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UNCONDITIONALLY STABLE CONSISTENT SPLITTING SCHEMES FOR THE NAVIER–STOKES EQUATIONS WITH C^0 FINITE ELEMENTS

XU LI AND JIE SHEN

ABSTRACT. We construct in this paper first- and second-order (in time) decoupled and unconditionally stable schemes for the Navier–Stokes equations based on the consistent splitting methods and C^0 finite elements, where spatial discontinuity of the gradient of the discrete velocity is treated using a discontinuous Galerkin framework. The challenge for their stability analysis is that the gradient of discrete velocity is discontinuous so that the estimate on the continuous Stokes pressure by Liu, Liu, and Pego [Comm. Pure Appl. Math. 60 (2007), pp. 1443–1487] cannot be applied in the space discrete case. We first extend the estimate on the Stokes pressure to the space discrete case, and then use it to prove the unconditional stability and carry out an error analysis for the proposed fully discrete schemes. Numerical results are presented to validate our analytical results.

1. Introduction

We consider in this paper numerical approximation of the following incompressible Navier–Stokes equations (NSEs)

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad J\times\Omega, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in} \quad J\times\Omega, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad [0,T]\times\Gamma, \tag{1.3}$$

$$\boldsymbol{u}(0,\bullet) = \boldsymbol{u}^0 \quad \text{in} \quad \Omega, \tag{1.4}$$

where the unknowns are velocity $\boldsymbol{u}$ and pressure p, $\nu > 0$ denotes the constant viscosity coefficient, $\boldsymbol{f}$ denotes the unit external body force, $\boldsymbol{u}^0$ is a given initial velocity satisfying $\nabla\cdot\boldsymbol{u}^0 = 0$, $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded and connected domain with Lipschitz boundary Γ, and $J = (0, T]$ with a given time $T > 0$.

The main objective of this paper is to construct a class of fully discrete and decoupled schemes for the NSEs using C^0 finite element methods in space and the consistent splitting methods in time [10, 25].

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Assuming that the nonlinear term is treated explicitly (resp. semi-implicitly), decoupled methods for NSEs, such as projection type methods and consistent splitting methods [9], only require solving a sequence of Poisson (resp. elliptic) equations at each time step, thus they are very efficient and can be easily implemented. Over the years, there has been an enormous amount of work in developing efficient and accurate decoupled schemes for NSEs, including projection type methods (see, e.g., [6, 9, 14–16, 20, 24, 29, 32]), consistent splitting methods [10, 18, 19, 23, 25, 26, 34] (see also the Gauge method [5, 27, 33]), and artificial compressibility methods (see, e.g., [11–13]), to name just a few.

A main issue with the projection type methods is that only schemes with first-order pressure (or velocity) extrapolation are proven to be unconditionally stable, limiting their accuracy to weakly second-order [9]. On the other hand, The consistent splitting methods are based on an equivalent system of NSEs (cf. [10, 25]). More precisely, the continuity equation (1.2) is replaced by the following equivalent pressure-Poison equation

$$(\nabla p, \nabla q) = (\boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\nabla\times\nabla\times\boldsymbol{u}, \nabla q) \quad \forall q \in H^1(\Omega), \tag{1.5}$$

which is obtained by testing (1.1) with ∇q and using the divergence-free condition (1.2) and the boundary condition (1.3). It has been observed that decoupled schemes based on this equivalent formulation do not produce splitting errors and can be naturally extended to higher order temporal discretizations. In addition, they do not require the inf-sup condition between velocity and pressure spaces.

The unconditional stability of the first-order semi-discrete consistent splitting schemes was proved in [10, 25]. However, the stability analysis of any second- or higher-order consistent splitting scheme remained open until very recently Huang and Shen constructed a generalized BDF2 formula in [18] which is more stiffly stable than the usual BDF2 scheme, applied it to construct a new second-order consistent splitting scheme for the NSEs, and proved its unconditional stability. A main tool in the stability analysis of consistent splitting methods in [18, 25] is the following estimate of the Stokes pressure $p_{\mathrm{S}}(\boldsymbol{u})$ [25]:

$$\|\nabla p_{\mathrm{s}}(\boldsymbol{u})\|^2 \leqslant (\frac{1}{2}+\varepsilon)\|\Delta\boldsymbol{u}\|^2 + C\|\nabla\boldsymbol{u}\|^2 \quad \text{for all } \boldsymbol{u}\in\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega),\ \partial\Omega\in C^3, \tag{1.6}$$

where the Stokes pressure is defined as

$$\nabla p_{\mathrm{S}}(\boldsymbol{u}) = (\Delta\mathcal{P} - \mathcal{P}\Delta)\boldsymbol{u},$$

with $\mathcal{P}$ being the Leray-Helmholtz projector from $\boldsymbol{L}^2(\Omega)$ on to $\boldsymbol{H} = \{\boldsymbol{v}\in L^2(\Omega) : \nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0\}$. Then, it is shown in [25] that for a given $\boldsymbol{u}\in\boldsymbol{H}^2(\Omega)$, we have

$$(\nabla p_{\mathrm{S}}(\boldsymbol{u}), \nabla q) = -(\nabla\times\nabla\times\boldsymbol{u}, \nabla q) = \int_\Gamma \nabla\times\boldsymbol{u}\cdot(\nabla q\times\boldsymbol{n})\,ds \quad \forall q\in H^1(\Omega)/\mathbb{R}. \tag{1.7}$$

In order to use the estimate (1.6), one has to test the momentum equation (1.1) with $-\Delta\boldsymbol{v}$ for some $\boldsymbol{v}\in\boldsymbol{H}^2(\Omega)$. While this is not a problem in space continuous case, it does create a significant issue for spatial discretization as a conforming approximation would require C^1 elements which are not easy to use. In fact, the fully discrete schemes considered in [25, 26] used C^1 finite elements. To the best of our knowledge, there is no unconditionally stable schemes using C^0 finite elements and consistent splitting methods.

In this paper, we propose and analyze a class of consistent splitting schemes, using C^0 discontinuous Galerkin (DG) finite elements for both velocity and pressure. More precisely, for the spatial discretization we use the interior penalty strategy from the DG method (see the review [1]) to penalize the discontinuity of gradient of the discrete velocity. In particular, this pair of approximation spaces for velocity and pressure does not require to satisfy the usual inf-sup condition.

However, due to the discontinuity of gradient of the discrete velocity, the stability and error analyses are very challenging since the estimate (1.6) is no longer valid for C^0 DG finite elements. Thus, a key in proving the unconditional stability and deriving an *a priori* error estimate in the fully discrete case is to extend the inequality (1.6) to the case $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega) + \boldsymbol{V}_h$, where $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega)$ is the discrete velocity space such as the standard (vector-valued) C^0 Lagrange element space.

The main contributions of this paper include

- Construction of first- and second-order (in time) consistent splitting fully discrete schemes with C^0 finite elements for the NSEs;
- Extension of the estimate (1.6) on the Stokes pressure in the continuous case to the discrete case (see Theorem 3.4): this is a key result for the analysis of fully discrete consistent splitting schemes;
- Stability and error analysis of the fully discrete consistent splitting schemes in the absence of nonlinear term. We believe that this is the first stability and error analysis for fully discrete consistent splitting schemes with C^0 finite elements.

As pointed out in [18], we emphasize that the main difficulty in the stability and error analysis of consistent splitting schemes is to deal with the Stokes operator. This is because the analysis related to Stokes pressure is only connected to the diffusion term and temporal derivative term. Therefore, in order to simplify the presentation, we limit our attention in the analysis to the unsteady Stokes equations. But it is expected that our analysis can be extended, albeit tedious, to the full NSEs.

This paper is organized as follows. In Section 2 we give some background of the semi-discrete (in time) consistent splitting schemes and propose a class of fully discrete schemes with pure C^0 elements for them, where the DG methods are employed to treat the discontinuity of the gradient of the velocity field. In Section 3, we extend the estimate (1.6) on the Stokes pressure in the space continuous case to the space discrete case. In Sections 4 and 5, we consider the unsteady Stokes problem, and use the estimate of the discrete Stokes pressure to prove unconditional stability and carry out error analysis, respectively, for the fully discrete schemes. In Section 6, we present some numerical experiments to validate our error estimates. Finally we make some concluding remarks in Section 7.

Throughout the paper we use C and ε, with or without subscripts, to denote a generic positive constant and a generic small positive constant, respectively. For any subdomain $D \subseteq \Omega$, the inner product and norm in $L^2(D)$ (or $\boldsymbol{L}^2(D)$) are denoted by $(\bullet, \bullet)_D$ and $\| \bullet \|_D$, respectively. When $D = \Omega$, the subscript is omitted. For any $(d-1)$-dimensional linear manifold F, let $| \bullet |_F$ and $| \bullet |_{\pm\frac{1}{2},F}$ denote the $L^2(F)$ norm and $H^{\pm\frac{1}{2}}(F)$ norms (see, e.g., [8, pp. 6–8]), respectively.

2. Consistent splitting C^0 finite element schemes

In this section, we shall construct consistent splitting C^0 finite element schemes for the Navier–Stokes equations.

2.1. Fundamental framework for a class of C^0 consistent splitting methods.

For the sake of clarity, we start with the following equivalent formulation of the time dependent Stokes equations:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in} \quad J \times \Omega, \tag{2.1}$$

$$(\nabla p, \nabla q) = (\boldsymbol{f} - \nu\nabla \times \nabla \times \boldsymbol{u}, \nabla q) \quad \forall q \in H^1(\Omega). \tag{2.2}$$

By integration by parts, one has

$$(\nabla \times \nabla \times \boldsymbol{u}, \nabla q) = -\int_\Gamma \nabla \times \boldsymbol{u} \cdot (\nabla q \times \boldsymbol{n})\, ds.$$

Therefore (2.2) is also equivalent to

$$(\nabla p, \nabla q) = (\boldsymbol{f}, \nabla q) + \nu \int_\Gamma \nabla \times \boldsymbol{u} \cdot (\nabla q \times \boldsymbol{n})\, ds \quad \forall q \in H^1(\Omega). \tag{2.3}$$

One can then construct decoupled consistent splitting schemes for (2.1)–(2.2). For instance, a first-order consistent splitting scheme is:

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu\Delta\boldsymbol{u}^{n+1} = \boldsymbol{f}^{n+1} - \nabla p^n, \tag{2.4}$$

$$(\nabla p^{n+1}, \nabla q) = (\boldsymbol{f}^{n+1}, \nabla q) + \nu \int_\Gamma \nabla \times \boldsymbol{u}^{n+1} \cdot (\nabla q \times \boldsymbol{n})\, ds \quad \forall q \in H^1(\Omega). \tag{2.5}$$

The unconditional stability of the above scheme has been proved in [10, 25]. However, the stability of a direct extension to second-order with BDF2 and second-order extrapolation for the pressure is still an open problem. Recently, a stable second-order consistent splitting method was constructed using a generalized BDF2 scheme proposed in [18],

$$\begin{aligned} \frac{(2\beta+1)\boldsymbol{u}^{n+1} - 4\beta\boldsymbol{u}^n + (2\beta-1)\boldsymbol{u}^{n-1}}{2\Delta t} &- \nu\Delta\left(\beta\boldsymbol{u}^{n+1} - (\beta-1)\boldsymbol{u}^n\right) \\ &= \boldsymbol{f}^{n+\beta} - \nabla\left((\beta+1)p^n - \beta p^{n-1}\right), \end{aligned} \tag{2.6}$$

where $\beta \geqslant 1$ is a parameter and the case $\beta = 1$ reduces to the usual BDF2. It is shown in [18] that the scheme (2.6)–(2.5) is unconditionally stable for $\beta \geqslant 5$. Below, we shall construct a stable fully discretization of this scheme using C^0 finite elements.

To simplify the notations, we introduce two operators d_β and $\mathcal{L}_\beta$ such that $d_\beta w^{n+1} = ((2\beta+1)w^{n+1} - 4\beta w^n + (2\beta-1)w^{n-1})/2$ and $\mathcal{L}_\beta w^{n+1} = \beta w^{n+1} - (\beta-1)w^n$ for arbitrary functions w. Therefore (2.6) can be rewritten as

$$\frac{d_\beta\boldsymbol{u}^{n+1}}{\Delta t} - \nu\Delta\mathcal{L}_\beta\boldsymbol{u}^{n+1} = \boldsymbol{f}^{n+\beta} - \nabla\mathcal{L}_{\beta+1}p^n.$$

It can be easily shown by Taylor expansion that, for a smooth function $g(t)$, one has

$$\frac{d_\beta g^{n+1}}{\Delta t} = g'\left(t^{n+\beta}\right) + \frac{1-3\beta^2}{6} g'''\left(t^{n+\beta}\right)\Delta t^2 + \mathcal{O}\left(\Delta t^3\right), \tag{2.7}$$

$$\mathcal{L}_\beta g^{n+1} = g^{n+\beta} - \frac{\beta(\beta-1)}{2} g''\left(t^{n+\beta}\right)\Delta t^2 + \mathcal{O}\left(\Delta t^3\right), \tag{2.8}$$

$$\mathcal{L}_{\beta+1} g^{n} = g^{n+\beta} - \frac{\beta(\beta+1)}{2} g''\left(t^{n+\beta}\right)\Delta t^2 + \mathcal{O}\left(\Delta t^3\right). \tag{2.9}$$

Denote by $\boldsymbol{V} := \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ the solution space of $\boldsymbol{u}$. To prove the stability of the above schemes via the estimate (1.6), variational formulations obtained by testing (2.6) with $-\Delta \boldsymbol{v}$ are considered in [18], i.e.,

$$\frac{(\nabla d_\beta \boldsymbol{u}^{n+1}, \nabla \boldsymbol{v})}{\Delta t} + \nu(\Delta \mathcal{L}_\beta \boldsymbol{u}^{n+1}, \Delta \boldsymbol{v}) = (\boldsymbol{f}^{n+\beta} - \nabla \mathcal{L}_{\beta+1} p^n, -\Delta \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.10}$$

Therefore, a straightforward conforming spatial discretization for (2.10) would require C^1 elements for the velocity approximation (see, e.g., [25, 26]). Since C^1 elements are cumbersome to use in practice, we shall construct below a C^0 finite element scheme.

2.2. **Full discretization using C^0 finite elements.** Let $\mathcal{T}$ be a shape-regular partition of Ω (not necessary to be simplicial). The sets of faces, boundary faces and interior faces are denoted by $\mathcal{F}$, $\mathcal{F}^\partial$ and $\mathcal{F}^0$, respectively. Note that $\mathcal{F} = \mathcal{F}^\partial \cup \mathcal{F}^0$. We use h_K and h_F to denote the diameter of any $K \in \mathcal{T}$ and $F \in \mathcal{F}$, respectively, and set $h := \max_{K\in\mathcal{T}} h_K$ and $h_\mathcal{T} \in L^2(\Omega)$ such that $h_\mathcal{T}|_K := h_K$ for all $K \in \mathcal{T}$. For $m > 0$ we define

$$H^m(\mathcal{T}) := \{q \in L^2(\Omega) : q|_K \in H^m(K) \text{ for all } K \in \mathcal{T}\}.$$

The space $H^m(\mathcal{F}^0)$ can be defined similarly. Like in many DG methods, we introduce the jump operator $[\![\bullet]\!] : H^2(\mathcal{T}) \to H^{\frac{3}{2}}(\mathcal{F}^0)$ and the average operator $\{\{\bullet\}\} : H^2(\mathcal{T}) \to H^{\frac{3}{2}}(\mathcal{F}^0)$ as follows. On an interior face $F \in \mathcal{F}^0$ which is shared by two elements K_1 and K_2, we define

$$[\![q]\!] := q|_{K_1} - q|_{K_2}, \quad \{\{q\}\} = \frac{1}{2}(q|_{K_1} + q|_{K_2}).$$

In addition, we extend the definition of $[\![\bullet]\!]$ and $\{\{\bullet\}\}$ to $[H^2(\mathcal{T})]^d$ and $[H^2(\mathcal{T})]^{d\times d}$ component-wise. The following well-known identity can be found in [1],

$$\sum_{K\in\mathcal{T}} \int_{\partial K} pq \, ds = \sum_{F\in\mathcal{F}^0} \int_F \{\{p\}\}[\![q]\!] \, ds + \sum_{F\in\mathcal{F}^0} \int_F [\![p]\!]\{\{q\}\} \, ds + \sum_{F\in\mathcal{F}^\partial} \int_F pq \, ds. \tag{2.11}$$

Denote by $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega)$ and $Q_h \subset H^1(\Omega)/\mathbb{R}$ some pair of finite element spaces of velocity and pressure. They do not need to satisfy the inf-sup condition. Equations (2.3) and (2.5) motivate us to use the following scheme to update the pressure: for given $\boldsymbol{u}_h^{n+1} \in \boldsymbol{V}_h$, compute $p_h^{n+1} \in Q_h$ by

$$(\nabla p_h^{n+1}, \nabla q_h) = (\boldsymbol{f}^{n+1}, \nabla q_h) + \sum_{F\in\mathcal{F}^\partial} \nu \int_F \nabla \times \boldsymbol{u}_h^{n+1} \cdot (\nabla q_h \times \boldsymbol{n}_F) \, ds \quad \forall q_h \in Q_h. \tag{2.12}$$

Next, let us investigate the discretizations of the velocity-updating procedure. We consider a general form of the momentum equation,

$$\boldsymbol{u}_t + \mathcal{M} = \boldsymbol{0}.$$

When $\mathcal{M} = -\nu\Delta\boldsymbol{u} + \nabla p - \boldsymbol{f}$, it is exactly (2.1). Similarly, if we take $\mathcal{M} = -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\Delta)\boldsymbol{u} + \nabla p - \boldsymbol{f}$, it becomes the momentum equation (1.1).

Assume that $\mathcal{M} = -\boldsymbol{u}_t \in \boldsymbol{H}^{\frac{3}{2}+\varepsilon}(\Omega)$. Instead of testing the momentum equation with $\Delta\boldsymbol{v}$ for $\boldsymbol{v} \in \boldsymbol{V}_h$ (since $\Delta\boldsymbol{v}$ is not well-defined on $\boldsymbol{V}_h$), we take gradient on both sides of the above equation first to get

$$\nabla\boldsymbol{u}_t + \nabla\mathcal{M} = \boldsymbol{0}.$$

Testing the above equation with $\nabla\boldsymbol{v}$ for any $\boldsymbol{v} \in \boldsymbol{V}_h$, integration by parts and using the identity (2.11) give

$$\begin{aligned}(\nabla\boldsymbol{u}_t, \nabla\boldsymbol{v}) - (\mathcal{M}, \Delta_h\boldsymbol{v}) + \sum_{F\in\mathcal{F}^0}\int_F \{\{\mathcal{M}\}\}\cdot([\![\nabla\boldsymbol{v}]\!]\boldsymbol{n}_F)\,ds \\ + \sum_{F\in\mathcal{F}^0}\int_F [\![\mathcal{M}]\!]\cdot(\{\{\nabla\boldsymbol{v}\}\}\boldsymbol{n}_F)\,ds + \sum_{F\in\mathcal{F}^\partial}\int_F \mathcal{M}\cdot(\nabla\boldsymbol{v}\boldsymbol{n}_F)\,ds = 0,\end{aligned}$$

where Δ_h is the broken Laplace operator, meaning

$$(\mathcal{M}, \Delta_h\boldsymbol{v}) := \sum_{K\in\mathcal{T}}(\mathcal{M}, \Delta\boldsymbol{v})_K \quad \text{for any } \boldsymbol{v} \in \boldsymbol{V}_h,$$

and $\boldsymbol{n}_F$ is a unit normal vector of F. Since $\mathcal{M} \in \boldsymbol{H}^{\frac{3}{2}+\varepsilon}(\Omega)$ (implying that $[\![\mathcal{M}]\!] = \boldsymbol{0}$ on $\mathcal{F}^0$) and $\mathcal{M} = -\boldsymbol{u}_t$, one has

$$(\nabla\boldsymbol{u}_t, \nabla\boldsymbol{v}) - (\mathcal{M}, \Delta_h\boldsymbol{v}) + \sum_{F\in\mathcal{F}^0}\int_F \{\{\mathcal{M}\}\}\cdot([\![\nabla\boldsymbol{v}]\!]\boldsymbol{n}_F)\,ds = \sum_{F\in\mathcal{F}^\partial}\int_F \boldsymbol{u}_t\cdot(\nabla\boldsymbol{v}\boldsymbol{n}_F)\,ds.$$

After adding a penalty term which is inspired by DG methods (see [1]), we finally get

(2.13)
$$\begin{aligned}(\nabla\boldsymbol{u}_t, \nabla\boldsymbol{v}) - (\mathcal{M}, \Delta_h\boldsymbol{v}) + \sum_{F\in\mathcal{F}^0}\int_F \{\{\mathcal{M}\}\}\cdot([\![\nabla\boldsymbol{v}]\!]\boldsymbol{n}_F)\,ds \\ + \sum_{F\in\mathcal{F}^0}\frac{\gamma_F}{h_F}\int_F([\![\nabla\boldsymbol{u}]\!]\boldsymbol{n}_F)\cdot([\![\nabla\boldsymbol{v}]\!]\boldsymbol{n}_F)\,ds = \sum_{F\in\mathcal{F}^\partial}\int_F \boldsymbol{u}_t\cdot(\nabla\boldsymbol{v}\boldsymbol{n}_F)\,ds,\end{aligned}$$

with $\gamma_F > 0, F \in \mathcal{F}^0$, being the penalty parameters. Note that for functions in $\boldsymbol{V}_h$, the interior jump of their tangent components vanishes. Therefore one has

$$\int_F([\![\nabla\boldsymbol{u}]\!]\boldsymbol{n}_F)\cdot([\![\nabla\boldsymbol{v}]\!]\boldsymbol{n}_F)\,ds = \int_F [\![\nabla\boldsymbol{u}]\!]\cdot[\![\nabla\boldsymbol{v}]\!]\,ds \quad \text{for all } F \in \mathcal{F}^0.$$

So these two forms are equivalent in the context of our study.

Taking $\mathcal{M} = -\nu\Delta\boldsymbol{u} + \nabla p - \boldsymbol{f}$, we can construct fully discrete schemes by combining semi-discretizations in time (see (2.4) and (2.6)) and the semi-discretization in space (based on (2.13)). To simplify the notation, we define

$$a_{\mathrm{Lap}}(\boldsymbol{u}^{n+1}, \boldsymbol{v}) := \nu(\Delta_h \boldsymbol{u}^{n+1}, \Delta_h \boldsymbol{v}) - \sum_{F\in\mathcal{F}^0} \int_F \{\{\nu\Delta_h \boldsymbol{u}^{n+1}\}\} \cdot ([\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F)\, ds$$
$$- \sum_{F\in\mathcal{F}^0} \int_F ([\![\nabla \boldsymbol{u}^{n+1}]\!]\boldsymbol{n}_F) \cdot \{\{\nu\Delta_h \boldsymbol{v}\}\}\, ds$$
$$+ \sum_{F\in\mathcal{F}^0} \frac{\gamma_F}{h_F} \int_F ([\![\nabla \boldsymbol{u}^{n+1}]\!]\boldsymbol{n}_F) \cdot ([\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F)\, ds,$$

and

$$a_{\mathrm{Lap}-\beta}(\boldsymbol{u}^{n+1}, \boldsymbol{v}) := \nu(\Delta_h \mathcal{L}_\beta \boldsymbol{u}^{n+1}, \Delta_h \boldsymbol{v})$$
$$- \sum_{F\in\mathcal{F}^0} \int_F \{\{\nu\Delta_h \mathcal{L}_\beta \boldsymbol{u}^{n+1}\}\} \cdot ([\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F)\, ds$$
$$- \sum_{F\in\mathcal{F}^0} \int_F ([\![\nabla \mathcal{L}_\beta \boldsymbol{u}^{n+1}]\!]\boldsymbol{n}_F) \cdot \{\{\nu\Delta_h \boldsymbol{v}\}\}\, ds$$
$$+ \sum_{F\in\mathcal{F}^0} \frac{\gamma_F}{h_F} \int_F ([\![\nabla \mathcal{L}_{\beta+1} \boldsymbol{u}^{n+1}]\!]\boldsymbol{n}_F) \cdot ([\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F)\, ds,$$

where the third terms in the right-hand side of above two formulations are also artificial terms to guarantee the symmetry of the coefficient matrices. Suppose that $\boldsymbol{u}_h^0 \in \boldsymbol{V}_h$ is some interpolation of $\boldsymbol{u}^0$ and p_h^0 is obtained via (2.12).

Then, a first-order (in time) fully discrete scheme is

Scheme I Given $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times Q_h$, find $\boldsymbol{u}_h^{n+1} \in \boldsymbol{V}_h$ such that for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$

(2.14)
$$\frac{(\nabla \boldsymbol{u}_h^{n+1} - \nabla \boldsymbol{u}_h^n, \nabla \boldsymbol{v}_h)}{\Delta t} + a_{\mathrm{Lap}}(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)$$
$$= - \sum_{F\in\mathcal{F}^0} \int_F \{\{\nabla p_h^n - \boldsymbol{f}^{n+1}\}\} \cdot ([\![\nabla \boldsymbol{v}_h]\!]\boldsymbol{n}_F)\, ds$$
$$- (\boldsymbol{f}^{n+1} - \nabla p_h^n, \Delta_h \boldsymbol{v}_h) + \sum_{F\in\mathcal{F}^\partial} \int_F \boldsymbol{u}_t(t^{n+1}) \cdot (\nabla \boldsymbol{v}_h \boldsymbol{n}_F)\, ds,$$

and then update p_h^{n+1} by (2.12). Similarly, a second-order (in time) fully discrete scheme is

Scheme II Given $(\boldsymbol{u}_h^i, p_h^i) \in \boldsymbol{V}_h \times Q_h$ with $i = n, n-1$, find $\boldsymbol{u}_h^{n+1} \in \boldsymbol{V}_h$ such that for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$

(2.15)
$$\frac{(\nabla d_\beta \boldsymbol{u}_h^{n+1}, \nabla \boldsymbol{v}_h)}{\Delta t} + a_{\mathrm{Lap}-\beta}(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)$$
$$= - \sum_{F\in\mathcal{F}^0} \int_F \{\{\nabla \mathcal{L}_{\beta+1} p_h^n - \boldsymbol{f}^{n+\beta}\}\} \cdot ([\![\nabla \boldsymbol{v}_h]\!]\boldsymbol{n}_F)\, ds$$
$$- (\boldsymbol{f}^{n+\beta} - \mathcal{L}_{\beta+1} \nabla p_h^n, \Delta_h \boldsymbol{v}_h) + \sum_{F\in\mathcal{F}^\partial} \int_F \boldsymbol{u}_t(t^{n+\beta}) \cdot (\nabla \boldsymbol{v}_h \boldsymbol{n}_F)\, ds,$$

and then update p_h^{n+1} by (2.12).

Here $\boldsymbol{u}_t$ represents the time derivative of the true velocity solution. Note that $\boldsymbol{u}_t|_\Gamma$ is known from the boundary conditions of $\boldsymbol{u}$ (indeed we have $\boldsymbol{u}_t|_\Gamma = 0$ for homogeneous boundary conditions) and thus the last terms in (2.14) and (2.15) are computable. If $\boldsymbol{u}_t|_\Gamma \cdot \boldsymbol{n}$ is nonzero, the way updating pressure should also be modified as follows (cf. [25]):

$$(\nabla p_h^{n+1}, \nabla q_h) = (\boldsymbol{f}^{n+1}, \nabla q_h) + \sum_{F\in\mathcal{F}^\partial} \nu \int_F \nabla \times \boldsymbol{u}_h^{n+1} \cdot (\nabla q_h \times \boldsymbol{n}_F)\, ds - \int_{\partial\Omega} \boldsymbol{u}_t \cdot \boldsymbol{n} q_h \, ds \quad \forall q_h \in Q_h.$$

Remark 2.1. The methods (2.14) and (2.15) can be regarded as two continuous interior penalty (CIP) methods (see, e.g., [2,4,7]) for discretizing a formal fourth-order problem of the velocity $\boldsymbol{u}$ (i.e., $-\Delta \boldsymbol{u}_t - \Delta\mathcal{M} = \boldsymbol{0}$) with the Dirichlet boundary condition $\boldsymbol{u}|_\Gamma = \boldsymbol{0}$ and Neumann boundary condition $\mathcal{M}|_\Gamma = -\boldsymbol{u}_t|_\Gamma$ (indeed it is also zero here). CIP methods use C^0 finite elements and penalize the interior continuity of derivatives through a DG framework (see a_{Lap} and $a_{\text{Lap}-\beta}$). For the penalty term in $a_{\text{Lap}-\beta}$, i.e.,

$$+ \sum_{F\in\mathcal{F}^0} \frac{\gamma_F}{h_F} \int_F ([\![\nabla \mathcal{L}_{\beta+1} \boldsymbol{u}^{n+1}]\!] \boldsymbol{n}_F) \cdot ([\![\nabla \boldsymbol{v}]\!] \boldsymbol{n}_F)\, ds,$$

the operator $\mathcal{L}_{\beta+1}$ can also be replaced by either $\mathcal{L}_\beta$ or $\mathcal{L}_1$ (identity operator, which is more commonly-used in CIP methods), which only slightly affects the lower bound requirement for γ_F in stability analysis (see Section 4). We use $\mathcal{L}_{\beta+1}$ here because the stability analysis requires testing the discrete problem with $\boldsymbol{v}_h = \mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}$, which resembles the semi-discrete case [18]. Such a choice can help us simplify the analysis.

2.3. **Extension to the Navier–Stokes equations.** For the NSEs, the velocity-updating procedure for the momentum equation (1.1) can be constructed similarly by taking $\mathcal{M} := -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \boldsymbol{f}$ in (2.13). More precisely, a second-order fully discrete scheme for the Navier–Stokes equations is

$$\begin{aligned} &\frac{(\nabla d_\beta \boldsymbol{u}_h^{n+1}, \nabla \boldsymbol{v}_h)}{\Delta t} + a_{\text{Lap}-\beta}(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) \\ &= -\sum_{F\in\mathcal{F}^0} \int_F \{\!\{(\mathcal{L}_{\beta+1}\boldsymbol{u}_h^n \cdot \nabla)\mathcal{L}_{\beta+1}\boldsymbol{u}_h^n + \nabla \mathcal{L}_{\beta+1} p_h^n - \boldsymbol{f}^{n+\beta}\}\!\} \cdot ([\![\nabla \boldsymbol{v}_h]\!]\boldsymbol{n}_F)\, ds \\ &\quad - (\boldsymbol{f}^{n+\beta} - (\mathcal{L}_{\beta+1}\boldsymbol{u}_h^n \cdot \nabla)\mathcal{L}_{\beta+1}\boldsymbol{u}_h^n - \mathcal{L}_{\beta+1}\nabla p_h^n, \Delta_h \boldsymbol{v}_h) \\ &\quad + \sum_{F\in\mathcal{F}^\partial} \int_F \boldsymbol{u}_t(t^{n+\beta}) \cdot (\nabla \boldsymbol{v}_h \boldsymbol{n}_F)\, ds; \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} (\nabla p_h^{n+1}, \nabla q_h) &= (\boldsymbol{f}^{n+1} - (\boldsymbol{u}_h^{n+1}\cdot\nabla)\boldsymbol{u}_h^{n+1}, \nabla q) \\ &\quad + \sum_{F\in\mathcal{F}^\partial} \nu \int_F \nabla \times \boldsymbol{u}_h^{n+1} \cdot (\nabla q_h \times \boldsymbol{n}_F)\, ds \quad \forall q_h \in Q_h. \end{aligned} \tag{2.17}$$

3. Estimate of the discrete Stokes pressure

This section extends the estimate of the Stokes pressure (1.6) from $\boldsymbol{V} = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ to $\boldsymbol{V}(h) := \boldsymbol{V} + \boldsymbol{V}_h$. Similarly to the space continuous case (see [18, 25]), the numerical analysis is limited to domains with C^3 boundary. We first enlarge the domain of the Stokes pressure p_{S} as follows: For any $\boldsymbol{u} \in \boldsymbol{V}(h)$,

$$(\nabla p_{\mathrm{S}}(\boldsymbol{u}), \nabla q) = \sum_{F \in \mathcal{F}^{\partial}} \int_F \nabla \times \boldsymbol{u} \cdot (\nabla q \times \boldsymbol{n}_F)\, ds \quad \forall q \in H^1(\Omega)/\mathbb{R}. \tag{3.1}$$

For arbitrary $\boldsymbol{u} \in \boldsymbol{V}(h)$, since $(\nabla p_{\mathrm{S}}(\boldsymbol{u}), \nabla q) = 0$ for all $q \in H_0^1(\Omega)$ (as $\nabla q \times \boldsymbol{n}_F$ also vanishes on all boundary faces), we have $\nabla p_{\mathrm{S}}(\boldsymbol{u}) \in H(\mathrm{div}; \Omega)$ and $\Delta p_{\mathrm{S}}(\boldsymbol{u}) = 0$. Then it follows from a trace theorem (see, e.g., [8, Theorem 2.5]) that $\frac{\partial p_{\mathrm{S}}(\boldsymbol{u})}{\partial n}$ belongs to $H^{-1/2}(\Gamma)$. These imply that $p_{\mathrm{S}}(\boldsymbol{u})$ is a harmonic function satisfying the following Neumann boundary value problem

$$\Delta p_{\mathrm{S}}(\boldsymbol{u}) = 0 \quad \text{in } \Omega, \qquad \frac{\partial p_{\mathrm{S}}(\boldsymbol{u})}{\partial n} = -(\nabla \times \nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} \quad \text{on } \Gamma,$$

where $(\nabla \times \nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} \in H^{-1/2}(\Gamma)$ should be understood in the sense that

$$\int_\Gamma (\nabla \times \nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} q \, ds = -\sum_{F \in \mathcal{F}^{\partial}} \int_F \nabla \times \boldsymbol{u} \cdot (\nabla q \times \boldsymbol{n})\, ds \quad \text{for all } q \in H^1(\Omega)/\mathbb{R}.$$

Let $\Phi(x) := \mathrm{dist}(\boldsymbol{x}, \Gamma)$ describe the distance between the point $\boldsymbol{x} \in \Omega$ and Γ. For any $s > 0$, the set of points $\boldsymbol{x} \in \Omega$ within distance s from Γ and its complementary set are denoted by

$$\Omega_s := \{\boldsymbol{x} \in \Omega : \Phi(\boldsymbol{x}) < s\} \quad \text{and} \quad \Omega_s^{\mathrm{c}} := \Omega \backslash \Omega_s,$$

respectively. We also set $\Gamma_s = \{x \in \Omega : \Phi(\boldsymbol{x}) = s\}$. Suppose Γ is C^3 and compact, it is pointed out in [25] that $\Phi \in C^3(\bar{\Omega}_{s_0})$ for some $s_0 > 0$ and $\boldsymbol{n}(\boldsymbol{x}) = -\nabla \Phi$ is exactly the unit outward normal vector to $\Gamma_s = \Gamma_s^{\mathrm{c}}$ with $s = \Phi(\boldsymbol{x})$.

The proof of Lemma 3.1 is very similar to [25, Lemma 3.2]. So we omit it.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary Γ of class C^2. Let $\boldsymbol{u} \in \boldsymbol{V}(h)$ and suppose that*

$$\boldsymbol{u}_\parallel := \left(I - \boldsymbol{n}\boldsymbol{n}^\top\right) \boldsymbol{u} = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}, \quad \boldsymbol{u}_\perp := \boldsymbol{n}\boldsymbol{n}^\top \boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n},$$

in some neighborhood of Γ. Then the following are valid:

(i) *If $\boldsymbol{u} = 0$ on Γ, then $\nabla \cdot \boldsymbol{u}_\parallel = 0$ on Γ.*
(ii) *If $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, then $\nabla \times \boldsymbol{u}_\perp = 0$ on Γ.*

On the whole domain Ω, we define

$$\boldsymbol{u}_\parallel := \xi \left(I - \boldsymbol{n}\boldsymbol{n}^\top\right) \boldsymbol{u}, \quad \boldsymbol{u}_\perp := \xi \boldsymbol{n}\boldsymbol{n}^\top \boldsymbol{u} + (1 - \xi)\boldsymbol{u},$$

where ξ is a cutoff function satisfying $\xi(\boldsymbol{x}) = 1$ when $\Phi(\boldsymbol{x}) < \frac{1}{2}s$ and $\xi(\boldsymbol{x}) = 0$ when $\Phi(\boldsymbol{x}) > s$. Note that ξ can be C^3 for small s (see [25, Section 3.4]).

Lemma 3.2. *Let Ω be a bounded domain with boundary Γ of C^3, and let $\boldsymbol{u} \in \boldsymbol{V}(h)$ be arbitrary. Then for any $q \in H^1(\Omega)$ satisfying $\Delta q = 0$ and $\varepsilon > 0$, there exists $C > 0$ which scales like ε^{-1} such that*

$$|(\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u}), \nabla q)| \leqslant C C_{\mathrm{inv}} \sum_{F \in \mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!] \boldsymbol{n}_F|^2 \, ds + \varepsilon \|\nabla q\|^2,$$

where C_{inv} is a constant for the inverse inequality.

Proof. For any $\boldsymbol{u} \in \boldsymbol{V}(h)$, there exist $\boldsymbol{w} \in \boldsymbol{V}$ and $\boldsymbol{z} \in \boldsymbol{V}_h$ such that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{z}$. Define $\boldsymbol{w}_\parallel$ and $\boldsymbol{z}_\parallel$ in the same way as $\boldsymbol{u}_\parallel$. It has been proved in [25] that

$$(\Delta \boldsymbol{w}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{w}), \nabla q) = 0 \text{ if } q \in H^1(\Omega) \text{ satisfies } \Delta q = 0. \tag{3.2}$$

For the estimate related to $\boldsymbol{z}$, a combination of integration by parts, Lemma 3.1(i) and the fact $\Delta q = 0$ gives that

$$\begin{aligned}(\Delta_h \boldsymbol{z}_\parallel, \nabla q) &= (-\nabla_h \times \nabla \times \boldsymbol{z}_\parallel + \nabla_h \nabla \cdot \boldsymbol{z}_\parallel, \nabla q)\\ &= \sum_{F \in \mathcal{F}^\partial} \int_F \nabla \times \boldsymbol{z}_\parallel \cdot (\nabla q \times \boldsymbol{n})\, ds + \sum_{F \in \mathcal{F}^0} \int_F [\![\nabla \times \boldsymbol{z}_\parallel]\!] \cdot (\nabla q \times \boldsymbol{n}_F)\, ds\\ &\quad + \sum_{F \in \mathcal{F}^0} \int_F [\![\nabla \cdot \boldsymbol{z}_\parallel]\!] (\nabla q \cdot \boldsymbol{n}_F)\, ds.\end{aligned}$$

Further, by Lemma 3.1(ii) one has $\nabla \times \boldsymbol{z}_\parallel = \nabla \times \boldsymbol{z}$ on Γ, which implies that

$$\begin{aligned}(\Delta_h \boldsymbol{z}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{z}), \nabla q) = &\sum_{F \in \mathcal{F}^0} \int_F [\![\nabla \times \boldsymbol{z}_\parallel]\!] \cdot (\nabla q \times \boldsymbol{n}_F)\, ds\\ &+ \sum_{F \in \mathcal{F}^0} \int_F [\![\nabla \cdot \boldsymbol{z}_\parallel]\!] (\nabla q \cdot \boldsymbol{n}_F)\, ds.\end{aligned} \tag{3.3}$$

Note that

$$\begin{aligned}\|\nabla q\|_{\boldsymbol{H}(\mathrm{div};D)} &= (\|\nabla q\|_D^2 + \|\Delta q\|_D^2)^{\frac{1}{2}} = \|\nabla q\|_D,\\ \|\nabla q\|_{\boldsymbol{H}(\mathrm{curl};D)} &= (\|\nabla q\|_D^2 + \|\nabla \times \nabla q\|_D^2)^{\frac{1}{2}} = \|\nabla q\|_D,\end{aligned}$$

for arbitrary $D \subseteq \Omega$. By two trace theorems related to $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathrm{curl};\Omega)$ (see [8, Theorems 2.5 and 2.11]), we arrive at

$$\begin{aligned}&\Big|\sum_{F \in \mathcal{F}^0} \int_F [\![\nabla \times \boldsymbol{z}_\parallel]\!] \cdot (\nabla q \times \boldsymbol{n}_F)\, ds + \sum_{F \in \mathcal{F}^0} \int_F [\![\nabla \cdot \boldsymbol{z}_\parallel]\!] (\nabla q \cdot \boldsymbol{n}_F)\, ds\Big|\\ &\leqslant \sum_{F \in \mathcal{F}^0} (|[\![\nabla \times \boldsymbol{z}_\parallel]\!]|_{\frac{1}{2},F} |\nabla q \times \boldsymbol{n}_F|_{-\frac{1}{2},F} + |[\![\nabla \cdot \boldsymbol{z}_\parallel]\!]|_{\frac{1}{2},F} |\nabla q \cdot \boldsymbol{n}_F|_{-\frac{1}{2},F})\\ &\leqslant \sum_{F \in \mathcal{F}^0} (|[\![\nabla \times \boldsymbol{z}_\parallel]\!]|_{\frac{1}{2},F} \|\nabla q\|_{K_F} + |[\![\nabla \cdot \boldsymbol{z}_\parallel]\!]|_{\frac{1}{2},F} \|\nabla q\|_{K_F})\\ &\leqslant C \sum_{F \in \mathcal{F}^0} |[\![\nabla \boldsymbol{z}_\parallel]\!]|_{\frac{1}{2},F}^2 + \varepsilon \|\nabla q\|^2,\end{aligned} \tag{3.4}$$

where in the last step an ε-scaled Young's inequality is applied. Since $\boldsymbol{n}$ is uniformly bounded, it follows from an inverse inequality that

$$|[\![\nabla \boldsymbol{z}_\parallel]\!]|_{\frac{1}{2},F}^2 \leqslant C |[\![\nabla \boldsymbol{z}]\!]|_{\frac{1}{2},F}^2 \leqslant C C_{\mathrm{inv}} h_F^{-1} |[\![\nabla \boldsymbol{z}]\!]|_F^2\, ds = C C_{\mathrm{inv}} h_F^{-1} |[\![\nabla \boldsymbol{z}]\!] \boldsymbol{n}_F|_F^2\, ds. \tag{3.5}$$

Note that $\nabla p_{\mathrm{S}}(\boldsymbol{u}) = \nabla p_{\mathrm{S}}(\boldsymbol{w}) + \nabla p_{\mathrm{S}}(\boldsymbol{z})$ and $\boldsymbol{u}_\parallel = \boldsymbol{w}_\parallel + \boldsymbol{z}_\parallel$. A combination of (3.2)–(3.5), the triangle inequality, and the fact $[\![\nabla \boldsymbol{w}]\!] \boldsymbol{n}_F = \boldsymbol{0}$ over all $F \in \mathcal{F}^0$ (see, e.g., [4, Eq. (3.6)]) yields

$$\begin{aligned}|(\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u}), \nabla q)| &\leqslant |(\Delta \boldsymbol{w}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{w}), \nabla q)| + |(\Delta_h \boldsymbol{z}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{z}), \nabla q)| \\ &\leqslant C C_{\mathrm{inv}} \sum_{F\in\mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{z}]\!]\boldsymbol{n}_F|^2 \, ds + \varepsilon \|\nabla q\|^2 \\ &= C C_{\mathrm{inv}} \sum_{F\in\mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!]\boldsymbol{n}_F|^2 \, ds + \varepsilon \|\nabla q\|^2.\end{aligned}$$

This completes the proof. □

Lemma 3.2 implies that

$$\begin{aligned}\|\Delta_h \boldsymbol{u}\|^2 &= \|\Delta_h \boldsymbol{u}_\perp\|^2 + 2\left(\Delta_h \boldsymbol{u}_\perp, \Delta_h \boldsymbol{u}_\parallel\right) + \|\Delta_h \boldsymbol{u}_\parallel\|^2 \\ &= \|\Delta_h \boldsymbol{u}_\perp\|^2 + 2\left(\Delta_h \boldsymbol{u}_\perp, \Delta_h \boldsymbol{u}_\parallel\right) + \left\|\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u})\right\|^2 \\ &\quad + 2(\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u}), \nabla p_{\mathrm{S}}(\boldsymbol{u})) + \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2 \\ &\geqslant 2\left(\Delta_h \boldsymbol{u}_\perp, \Delta_h \boldsymbol{u}_\parallel\right) + \left\|\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u})\right\|^2 \\ &\quad - 2C C_{\mathrm{inv}} \sum_{F\in\mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!]\boldsymbol{n}_F|^2 \, ds + (1-2\varepsilon)\|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2.\end{aligned} \tag{3.6}$$

What remains is to estimate $(\Delta_h \boldsymbol{u}_\perp, \Delta_h \boldsymbol{u}_\parallel)$ and $\|\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2$.

Lemma 3.3. *With the same assumption as in Lemma* 3.2*, for any $\varepsilon > 0$, there exists C such that*

$$\left(\Delta_h \boldsymbol{u}_\perp, \Delta_h \boldsymbol{u}_\parallel\right) \geqslant -\varepsilon \|\Delta_h \boldsymbol{u}\|^2 - C\|\nabla \boldsymbol{u}\|^2,$$

$$\left\|\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u})\right\|^2 \geqslant (1-\varepsilon)\|\nabla p_{\mathrm{S}}\|^2 - C\|\nabla \boldsymbol{u}\|^2 - C C_{\mathrm{inv}} \sum_{F\in\mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!]\boldsymbol{n}_F|^2 \, ds.$$

Proof. See [25, Claim 1] for the first assertion. Let us prove the second assertion. Note that

$$\Delta_h \boldsymbol{u}_\parallel = \xi\left(I - \boldsymbol{n}\boldsymbol{n}^\top\right)\Delta_h \boldsymbol{u} + R_2, \tag{3.7}$$

where R_2 consists of the terms in which the derivatives of $\boldsymbol{u}$ are at most first order and therefore it fulfills $\|R_2\| \leqslant C\|\nabla \boldsymbol{u}\|$. Let $\boldsymbol{a} = \nabla p_{\mathrm{S}}$ and $\boldsymbol{b} = \Delta_h \boldsymbol{u}_\parallel$, and put

$$\boldsymbol{a}_\parallel = \left(I - \boldsymbol{n}\boldsymbol{n}^\top\right)\boldsymbol{a}, \quad \boldsymbol{a}_\perp = \left(\boldsymbol{n}\boldsymbol{n}^\top\right)\boldsymbol{a}, \quad \boldsymbol{b}_\parallel = \left(I - \boldsymbol{n}\boldsymbol{n}^\top\right)\boldsymbol{b}, \quad \boldsymbol{b}_\perp = \left(\boldsymbol{n}\boldsymbol{n}^\top\right)\boldsymbol{b}.$$

So it holds

$$\|\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u})\| = \|\boldsymbol{a}-\boldsymbol{b}\|^2 = \int_{\Omega_s^{\mathrm{c}}} |\boldsymbol{a}|^2 \, d\boldsymbol{x} + \int_{\Omega_s} |\boldsymbol{a}_\perp - \boldsymbol{b}_\perp|^2 \, d\boldsymbol{x} + \int_{\Omega_s} |\boldsymbol{a}_\parallel - \boldsymbol{b}_\parallel|^2 \, d\boldsymbol{x}.$$

First,

$$\int_{\Omega_s} |\boldsymbol{a}_\perp - \boldsymbol{b}_\perp|^2 \geqslant (1-\varepsilon)\int_{\Omega_s} |\boldsymbol{a}_\perp|^2 - C\int_{\Omega_s} |\boldsymbol{b}_\perp|^2.$$

Using Lemma 3.2 and the fact that $\boldsymbol{b}$ vanishes on Ω_s^{c}, one has

$$\begin{aligned}\int_\Omega \boldsymbol{a}\cdot(\boldsymbol{a}-\boldsymbol{b})\, d\boldsymbol{x} &= \int_{\Omega_s^{\mathrm{c}}} |\boldsymbol{a}|^2 \, d\boldsymbol{x} + \int_{\Omega_s} \boldsymbol{a}_\perp \cdot (\boldsymbol{a}_\perp - \boldsymbol{b}_\perp)\, d\boldsymbol{x} + \int_{\Omega_s} \boldsymbol{a}_\parallel \cdot (\boldsymbol{a}_\parallel - \boldsymbol{b}_\parallel)\, d\boldsymbol{x} \\ &\leqslant C C_{\mathrm{inv}} \sum_{F\in\mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!]\boldsymbol{n}_F|^2 \, ds + \varepsilon \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2,\end{aligned}$$

which implies

$$\begin{aligned}\int_{\Omega_s} |\boldsymbol{a}_\parallel - \boldsymbol{b}_\parallel|^2 + |\boldsymbol{a}_\parallel|^2 \, d\boldsymbol{x} &\geqslant -2 \int_{\Omega_s} \boldsymbol{a}_\parallel \cdot (\boldsymbol{a}_\parallel - \boldsymbol{b}_\parallel) \, d\boldsymbol{x} \\ &\geqslant 2 \int_{\Omega_s^{\mathrm{c}}} |\boldsymbol{a}|^2 \, d\boldsymbol{x} + 2 \int_{\Omega_s} \boldsymbol{a}_\perp \cdot (\boldsymbol{a}_\perp - \boldsymbol{b}_\perp) \, d\boldsymbol{x} \\ &\quad - CC_{\mathrm{inv}} \sum_{F \in \mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!] \boldsymbol{n}_F|^2 \, ds - \varepsilon \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2 \\ &\geqslant 2 \int_{\Omega_s^{\mathrm{c}}} |\boldsymbol{a}|^2 \, d\boldsymbol{x} + (2 - \varepsilon) \int_{\Omega_s} |\boldsymbol{a}_\perp|^2 \, d\boldsymbol{x} - C \int_{\Omega_s} |\boldsymbol{b}_\perp|^2 \, d\boldsymbol{x} \\ &\quad - CC_{\mathrm{inv}} \sum_{F \in \mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!] \boldsymbol{n}_F|^2 \, ds - \varepsilon \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\Omega_s} |\boldsymbol{a}_\parallel - \boldsymbol{b}_\parallel|^2 \geqslant (1-\varepsilon) \int_{\Omega_s} |\boldsymbol{a}_\parallel|^2 \, d\boldsymbol{x} + (2-\varepsilon) \int_{\Omega_s} (|\boldsymbol{a}_\perp|^2 - |\boldsymbol{a}_\parallel|^2) \, d\boldsymbol{x} - C \int_{\Omega_s} |\boldsymbol{b}_\perp|^2 \, d\boldsymbol{x} \\ - CC_{\mathrm{inv}} \sum_{F \in \mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!] \boldsymbol{n}_F|^2 \, ds - \varepsilon \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2.\end{aligned}$$

Collecting the estimates above yields

$$\begin{aligned}\int_{\Omega} |\boldsymbol{a} - \boldsymbol{b}|^2 \geqslant (1 - \varepsilon) \int_{\Omega} |\boldsymbol{a}|^2 \, d\boldsymbol{x} + (2 - \varepsilon) \int_{\Omega_s} (|\boldsymbol{a}_\perp|^2 - |\boldsymbol{a}_\parallel|^2) \, d\boldsymbol{x} - C \int_{\Omega_s} |\boldsymbol{b}_\perp|^2 \, d\boldsymbol{x} \\ - CC_{\mathrm{inv}} \sum_{F \in \mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!] \boldsymbol{n}_F|^2 \, ds - \varepsilon \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2.\end{aligned}$$

By (3.7), it holds

$$\int_{\Omega_s} |\boldsymbol{b}_\perp|^2 = \int_{\Omega_s} |\boldsymbol{n} \cdot R_2|^2 \leqslant C \int_{\Omega} |\nabla \boldsymbol{u}|^2.$$

Then it follows from [25, Lemma 3.1] that

$$\begin{aligned}\|\Delta_h \boldsymbol{u}_\parallel - \nabla p_{\mathrm{S}}(\boldsymbol{u})\| \geqslant (1 - \varepsilon - 2C_0 s) \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2 - C \|\nabla \boldsymbol{u}\|^2 \\ - CC_{\mathrm{inv}} \sum_{F \in \mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!] \boldsymbol{n}_F|^2 \, ds.\end{aligned}$$

□

Theorem 3.4. *Let Ω be a bounded domain with boundary Γ of C^3. For any $\varepsilon > 0$, there exist positive constants C_{g} and C_{j} such that*

(3.8)

$$\|\nabla p_{\mathrm{s}}(\boldsymbol{u})\|^2 \leqslant (\frac{1}{2} + \varepsilon) \|\Delta_h \boldsymbol{u}\|^2 + C_{\mathrm{g}} \|\nabla \boldsymbol{u}\|^2 + C_{\mathrm{j}} \sum_{F \in \mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!] \boldsymbol{n}_F|^2 \, ds \; \forall \boldsymbol{u} \in \boldsymbol{V}(h).$$

Proof. By (3.6) and Lemma 3.3, one has

$$\|\Delta_h \boldsymbol{u}\|^2 \geqslant -2\varepsilon \|\Delta_h \boldsymbol{u}\|^2 - C \|\nabla \boldsymbol{u}\|^2 - CC_{\mathrm{inv}} \sum_{F \in \mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{u}]\!] \boldsymbol{n}_F|^2 \, ds + (2 - 3\varepsilon) \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|^2.$$

Then (3.8) follows immediately. □

4. Unconditional stability analysis

This section is devoted to the stability analysis. For simplicity the analysis will be restricted to simplicial meshes. However, there is no fundamental difficulty for extension to more general partitions. To emphasize the main idea, we only consider the Stokes case with $\boldsymbol{f} = \mathbf{0}$. It seems that the analysis of the full NSEs does not involve additional essential difficulties (cf. [18]). Then Scheme I (note that $\boldsymbol{u}_t|_\Gamma = \mathbf{0}$) is reduced to: Update $\boldsymbol{u}_h^{n+1} \in \boldsymbol{V}_h$ by

$$\begin{aligned}&\frac{(\nabla \boldsymbol{u}_h^{n+1} - \nabla \boldsymbol{u}_h^n, \nabla \boldsymbol{v}_h)}{\Delta t} + a_{\mathrm{Lap}}(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)\\ &= -\sum_{F\in\mathcal{F}^0} \int_F \{\{\nabla p_h^n\}\} \cdot ([\![\nabla \boldsymbol{v}_h]\!]\boldsymbol{n}_F)\, ds + (\nabla p_h^n, \Delta_h \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\end{aligned} \tag{4.1}$$

and update $p_h^{n+1} \in Q_h$ by

$$(\nabla p_h^{n+1}, \nabla q_h) = \sum_{F\in\mathcal{F}^\partial} \nu \int_F \nabla \times \boldsymbol{u}_h^{n+1} \cdot (\nabla q_h \times \boldsymbol{n}_F)\, ds \quad \forall q_h \in Q_h. \tag{4.2}$$

Scheme II is reduced to: Update $\boldsymbol{u}_h^{n+1} \in \boldsymbol{V}_h$ by

$$\begin{aligned}&\frac{(\nabla d_\beta \boldsymbol{u}_h^{n+1}, \nabla \boldsymbol{v}_h)}{\Delta t} + a_{\mathrm{Lap}-\beta}(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)\\ &= -\sum_{F\in\mathcal{F}^0} \int_F \{\{\nabla \mathcal{L}_{\beta+1} p_h^n\}\} \cdot ([\![\nabla \boldsymbol{v}_h]\!]\boldsymbol{n}_F)\, ds + (\mathcal{L}_{\beta+1}\nabla p_h^n, \Delta_h \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\end{aligned} \tag{4.3}$$

and update $p_h^{n+1} \in Q_h$ by (4.2).

The discrete Stokes pressure $p_{h,\mathrm{S}} : \boldsymbol{V}(h) \to Q_h$ is defined by

$$(\nabla p_{h,\mathrm{S}}(\boldsymbol{u}), \nabla q_h) = \sum_{F\in\mathcal{F}^\partial} \int_F \nabla \times \boldsymbol{u} \cdot (\nabla q_h \times \boldsymbol{n}_F)\, ds \quad \forall q_h \in Q_h. \tag{4.4}$$

We derive immediately from the above and (1.7) that

$$\|\nabla p_{h,\mathrm{S}}(\boldsymbol{u})\| \leqslant \|\nabla p_{\mathrm{S}}(\boldsymbol{u})\|.$$

Let $(\boldsymbol{u}_h, p_h)$ be the discrete solution of (4.1)–(4.2) or (4.3)–(4.2). Then the following holds true:

$$\|\nabla p_h^n\|^2 = \nu^2 \|\nabla p_{h,\mathrm{S}}(\boldsymbol{u}_h^n)\|^2 \leqslant \nu^2 \|\nabla p_{\mathrm{S}}(\boldsymbol{u}_h^n)\|^2 \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Define the semi-norm $|\!|\!|\cdot|\!|\!|_{\mathrm{j}}$ on $\boldsymbol{V}(h)$ by

$$|\!|\!|\boldsymbol{v}|\!|\!|_{\mathrm{j}}^2 := \sum_{F\in\mathcal{F}^0} h_F^{-1} \int_F |[\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F|^2\, ds \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(h).$$

Lemma 4.1. *Assume $\mathcal{M}_h$ is a function satisfying the inverse estimate $|\mathcal{M}_h|_F \leqslant C_{\mathrm{inv}} h_F^{-\frac{1}{2}} \|\mathcal{M}_h\|_{K_F}$ for all $F \in \mathcal{F}^0$ and $K_F \in \mathcal{T}$ with $K_F \supset F$. Then it holds for any $\varepsilon_0 > 0$ that*

$$\Big|\sum_{F\in\mathcal{F}^0} \int_F \{\{\mathcal{M}_h\}\} \cdot ([\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F)\, ds\Big| \leqslant \varepsilon_0 (d+1) \|\mathcal{M}_h\|^2 + \frac{C_{\mathrm{inv}}^2}{4\varepsilon_0} |\!|\!|\boldsymbol{v}|\!|\!|_{\mathrm{j}}^2 \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{V}_h.$$

Proof. Since there are $d+1$ faces per simplex, a combination of the Cauchy–Schwarz inequality, Young's inequality and the inverse inequality gives

$$\begin{aligned}
&|\sum_{F\in\mathcal{F}^0}\int_F \{\{\mathcal{M}_h\}\}\cdot([\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F)\,ds| \\
&\quad\leqslant \sum_{F\in\mathcal{F}^0}(h_F^{1/2}C_{\rm inv}^{-1}|\{\{\mathcal{M}_h\}\}|_F)(h_F^{-1/2}C_{\rm inv}|[\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F|_F) \\
&\quad\leqslant \Big(\sum_{F\in\mathcal{F}^0}h_F C_{\rm inv}^{-2}|\{\{\mathcal{M}_h\}\}|_F^2\Big)^{\frac12}\Big(\sum_{F\in\mathcal{F}^0}h_F^{-1}C_{\rm inv}^{2}|[\![\nabla \boldsymbol{v}]\!]\boldsymbol{n}_F|_F^2\Big)^{\frac12} \\
&\quad\leqslant \Big(\sum_{F\in\mathcal{F}^0}(d+1)\|\mathcal{M}_h\|^2\Big)^{\frac12}\Big(\sum_{F\in\mathcal{F}^0}C_{\rm inv}^2\,|\!|\!|\boldsymbol{v}|\!|\!|_{\rm j}^2\Big)^{\frac12} \\
&\quad\leqslant \varepsilon_0(d+1)\|\mathcal{M}_h\|^2+\frac{C_{\rm inv}^2}{4\varepsilon_0}\,|\!|\!|\boldsymbol{v}|\!|\!|_{\rm j}^2\,.
\end{aligned}$$

□

In this paper, the following discrete Gronwall's inequality is applied; see [18, Lemma 2].

Lemma 4.2 (Discrete Gronwall's lemma)**.** *Let a_n, b_n, c_n, and d_n be four nonnegative series such that*

$$a_m+\tau\sum_{n=1}^m b_n \leqslant \tau\sum_{n=0}^{m-1}a_n d_n+\tau\sum_{n=0}^{m-1}c_n + C,\quad m\geqslant 1,$$

with C and τ being two positive constants. Then it holds

$$a_m+\tau\sum_{n=1}^m b_n \leqslant \exp\left(\tau\sum_{n=0}^{m-1}d_n\right)\left(\tau\sum_{n=0}^{m-1}c_n + C\right),\quad m\geqslant 1.$$

4.1. **Stability analysis for the first-order scheme** (4.1)–(4.2)**.** Let

$$I_1(\boldsymbol{v}_h) := \sum_{F\in\mathcal{F}^0}\int_F \{\{\nabla p_h^n\}\}\cdot([\![\nabla \boldsymbol{v}_h]\!]\boldsymbol{n}_F)\,ds.$$

Lemma 4.1 implies

$$|I_1(\boldsymbol{v}_h)|\leqslant \nu^{-1}\varepsilon_0(d+1)\|\nabla p_h^n\|^2+\frac{\nu C_{\rm inv}^2}{4\varepsilon_0}\,|\!|\!|\boldsymbol{v}_h|\!|\!|_{\rm j}^2\,,$$

and

$$a_{\rm Lap}(\boldsymbol{u}_h^{n+1},\boldsymbol{u}_h^{n+1})\geqslant \nu(1-2\varepsilon_0(d+1))\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2+(\gamma-\frac{\nu C_{\rm inv}^2}{2\varepsilon_0})\,|\!|\!|\boldsymbol{u}_h^{n+1}|\!|\!|_{\rm j}^2 \tag{4.6}$$

with $\gamma=\min_{F\in\mathcal{F}^0}\{\gamma_F\}$. Further, one has

$$|(\nabla p_h^n,\Delta_h\boldsymbol{v}_h)|\leqslant \frac{1}{2\nu}\|\nabla p_h^n\|^2+\frac{\nu}{2}\|\Delta_h\boldsymbol{v}_h\|^2.$$

Let $\mathcal{RHS}_1(\boldsymbol{v}_h) := -I_1(\boldsymbol{v}_h)+(\nabla p_h^n,\Delta_h\boldsymbol{v}_h)$ denote the right-hand side of (4.1). By Theorem 3.4 and (4.5), it holds

(4.7)

$$\begin{aligned}
|\mathcal{RHS}_1(\boldsymbol{v}_h)| &\leqslant \frac{\nu}{2}\|\Delta_h\boldsymbol{v}_h\|^2+(\varepsilon_0(d+1)/\nu+\frac{1}{2\nu})\|\nabla p_h^n\|^2+\frac{\nu C_{\rm inv}^2}{4\varepsilon_0}\,|\!|\!|\boldsymbol{v}_h|\!|\!|_{\rm j}^2 \\
&\leqslant \tau_1\|\Delta_h\boldsymbol{u}_h^n\|^2+\tau_0 C_{\rm g}\|\nabla\boldsymbol{u}_h^n\|^2+\tau_0 C_{\rm j}\,|\!|\!|\boldsymbol{u}_h^n|\!|\!|_{\rm j}^2+\frac{\nu}{2}\|\Delta_h\boldsymbol{v}_h\|^2+\frac{\nu C_{\rm inv}^2}{4\varepsilon_0}\,|\!|\!|\boldsymbol{v}_h|\!|\!|_{\rm j}^2\,,
\end{aligned}$$

where

$$\tau_0 := (\varepsilon_0(d+1) + \frac{1}{2})\nu, \quad \tau_1 := \tau_0(\frac{1}{2} + \varepsilon). \tag{4.8}$$

This implies Theorem 4.3.

Theorem 4.3. *Let τ_0, τ_1 be defined as (4.8) and $\tau_2 := \gamma - \frac{3\nu C_{\rm inv}^2}{4\varepsilon_0} - \tau_0 C_{\rm j}$ with $\gamma = \min_{F\in\mathcal{F}^0}\{\gamma_F\}$. Assume that $\gamma_F > \tau_0 C_{\rm j} + \frac{3\nu C_{\rm inv}^2}{4\varepsilon_0}$ for all $F \in \mathcal{F}^0$ with some $\varepsilon_0, \varepsilon$ satisfying $(\varepsilon_0(d+1)(\frac{5}{2}+\varepsilon) + \frac{\varepsilon}{2}) < \frac{1}{4}$ such that both $\frac{\nu}{2} - 2\nu\varepsilon_0(d+1) - \tau_1$ and τ_2 are positive. With the same assumption on Ω as in Theorem 3.4, scheme (4.1)–(4.2) is unconditionally stable in the sense that*

$$\begin{aligned}\|\nabla \boldsymbol{u}_h^{n+1}\|^2 &+ 2\Delta t(\frac{\nu}{2} - 2\nu\varepsilon_0(d+1) - \tau_1)\sum_{i=1}^{n+1}\|\Delta_h \boldsymbol{u}_h^i\|^2 + 2\Delta t\tau_2\sum_{i=1}^{n+1}|||\boldsymbol{u}_h^i|||_{\rm j}^2 \\ &\leqslant \exp(2\tau_0 C_{\rm g} t^{n+1})\left[\|\nabla \boldsymbol{u}_h^0\|^2 + 2\Delta t\tau_1\|\Delta_h \boldsymbol{u}_h^0\|^2 + 2\Delta t\tau_0 C_{\rm j}|||\boldsymbol{u}_h^0|||_{\rm j}^2\right].\end{aligned} \tag{4.9}$$

Proof. Taking $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1}$ and inserting (4.6) and (4.7) into (4.1) yield

$$\begin{aligned}&\frac{1}{2\Delta t}(\|\nabla \boldsymbol{u}_h^{n+1}\|^2 - \|\nabla \boldsymbol{u}_h^n\|^2 + \|\nabla \boldsymbol{u}_h^{n+1} - \nabla \boldsymbol{u}_h^n\|^2) + \nu(\frac{1}{2} - 2\varepsilon_0(d+1))\|\Delta_h \boldsymbol{u}_h^{n+1}\|^2 \\ &+ (\gamma - \frac{3\nu C_{\rm inv}^2}{4\varepsilon_0})|||\boldsymbol{u}_h^{n+1}|||_{\rm j}^2 \leqslant \tau_1\|\Delta_h \boldsymbol{u}_h^n\|^2 + \tau_0 C_{\rm g}\|\nabla \boldsymbol{u}_h^n\|^2 + \tau_0 C_{\rm j}|||\boldsymbol{u}_h^n|||_{\rm j}^2.\end{aligned}$$

Taking the summation of all time steps up to $n+1$ gives

$$\begin{aligned}&\frac{1}{2\Delta t}\|\nabla \boldsymbol{u}_h^{n+1}\|^2 + (\frac{\nu}{2} - 2\nu\varepsilon_0(d+1) - \tau_1)\sum_{i=1}^{n+1}\|\Delta_h \boldsymbol{u}_h^i\|^2 + \tau_2\sum_{i=1}^{n+1}|||\boldsymbol{u}_h^i|||_{\rm j}^2 \\ &\leqslant \frac{1}{2\Delta t}\|\nabla \boldsymbol{u}_h^0\|^2 + \tau_1\|\Delta_h \boldsymbol{u}_h^0\|^2 + \tau_0 C_{\rm g}\sum_{i=0}^{n}\|\nabla \boldsymbol{u}_h^i\|^2 + \tau_0 C_{\rm j}|||\boldsymbol{u}_h^0|||_{\rm j}^2.\end{aligned}$$

Then (4.9) follows by a discrete Gronwall inequality (see Lemma 4.2). This completes the proof. □

4.2. **Stability analysis for the second-order scheme** (4.3)–(4.2)**.** For any functions $\{w^i, i \in \mathbb{N}\}$, the following identities can be found in [18, Eqs. (3.8) & (3.16)], which will be used to prove the unconditional stability of (4.3) in case $\beta = 5$.

$$\begin{aligned}(\mathcal{L}_\beta w^{n+1}, \mathcal{L}_{\beta+1} w^{n+1}) &= \frac{\beta - 1}{\beta}\|\mathcal{L}_{\beta+1} w^{n+1}\|^2 + \frac{1}{\beta}\|w^{n+1}\|^2 \\ &\quad + \frac{1}{2}(\|w^{n+1}\|^2 - \|w^n\|^2 + \|w^{n+1} - w^n\|^2),\end{aligned} \tag{4.10}$$

(4.11)

$$\begin{aligned}(d_5 w^{n+1}, \mathcal{L}_6 w^{n+1}) &= \frac{1}{10}\left(\|w^{n+1}\|^2 - \|w^n\|^2\right) + 10\left\|\frac{9}{5}w^{n+1} - \frac{3}{2}w^n\right\|^2 \\ &- 10\left\|\frac{9}{5}w^n - \frac{3}{2}w^{n-1}\right\|^2 + 90\left\|\frac{1}{2}w^{n+1} - w^n + \frac{1}{2}w^{n-1}\right\|^2 \\ &+ \frac{13}{2}\|w^{n+1} - w^n\|^2 - \frac{9}{2}\|w^n - w^{n-1}\|^2 + \frac{9}{2}\|w^{n+1} - 2w^n + w^{n-1}\|^2 \\ &\geqslant E(w^{n+1}) - E(w^n),\end{aligned}$$

where $E(\bullet)$ is defined as

$$E(w^{n+1}) = \frac{1}{10}\|w^{n+1}\|^2 + 10\|\frac{9}{5}w^{n+1} - \frac{3}{2}w^n\|^2 + \frac{9}{2}\|w^{n+1} - w^n\|^2.$$

In addition, the following equality and inequality will be also used in analysis.

$$\begin{aligned}\|\mathcal{L}_{\beta+1}w^{n+1}\|^2 &= \|\mathcal{L}_\beta w^{n+1} + (w^{n+1} - w^n)\|^2 = \|\mathcal{L}_\beta w^{n+1}\|^2 \\ &\quad + \beta\|w^{n+1} - w^n\|^2 + \frac{1}{2}(\|w^{n+1}\|^2 - \|w^n\|^2 + \|w^{n+1} - w^n\|^2).\end{aligned} \tag{4.12}$$

$$\begin{aligned}\|\mathcal{L}_{\beta+1}w^{n+1}\|^2 &= \|w^{n+1} + \beta(w^{n+1} - w^n)\|^2 \\ &= \|w^{n+1}\|^2 + \beta^2\|w^{n+1} - w^n\|^2 + 2\beta(w^{n+1}, w^{n+1} - w^n) \\ &= (\beta+1)\|w^{n+1}\|^2 + \beta(\beta+1)\|w^{n+1} - w^n\|^2 - \beta\|w^n\|^2 \\ &\leqslant (\beta+1)\|w^{n+1}\|^2 + \beta(\beta+1)\|w^{n+1} - w^n\|^2.\end{aligned} \tag{4.13}$$

Let

$$\mathcal{RHS}_2(\boldsymbol{v}_h) := -\sum_{F\in\mathcal{F}^0}\int_F \{\!\{\nabla\mathcal{L}_{\beta+1}p_h^n\}\!\} \cdot ([\![\nabla\boldsymbol{v}_h]\!]\boldsymbol{n}_F)\,ds + (\mathcal{L}_{\beta+1}\nabla p_h^n, \Delta_h\boldsymbol{v}_h)$$

denote the right-hand side of (4.3). Similarly to (4.7), noting that $\mathcal{L}_{\beta+1}p_h^n = \nu\mathcal{L}_{\beta+1}(p_{h,\mathrm{S}}(\boldsymbol{u}_h^n)) = \nu p_{h,\mathrm{S}}(\mathcal{L}_{\beta+1}\boldsymbol{u}_h^n)$, one has

$$\begin{aligned}|\mathcal{RHS}_2(\boldsymbol{v}_h)| &\leqslant \tau_1\|\Delta_h\mathcal{L}_{\beta+1}\boldsymbol{u}_h^n\|^2 + \tau_0 C_{\mathrm{g}}\|\nabla\mathcal{L}_{\beta+1}\boldsymbol{u}_h^n\|^2 \\ &\quad + \tau_0 C_{\mathrm{j}}\,|\!|\!|\mathcal{L}_{\beta+1}\boldsymbol{u}_h^n|\!|\!|_{\mathrm{j}}^2 + \frac{\nu}{2}\|\Delta_h\boldsymbol{v}_h\|^2 + \frac{\nu C_{\mathrm{inv}}^2}{4\varepsilon_0}\,|\!|\!|\boldsymbol{v}_h|\!|\!|_{\mathrm{j}}^2.\end{aligned} \tag{4.14}$$

Also, Lemma 4.1, (4.12), and (4.10) imply

$$\begin{aligned}&a_{\mathrm{Lap}-\beta}(\boldsymbol{u}_h^{n+1}, \mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}) \geqslant \nu(\Delta_h\mathcal{L}_\beta\boldsymbol{u}_h^{n+1}, \Delta_h\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}) \\ &\quad - \nu\varepsilon_0(d+1)\|\Delta_h\mathcal{L}_\beta\boldsymbol{u}_h^{n+1}\|^2 - \nu\varepsilon_0(d+1)\|\Delta_h\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}\|^2 \\ &\quad + (\gamma - \frac{\nu C_{\mathrm{inv}}^2}{4\varepsilon_0})\,|\!|\!|\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}|\!|\!|_{\mathrm{j}}^2 - \frac{\nu C_{\mathrm{inv}}^2}{4\varepsilon_0}\,|\!|\!|\mathcal{L}_\beta\boldsymbol{u}_h^{n+1}|\!|\!|_{\mathrm{j}}^2 \\ &\quad \geqslant \nu(\Delta_h\mathcal{L}_\beta\boldsymbol{u}_h^{n+1}, \Delta_h\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}) - 2\nu\varepsilon_0(d+1)\|\Delta_h\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}\|^2 \\ &\quad + (\gamma - \frac{\nu C_{\mathrm{inv}}^2}{2\varepsilon_0})\,|\!|\!|\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}|\!|\!|_{\mathrm{j}}^2 + \frac{\nu\varepsilon_0(d+1)}{2}(\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2 - \|\Delta_h\boldsymbol{u}_h^n\|^2) \\ &\quad + \frac{\nu C_{\mathrm{inv}}^2}{8\varepsilon_0}(|\!|\!|\boldsymbol{u}_h^{n+1}|\!|\!|_{\mathrm{j}}^2 - |\!|\!|\boldsymbol{u}_h^n|\!|\!|_{\mathrm{j}}^2) \\ &\quad \geqslant \nu\Big[\frac{\beta-1}{\beta}\|\Delta_h\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}\|^2 + \frac{1}{\beta}\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2 + \frac{1}{2}(\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2 - \|\Delta_h\boldsymbol{u}_h^n\|^2)\Big] \\ &\quad - 2\nu\varepsilon_0(d+1)\|\Delta_h\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}\|^2 + (\gamma - \frac{\nu C_{\mathrm{inv}}^2}{2\varepsilon_0})\,|\!|\!|\mathcal{L}_{\beta+1}\boldsymbol{u}_h^{n+1}|\!|\!|_{\mathrm{j}}^2 \\ &\quad + \frac{\nu\varepsilon_0(d+1)}{2}(\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2 - \|\Delta_h\boldsymbol{u}_h^n\|^2) + \frac{\nu C_{\mathrm{inv}}^2}{8\varepsilon_0}(|\!|\!|\boldsymbol{u}_h^{n+1}|\!|\!|_{\mathrm{j}}^2 - |\!|\!|\boldsymbol{u}_h^n|\!|\!|_{\mathrm{j}}^2).\end{aligned} \tag{4.15}$$

Theorem 4.4. *Let τ_0, τ_1 be defined as (4.8) and $\tau_2 := \gamma - \frac{3\nu C_{\mathrm{inv}}^2}{4\varepsilon_0} - \tau_0 C_{\mathrm{j}}$ with $\gamma = \min_{F\in\mathcal{F}^0}\{\gamma_F\}$. Assume that $\gamma_F > \tau_0 C_{\mathrm{j}} + \frac{3\nu C_{\mathrm{inv}}^2}{4\varepsilon_0}$ for all $F \in \mathcal{F}^0$ with some $\varepsilon_0, \varepsilon$ satisfying $(\varepsilon_0(d+1)(\frac{5}{2}+\varepsilon) + \frac{\varepsilon}{2}) < \frac{1}{20}$ such that both $\frac{3\nu}{10} - 2\nu\varepsilon_0(d+1) - \tau_1$ and τ_2 are* *positive. With the same assumption on* Ω *as in Theorem* 3.4*, scheme* (4.3)–(4.2) *is unconditionally stable for* $\beta = 5$ *in the sense that*

(4.16)
$$\begin{aligned}
&E(\nabla \boldsymbol{u}_h^{n+1}) + \Delta t(\frac{3\nu}{10} - 2\nu\varepsilon_0(d+1) - \tau_1)\sum_{i=2}^{n+1}\|\Delta_h\mathcal{L}_6\boldsymbol{u}_h^i\|^2 + \frac{\nu\Delta t}{5}\sum_{i=2}^{n+1}\|\Delta_h\boldsymbol{u}_h^i\|^2\\
&+\frac{\nu\Delta t}{2}(1+\varepsilon_0(d+1))\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2 + \Delta t\tau_2\sum_{i=2}^{n+1}|||\mathcal{L}_6\boldsymbol{u}_h^i|||_{\mathrm{j}}^2 + \frac{\nu C_{\mathrm{inv}}^2\Delta t}{8\varepsilon_0}|||\boldsymbol{u}_h^{n+1}|||_{\mathrm{j}}^2\\
&\leqslant \exp(60\tau_0 C_{\mathrm{g}} t^n)\Bigg[E(\nabla\boldsymbol{u}_h^1) + \Delta t\tau_1\|\Delta_h\mathcal{L}_6\boldsymbol{u}_h^1\|^2 + \frac{\nu\Delta t}{2}(1+\varepsilon_0(d+1))\|\Delta_h\boldsymbol{u}_h^1\|^2\\
&+\Delta t\tau_0 C_{\mathrm{j}}|||\mathcal{L}_6\boldsymbol{u}_h^1|||_{\mathrm{j}}^2 + \frac{\nu C_{\mathrm{inv}}^2\Delta t}{8\varepsilon_0}|||\boldsymbol{u}_h^1|||_{\mathrm{j}}^2\Bigg].
\end{aligned}$$

Proof. Inserting (4.14) and (4.15) into (4.3), taking $\boldsymbol{v}_h = \mathcal{L}_6\boldsymbol{u}_h^{n+1}$ and using (4.11) yield

$$\begin{aligned}
&\frac{1}{\Delta t}(E(\nabla\boldsymbol{u}_h^{n+1}) - E(\nabla\boldsymbol{u}_h^n)) + (\frac{3\nu}{10} - 2\nu\varepsilon_0(d+1))\|\Delta_h\mathcal{L}_6\boldsymbol{u}_h^{n+1}\|^2 + \frac{\nu}{5}\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2\\
&+\frac{\nu}{2}(1+\varepsilon_0(d+1))(\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2 - \|\Delta_h\boldsymbol{u}_h^n\|^2) + (\gamma - \frac{3\nu C_{\mathrm{inv}}^2}{4\varepsilon_0})|||\mathcal{L}_6\boldsymbol{u}_h^{n+1}|||_{\mathrm{j}}^2\\
&+\frac{\nu C_{\mathrm{inv}}^2}{8\varepsilon_0}(|||\boldsymbol{u}_h^{n+1}|||_{\mathrm{j}}^2 - |||\boldsymbol{u}_h^n|||_{\mathrm{j}}^2) \leqslant \tau_1\|\Delta_h\mathcal{L}_6\boldsymbol{u}_h^n\|^2 + \tau_0 C_{\mathrm{g}}\|\nabla\mathcal{L}_6\boldsymbol{u}_h^n\|^2 + \tau_0 C_{\mathrm{j}}|||\mathcal{L}_6\boldsymbol{u}_h^n|||_{\mathrm{j}}^2.
\end{aligned}$$

Taking the summation of all the time steps from 2 to $n+1$ gives

$$\begin{aligned}
&\frac{1}{\Delta t}E(\nabla\boldsymbol{u}_h^{n+1}) + (\frac{3\nu}{10} - 2\nu\varepsilon_0(d+1) - \tau_1)\sum_{i=2}^{n+1}\|\Delta_h\mathcal{L}_6\boldsymbol{u}_h^i\|^2 + \frac{\nu}{5}\sum_{i=2}^{n+1}\|\Delta_h\boldsymbol{u}_h^i\|^2\\
&+\frac{\nu}{2}(1+\varepsilon_0(d+1))\|\Delta_h\boldsymbol{u}_h^{n+1}\|^2 + \tau_2\sum_{i=2}^{n+1}|||\mathcal{L}_6\boldsymbol{u}_h^i|||_{\mathrm{j}}^2 + \frac{\nu C_{\mathrm{inv}}^2}{8\varepsilon_0}|||\boldsymbol{u}_h^{n+1}|||_{\mathrm{j}}^2\\
&\leqslant \frac{1}{\Delta t}E(\nabla\boldsymbol{u}_h^1) + \tau_1\|\Delta_h\mathcal{L}_6\boldsymbol{u}_h^1\|^2 + \frac{\nu}{2}(1+\varepsilon_0(d+1))\|\Delta_h\boldsymbol{u}_h^1\|^2\\
&+\tau_0 C_{\mathrm{g}}\sum_{i=1}^{n}\|\nabla\mathcal{L}_6\boldsymbol{u}_h^i\|^2 + \tau_0 C_{\mathrm{j}}|||\mathcal{L}_6\boldsymbol{u}_h^1|||_{\mathrm{j}}^2 + \frac{\nu C_{\mathrm{inv}}^2}{8\varepsilon_0}|||\boldsymbol{u}_h^1|||_{\mathrm{j}}^2.
\end{aligned} \tag{4.17}$$

By (4.13) one has

$$\|\nabla\mathcal{L}_6\boldsymbol{u}_h^n\|^2 \leqslant 6\|\nabla\boldsymbol{u}_h^n\|^2 + 30\|\nabla(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1})\|^2 \leqslant 60E(\nabla\boldsymbol{u}_h^n).$$

Then (4.16) follows immediately from Lemma 4.2. □

5. Error estimates

For simplicity we only analyze the second order scheme with $\beta = 5$ and Lagrange finite elements on simplicial elements. For the true solution $\boldsymbol{u}$ of (2.1)–(2.2)-(1.3)–(1.4), we assume that $\boldsymbol{u}(t,\bullet) \in \boldsymbol{V}$ for all $t \in [0,T]$ such that $[\![\nabla\boldsymbol{u}]\!]\boldsymbol{n}_F = \boldsymbol{0}$ over all $F \in \mathcal{F}^0$ (this means the artificial terms such as the penalty term in a_{Lap} and $a_{\mathrm{Lap}-\beta}$ do not produce consistency errors) and the initial velocity satisfies the compatibility conditions implied by consistent splitting methods (see, e.g., [25, Theorem 4.1] and

[18, Theorem 8]). Let $\boldsymbol{e}_u^n := \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$ and $e_p^n := p(t_n) - p_h^n$. Let $z_h^n \in Q_h$ be determined by

$$(\nabla z_h^n, \nabla q_h) = (\nabla p(t_n), \nabla q_h) \quad \forall q_h \in Q_h,$$

and let $\boldsymbol{w}_h^n$ be the classical Lagrange element interpolation of $\boldsymbol{u}(t_n)$. We decompose the errors as

$$\begin{aligned} \boldsymbol{e}_u^n &= \boldsymbol{\eta}_u^n + \boldsymbol{\phi}_u^n = \boldsymbol{u}(t_n) - \boldsymbol{w}_h^n + \boldsymbol{w}_h^n - \boldsymbol{u}_h^n, \\ e_p^n &= \eta_p^n + \phi_p^n = p(t_n) - z_h^n + z_h^n - p_h^n. \end{aligned}$$

They satisfy the following error equations:

(5.1)

$$\begin{aligned} \frac{(\nabla d_\beta \boldsymbol{\phi}_u^{n+1}, \nabla \boldsymbol{v}_h)}{\Delta t} + a_{\mathrm{Lap}-\beta}(\boldsymbol{\phi}_u^{n+1}, \boldsymbol{v}_h) &= -\sum_{F\in\mathcal{F}^0} \int_F \{\{\nabla \mathcal{L}_{\beta+1} \phi_p^n\}\} \cdot ([\![\nabla \boldsymbol{v}_h]\!] \boldsymbol{n}_F)\, ds \\ &+ (\mathcal{L}_{\beta+1} \nabla \phi_p^n, \Delta_h \boldsymbol{v}_h) + \mathcal{RHS}_\eta(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{aligned}$$

(5.2)

$$(\nabla \phi_p^{n+1}, \nabla q_h) = (\nabla e_p^{n+1}, \nabla q_h) = \nu \sum_{F\in\mathcal{F}^\partial} \int_F \nabla \times \boldsymbol{e}_u^{n+1} \cdot (\nabla q_h \times \boldsymbol{n}_F)\, ds \quad \forall q_h \in Q_h,$$

where

$$\begin{aligned} \mathcal{RHS}_\eta(\boldsymbol{v}_h) = &-(\nabla \boldsymbol{u}_t(t^{n+\beta}) - \frac{\nabla d_\beta \boldsymbol{w}_h^{n+1}}{\Delta t}, \nabla \boldsymbol{v}_h) - \nu(\Delta \boldsymbol{u}(t^{n+\beta}) - \Delta_h \mathcal{L}_\beta \boldsymbol{w}_h^{n+1}, \Delta_h \boldsymbol{v}_h) \\ &+ \sum_{F\in\mathcal{F}^0} \int_F ([\![\nabla \mathcal{L}_\beta \boldsymbol{\eta}_u^{n+1}]\!] \boldsymbol{n}_F) \cdot \{\{\nu \Delta_h \boldsymbol{v}\}\}\, ds \\ &- \sum_{F\in\mathcal{F}^0} \frac{\gamma_F}{h_F} \int_F ([\![\nabla \mathcal{L}_{\beta+1} \boldsymbol{\eta}_u^{n+1}]\!] \boldsymbol{n}_F) \cdot ([\![\nabla \boldsymbol{v}_h]\!] \boldsymbol{n}_F)\, ds \\ &+ \sum_{F\in\mathcal{F}^0} \int_F \{\{\nu \Delta \boldsymbol{u}(t^{n+\beta}) - \nabla p(t^{n+\beta}) - \nu \Delta_h \mathcal{L}_\beta \boldsymbol{w}_h^{n+1} + \nabla \mathcal{L}_{\beta+1} z_h^n\}\} \\ &\qquad \cdot ([\![\nabla \boldsymbol{v}_h]\!] \boldsymbol{n}_F)\, ds \\ &+ (\nabla p(t^{n+\beta}) - \mathcal{L}_{\beta+1} \nabla z_h^n, \Delta_h \boldsymbol{v}_h). \end{aligned}$$

Relationships (5.2) imply that $\phi_p^n = p_{h,\mathrm{S}}(\boldsymbol{e}_u^n)$ for all $n \in \mathbb{N}^+$. Theorem 3.4 and (4.5) imply the following estimate of ϕ_p^n.

Lemma 5.1. *With the same assumption as in Theorem* 3.4*, for any* $\varepsilon_2 > 0$ *there exists* C *such that*

$$\begin{aligned} \|\nabla \phi_p^n\|^2 &\leqslant \nu^2 [(\frac{1}{2} + \varepsilon) \|\Delta_h \boldsymbol{e}_u^n\|^2 + C_{\mathrm{g}} \|\nabla \boldsymbol{e}_u^n\|^2 + C_{\mathrm{j}} \, |\!|\!| \boldsymbol{e}_u^n |\!|\!|_{\mathrm{j}}^2] \\ &\leqslant \nu^2 [(\frac{1}{2} + \varepsilon + \varepsilon_2) \|\Delta_h \boldsymbol{\phi}_u^n\|^2 + (C_{\mathrm{g}} + \varepsilon_2) \|\nabla \boldsymbol{\phi}_u^n\|^2 + (C_{\mathrm{j}} + \varepsilon_2) \, |\!|\!| \boldsymbol{\phi}_u^n |\!|\!|_{\mathrm{j}}^2 \\ &+ C(\|\Delta_h \boldsymbol{\eta}_u^n\|^2 + \|\nabla \boldsymbol{\eta}_u^n\|^2 + |\!|\!| \boldsymbol{\eta}_u^n |\!|\!|_{\mathrm{j}}^2]. \end{aligned}$$

Proof. The first inequality follows immediately from a combination of Theorem 3.4 and (4.5). The second inequality follows from the following inequality,

$$(a + b)^2 = a^2 + 2ab + b^2 \leqslant (1 + \tilde{\varepsilon}) a^2 + (1 + \frac{1}{\tilde{\varepsilon}}) b^2.$$

Taking $\varepsilon_2 = \max\{\frac{1}{2} + \varepsilon, C_{\mathrm{g}}, C_{\mathrm{j}}\} \tilde{\varepsilon}$ makes the second inequality in the lemma. □

Define the semi-norm $|||\cdot|||_{\mathrm{a}}$ on $\boldsymbol{H}^{\frac{1}{2}}(\mathcal{T})$ as

$$|||\boldsymbol{z}|||_{\mathrm{a}}^2 := \sum_{F\in\mathcal{F}^0} h_F \int_F |\{\{\boldsymbol{z}\}\}|^2 \, ds.$$

Note that for any $\boldsymbol{v} \in \boldsymbol{V}_h$, one has

$$\begin{aligned}
\|\nabla \boldsymbol{v}\|^2 &= (-\Delta_h \boldsymbol{v}, \boldsymbol{v}) + \sum_{F\in\mathcal{F}^0} \int_F [\![\nabla \boldsymbol{v}\boldsymbol{n}]\!] \cdot \boldsymbol{v} \, ds \\
&\leqslant C \left(\|\Delta_h \boldsymbol{v}\|^2 + |||\boldsymbol{v}|||_{\mathrm{j}}^2 \right)^{\frac{1}{2}} \left(\|\boldsymbol{v}\|^2 + \sum_{F\in\mathcal{F}^0} h_F \int_F |\boldsymbol{v}|^2 \, ds \right)^{\frac{1}{2}} \\
&\leqslant C \left(\|\Delta_h \boldsymbol{v}\|^2 + |||\boldsymbol{v}|||_{\mathrm{j}}^2 \right)^{\frac{1}{2}} \|\boldsymbol{v}\| \leqslant C \left(\|\Delta_h \boldsymbol{v}\|^2 + |||\boldsymbol{v}|||_{\mathrm{j}}^2 \right)^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|,
\end{aligned}$$

where in the last two inequalities an inverse trace inequality and Poincaré's inequality are employed. This implies

$$\|\nabla \boldsymbol{v}\|^2 \leqslant C(\|\Delta_h \boldsymbol{v}\|^2 + |||\boldsymbol{v}|||_{\mathrm{j}}^2) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}_h.$$

It can be verified from Young's inequality and the above inequality that, for any $\varepsilon_3 > 0$ there exists C such that

$$\begin{aligned}
&|\mathcal{RHS}_\eta(\boldsymbol{v}_h)| \\
&\quad \leqslant \varepsilon_3 \left(\|\Delta_h \boldsymbol{v}_h\|^2 + |||\boldsymbol{v}_h|||_{\mathrm{j}}^2 \right) \\
&\quad + C\Big\{ \|\nabla \boldsymbol{u}_t(t^{n+\beta}) - \nabla d_\beta \boldsymbol{w}_h^{n+1}/\Delta t\|^2 + \nu \|\Delta \boldsymbol{u}(t^{n+\beta}) - \Delta_h \mathcal{L}_\beta \boldsymbol{w}_h^{n+1}\|^2 \\
&\qquad + |||\mathcal{L}_{\beta+1} \boldsymbol{\eta}_u^{n+1}|||_{\mathrm{j}}^2 + |||\nu \Delta \boldsymbol{u}(t^{n+\beta}) - \nu \Delta_h \mathcal{L}_\beta \boldsymbol{w}_h^{n+1} - \nabla p(t^{n+\beta}) + \nabla \mathcal{L}_{\beta+1} z_h^n|||_{\mathrm{a}}^2 \\
&\qquad + \|\nabla p(t^{n+\beta}) - \mathcal{L}_{\beta+1} \nabla z_h^n\|^2 \Big\}.
\end{aligned}$$

The following inequality can translate a face-based norm to cell-based norms (see, e.g., [1, Eq. (4.20)]): for any $q \in H^1(K)$ and for any face $F \subset \partial K$, there exists C such that

$$|q|_F^2 \leqslant C(h_F^{-1} \|q\|_K^2 + h_F \|\nabla q\|_K^2) \quad \text{for all } K \in \mathcal{T}. \tag{5.3}$$

Let m and m' ($m \geqslant 2, m' \geqslant 1$) be the polynomial orders of the velocity and pressure spaces, respectively. Suppose $(\boldsymbol{u}, p) \in C^2([0,T]; H^{m+1}(\Omega)) \times C^2([0,T]; H^{m'+1}(\Omega))$. Then a combination of (5.3) and the approximation errors in time for d_β and $\mathcal{L}_{\beta+1}$ (see (2.7) and (2.9)), and in space (see, e.g., [3, Theorem 4.4.4]) yields

$$|\mathcal{RHS}_\eta(\boldsymbol{v}_h)| \leqslant \varepsilon_3 \left(\|\Delta_h \boldsymbol{v}_h\|^2 + |||\boldsymbol{v}_h|||_{\mathrm{j}}^2 \right) + C(\Delta t^4 + h^{\min\{2m-2, 2m'\}}) + C\nu(\Delta t^4 + h^{2m-2}).$$

Similarly to (4.17), taking $\boldsymbol{v} = \mathcal{L}_{\beta+1}\boldsymbol{\phi}_u^{n+1}$, one obtains

$$\begin{aligned}
&\frac{1}{\Delta t}E(\nabla\boldsymbol{\phi}_h^{n+1}) + (\frac{3\nu}{10} - 2\nu\varepsilon_0(d+1) - \tau_1 - \varepsilon_3)\sum_{i=2}^{n+1}\|\Delta_h\mathcal{L}_6\boldsymbol{\phi}_h^i\|^2 \\
&\quad + \frac{\nu}{5}\sum_{i=2}^{n+1}\|\Delta_h\boldsymbol{\phi}_h^i\|^2 + \frac{\nu}{2}(1+\varepsilon_0(d+1))\|\Delta_h\boldsymbol{\phi}_h^{n+1}\|^2 \\
&\quad + (\tau_2 - \varepsilon_3)\sum_{i=2}^{n+1}|||\mathcal{L}_6\boldsymbol{\phi}_h^i|||_{\mathrm{j}}^2 + \frac{\nu C_{\mathrm{inv}}^2}{8\varepsilon_0}|||\boldsymbol{\phi}_h^{n+1}|||_{\mathrm{j}}^2 \\
&\leqslant \frac{1}{\Delta t}E(\nabla\boldsymbol{\phi}_h^1) + \tau_1\|\Delta_h\mathcal{L}_6\boldsymbol{\phi}_h^1\|^2 + \frac{\nu}{2}(1+\varepsilon_0(d+1))\|\Delta_h\boldsymbol{\phi}_h^1\|^2 + \tau_0 C_{\mathrm{g}}\sum_{i=1}^{n}\|\nabla\mathcal{L}_6\boldsymbol{\phi}_h^i\|^2 \\
&+\tau_0 C_{\mathrm{j}}|||\mathcal{L}_6\boldsymbol{\phi}_h^1|||_{\mathrm{j}}^2 + \frac{\nu C_{\mathrm{inv}}^2}{8\varepsilon_0}|||\boldsymbol{\phi}_h^1|||_{\mathrm{j}}^2 + Cn(\Delta t^4 + h^{\min\{2m-2,2m'\}}) + C\nu n(\Delta t^4 + h^{2m-2}).
\end{aligned}$$

By Lemma 4.2, we have

$$\begin{aligned}
&E(\nabla\boldsymbol{\phi}_h^{n+1}) + \Delta t(\frac{3\nu}{10} - 2\nu\varepsilon_0(d+1) - \tau_1 - \varepsilon_3)\sum_{i=2}^{n+1}\|\Delta_h\mathcal{L}_6\boldsymbol{\phi}_h^i\|^2 + \frac{\nu\Delta t}{5}\sum_{i=2}^{n+1}\|\Delta_h\boldsymbol{\phi}_h^i\|^2 \\
&+ \frac{\nu\Delta t}{2}(1+\varepsilon_0(d+1))\|\Delta_h\boldsymbol{\phi}_h^{n+1}\|^2 + \Delta t(\tau_2 - \varepsilon_3)\sum_{i=2}^{n+1}|||\mathcal{L}_6\boldsymbol{\phi}_h^i|||_{\mathrm{j}}^2 + \frac{\nu\Delta t C_{\mathrm{inv}}^2}{8\varepsilon_0}|||\boldsymbol{\phi}_h^{n+1}|||_{\mathrm{j}}^2 \\
&\quad\leqslant \exp(60\tau_0 C_{\mathrm{g}} t^n)\Big[E(\nabla\boldsymbol{\phi}_h^1) + \Delta t\tau_1\|\Delta_h\mathcal{L}_6\boldsymbol{\phi}_h^1\|^2 + \frac{\nu\Delta t}{2}(1+\varepsilon_0(d+1))\|\Delta_h\boldsymbol{\phi}_h^1\|^2 \\
&\qquad\qquad + \Delta t\tau_0 C_{\mathrm{j}}|||\mathcal{L}_6\boldsymbol{\phi}_h^1|||_{\mathrm{j}}^2 + \frac{\nu\Delta t C_{\mathrm{inv}}^2}{8\varepsilon_0}|||\boldsymbol{\phi}_h^1|||_{\mathrm{j}}^2 \\
&\qquad\qquad + Ct^n(\Delta t^4 + h^{\min\{2m-2,2m'\}} + \nu\Delta t^4 + \nu h^{2m-2})\Big].
\end{aligned}$$

Assume that the $(\boldsymbol{u}_h^1, p_h^1)$ is obtained via (2.14)–(2.12). One can check that it satisfies the error equation,

$$\begin{aligned}
&(\nabla\boldsymbol{e}_u^1, \nabla\boldsymbol{v}_h) + \Delta t a_{\mathrm{Lap}}(\boldsymbol{e}_u^1, \boldsymbol{v}_h) = (\nabla\boldsymbol{e}_u^0, \nabla\boldsymbol{v}_h) \\
&\qquad - \Delta t\sum_{F\in\mathcal{F}^0}\int_F \{\{\nabla e_p^0\}\}\cdot([\![\nabla\boldsymbol{v}_h]\!]\boldsymbol{n}_F)\,ds + \Delta t(\nabla e_p^0, \Delta_h\boldsymbol{v}_h) + (\boldsymbol{g}\Delta t^2, \nabla\boldsymbol{v}_h)
\end{aligned}$$

and (5.2) with $n = 0$, where $\|\boldsymbol{g}\Delta t^2\| = \mathcal{O}(\Delta t^2)$ depicts the truncation error. Thus it can be proven that

$$\begin{aligned}
&\|\nabla\boldsymbol{\phi}_h^1\| + \Delta t(\|\Delta_h\boldsymbol{\phi}_h^1\| + |||\boldsymbol{\phi}_h^1|||_{\mathrm{j}}) \\
&\qquad\leqslant C(\Delta t^2 + h^{\min\{m-1,m'\}} + \|\nabla\boldsymbol{e}_u^0\| + \Delta t(\|\nabla\boldsymbol{e}_u^0\| + \|\Delta_h\boldsymbol{e}_u^0\| + |||\boldsymbol{e}_u^0|||_{\mathrm{j}})).
\end{aligned}$$

Together with the triangle inequality and Lemma 5.1, this implies the following estimate.

Theorem 5.2. *Let m and m' ($m \geqslant 2, m' \geqslant 1$) be the polynomial orders of the velocity and pressure spaces, respectively. Let $(\boldsymbol{u}_h, p_h)$ be the solution of* (2.15) *with $\beta = 5$. Suppose $(\boldsymbol{u}, p) \in C^2([0,T]; H^{m+1}(\Omega)) \times C^2([0,T]; H^{m'+1}(\Omega))$ and $(\boldsymbol{u}_h^1, p_h^1)$ is obtained via* (2.14)–(2.12). *With the same assumption as in Theorem 4.4, the* *following holds true:*

$$\sup_{2\leqslant \ell\leqslant n+1} \|\nabla(\boldsymbol{u}(t^\ell) - \boldsymbol{u}_h^\ell)\|^2 + \nu\Delta t \sum_{\ell=2}^{n+1} \|\Delta_h(\boldsymbol{u}(t^\ell) - \boldsymbol{u}_h^\ell)\|^2 + \frac{\Delta t}{\nu} \sum_{\ell=2}^{n+1} \|\nabla(p(t^\ell) - p_h^\ell)\|^2$$
$$\leqslant C(\Delta t^4 + h^{\min\{2m-2,2m'\}} + \|\nabla \boldsymbol{e}_u^0\|^2 + \Delta t(\|\nabla \boldsymbol{e}_u^0\|^2 + \|\Delta_h \boldsymbol{e}_u^0\|^2 + \|\!|\boldsymbol{e}_u^0|\!\|_{\mathrm{j}}^2)).$$

With the approximation properties of Lagrange elements (see, e.g., [3, Theorem 4.4.4]) and (5.3), it is not hard to see that the right-hand side of above estimate is $\mathcal{O}(\Delta t^4 + h^{\min\{2m-2,2m'\}})$.

6. Numerical experiments

In this section we test the convergence rates of the methods (2.15)–(2.12) and (2.16)–(2.17) for the Stokes equations and Navier–Stokes equations, respectively. On $\Omega = (0,1)^2$, the true solutions are prescribed as

$$\boldsymbol{u}(x,y,t) = \frac{1}{2\pi}\sin(\pi t)\begin{pmatrix} \sin(\pi x)^2 \sin(\pi y)\cos(\pi y) \\ -\sin(\pi y)^2 \sin(\pi x)\cos(\pi x)\end{pmatrix},$$
$$p(x,y,t) = \frac{1}{2\pi}\sin(\pi t)\cos(\pi x)\cos(\pi y).$$

The right-hand side $\boldsymbol{f}$ is chosen such that $(\boldsymbol{u}, p)$ solves the instationary (Navier–) Stokes problem with $\nu = 1$. The stabilization parameter γ_F is set to be 20 for all $F \in \mathcal{F}^0$.

To verify the temporal convergence rates, we use the $\boldsymbol{P}_4 \times P_3$ Lagrange element pair on the fourth refinement of a grid with $h = 0.25$ (see Figure 6.1) for spatial discretization, whose spatial errors should be very small such that the temporal errors dominate. The initial time step is $\Delta t = 0.025$, and the ending time is $T = 0.5$. Some numerical results for $\beta = 5$ are shown in Figure 6.2, from which one can see both the schemes for the Stokes equations and Navier–Stokes equations have second-order convergence rates in time with respect to all the shown norms.

To verify the spatial convergence rates, we try several Lagrange element pairs with $\Delta t = 10^{-5}$ and $T = 0.1$, including the inf-sup stable pairs $\boldsymbol{P}_2 \times P_1$ and $\boldsymbol{P}_3 \times P_2$, and the non-inf-sup stable equal-order pairs $\boldsymbol{P}_2 \times P_2$ and $\boldsymbol{P}_3 \times P_3$. The initial grid is shown in Figure 6.1. Some numerical results can be found in Tables 6.1–6.4. The error tables for the Navier–Stokes equations are not shown here because they are almost the same as the ones for the Stokes equations.

There are several interesting phenomena. First, the convergence rates of all the terms are exactly or better than the predicted ones in Theorem 5.2. Second, for the convergence rates of the L^2 errors of velocity, it is only suboptimal for $\boldsymbol{P}_2$ velocity elements but optimal for $\boldsymbol{P}_3$ velocity elements. Finally, the non-inf-sup stable pairs also work for our method and they usually have better convergence rates with respect to pressure errors, compared to the inf-sup stable pairs with the same velocity element.

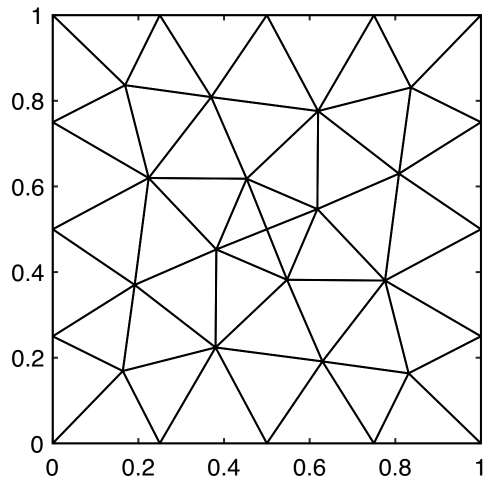

FIGURE 6.1. Initial grid

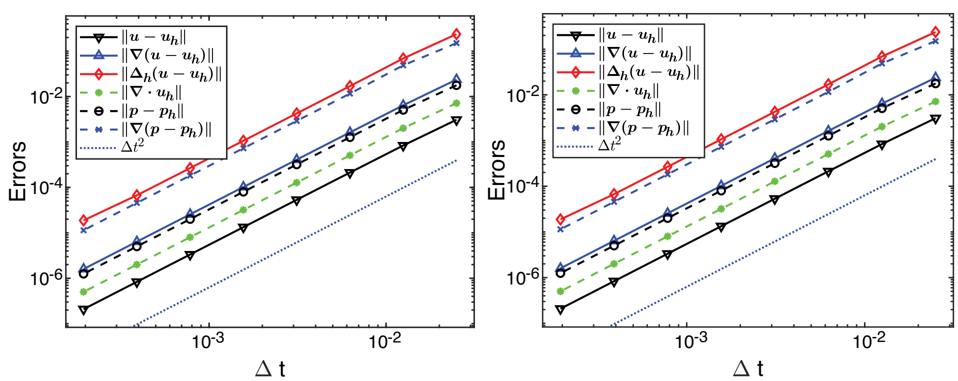

FIGURE 6.2. Shown above are errors versus time step at $T = 0.5$ by (2.15)–(2.12) with $\beta = 5$ for the Stokes equations (left) and (2.16)–(2.17) with $\beta = 5$ for the Navier–Stokes equations (right), respectively. The $\boldsymbol{P}_4 \times P_3$ element pair is used.

TABLE 6.1. Errors and convergence rates by (2.15)–(2.12) using $\boldsymbol{P}_2 \times P_1$ finite elements with $\beta = 5$ at $T = 0.1$ ($\Delta t = 10^{-5}$)

h	$\|\boldsymbol{u} - \boldsymbol{u}_h\|$		$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|$		$\|\Delta_h(\boldsymbol{u} - \boldsymbol{u}_h)\|$		$\|\nabla \cdot \boldsymbol{u}_h\|$		$\|p - p_h\|$		$\|\nabla(p - p_h)\|$	
2^{-2}	2.66e-3		2.40e-2		3.76e-1		8.64e-3		4.22e-3		5.21e-2	
2^{-3}	8.62e-4	1.63	8.45e-3	1.51	1.92e-1	0.97	3.98e-3	1.12	2.05e-3	1.05	2.69e-2	0.96
2^{-4}	2.48e-4	1.80	2.59e-3	1.70	9.21e-2	1.06	1.39e-3	1.52	8.09e-4	1.34	1.18e-2	1.18
2^{-5}	6.64e-5	1.90	7.16e-4	1.86	4.43e-2	1.06	4e-4	1.79	2.51e-4	1.69	4.60e-3	1.36
2^{-6}	1.70e-5	1.96	1.86e-4	1.95	2.18e-2	1.03	1.06e-4	1.92	6.83e-5	1.88	1.89e-3	1.28

TABLE 6.2. Errors and convergence rates by (2.15)–(2.12) using $\boldsymbol{P}_2 \times P_2$ finite elements with $\beta = 5$ at $T = 0.1$ ($\Delta t = 10^{-5}$)

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|$		$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|$		$\|\Delta_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$		$\|\nabla\cdot\boldsymbol{u}_h\|$		$\|p-p_h\|$		$\|\nabla(p-p_h)\|$	
2^{-2}	2.69e-3		2.42e-2		3.76e-1		8.75e-3		4.70e-3		5.59e-2	
2^{-3}	8.68e-4	1.63	8.54e-3	1.51	1.92e-1	0.97	4.08e-3	1.10	2.23e-3	1.08	2.81e-2	0.99
2^{-4}	2.49e-4	1.80	2.61e-3	1.71	9.22e-2	1.06	1.40e-3	1.54	8.26e-4	1.43	1.06e-2	1.40
2^{-5}	6.64e-5	1.91	7.17e-4	1.86	4.43e-2	1.06	4.02e-4	1.80	2.51e-4	1.72	3.32e-3	1.68
2^{-6}	1.70e-5	1.96	1.86e-4	1.95	2.18e-2	1.03	1.06e-4	1.93	6.77e-5	1.89	9.39e-4	1.82

TABLE 6.3. Errors and convergence rates by (2.15)–(2.12) using $\boldsymbol{P}_3 \times P_2$ finite elements with $\beta = 5$ at $T = 0.1$ ($\Delta t = 10^{-5}$)

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|$		$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|$		$\|\Delta_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$		$\|\nabla\cdot\boldsymbol{u}_h\|$		$\|p-p_h\|$		$\|\nabla(p-p_h)\|$	
2^{-2}	9.69e-5		1.97e-3		7.70e-2		8.84e-4		2.48e-4		4.98e-3	
2^{-3}	6.56e-6	3.88	2.48e-4	2.99	1.98e-2	1.96	1.07e-4	3.04	3.70e-5	2.75	1.15e-3	2.12
2^{-4}	4.27e-7	3.94	3.01e-5	3.04	4.89e-3	2.02	1.33e-5	3.01	3.44e-6	3.43	2.36e-4	2.29
2^{-5}	2.73e-8	3.97	3.70e-6	3.03	1.21e-3	2.02	1.66e-6	3.00	3.02e-7	3.51	5.39e-5	2.13
2^{-6}	1.82e-9	3.91	4.59e-7	3.01	3e-4	2.01	2.07e-7	3.00	2.92e-8	3.37	1.31e-5	2.04

TABLE 6.4. Errors and convergence rates by (2.15)–(2.12) using $\boldsymbol{P}_3 \times P_3$ finite elements with $\beta = 5$ at $T = 0.1$ ($\Delta t = 10^{-5}$)

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|$		$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|$		$\|\Delta_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$		$\|\nabla\cdot\boldsymbol{u}_h\|$		$\|p-p_h\|$		$\|\nabla(p-p_h)\|$	
2^{-2}	9.69e-5		1.97e-3		7.69e-2		8.81e-4		2.95e-4		6.52e-3	
2^{-3}	6.57e-6	3.88	2.48e-4	2.99	1.98e-2	1.96	1.07e-4	3.04	3.89e-5	2.92	1.29e-3	2.34
2^{-4}	4.28e-7	3.94	3.01e-5	3.04	4.89e-3	2.02	1.33e-5	3.01	3.41e-6	3.51	2.17e-4	2.57
2^{-5}	2.73e-8	3.97	3.70e-6	3.03	1.21e-3	2.02	1.66e-6	3.00	2.66e-7	3.68	3.65e-5	2.57
2^{-6}	1.82e-9	3.91	4.59e-7	3.01	3e-4	2.01	2.07e-7	3.00	2e-8	3.73	6.28e-6	2.54

7. CONCLUDING REMARKS

We constructed in this paper first- and second-order (in time) pure C^0 consistent splitting finite element schemes for the Navier–Stokes equations, for which one only needs to solve a sequence of Poisson type equations at each time step, and the inf-sup condition between velocity and pressure finite element spaces is not required. To avoid using C^1 conforming elements, the spatial discretization is constructed in a discontinuous Galerkin (DG) framework. Therefore, the proposed schemes are very efficient and easy to implement.

As a key result, we extended the estimate on the Stokes pressure in the space continuous case to the space discontinuous case with the C^0 DG finite elements. With the help of this key estimate, we established unconditional stability and carried out error analysis for the fully discrete schemes in the absence of nonlinear terms. Finally, we presented numerical experiments to validate our theoretical analysis with both inf-sup stable and non-inf-sup stable C^0 finite elements.

A key assumption for the estimate of the Stokes pressure in (1.6), which is crucial for the stability and error analysis, is $\partial\Omega \in C^3$ which excludes polygonal domains. Numerical results in this paper and in [18,26] indicate that the proved stability and convergence results still hold in rectangular domains. However, it is not clear and beyond the scope of this paper whether the proof can be extended to polygonal domains.

On the other hand, the derivation of error estimates requires the solution to be sufficiently smooth at $t = 0$ which requires that the data satisfy certain compatibility conditions [31]. It is possible to relax the smoothness assumption at $t = 0$ by using the so-called smoothing properties as in [17, 28, 30] or with graded variable time steps near $t = 0$ [21, 22, 28], but the process is delicately tedious and beyond the scope of the current paper.

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Eastern Institute for Advanced Study, Eastern Institute of Technology, Tongxin Road 568, Ningbo 315200, People's Republic of China; and School of Mathematical Sciences, University of Science and Technology of China, Jinzhai Road 96, Hefei 230026, People's Republic of China

Email address: xuli@eitech.edu.cn

Eastern Institute for Advanced Study, Eastern Institute of Technology, Tongxin Road 568, Ningbo 315200, People's Republic of China

Email address: jshen@eitech.edu.cn