1/2}, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} T_{3,i+1/2,j+1/2}^{n+1/2} &= \Delta\phi_{i+1/2,j+1/2}^{n+1/2} - \left[d_x \frac{\partial \phi}{\partial x} + d_y \frac{\partial \phi}{\partial y}\right]_{i+1/2,j+1/2}^{n+1/2} \\ &\le C(h^2 + k^2)\|\phi\|_{L^\infty(J;W_\infty^4(\Omega))}. \end{aligned} \tag{4.23}$$

Subtracting Eq. (3.1c) from Eq. (3.19c) gives that

$$\begin{aligned} d_t e_r^{n+1} &= \frac{1}{2\sqrt{E_1^h(\tilde{Z}^{n+1/2}) + \delta}} (F'(\tilde{Z}^{n+1/2}), d_t Z^{n+1})_{l^2,M} \\ &\quad - \frac{1}{2\sqrt{E_1(\phi^{n+1/2}) + \delta}} \int_\Omega F'(\phi^{n+1/2}) \phi_t^{n+1/2} d\mathbf{x} + T_4^{n+1/2}, \end{aligned} \tag{4.24}$$

where

$$T_4^{n+1/2} = r_t^{n+1/2} - d_t r^{n+1} \le C\|r\|_{W_\infty^3(J)} \Delta t^2. \tag{4.25}$$

Multiplying Eq. (4.19) by $e_{\mu,i+1/2,j+1/2}^{n+1/2} hk$, and making summation on i, j for $0 \le i \le N_x - 1,\ 0 \le j \le N_y - 1$, we have

$$\begin{aligned} &(d_t e_\phi^{n+1}, e_\mu^{n+1/2})_{l^2,M} \\ &\quad = M\left(d_x(D_x e_\mu + \delta_x(\mu))^{n+1/2} + d_y(D_y e_\mu + \delta_y(\mu))^{n+1/2}, e_\mu^{n+1/2}\right)_{l^2,M} \\ &\qquad - (\mathcal{P}_h^y \mathcal{P}_h^x [U_1 D_x \tilde{Z} + U_2 D_y \tilde{Z}]^{n+1/2} - \mathbf{u}^{n+1/2} \cdot \nabla \phi^{n+1/2}, e_\mu^{n+1/2})_{l^2,M} \\ &\qquad + (T_1^{n+1/2}, e_\mu^{n+1/2})_{l^2,M} + (T_2^{n+1/2}, e_\mu^{n+1/2})_{l^2,M}. \end{aligned} \tag{4.26}$$

Recalling Lemma 3.1, the first term on the right-hand side of (4.26) can be estimated as follows:

$$\begin{aligned} &M\left(d_x(D_x e_\mu + \delta_x(\mu))^{n+1/2} + d_y(D_y e_\mu + \delta_y(\mu))^{n+1/2}, e_\mu^{n+1/2}\right)_{l^2,M} \\ &\quad = -M\left((D_x e_\mu + \delta_x(\mu))^{n+1/2}, D_x e_\mu^{n+1/2}\right)_{l^2,T,M} \\ &\qquad - M\left((D_y e_\mu + \delta_y(\mu))^{n+1/2}, D_y e_\mu^{n+1/2}\right)_{l^2,M,T} \end{aligned}$$

$$= -M\|\mathbf{D}e_\mu^{n+1/2}\|_{l^2}^2 - M(\delta_x(\mu)^{n+1/2}, D_x e_\mu^{n+1/2})_{l^2,T,M} \\ -\ M(\delta_y(\mu)^{n+1/2}, D_y e_\mu^{n+1/2})_{l^2,M,T}. \tag{4.27}$$

With the aid of Cauchy–Schwarz inequality, the last two terms on the right-hand side of (4.27) can be transformed into:

$$-M(\delta_x(\mu)^{n+1/2}, D_x e_\mu^{n+1/2})_{l^2,M,T} - M(\delta_y(\mu)^{n+1/2}, D_y e_\mu^{n+1/2})_{l^2,T,M} \\ \leq \frac{M}{6}\|\mathbf{D}e_\mu^{n+1/2}\|_{l^2}^2 + C\|\mu\|_{L^\infty(J;W_\infty^3(\Omega))}^2(h^4+k^4). \tag{4.28}$$

The second term on the right-hand side of (4.26) can be transformed into

$$-(\mathcal{P}_h^y\mathcal{P}_h^x[U_1D_x\tilde{Z} + U_2D_y\tilde{Z}]^{n+1/2} - \mathbf{u}^{n+1/2}\cdot\nabla\phi^{n+1/2}, e_\mu^{n+1/2})_{l^2,M} \\ = -(\mathcal{P}_h^y\mathcal{P}_h^x[U_1D_x\tilde{Z} + U_2D_y\tilde{Z}]^{n+1/2} \\ - \mathcal{P}_h^y\mathcal{P}_h^x[\widehat{U}_1D_x\tilde{Z} + \widehat{U}_2D_y\tilde{Z}]^{n+1/2}, e_\mu^{n+1/2})_{l^2,M} \\ - (\mathcal{P}_h^y\mathcal{P}_h^x[\widehat{U}_1D_x\tilde{Z} + \widehat{U}_2D_y\tilde{Z}]^{n+1/2} \\ - \mathcal{P}_h^y\mathcal{P}_h^x[u_1D_x\tilde{Z} + u_2D_y\tilde{Z}]^{n+1/2}, e_\mu^{n+1/2})_{l^2,M} \\ - (\mathcal{P}_h^y\mathcal{P}_h^x[u_1D_x\tilde{Z} + u_2D_y\tilde{Z}]^{n+1/2} - \mathbf{u}^{n+1/2}\cdot\nabla\phi^{n+1/2}, e_\mu^{n+1/2})_{l^2,M}. \tag{4.29}$$

Then, taking note of the definition of interpolations $\mathcal{P}_h^x$ and $\mathcal{P}_h^y$, the first term on the right-hand side of (4.29) can be bounded by

$$-(\mathcal{P}_h^y\mathcal{P}_h^x[U_1D_x\tilde{Z} + U_2D_y\tilde{Z}]^{n+1/2} - \mathcal{P}_h^y\mathcal{P}_h^x[\widehat{U}_1D_x\tilde{Z} + \widehat{U}_2D_y\tilde{Z}]^{n+1/2}, e_\mu^{n+1/2})_{l^2,M} \\ \leq C\|\mathbf{D}\tilde{Z}\|_\infty^2\|\widehat{e}_{\mathbf{u}}^{n+1/2}\|_{l^2}^2 + C\|e_\mu^{n+1/2}\|_{l^2,M}^2. \tag{4.30}$$

Similarly noting Lemma 4.1, the second term on the right-hand side of (4.29) can be estimated by

$$-\ (\mathcal{P}_h^y\mathcal{P}_h^x[\widehat{U}_1D_x\tilde{Z} + \widehat{U}_2D_y\tilde{Z}]^{n+1/2} - \mathcal{P}_h^y\mathcal{P}_h^x[u_1D_x\tilde{Z} + u_2D_y\tilde{Z}]^{n+1/2}, e_\mu^{n+1/2})_{l^2,M} \\ \leq C\|\mathbf{D}\tilde{Z}\|_\infty^2\|\widetilde{e}_{\mathbf{u}}^{n+1/2}\|_{l^2}^2 + C\|e_\mu^{n+1/2}\|_{l^2,M}^2 \\ \leq C\|e_\mu^{n+1/2}\|_{l^2,M}^2 + C(\Delta t^4 + h^4 + k^4). \tag{4.31}$$

Supposing that $\phi \in W^{2,\infty}(J;L^\infty(\Omega))$, the last term on the right-hand side of (4.29) can be estimated by

$$-\ (\mathcal{P}_h^y\mathcal{P}_h^x[u_1D_x\tilde{Z} + u_2D_y\tilde{Z}]^{n+1/2} - \mathbf{u}^{n+1/2}\cdot\nabla\phi^{n+1/2}, e_\mu^{n+1/2})_{l^2,M} \\ \leq C\|e_\mu^{n+1/2}\|_{l^2,M}^2 + C\|\mathbf{D}e_\phi^n\|_{l^2,M}^2 + C\|\mathbf{D}e_\phi^{n-1}\|_{l^2,M}^2 \\ +\ C\|\phi\|_{W_\infty^2(J;L^\infty(\Omega))}^2\Delta t^4. \tag{4.32}$$

Multiplying Eq. (4.22) by $d_t e^{n+1}_{\phi,i+1/2,j+1/2} hk$, and making summation on i, j for $0 \le i \le N_x - 1,\ 0 \le j \le N_y - 1$, we have

$$
\begin{aligned}
&(e^{n+1/2}_{\mu}, d_t e^{n+1}_{\phi})_{l^2,M} \\
&\quad = -\lambda(d_x(D_x e_\phi + \delta_x(\phi))^{n+1/2} + d_y(D_y e_\phi + \delta_y(\phi))^{n+1/2}, d_t e^{n+1}_{\phi})_{l^2,M} \\
&\qquad + \lambda\left(\frac{R^{n+1/2}}{\sqrt{E^h_1(\tilde{Z}^{n+1/2}) + \delta}} F'(\tilde{Z}^{n+1/2}) \right. \\
&\qquad \left. - \frac{r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2}) + \delta}} F'(\phi^{n+1/2}), d_t e^{n+1}_{\phi}\right)_{l^2,M} \\
&\qquad + \lambda(T^{n+1/2}_3, d_t e^{n+1}_{\phi})_{l^2,M}.
\end{aligned}
\tag{4.33}
$$

Similar to the estimate of Eq. (3.24), the first term on the right-hand side of Eq. (4.33) can be transformed into the following:

$$
\begin{aligned}
&-\lambda(d_x(D_x e_\phi + \delta_x(\phi))^{n+1/2} + d_y(D_y e_\phi + \delta_y(\phi))^{n+1/2}, d_t e^{n+1}_{\phi})_{l^2,M} \\
&\quad = \lambda(D_x e^{n+1/2}_{\phi}, d_t D_x e^{n+1}_{\phi})_{l^2,T,M} + \lambda(D_y e^{n+1/2}_{\phi}, d_t D_y e^{n+1}_{\phi})_{l^2,M,T} \\
&\qquad + \lambda(\delta_x(\phi)^{n+1/2}, d_t D_x e^{n+1/2}_{\phi})_{l^2,T,M} \\
&\qquad + \lambda(\delta_y(\phi)^{n+1/2}, d_t D_y e^{n+1/2}_{\phi})_{l^2,M,T} \\
&\quad = \lambda \frac{\|\mathbf{D} e^{n+1}_{\phi}\|^2_{l^2} - \|\mathbf{D} e^{n}_{\phi}\|^2_{l^2}}{2\Delta t} + \lambda(\delta_x(\phi)^{n+1/2}, d_t D_x e^{n+1/2}_{\phi})_{l^2,T,M} \\
&\qquad + \lambda(\delta_y(\phi)^{n+1/2}, d_t D_y e^{n+1/2}_{\phi})_{l^2,M,T}.
\end{aligned}
\tag{4.34}
$$

The second term on the right-hand side of Eq. (4.33) can be rewritten as follows:

$$
\begin{aligned}
&\lambda\left(\frac{R^{n+1/2}}{\sqrt{E^h_1(\tilde{Z}^{n+1/2}) + \delta}} F'(\tilde{Z}^{n+1/2}) - \frac{r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2}) + \delta}} F'(\phi^{n+1/2}), d_t e^{n+1}_{\phi}\right)_{l^2,M} \\
&\quad = \lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E^h_1(\tilde{Z}^{n+1/2}) + \delta}} - \frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E^h_1(\tilde{\phi}^{n+1/2}) + \delta}}, d_t e^{n+1}_{\phi}\right)_{l^2,M} \\
&\qquad + \lambda r^{n+1/2}\left(\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E^h_1(\tilde{\phi}^{n+1/2}) + \delta}} - \frac{F'(\phi^{n+1/2})}{\sqrt{E_1(\phi^{n+1/2}) + \delta}}, d_t e^{n+1}_{\phi}\right)_{l^2,M} \\
&\qquad + \lambda e^{n+1/2}_{r}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E^h_1(\tilde{Z}^{n+1/2}) + \delta}}, d_t e^{n+1}_{\phi}\right)_{l^2,M}.
\end{aligned}
\tag{4.35}
$$

Taking note of (4.19), the first term on the right-hand side of (4.35) can be transformed into the following:

$$\begin{aligned}
&\lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}-\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E_1^h(\tilde{\phi}^{n+1/2})+\delta}}, d_t e_\phi^{n+1}\right)_{l^2,M}\\
&\quad= M\lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}-\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E_1^h(\tilde{\phi}^{n+1/2})+\delta}}, d_x(D_x e_\mu+\delta_x(\mu))^{n+1/2}\right)_{l^2,M}\\
&\qquad+ M\lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}-\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E_1^h(\tilde{\phi}^{n+1/2})+\delta}}, d_y(D_y e_\mu+\delta_y(\mu))^{n+1/2}\right)_{l^2,M}\\
&\qquad- \lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}-\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E_1^h(\tilde{\phi}^{n+1/2})+\delta}}, \mathcal{P}_h[U_1 D_x\tilde{Z}+U_2 D_y\tilde{Z}]^{n+1/2}\right.\\
&\qquad\left.- \mathbf{u}_{i+1/2,j+1/2}^{n+1/2}\cdot\nabla\phi^{n+1/2}\right)_{l^2,M}\\
&\qquad+ \lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}-\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E_1^h(\tilde{\phi}^{n+1/2})+\delta}}, T_1^{n+1/2}+T_2^{n+1/2}\right)_{l^2,M}. \qquad (4.36)
\end{aligned}$$

Similar to the estimates in Ref. 14, and using the Cauchy–Schwartz inequality, we can deduce that

$$\begin{aligned}
&M\lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}-\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E_1^h(\tilde{\phi}^{n+1/2})+\delta}}, d_x(D_x e_\mu+\delta_x(\mu))^{n+1/2}\right)_{l^2,M}\\
&\quad= -M\lambda r^{n+1/2}\left(\frac{D_xF'(\tilde{Z}^{n+1/2})}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}-\frac{D_xF'(\tilde{\phi}^{n+1/2})}{\sqrt{E_1^h(\tilde{\phi}^{n+1/2})+\delta}}, (D_x e_\mu+\delta_x(\mu))^{n+1/2}\right)_{l^2,M}\\
&\quad\le \frac{M}{6}\|D_x e_\mu^{n+1/2}\|^2_{l^2,T,M}+C\|r\|^2_{L^\infty(J)}(\|e_\phi^n\|^2_m+\|e_\phi^{n-1}\|^2_{l^2,M})\\
&\qquad+C\|r\|^2_{L^\infty(J)}(\|D_x e_\phi^n\|^2_{l^2,T,M}+\|D_x e_\phi^{n-1}\|^2_{l^2,T,M})\\
&\qquad+C\|\mu\|^2_{L^\infty(J;W^3_\infty(\Omega))}(h^4+k^4). \qquad (4.37)
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&M\lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})+\delta}}-\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E_1^h(\tilde{\phi}^{n+1/2})+\delta}}, d_y(D_y e_\mu+\delta_y(\mu))^{n+1/2}\right)_{l^2,M}\\
&\quad\le \frac{M}{6}\|D_y e_\mu^{n+1/2}\|^2_{l^2,M,T}+C\|r\|^2_{L^\infty(J)}(\|e_\phi^n\|^2_{l^2,M}+\|e_\phi^{n-1}\|^2_{l^2,M})
\end{aligned}$$

$$+ C\|r\|^2_{L^\infty(J)}(\|D_y e^n_\phi\|^2_{l^2,M,T} + \|D_y e^{n-1}_\phi\|^2_{l^2,M,T})$$
$$+ C\|\mu\|^2_{L^\infty(J;W^3_\infty(\Omega))}(h^4 + k^4). \tag{4.38}$$

Then Eq. (4.36) can be estimated by:

$$\begin{aligned}
&\lambda r^{n+1/2}\left(\frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E^h_1(\tilde{Z}^{n+1/2})+\delta}} - \frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E^h_1(\tilde{\phi}^{n+1/2})+\delta}}, d_t e^{n+1}_\phi\right)_{l^2,M}\\
&\quad\le \frac{M}{6}\|\mathbf{D}e^{n+1/2}_\mu\|^2_{l^2} + C\|r\|_{L^\infty(J)}(\|e^n_\phi\|^2_{l^2,M} + \|e^{n-1}_\phi\|_{l^2,M^2})\\
&\qquad + C\|r\|_{L^\infty(J)}(\|\mathbf{D}e^n_\phi\|^2_{l^2} + \|\mathbf{D}e^{n-1}_\phi\|^2_{l^2}) + C\|\mathbf{D}\tilde{Z}\|^2_\infty\|\hat{e}^{n+1/2}_{\mathbf{u}}\|^2_{l^2}\\
&\qquad + C\|\mu\|^2_{L^\infty(J;W^4_\infty(\Omega))}(h^4+k^4) + C\|\phi\|^2_{W^3_\infty(J;L^\infty(\Omega))}\Delta t^4.
\end{aligned} \tag{4.39}$$

Similar to the estimates of (4.36), the second term on the right-hand side of (4.35) can be controlled by:

$$\begin{aligned}
&\lambda r^{n+1/2}\left(\frac{F'(\tilde{\phi}^{n+1/2})}{\sqrt{E^h_1(\tilde{\phi}^{n+1/2})+\delta}} - \frac{F'(\phi^{n+1/2})}{\sqrt{E_1(\phi^{n+1/2})+\delta}}, d_t e^{n+1}_\phi\right)_{l^2,M}\\
&\quad\le \frac{M}{6}\|\mathbf{D}e^{n+1/2}_\mu\|^2_{l^2} + C\|\mathbf{D}e^n_\phi\|^2_{l^2,M} + C\|\mathbf{D}e^{n-1}_\phi\|^2_{l^2,M}\\
&\qquad + C\|\mathbf{D}\tilde{Z}\|^2_\infty\|\hat{e}^{n+1/2}_{\mathbf{u}}\|^2_{l^2} + C\|\phi\|^2_{W^3_\infty(J;W^1_\infty(\Omega))}\Delta t^4\\
&\qquad + C(\|\mu\|^2_{L^\infty(J;W^4_\infty(\Omega))} + \|\phi\|^2_{L^\infty(J;W^2_\infty(\Omega))})(h^4+k^4).
\end{aligned} \tag{4.40}$$

Multiplying Eq. (4.24) by $\lambda(e^{n+1}_r + e^n_r)$ leads to

$$\begin{aligned}
\lambda\frac{(e^{n+1}_r)^2 - (e^n_r)^2}{\Delta t} &= \lambda\frac{e^{n+1/2}_r}{\sqrt{E^h_1(\tilde{Z}^{n+1/2})+\delta}}(F'(\tilde{Z}^{n+1/2}), d_t Z^{n+1})_{l^2,M}\\
&\quad - \lambda\frac{e^{n+1/2}_r}{\sqrt{E_1(\phi^{n+1/2})+\delta}}\int_\Omega F'(\phi^{n+1/2})\phi^{n+1/2}_t d\mathbf{x}\\
&\quad + \lambda T^{n+1/2}_4\cdot(e^{n+1}_r + e^n_r).
\end{aligned} \tag{4.41}$$

Then similar to the estimates in Ref. 14, we have

$$\begin{aligned}
&\lambda\frac{(e^{n+1}_r)^2 - (e^n_r)^2}{\Delta t}\\
&\quad\le \lambda\frac{e^{n+1/2}_r}{\sqrt{E^h_1(\tilde{Z}^{n+1/2})+\delta}}(F'(\tilde{Z}^{n+1/2}), d_t e^{n+1}_\phi)_{l^2,M} + \lambda T^{n+1/2}_4\cdot(e^{n+1}_r + e^n_r)
\end{aligned}$$

$$
\begin{aligned}
&+C(e_r^{n+1/2})^2+C\|\phi\|^2_{W^1_\infty(J;L^\infty(\Omega))}(\|e_\phi^n\|^2_{l^2,M}+\|e_\phi^{n-1}\|^2_{l^2,M})\\
&+C\|\phi\|^2_{W^1_\infty(J;W^2_\infty(\Omega))}(h^4+k^4).
\end{aligned}
\tag{4.42}
$$

Combining the above equations and using Cauchy–Schwarz inequality leads to

$$
\begin{aligned}
&\lambda\frac{(e_r^{n+1})^2-(e_r^n)^2}{\Delta t}+\lambda\frac{\|\mathbf{D}e_\phi^{n+1}\|^2_{l^2}-\|\mathbf{D}e_\phi^n\|^2_{l^2}}{2\Delta t}+M\|\mathbf{D}e_\mu^{n+1/2}\|^2_{l^2}\\
&\quad\le\frac{M}{2}\|\mathbf{D}e_\mu^{n+1/2}\|^2_{l^2}+C\|e_\mu^{n+1/2}\|^2_{l^2,M}+C\|r\|^2_{L^\infty(J)}(\|e_\phi^n\|^2_{l^2,M}+\|e_\phi^{n-1}\|^2_{l^2,M})\\
&\qquad+C\|\mathbf{D}\tilde{Z}\|^2_\infty\|\hat{e}_{\mathbf{u}}^{n+1/2}\|^2_{l^2}+C\|r\|^2_{L^\infty(J)}(\|\mathbf{D}e_\phi^n\|^2_{l^2}+\|\mathbf{D}e_\phi^{n-1}\|^2_{l^2})\\
&\qquad-\lambda(\delta_x(\phi)^{n+1/2},d_tD_xe_\phi^{n+1/2})_{l^2,T,M}-\lambda(\delta_y(\phi)^{n+1/2},d_tD_ye_\phi^{n+1/2})_{l^2,M,T}\\
&\qquad+\lambda(T_3^{n+1/2},d_te_\phi^{n+1})_{l^2,M}+\lambda T_4^{n+1/2}\cdot(e_r^{n+1}+e_r^n)\\
&\qquad+C(e_r^{n+1/2})^2+C\|\phi\|^2_{W^1_\infty(J;L^\infty(\Omega))}(\|e_\phi^n\|^2_{l^2,M}+\|e_\phi^{n-1}\|^2_{l^2,M})\\
&\qquad+C(\|\phi\|^2_{W^1_\infty(J;W^2_\infty(\Omega))}+\|\mu\|^2_{L^\infty(J;W^4_\infty(\Omega))})(h^4+k^4)\\
&\qquad+C\|\phi\|^2_{W^3_\infty(J;W^1_\infty(\Omega))}\Delta t^4.
\end{aligned}
\tag{4.43}
$$

Taking note of that

$$
\sum_{n=0}^{k}\Delta t(f^n,d_tg^{n+1})=-\sum_{n=1}^{k}\Delta t(d_tf^n,g^n)+(f^k,g^{k+1})+(f^0,g^0). \tag{4.44}
$$

Using the above equation and multiplying Eq. (4.43) by Δt, summing over n from 1 to m result in

$$
\begin{aligned}
&\lambda(e_r^{m+1})^2+\frac{\lambda}{2}\|\mathbf{D}e_\phi^{m+1}\|^2_{l^2}+\frac{M}{2}\sum_{n=0}^{m}\Delta t\|\mathbf{D}e_\mu^{n+1/2}\|^2_{l^2}\\
&\quad\le C\sum_{n=0}^{m+1}\Delta t\|\mathbf{D}e_\phi^n\|^2_{l^2}+\frac{M}{2}\sum_{n=0}^{k+1}\Delta t\|e_\mu^{n+1/2}\|^2_{l^2,M}\\
&\qquad+C\sum_{n=0}^{m+1}\Delta t\|\hat{e}_{\mathbf{u}}^{n+1/2}\|^2_{l^2}+C\sum_{n=0}^{m+1}\Delta t\|e_\phi^n\|^2_{l^2,M}\\
&\qquad+C\sum_{n=0}^{m+1}\Delta t(e_r^n)^2+C\|\phi\|^2_{W^3_\infty(J;W^{1,\infty}(\Omega))}\Delta t^4\\
&\qquad+C(\|\phi\|^2_{W^1_\infty(J;W^4_\infty(\Omega))}+\|\mu\|^2_{L^\infty(J;W^4_\infty(\Omega))})(h^4+k^4).
\end{aligned}
\tag{4.45}
$$

To proceed to the following error estimate, we should consider the second term on the right-hand side of (4.45). Multiplying (4.19) by $e^{n+1/2}_{\phi,i+1/2,j+1/2}hk$, and making summation on i, j for $0 \le i \le N_x - 1,\ 0 \le j \le N_y - 1$, we have

$$\begin{aligned}
&(d_t e_\phi^{n+1}, e_\phi^{n+1/2})_{l^2,M} \\
&\quad = M\left(d_x(D_x e_\mu + \delta_x(\mu))^{n+1/2} + d_y(D_y e_\mu + \delta_y(\mu))^{n+1/2}, e_\phi^{n+1/2}\right)_{l^2,M} \\
&\qquad - (\mathcal{P}_h[U_1 D_x \tilde{Z} + U_2 D_y \tilde{Z}]^{n+1/2} - \mathbf{u}^{n+1/2} \cdot \nabla\phi^{n+1/2}, e_\phi^{n+1/2})_{l^2,M} \\
&\qquad + (T_1^{n+1/2}, e_\phi^{n+1/2})_{l^2,M} + (T_2^{n+1/2}, e_\phi^{n+1/2})_{l^2,M}.
\end{aligned} \tag{4.46}$$

The first term on the right-hand side of (4.46) can be bounded by

$$\begin{aligned}
&M\left(d_x(D_x e_\mu + \delta_x(\mu))^{n+1/2} + d_y(D_y e_\mu + \delta_y(\mu))^{n+1/2}, e_\phi^{n+1/2}\right)_{l^2,M} \\
&\quad = -M\left((D_x e_\mu + \delta_x(\mu))^{n+1/2}, D_x e_\phi^{n+1/2}\right)_{l^2,T,M} \\
&\qquad - M\left((D_y e_\mu + \delta_y(\mu))^{n+1/2}, D_y e_\phi^{n+1/2}\right)_{l^2,M,T} \\
&\quad \le M\left(e_\mu^{n+1/2}, d_x(D_x e_\phi + \delta_x(\phi))^{n+1/2} + d_y(D_y e_\phi + \delta_y(\phi))^{n+1/2}\right)_{l^2,M} \\
&\qquad + \frac{M}{4}\|\mathbf{D}e_\mu^{n+1/2}\|_{l^2}^2 + C\|\mathbf{D}e_\phi^{n+1/2}\|_{l^2}^2 \\
&\qquad + C(\|\mu\|^2_{L^\infty(J;W^3_\infty(\Omega))} + \|\phi\|^2_{L^\infty(J;W^3_\infty(\Omega))})(h^4 + k^4) \\
&\quad \le -\frac{M}{2}\|e_\mu^{n+1/2}\|^2_{l^2,M} + C(e_r^{n+1} + e_r^n)^2 + C(\|e_\phi^n\|^2_{l^2,M} + \|e_\phi^{n-1}\|^2_{l^2,M}) \\
&\qquad + \frac{M}{4}\|\mathbf{D}e_\mu^{n+1/2}\|_{l^2}^2 + C\|\mathbf{D}e_\phi^{n+1/2}\|_{l^2}^2 + C\|\phi\|^2_{L^\infty(J;W^4_\infty(\Omega))}(h^4 + k^4) \\
&\qquad + C(\|\mu\|^2_{L^\infty(J;W^3_\infty(\Omega))} + \|\phi\|^2_{L^\infty(J;W^3_\infty(\Omega))})(h^4 + k^4).
\end{aligned} \tag{4.47}$$

The second term on the right-hand side of (4.46) can be estimated by

$$\begin{aligned}
&-(\mathcal{P}_h[U_1 D_x \tilde{Z} + U_2 D_y \tilde{Z}]^{n+1/2} - \mathbf{u}^{n+1/2} \cdot \nabla\phi^{n+1/2}, e_\phi^{n+1/2})_{l^2,M} \\
&\quad \le C\|\mathbf{D}\tilde{Z}\|_\infty^2 \|\widehat{e}_{\mathbf{u}}^{n+1/2}\|_{l^2}^2 + C\|\mathbf{D}e_\phi^n\|^2_{l^2,M} + C\|\mathbf{D}e_\phi^{n-1}\|^2_{l^2,M} \\
&\qquad + C\|e_\phi^{n+1/2}\|^2_{l^2,M} + C(\Delta t^4 + h^4 + k^4).
\end{aligned} \tag{4.48}$$

Combining (4.46) with (4.47) and (4.48), multiplying by $2\Delta t$, and summing over n from 1 to m give that

$$\begin{aligned}
&\|e_\phi^{m+1}\|^2_{l^2,M} + M\sum_{n=0}^{m} \Delta t \|e_\mu^{n+1/2}\|^2_{l^2,M} \\
&\quad \le C\sum_{n=0}^{m} \Delta t (e_r^{n+1})^2 + C\sum_{n=0}^{m} \Delta t \|e_\phi^{n+1}\|^2_{l^2,M} + C\sum_{n=0}^{m} \Delta t \|\widehat{e}_{\mathbf{u}}^{n+1/2}\|_{l^2}^2
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{M}{4}\sum_{n=0}^{k}\Delta t\|\mathbf{D}e_{\mu}^{n+1/2}\|_{l^2}^2 + C\sum_{n=0}^{k}\Delta t\|\mathbf{D}e_{\phi}^{n+1/2}\|_{l^2}^2 \\
&+ C(\|\mu\|^2_{L^\infty(J;W^4_\infty(\Omega))} + \|\phi\|^2_{L^\infty(J;W^4_\infty(\Omega))})(h^4+k^4) \\
&+ C\|\phi\|^2_{W^3_\infty(J;L^\infty(\Omega))}\Delta t^4.
\end{aligned} \tag{4.49}
$$

Combining (4.45) with the above equation leads to

$$
\begin{aligned}
&\|e_{\phi}^{m+1}\|^2_{l^2,M} + \frac{M}{2}\sum_{n=0}^{m}\Delta t\|e_{\mu}^{n+1/2}\|^2_{l^2,M} + \lambda(e_r^{m+1})^2 \\
&\quad + \frac{\lambda}{2}\|\mathbf{D}e_{\phi}^{m+1}\|^2_{l^2} + \frac{M}{4}\sum_{n=0}^{m}\Delta t\|\mathbf{D}e_{\mu}^{n+1/2}\|^2_{l^2} \\
&\quad \le C\sum_{n=0}^{m+1}\Delta t\|\mathbf{D}e_{\phi}^{n}\|^2_{l^2} + C\sum_{n=0}^{m}\Delta t\|\widehat{e}_{\mathbf{u}}^{n+1/2}\|^2_{l^2} \\
&\quad + C\sum_{n=0}^{m+1}\Delta t\|e_{\phi}^{n}\|^2_{l^2,M} + C\sum_{n=0}^{m+1}\Delta t(e_r^n)^2 \\
&\quad + C(\Delta t^4 + h^4 + k^4).
\end{aligned} \tag{4.50}
$$

□

Lemma 4.4. *Suppose that the hypotheses* (4.12) *hold, and* $\phi \in W^3_\infty(J;W^4_\infty(\Omega))$, $\mu \in L^\infty(J;W^4_\infty(\Omega))$, $\boldsymbol{u} \in W^3_\infty(J;W^4_\infty(\Omega))^2$, $p \in W^3_\infty(J;W^3_\infty(\Omega))$, *then for the case of Stokes equation, the approximate errors of discrete velocity and pressure satisfy*

$$
\begin{aligned}
&\|\widehat{e}_{\mathbf{u}}^{m+1}\|^2_{l^2} + \|\boldsymbol{D}\widehat{e}_{\mathbf{u}}^{m+1}\|^2 + \sum_{n=0}^{m}\Delta t\|\widehat{e}_{p}^{n+1/2}\|^2_{l^2,M} \\
&\quad \le C\sum_{n=0}^{m}\Delta t\|e_{\mu}^{n+1/2}\|^2_{l^2,M} + C\sum_{n=0}^{m}\Delta t\|e_{\phi}^{n}\|^2_{l^2,M} \\
&\quad + C(\Delta t^4 + h^4 + k^4), \qquad m \le N,
\end{aligned} \tag{4.51}
$$

where the positive constant C *is independent of* h, k *and* Δt.

Proof. Subtracting (4.2) from (3.19d) for the case of Stokes equation with $\gamma = 0$, we can obtain

$$
\begin{aligned}
&d_t\widehat{e}^{n+1}_{\mathbf{u},1,i,j+1/2} - \nu\frac{d_x\widehat{e}^{n+1/2}_{\mathbf{u},1,i+1/2,j+1/2} - d_x\widehat{e}^{n+1/2}_{\mathbf{u},1,i-1/2,j+1/2}}{h_i} \\
&\quad - \nu\frac{D_y\widehat{e}^{n+1/2}_{\mathbf{u},1,i,j+1} - D_y\widehat{e}^{n+1/2}_{\mathbf{u},1,i,j}}{k_{j+1/2}} + D_x\widehat{e}^{n+1/2}_{p,i,j+1/2}
\end{aligned}
$$

$$= \mathcal{P}_h W^{n+1/2}_{i,j+1/2}[D_x\tilde{Z}]^{n+1/2}_{i,j+1/2} - \mu^{n+1/2}_{i,j+1/2}\frac{\partial \phi}{\partial x}^{n+1/2}_{i,j+1/2}$$
$$+ \frac{\partial u_1}{\partial t}\Big|^{n+1/2}_{i,j+1/2} - [d_t\widehat{U}_1]^{n+1}_{i,j+1/2}. \tag{4.52}$$

For a discrete function $\{v^n_{1,i,j+1/2}\}$ such that $v^n_{1,i,j+1/2}|_{\partial\Omega} = 0$, multiplying (4.52) by times $v^n_{1,i,j+1/2}hk$ and make summation for i, j with $i = 1, \ldots, N_x - 1$, $j = 0, \ldots, N_y - 1$, and recalling Lemma 3.1 lead to

$$\begin{aligned}
&(d_t\widehat{e}^{n+1}_{\mathbf{u},1}, v^n_1)_{l^2,T,M} + \nu(d_x\widehat{e}^{n+1/2}_{\mathbf{u},1}, d_x v^n_1)_{l^2,M}\\
&\quad + \nu(D_y\widehat{e}^{n+1/2}_{\mathbf{u},1}, D_y v^n_1)_{l^2,T_y} - (\widehat{e}^{n+1/2}_p, d_x v^n_1)_{l^2,M}\\
&= (\mathcal{P}_h W^{n+1/2}[D_x\tilde{Z}]^{n+1/2} - \mu^{n+1/2}\frac{\partial \phi^{n+1/2}}{\partial x}, v^n_1)_{l^2,T,M}\\
&\quad + \left(\frac{\partial u^{n+1/2}_1}{\partial t} - d_t\widehat{U}^{n+1}_1, v^n_1\right)_{l^2,T,M}.
\end{aligned} \tag{4.53}$$

Similarly in the y direction, we have

$$\begin{aligned}
&(d_t\widehat{e}^{n+1}_{\mathbf{u},2}, v^n_2)_{l^2,M,T} + \nu(d_y\widehat{e}^{n+1/2}_{\mathbf{u},2}, d_y v^n_2)_{l^2,M}\\
&\quad + \nu(D_x\widehat{e}^{n+1/2}_{\mathbf{u},2}, D_x v^n_2)_{l^2,T_x} - (\widehat{e}^{n+1/2}_p, d_y v^n_2)_{l^2,M}\\
&= \left(\mathcal{P}_h W^{n+1/2}[D_y\tilde{Z}]^{n+1/2} - \mu^{n+1/2}\frac{\partial \phi^{n+1/2}}{\partial y}, v^n_2\right)_{l^2,M,T}\\
&\quad + \left(\frac{\partial u^{n+1/2}_2}{\partial t} - d_t\widehat{U}^{n+1}_2, v^n_2\right)_{l^2,M,T}.
\end{aligned} \tag{4.54}$$

Adding (4.53) and (4.54) results in

$$\begin{aligned}
&(d_t\widehat{e}^{n+1}_{\mathbf{u},1}, v^n_1)_{l^2,T,M} + (d_t\widehat{e}^{n+1}_{\mathbf{u},2}, v^n_2)_{l^2,M,T} + \nu(d_x\widehat{e}^{n+1/2}_{\mathbf{u},1}, d_x v^n_1)_{l^2,M}\\
&\quad + \nu(D_y\widehat{e}^{n+1/2}_{\mathbf{u},1}, D_y v^n_1)_{l^2,T_y} + \nu(d_y\widehat{e}^{n+1/2}_{\mathbf{u},2}, d_y v^n_2)_{l^2,M}\\
&\quad + \nu(D_x\widehat{e}^{n+1/2}_{\mathbf{u},2}, D_x v^n_2)_{l^2,T_x} - (\widehat{e}^{n+1/2}_p, d_x v^n_1 + d_y v^n_2)_{l^2,M}\\
&= \left(\mathcal{P}_h W^{n+1/2}[D_x\tilde{Z}]^{n+1/2} - \mu^{n+1/2}\frac{\partial \phi^{n+1/2}}{\partial x}, v^n_1\right)_{l^2,T,M}\\
&\quad + \left(\mathcal{P}_h W^{n+1/2}[D_y\tilde{Z}]^{n+1/2} - \mu^{n+1/2}\frac{\partial \phi^{n+1/2}}{\partial y}, v^n_2\right)_{l^2,M,T}\\
&\quad + \left(\frac{\partial u^{n+1/2}_1}{\partial t} - d_t\widehat{U}^{n+1}_1, v^n_1\right)_{l^2,T,M}\\
&\quad + \left(\frac{\partial u^{n+1/2}_2}{\partial t} - d_t\widehat{U}^{n+1}_2, v^n_2\right)_{l^2,M,T}.
\end{aligned} \tag{4.55}$$

Recalling the definition of the interpolation operator $\mathcal{P}_h$ and assuming that (4.12b) holds, the first term on the right-hand side of (4.55) can be transformed into the following:

$$\begin{aligned}
&\left(\mathcal{P}_h W^{n+1/2}[D_x\tilde{Z}]^{n+1/2} - \mu^{n+1/2}\frac{\partial\phi^{n+1/2}}{\partial x}, v_1^n\right)_{l^2,T,M}\\
&= ((\mathcal{P}_h W^{n+1/2} - \mathcal{P}_h\mu^{n+1/2})[D_x\tilde{Z}]^{n+1/2}, v_1^n)_{l^2,T,M}\\
&\quad + ((\mathcal{P}_h\mu^{n+1/2} - \mu^{n+1/2})[D_x\tilde{Z}]^{n+1/2}, v_1^n)_{l^2,T,M}\\
&\quad + \left(\mu^{n+1/2}([D_x\tilde{Z}]^{n+1/2} - \frac{\partial\phi^{n+1/2}}{\partial x}\right), v_1^n)_{l^2,T,M}\\
&\le C\|e_\mu^{n+1/2}\|_{l^2,M}^2 + C\|e_\phi^n\|_{l^2,M}^2 + C\|e_\phi^{n-1}\|_{l^2,M}^2\\
&\quad + \frac{1}{4}\|v_1^n\|_{l^2,T,M}^2 + C(\Delta t^4 + h^4 + k^4).
\end{aligned} \tag{4.56}$$

Similarly the second term on the right-hand side of (4.55) can be estimated by

$$\begin{aligned}
&\left(\mathcal{P}_h W^{n+1/2}[D_y\tilde{Z}]^{n+1/2} - \mu^{n+1/2}\frac{\partial\phi^{n+1/2}}{\partial y}, v_2^n\right)_{l^2,M,T}\\
&\le C\|e_\mu^{n+1/2}\|_{l^2,M}^2 + C\|e_\phi^n\|_{l^2,M}^2 + C\|e_\phi^{n-1}\|_{l^2,M}^2\\
&\quad + \frac{1}{4}\|v_2^n\|_{l^2,M,T}^2 + C(\Delta t^4 + h^4 + k^4).
\end{aligned} \tag{4.57}$$

Taking note of Lemma 4.1 and using Cauchy–Schwarz inequality, the last two terms on the right-hand side of (4.55) can be controlled by

$$\begin{aligned}
&\left(\frac{\partial u_1^{n+1/2}}{\partial t} - d_t\widehat{U}_1^{n+1}, v_1^n\right)_{l^2,T,M} + \left(\frac{\partial u_2^{n+1/2}}{\partial t} - d_t\widehat{U}_2^{n+1}, v_2^n\right)_{l^2,M,T}\\
&\le \frac{1}{4}\|\mathbf{v}^n\|_{l^2}^2 + C(\Delta t^4 + h^4 + k^4).
\end{aligned} \tag{4.58}$$

Using Lemma 4.2 and the discrete Poincaré inequality, we can obtain

$$\begin{aligned}
\beta\|\widehat{e}_p^{n+1/2}\|_{l^2,M} &\le \sup_{\mathbf{v}\in\mathbf{V}_h}\frac{(\widehat{e}_p^{n+1/2}, d_x v_1^n + d_y v_2^n)_{l^2,M}}{\|D\mathbf{v}\|}\\
&\le C(\|d_t\widehat{e}_{\mathbf{u},1}^n\|_{l^2,T,M} + \|d_t\widehat{e}_{\mathbf{u},2}^n\|_{l^2,M,T} + \|d_x\widehat{e}_{\mathbf{u},1}^{n+1/2}\|_{l^2,M}\\
&\quad + \|D_y\widehat{e}_{\mathbf{u},1}^{n+1/2}\|_{l^2,T_y} + \|d_y\widehat{e}_{\mathbf{u},2}^{n+1/2}\|_{l^2,M} + \|D_x\widehat{e}_{\mathbf{u},2}^{n+1/2}\|_{l^2,T_x})\\
&\quad + C\|e_\mu^{n+1/2}\|_{l^2,M} + C\|e_\phi^n\|_{l^2,M} + C\|e_\phi^{n-1}\|_{l^2,M}\\
&\quad + O(\Delta t^2 + h^2 + k^2).
\end{aligned} \tag{4.59}$$

Setting $v^n_{1,i,j+1/2} = d_t\widehat{e}^{n+1}_{\mathbf{u},1,i,j+1/2}$, $v^n_{2,i+1/2,j} = d_t\widehat{e}^{n+1}_{\mathbf{u},2,i+1/2,j}$ in (4.55) leads to

$$\begin{aligned}
&\|d_t\widehat{e}^{n+1}_{\mathbf{u},1}\|^2_{l^2,T,M} + \|d_t\widehat{e}^{n+1}_{\mathbf{u},2}\|^2_{l^2,M,T} + \nu\frac{\|\mathbf{D}\widehat{e}^{n+1}_{\mathbf{u}}\|^2 - \|\mathbf{D}\widehat{e}^{n}_{\mathbf{u}}\|^2}{2\Delta t}\\
&\quad = (\mathcal{P}_h W^{n+1/2}[D_x\tilde{Z}]^{n+1/2} - \mu^{n+1/2}\frac{\partial\phi^{n+1/2}}{\partial x}, d_t\widehat{e}^{n+1}_{\mathbf{u},1})_{l^2,T,M}\\
&\qquad + \left(\mathcal{P}_h W^{n+1/2}[D_y\tilde{Z}]^{n+1/2} - \mu^{n+1/2}\frac{\partial\phi^{n+1/2}}{\partial y}, d_t\widehat{e}^{n+1}_{\mathbf{u},2}\right)_{l^2,M,T}\\
&\qquad + \left(\frac{\partial u_1^{n+1/2}}{\partial t} - d_t\widehat{U}^{n+1}_1, d_t\widehat{e}^{n+1}_{\mathbf{u},1}\right)_{l^2,T,M}\\
&\qquad + \left(\frac{\partial u_2^{n+1/2}}{\partial t} - d_t\widehat{U}^{n+1}_2, d_t\widehat{e}^{n+1}_{\mathbf{u},2}\right)_{l^2,M,T}.
\end{aligned} \tag{4.60}$$

Noting (4.56)–(4.58), we have

$$\begin{aligned}
&\|d_t\widehat{e}^{n+1}_{\mathbf{u},1}\|^2_{l^2,T,M} + \|d_t\widehat{e}^{n+1}_{\mathbf{u},2}\|^2_{l^2,M,T} + \nu\frac{\|\mathbf{D}\widehat{e}^{n+1}_{\mathbf{u}}\|^2 - \|\mathbf{D}\widehat{e}^{n}_{\mathbf{u}}\|^2}{2\Delta t}\\
&\quad \le C\|e^{n+1/2}_{\mu}\|^2_{l^2,M} + C\|e^{n}_{\phi}\|^2_{l^2,M} + C\|e^{n-1}_{\phi}\|^2_{l^2,M}\\
&\qquad + \frac{1}{2}\|d_t\widehat{e}^{n+1}_{\mathbf{u},1}\|^2_{l^2,T,M} + \frac{1}{2}\|d_t\widehat{e}^{n+1}_{\mathbf{u},2}\|^2_{l^2,M,T}\\
&\qquad + C(\Delta t^4 + h^4 + k^4).
\end{aligned} \tag{4.61}$$

Multiplying (4.61) by $2\Delta t$, and summing over n from 1 to m result in

$$\begin{aligned}
&\sum_{n=0}^{m}\Delta t(\|d_t\widehat{e}^{n+1}_{\mathbf{u},1}\|^2_{l^2,T,M} + \|d_t\widehat{e}^{n+1}_{\mathbf{u},2}\|^2_{l^2,M,T})\\
&\qquad + \nu\|\mathbf{D}\widehat{e}^{m+1}_{\mathbf{u}}\|^2 - \nu\|\mathbf{D}\widehat{e}^{0}_{\mathbf{u}}\|^2\\
&\qquad \le C\sum_{n=0}^{m}\Delta t\|e^{n+1/2}_{\mu}\|^2_{l^2,M} + C\sum_{n=0}^{m}\Delta t\|e^{n}_{\phi}\|^2_{l^2,M}\\
&\qquad + C(\Delta t^4 + h^4 + k^4).
\end{aligned} \tag{4.62}$$

Since $\widehat{e}^n_{\mathbf{u},1,0,j+1/2} = \widehat{e}^n_{\mathbf{u},1,N_x,j+1/2}$ and $\widehat{e}^n_{\mathbf{u},2,i+1/2,0} = \widehat{e}^n_{\mathbf{u},2,i+1/2,N_y}$, then we can obtain the following discrete Poincaré inequality.

$$\begin{aligned}
\|\widehat{e}^{m+1}_{\mathbf{u}}\|^2_{l^2} \le C\|\mathbf{D}\widehat{e}^{m+1}_{\mathbf{u}}\|^2 \le C\sum_{n=0}^{m}\Delta t\|e^{n+1/2}_{\mu}\|^2_{l^2,M} + C\sum_{n=0}^{m}\Delta t\|e^{n}_{\phi}\|^2_{l^2,M}&\\
+ C(\Delta t^4 + h^4 + k^4).&
\end{aligned} \tag{4.63}$$

Recalling (4.59), we have

$$\sum_{n=0}^{m} \Delta t \|\widehat{e}_p^{n+1/2}\|_{l^2,M} \leq C \sum_{n=0}^{m} \Delta t \|e_\mu^{n+1/2}\|_{l^2,M}^2 + C \sum_{n=0}^{m} \Delta t \|e_\phi^n\|_{l^2,M}^2 + C(\Delta t^4 + h^4 + k^4), \tag{4.64}$$

which leads to the desired result (4.51). □

4.4. *Verification of the hypotheses* (4.12) *and the main results*

In this section, we derive the final results.

Lemma 4.5. *Suppose that* $\phi \in W_\infty^1(J; W_\infty^4(\Omega)) \cap W_\infty^3(J; W_\infty^1(\Omega)), \mu \in L^\infty(J; W_\infty^4(\Omega))$, *and* $\boldsymbol{u} \in W_\infty^3(J; W_\infty^4(\Omega))^2$, $p \in W_\infty^3(J; W_\infty^3(\Omega))$ *and* $\Delta t \leq C(h+k)$, *then the hypotheses* (4.12) *holds.*

Proof. The proof of (4.12a) is essentially identical with the estimates in Ref. 14. Thus, we only provide a detail proof for (4.12b) below.

Step 1. (Definition of C^*): Using the scheme (3.19a)–(3.19f) for $n = 0$, Lemmas 4.3 and 4.4, and the inverse assumption, we can get the approximation $\mathbf{D}Z^1$ and the following property:

$$\begin{aligned}
\|\mathbf{D}Z^1\|_\infty &= \|\mathbf{D}Z^1 - \mathbf{I}_h\mathbf{D}\phi^1\|_\infty + \|\mathbf{I}_h\mathbf{D}\phi^1 - \mathbf{D}\phi^1\|_\infty + \|\mathbf{D}\phi^1\|_\infty \\
&\leq C\hat{h}^{-1}\|\mathbf{D}Z^1 - \mathbf{I}_h\mathbf{D}\phi^1\|_{l^2} + \|\mathbf{I}_h\mathbf{D}\phi^1 - \mathbf{D}\phi^1\|_\infty + \|\mathbf{D}\phi^1\|_\infty \\
&\leq C\hat{h}^{-1}(\|\mathbf{D}e_\phi^1\|_{l^2} + \|\mathbf{I}_h\mathbf{D}\phi^1 - \mathbf{D}\phi^1\|_{l^2}) + \|\mathbf{I}_h\mathbf{D}\phi^1 - \mathbf{D}\phi^1\|_\infty + \|\mathbf{D}\phi^1\|_\infty \\
&\leq C\hat{h}^{-1}(\Delta t^2 + \hat{h}^2) + \|\mathbf{D}\phi^1\|_\infty \leq C,
\end{aligned}$$

where $\hat{h}$ and Δt are selected such that $\hat{h}^{-1}\Delta t^2$ is sufficiently small.

Thus, define the positive constant C^* independent of $\hat{h}$ and Δt such that

$$C^* \geq \max\{\|\mathbf{D}Z^1\|_\infty, 2\|\mathbf{D}\phi(t)\|_\infty\}.$$

Step 2. (Induction): By the definition of C^*, it is trivial that hypothesis (4.12b) holds true for $l = 1$. Supposing that $\|\mathbf{D}Z^{l-1}\|_\infty \leq C^*$ holds true for an integer $l = 1, \ldots, N-1$, by Lemmas 4.3 and 4.4 with $m = l$, we have that

$$\|\mathbf{D}e_\phi^l\|_{l^2} \leq C(\hat{h}^2 + \Delta t^2).$$

Next we prove that $\|\mathbf{D}Z^l\|_\infty \leq C^*$ holds true. Since

$$\begin{aligned}
\|\mathbf{D}Z^l\|_\infty &= \|\mathbf{D}Z^l - \mathbf{I}_h\mathbf{D}\phi^l\|_\infty + \|\mathbf{I}_h\mathbf{D}\phi^l - \mathbf{D}\phi^l\|_\infty + \|\mathbf{D}\phi^l\|_\infty \\
&\leq C\hat{h}^{-1}(\|\mathbf{D}e_\phi^l\|_{l^2} + \|\mathbf{I}_h\mathbf{D}\phi^l - \mathbf{D}\phi^l\|_{l^2}) \\
&\quad + \|\mathbf{I}_h\mathbf{D}\phi^l - \mathbf{D}\phi^l\|_\infty + \|\mathbf{D}\phi^l\|_\infty \\
&\leq C_1\hat{h}^{-1}(\Delta t^2 + \hat{h}^2) + \|\mathbf{D}\phi^l\|_\infty.
\end{aligned} \tag{4.65}$$

Let $\Delta t \leq C_2\hat{h}$ and a positive constant $\hat{h}_1$ be small enough to satisfy

$$C_1(1+C_2^2)\hat{h}_1 \leq \frac{C^*}{2}.$$

Then for $\hat{h} \in (0, \hat{h}_1]$, Eq. (4.65) can be bounded by

$$\begin{aligned}\|\mathbf{D}Z^l\|_\infty &\leq C_1\hat{h}^{-1}(\Delta t^2 + \hat{h}^2) + \|\mathbf{D}\phi^l\|_\infty \\ &\leq C_1(1+C_2^2)\hat{h}_1 + \frac{C^*}{2} \leq C^*.\end{aligned} \tag{4.66}$$

Then, the proof of induction hypothesis (4.12b) ends. □

Recalling (4.63), we can transform (4.18) into the following:

$$\begin{aligned}&\|e_\phi^{m+1}\|_{l^2,M}^2 + \frac{M}{2}\sum_{n=0}^{m}\Delta t\|e_\mu^{n+1/2}\|_{l^2,M}^2 + \lambda(e_r^{m+1})^2 \\ &\quad + \frac{\lambda}{2}\|\mathbf{D}e_\phi^{m+1}\|_{l^2}^2 + \frac{M}{4}\sum_{n=0}^{m}\Delta t\|\mathbf{D}e_\mu^{n+1/2}\|_{l^2}^2 \\ &\leq C\sum_{n=0}^{m+1}\Delta t\|\mathbf{D}e_\phi^n\|_{l^2}^2 + C\sum_{n=0}^{m}\Delta t\|D\widehat{e}_{\mathbf{u}}^{n+1/2}\|^2 \\ &\quad + C\sum_{n=0}^{m+1}\Delta t\|e_\phi^n\|_{l^2,M}^2 + C\sum_{n=0}^{m+1}\Delta t(e_r^n)^2 \\ &\quad + C(\Delta t^4 + h^4 + k^4), \quad m \leq N,\end{aligned} \tag{4.67}$$

Multiplying (4.67) and (4.51) by $4C$ and M, respectively, and using Gronwall's inequality, we can deduce that

$$\begin{aligned}&\|e_\phi^{m+1}\|_{l^2,M}^2 + \sum_{n=0}^{m}\Delta t\|e_\mu^{n+1/2}\|_{l^2,M}^2 + (e_r^{m+1})^2 \\ &\quad + \|\mathbf{D}e_\phi^{m+1}\|_{l^2}^2 + \sum_{n=0}^{m}\Delta t\|\mathbf{D}e_\mu^{n+1/2}\|_{l^2}^2 + \|\widehat{e}_{\mathbf{u}}^{m+1}\|_{l^2}^2 \\ &\quad + \|\mathbf{D}\widehat{e}_{\mathbf{u}}^{m+1}\|^2 + \sum_{n=0}^{m}\Delta t\|\widehat{e}_p^{n+1/2}\|_{l^2,M}^2 \\ &\leq C(\Delta t^4 + h^4 + k^4), \quad m \leq N.\end{aligned} \tag{4.68}$$

Thus, we have

$$\begin{aligned}&\|Z^{m+1} - \phi^{m+1}\|_{l^2,M} + \|\mathbf{D}Z^{m+1} - \mathbf{D}\phi^{m+1}\|_{l^2} + |R^{m+1} - r^{m+1}| \\ &\quad + \left(\sum_{n=0}^{m}\Delta t\|\mathbf{D}W^{n+1/2} - \mathbf{D}\mu^{n+1/2}\|_{l^2}^2\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&+ \left(\sum_{n=0}^{m} \Delta t \|W^{n+1/2} - \mu^{n+1/2}\|_{l^2,M}^2\right)^{1/2} \\
&\leq C(\|\phi\|_{W_\infty^1(J;W_\infty^4(\Omega))} + \|\mu\|_{L^\infty(J;W_\infty^4(\Omega))})(h^2 + k^2) \\
&+ C\|\phi\|_{W_\infty^3(J;W_\infty^1(\Omega))} \Delta t^2.
\end{aligned} \tag{4.69}
$$

Recalling Lemma 4.1, we can obtain that

$$
\|d_x(U_1^m - u_1^m)\|_{l^2,M} + \|d_y(U_2^m - u_2^m)\|_{l^2,M} \leq O(\Delta t^2 + h^2 + k^2), \tag{4.70}
$$

$$
\begin{aligned}
&\|U_1^m - u_1^m\|_{l^2,T,M} + \|U_2^m - u_2^m\|_{l^2,M,T} + \left(\sum_{l=1}^{m} \Delta t \|(P-p)^{l-1/2}\|_{l^2,M}^2\right)^{1/2} \\
&\leq O(\Delta t^2 + h^2 + k^2),
\end{aligned} \tag{4.71}
$$

$$
\|D_y(U_1^m - u_1^m)\|_{l^2,T_y} \leq O(\Delta t^2 + h^2 + k^{3/2}), \tag{4.72}
$$

$$
\|D_x(U_2^m - u_2^m)\|_{l^2,T_x} \leq O(\Delta t^2 + h^{3/2} + k^2). \tag{4.73}
$$

Combing the above results together, we finally obtain our main results:

Theorem 4.1. *Suppose that $\phi \in W_\infty^1(J;W_\infty^4(\Omega)) \cap W_\infty^3(J;W_\infty^1(\Omega)), \mu \in L^\infty(J;W_\infty^4(\Omega))$, and $\boldsymbol{u} \in W_\infty^3(J;W_\infty^4(\Omega))^2$, $p \in W_\infty^3(J;W_\infty^3(\Omega))$ and $\Delta t \leq C(h+k)$, then for the Cahn–Hilliard–Stokes system, there exists a positive constant C independent of h, k and Δt such that*

$$
\begin{aligned}
&\|Z^{m+1} - \phi^{m+1}\|_{l^2,M} + \|\boldsymbol{D}Z^{m+1} - \boldsymbol{D}\phi^{m+1}\|_{l^2} + |R^{m+1} - r^{m+1}| \\
&+ \left(\sum_{n=0}^{m} \Delta t \|\boldsymbol{D}W^{n+1/2} - \boldsymbol{D}\mu^{n+1/2}\|_{l^2}^2\right)^{1/2} \\
&+ \left(\sum_{n=0}^{m} \Delta t \|W^{n+1/2} - \mu^{n+1/2}\|_{l^2,M}^2\right)^{1/2} \\
&\leq C(\|\phi\|_{W_\infty^1(J;W_\infty^4(\Omega))} + \|\mu\|_{L^\infty(J;W_\infty^4(\Omega))})(h^2 + k^2) \\
&+ C\|\phi\|_{W_\infty^3(J;W_\infty^1(\Omega))} \Delta t^2, \quad m \leq N,
\end{aligned} \tag{4.74}
$$

$$
\begin{aligned}
&\|d_x(U_1^m - u_1^m)\|_{l^2,M} + \|d_y(U_2^m - u_2^m)\|_{l^2,M} \\
&\leq O(\Delta t^2 + h^2 + k^2), \quad m \leq N,
\end{aligned} \tag{4.75}
$$

$$
\begin{aligned}
&\|\boldsymbol{U}^m - \boldsymbol{u}^m\|_{l^2} + \left(\sum_{l=1}^{m} \Delta t \|(P-p)^{l-1/2}\|_{l^2,M}^2\right)^{1/2} \\
&\leq O(\Delta t^2 + h^2 + k^2), \quad m \leq N,
\end{aligned} \tag{4.76}
$$

$$\|D_y(U_1^m - u_1^m)\|_{l^2,T_y} \le O(\Delta t^2 + h^2 + k^{3/2}), \qquad m \le N, \tag{4.77}$$

$$\|D_x(U_2^m - u_2^m)\|_{l^2,T_x} \le O(\Delta t^2 + h^{3/2} + k^2), \qquad m \le N. \tag{4.78}$$

5. Numerical Experiments

In this section, we provide some 2D numerical experiments to gauge the SAV/CN-FD method developed in the previous sections.

We transform (2.2) as

$$E(\phi) = \int_\Omega \left\{ \frac{1}{2}|\mathbf{u}|^2 + \lambda \left(\frac{1}{2}|\nabla\phi|^2 + \frac{\beta}{2\epsilon^2}\phi^2 + \frac{1}{4\epsilon^2}(\phi^2 - 1 - \beta)^2 - \frac{\beta^2 + 2\beta}{4\epsilon^2} \right) \right\} d\mathbf{x}, \tag{5.1}$$

where β is a positive number to be chosen. To apply our scheme (3.19a)–(3.19f) to the system (2.1), we drop the constant in the free energy and specify $E_1(\phi) = \frac{1}{4\epsilon^2}\int_\Omega (\phi^2 - 1 - \beta)^2 d\mathbf{x}$, and modify (3.19b) into

$$\begin{aligned} W^{n+1/2}_{i+1/2,j+1/2} &= -\lambda[d_x D_x Z + d_y D_y Z]^{n+1/2}_{i+1/2,j+1/2} + \frac{\lambda\beta}{\epsilon^2} Z^{n+1/2}_{i+1/2,j+1/2} \\ &\quad + \lambda \frac{R^{n+1/2}}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})}} F'(\tilde{Z}^{n+1/2}_{i,j}). \end{aligned} \tag{5.2}$$

Then, we can obtain

$$F'(\phi) = \frac{\delta E_1}{\delta \phi} = \frac{1}{\epsilon^2}\phi(\phi^2 - 1 - \beta). \tag{5.3}$$

For simplicity, we define

$$\begin{cases} \|f - g\|_{\infty,2} = \max\limits_{0\le n\le m} \left\{ \|f^{n+q} - g^{n+q}\|_X \right\}, \\ \|f - g\|_{2,2} = \left(\sum\limits_{n=0}^{m} \Delta t \left\| f^{n+q} - g^{n+q} \right\|_X^2 \right)^{1/2}, \\ \|R - r\|_\infty = \max\limits_{0\le n\le m} \{R^{n+1} - r^{n+1}\}, \end{cases}$$

where $q = \frac{1}{2}$, 1 and X is the corresponding discrete L^2 norm. In the following simulations, we choose $\Omega = (0,1)\times(0,1)$, $\beta = 5$ and $\gamma = 1$.

5.1. *Convergence rates of the SAV-MAC scheme for the Cahn–Hilliard–Navier–Stokes phase field model*

In this Example 1, we take $T = 0.1$, $\Delta t = 1E-4$, $\lambda = 0.1$, $\nu = 0.1$, $\epsilon^2 = 0.1$, $M = 0.001$, and the initial solution $\phi_0 = \cos(\pi x)\cos(\pi y)$, $u_1(x,y) = -x^2(x-1)^2(y-1)(2y-1)y/128$ and $u_2(x,y) = -u_1(y,x)$. We measure Cauchy error to get around

Table 1. Errors and convergence rates of the phase function and auxiliary scalar function for Example 1.

h	$\|e_Z\|_{\infty,2}$	Rate	$\|e_{\mathbf{D}Z}\|_{\infty,2}$	Rate	$\|e_R\|_\infty$	Rate
1/10	3.09E-3	—	1.37E-2	—	2.69E-5	—
1/20	7.74E-4	2.00	3.43E-3	1.99	6.76E-6	1.99
1/40	1.93E-4	2.00	8.60E-4	2.00	1.69E-6	2.00
1/80	4.84E-5	2.00	2.15E-4	2.00	4.23E-7	2.00

Table 2. Errors and convergence rates of the chemical potential and velocity for Example 1.

h	$\|e_W\|_{2,2}$	Rate	$\|e_{\mathbf{D}W}\|_{2,2}$	Rate	$\|e_{\mathbf{U}}\|_{\infty,2}$	Rate
1/10	1.59E-3	—	1.57E-2	—	1.67E-4	—
1/20	4.01E-4	1.98	4.09E-3	1.94	3.67E-5	2.19
1/40	1.01E-4	2.00	1.03E-3	1.99	8.88E-6	2.05
1/80	2.51E-5	2.00	2.59E-4	2.00	2.20E-6	2.01

Table 3. Errors and convergence rates of the velocity and pressure for Example 1.

h	$\|e_{d_xU_1}\|_{\infty,2}$	Rate	$\|e_{D_yU_1}\|_{\infty,2}$	Rate	$\|e_P\|_{2,2}$	Rate
1/10	9.14E-4	—	1.54E-3	—	1.06E-3	—
1/20	2.05E-4	2.16	4.28E-4	1.85	2.63E-4	2.01
1/40	4.99E-5	2.04	1.36E-4	1.66	6.56E-5	2.00
1/80	1.24E-5	2.01	4.56E-5	1.57	1.64E-5	2.00

the fact that we do not have possession of exact solution. Specifically, the error between two different grid spacings h and $\frac{h}{2}$ is calculated by $\|e_\zeta\| = \|\zeta_h - \zeta_{h/2}\|$.

The numerical results are listed in Tables 1–3 and give solid supporting evidence for the expected second-order convergence of the SAV/CN-FD scheme for the Cahn–Hilliard–Navier–Stokes phase-field model, which are consistent with the error estimates in Theorem 4.1. Here we only present the results for u_1 since the results for u_2 are similar to u_1.

5.2. *The dynamics of a square shape fluid*

In this Example 2, the evolution of a square shaped fluid bubble is simulated by using the following parameters:

$$\epsilon = 0.01, \quad \nu = 1, \quad \lambda = 0.01, \quad M = 0.002, \quad \hat{h} = 1/100, \quad \Delta t = 1E-3.$$

The initial velocity and pressure are set to zero. The initial phase function is chosen to be a rectangular bubble, i.e. $\phi = 1$ inside the bubble and $\phi = -1$ outside the bubble. Snapshots of the phase evolution at time $t = 0, 5, 6, 8, 10$, respectively, are presented in Fig. 2. As we can see, the rectangular bubble deforms into a circular bubble due to the surface tension.

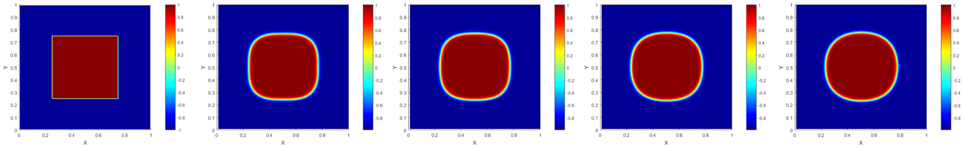


Fig. 2. Snapshots of the phase function in example 2 at $t = 0, 5, 6, 8, 10$, respectively.

5.3. *Buoyancy-driven flow*

In this Example 3, as the test of buoyancy-driven flow, we consider the case of a single bubble rising in a rectangular box. Similar to Ref. 5, we modify the Navier–Stokes Eq. (2.1c) as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mu \nabla \phi + \mathbf{b}, \tag{5.4}$$

where $\mathbf{b}$ is a buoyancy term that depends on the mass density ρ. We assume that the mass density depends on ϕ, and the following Boussinesq type approximation is applied:

$$\mathbf{b} = (0, -b(\phi))^t, \quad b(\phi) = \chi(\phi - \phi_0), \tag{5.5}$$

where ϕ_0 is a constant (usually the average value of ϕ), and χ is a constant. In this example, the numerical and physical parameters are given as follows:

$$\begin{cases} \hat{h} = 1/100, \ \Delta t = 5E - 4, \ M = 0.01, \\ \epsilon = 0.01, \ \nu = 1, \ \lambda = 0.001, \\ \phi_0 = -0.05, \ \chi = 40. \end{cases}$$

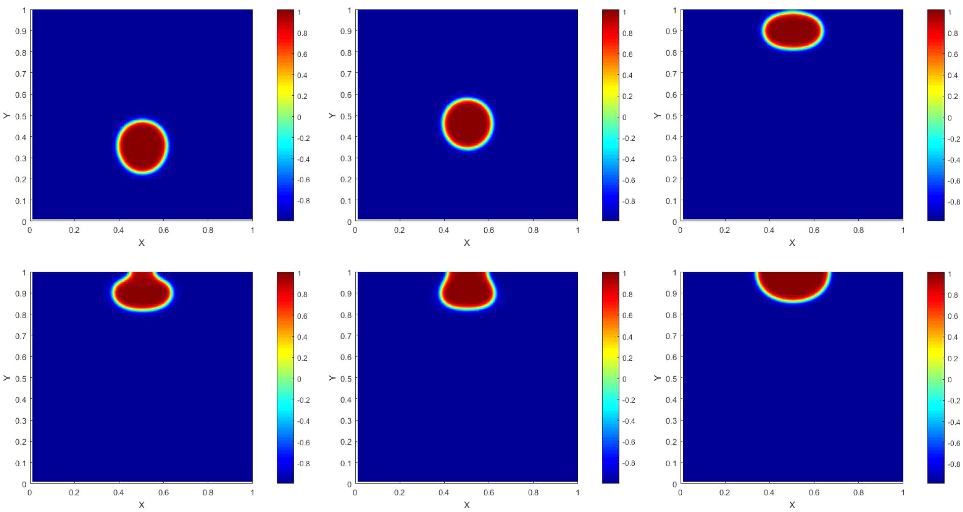


Fig. 3. Snapshots of the phase function in Example 3 at $t = 0.5, 1, 4, 4.1, 4.2, 5$, respectively.

The initial condition for the phase function is chosen to be a circular bubble that centered at $(\frac{1}{2}, \frac{1}{4})$, and the initial data for the velocity is taken as $\mathbf{u}^0 = 0$. Snapshots of the phase evolution at time $t = 0.5, 1, 4, 4.1, 4.2, 5$, respectively, are presented in Fig. 3. It starts as a circular bubble near the bottom of the domain. The density of the bubble is lighter than the density of the surrounding fluid. As expected, the bubble rises, reaching an elliptical shape, and then deforms as it approaches the upper boundary.

6. Conclusion

We developed a second-order fully discrete SAV-MAC scheme for the Cahn–Hilliard–Navier–Stokes phase-field model, and proved that it is unconditionally energy stable. We also carried out a rigorous error analysis for the Cahn–Hilliard–Stokes system and derived second-order error estimates both in time and space for phase-field variable, chemical potential, velocity and pressure in different discrete norms.

The SAV-MAC scheme, with an explicit treatment of the convective term in the phase equation, is extremely efficient as it leads to, at each time step, a sequence of Poisson type equations that can be solved by using fast Fourier transforms. We provided several numerical results to demonstrate the robustness and accuracy of the SAV-MAC scheme for the Cahn–Hilliard–Navier–Stokes phase-field model.

We only carried out an error analysis for the Cahn–Hilliard–Stokes system. To derive corresponding error estimates for the Cahn–Hilliard–Navier–Stokes system, one needs to use new discretizing techniques such as a high order upwind method to deal with the nonlinear term. This will be a subject of future research.

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