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ERROR ANALYSIS OF THE SAV-MAC SCHEME FOR THE NAVIER–STOKES EQUATIONS*

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Abstract. An efficient numerical scheme based on the scalar auxiliary variable (SAV) and marker and cell (MAC) scheme is constructed for the Navier–Stokes equations. A particular feature of the scheme is that the nonlinear term is treated explicitly while being unconditionally energy stable. A rigorous error analysis is carried out to show that both velocity and pressure approximations are second-order accurate in time and space. Numerical experiments are presented to verify the theoretical results.

Key words. MAC scheme, scalar auxiliary variable (SAV), energy stability, error estimate, numerical experiments

AMS subject classifications. 65M06, 65M12, 65M15, 76D07

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1. Introduction.

We consider in this paper the following incompressible Navier–Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega\times J, \tag{1.1a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times J, \tag{1.1b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega\times J, \tag{1.1c}$$

where Ω is an open bounded domain in $\mathbb{R}^2$, $J=(0,T]$, $(\mathbf{u},p)$ represent the unknown velocity and pressure, $\mathbf{f}$ is an external body force, $\nu>0$ is the viscosity coefficient, and $\mathbf{n}$ is the unit outward normal of the domain Ω.

Numerical solution of the Navier–Stokes equations plays an important role in computational fluid dynamics, and an enormous amount of work has been devoted to the design, analysis, and implementation of numerical schemes for the Navier–Stokes equations; see [23, 5, 6] and the references therein.

One of the main difficulties in numerically solving Navier–Stokes equations is the treatment of the nonlinear term. There are essentially three types of treatment: (i) fully implicit, which leads to a nonlinear system to solve at each time step; (ii) semi-implicit, which needs to solve a coupled elliptic equation with variable coefficients at each time step; and (iii) explicit, which only has to solve a generalized Stokes system, or even decoupled Poisson-type equations, at each time step but suffers from a CFL time step constraint at intermediate or large Reynolds numbers.

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From a computational point of view, it would be ideal to be able to treat the nonlinear term explicitly without any stability constraint. In a recent work [12], Lin, Yang, and Dong constructed such a scheme by introducing an auxiliary variable. The scheme was inspired by the recently introduced scalar auxiliary variable (SAV) approach [22, 20, 21] which can lead to linear, second-order, unconditionally energy stable schemes that require solving only decoupled elliptic equations with constant coefficients at each time step for a large class of gradient flows. The scheme constructed in [12] for Navier–Stokes equations requires solving two generalized Stokes equations (with constant coefficient) plus a nonlinear algebraic equation for the auxiliary variable at each time step. Hence, it is very efficient compared with other existing schemes. Ample numerical results presented in [12] indicate that the scheme is very effective for a variety of situations.

However, the nonlinear algebraic equation for the auxiliary variable has multiple solutions, and it is not clear whether all solutions converge to the exact solution or how to choose the right solution. This question can only be fully answered with a rigorous convergence analysis. But due to the explicit treatment of the nonlinear term and the nonlinear algebraic equation associated to the auxiliary variable, its convergence and error analysis cannot be obtained using a standard procedure. More precisely, two of the main difficulties for convergence and error analysis are (i) deriving a uniform L^∞ bound for the numerical solution from the modified energy stability, and (ii) dealing with the nonlinear algebraic equation for the auxiliary variable.

In this paper, we shall construct a fully discrete SAV scheme for the Navier–Stokes equations with the marker and cell (MAC) method [24, 26] for the spatial discretization. The MAC scheme has been widely used in engineering applications due to its simplicity while satisfying the discrete incompressibility constraint as well as locally conserving the mass, momentum, and kinetic energy [15, 16]. The stability and error estimates for the MAC scheme have been well studied; see, for instance, [4, 1, 8, 7] and the references therein. Most of the error estimates are only first order for both the velocity and the pressure, although Nicolaides [14] pointed out that numerical results suggest that the velocity is second-order convergent without proof. Inspired by the techniques in [19, 13] for Darcy–Forchheimer and Maxwell's equations, Rui and Li established the discrete LBB condition for the MAC method and derived second-order error estimates for both the velocity and the pressure in discrete L^2 norms for the Stokes equations in [18, 11] and for the Navier–Stokes equations in [10].

The main purposes of this paper are (i) to construct a SAV-MAC scheme for the Navier–Stokes equations, establish its energy stability, and present an efficient algorithm for solving the resulting system which is weakly nonlinear; and (ii) to carry out a rigorous error analysis for the SAV-MAC scheme. In particular, at each time step, our SAV-MAC scheme leads to two discrete MAC schemes for the generalized Stokes system that can be efficiently solved by using the usual techniques developed for the MAC scheme, and to a quadratic algebraic equation for the auxiliary variable.

The main contribution of this paper is a rigorous error analysis with second-order error estimates in time and space for both the velocity and pressure. This is achieved by using a bootstrap argument to establish the uniform bound for the approximate solution, followed by a sequence of delicate estimates. Our results show, in particular, that at least one solution of the quadratic algebraic equation for the auxiliary variable will converge to the exact solution. To the best of our knowledge, this is the first rigorous error analysis for an unconditionally energy stable scheme for the Navier–Stokes equations where the nonlinear term is treated explicitly.

The paper is organized as follows. In section 2, we present the semidiscrete

SAV scheme and fully discrete SAV-MAC scheme, establish the energy stability. and show how to numerically solve them efficiently. In section 3, we carry out a rigorous error analysis to establish second-order error estimates for the fully discrete SAV-MAC scheme. Numerical results are presented in section 4 to validate our theoretical results.

We now present some notation and conventions used in what follows. Throughout the paper we use C, with or without subscript, to denote a positive constant, which could have different values at different places.

Let $L^m(\Omega)$ be the standard Banach space with norm $\|v\|_{L^m(\Omega)} = \left(\int_\Omega |v|^m d\Omega\right)^{1/m}$, and set $\|v\|_\infty = \|v\|_{L^\infty(\Omega)}$. We denote by $(f,g) = \int_\Omega fg\, d\mathbf{x}$ the $L^2(\Omega)$ inner product, and set $\|f\| = (f,f)^{\frac{1}{2}}$. Let $W_p^k(\Omega)$ be the standard Sobolev space

$$W_p^k(\Omega) = \{g: \ \|g\|_{W_p^k(\Omega)} < \infty\},$$

where

$$\|g\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha g\|_{L^p(\Omega)}^p\right)^{1/p}. \tag{1.2}$$

We shall use the notation $W_k^p(J; W_l^q(\Omega))$ to represent the space with functions $f(t,\mathbf{x})$ with $t \in J$ and $\mathbf{x} \in \Omega$ such that $f(t,\cdot) \in W_l^q(\Omega)$ for a.e. $t \in J$, and $\|f(t,\cdot)\|_{W_l^q(\Omega)} \in W_k^p(J)$.

2. The SAV-MAC scheme. In this section, we construct the second-order MAC scheme based on the SAV approach for the Navier–Stokes equation.

Define the scalar auxiliary variable $q(t)$ by

$$q(t) = \sqrt{E(\mathbf{u}) + \delta}, \tag{2.1}$$

where $E(\mathbf{u}) = \int_\Omega \frac{1}{2}|\mathbf{u}|^2$ is the total energy of the system, and δ is an arbitrarily small positive constant. Then we have

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{2q}\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot \mathbf{u} d\mathbf{x} + \frac{1}{2\sqrt{E(\mathbf{u})+\delta}}\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u} d\mathbf{x}. \tag{2.2}$$

Following [12], we rewrite the governing system in the following equivalent form:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \dfrac{q(t)}{\sqrt{E(\mathbf{u})+\delta}}\mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & (2.3)\\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{1}{2q}\displaystyle\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{u} d\mathbf{x} + \frac{1}{2\sqrt{E(\mathbf{u})+\delta}}\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u} d\mathbf{x}, & (2.4)\\ \nabla\cdot\mathbf{u} = 0. & (2.5)\end{cases}$$

Remark 2.1. Note that in the case of inhomogeneous Dirichlet boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{g}$, (2.4) should be replaced by

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{2q}\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{u} d\mathbf{x} + \frac{1}{2\sqrt{E(\mathbf{u})+\delta}}\left(\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u} d\mathbf{x} - \int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{g})\cdot\frac{1}{2}|\mathbf{g}|^2 ds\right). \tag{2.6}$$

2.1. The semidiscrete case. For the reader's convenience, we shall first construct a second-order semidiscrete SAV scheme based on the Crank–Nicolson method, although we are mainly concerned with the analysis of a fully discrete scheme in this paper.

Set

$$\Delta t = T/N,\ t^n = n\Delta t \quad \text{for n} \leq \text{N},$$

and define

$$[d_t f]^n = \frac{f^n - f^{n-1}}{\Delta t}, \quad f^{n+1/2} = \frac{f^n + f^{n+1}}{2}.$$

Then the SAV scheme based on the Crank–Nicolson method is

$$\begin{cases} \dfrac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \dfrac{q^{n+1/2}}{\sqrt{E(\tilde{\mathbf{u}}^{n+1/2}) + \delta}} \tilde{\mathbf{u}}^{n+1/2} \\ \quad \cdot \nabla \tilde{\mathbf{u}}^{n+1/2} - \nu \Delta \mathbf{u}^{n+1/2} + \nabla p^{n+1/2} = \mathbf{f}^{n+1/2}, & (2.7) \\ \dfrac{q^{n+1} - q^n}{\Delta t} = \dfrac{1}{2q^{n+1/2}} \left(\dfrac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}, \mathbf{u}^{n+1/2} \right) \\ \qquad + \dfrac{1}{2\sqrt{E(\tilde{\mathbf{u}}^{n+1/2}) + \delta}} (\tilde{\mathbf{u}}^{n+1/2} \cdot \nabla \tilde{\mathbf{u}}^{n+1/2}, \mathbf{u}^{n+1/2}), & (2.8) \\ \nabla \cdot \mathbf{u}^{n+1} = 0, & (2.9) \end{cases}$$

where $\tilde{\mathbf{u}}^{n+1/2} = (3\mathbf{u}^n - \mathbf{u}^{n-1})/2$ with $n \geq 1$, and we compute $\tilde{\mathbf{u}}^{1/2}$ by the following simple first-order scheme:

$$\frac{\tilde{\mathbf{u}}^{1/2} - \mathbf{u}^0}{\Delta t/2} + \mathbf{u}^0 \cdot \nabla \mathbf{u}^0 - \nu \Delta \tilde{\mathbf{u}}^{1/2} + \nabla p^{1/2} = \mathbf{f}^{1/2}, \tag{2.10}$$

which has a local truncation error of $O(\Delta t^2)$.

Remark 2.2. In the case of inhomogeneous Dirichlet boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{g}$, (2.8) should be replaced by

$$\begin{aligned} &\frac{q^{n+1} - q^n}{\Delta t} = \frac{1}{2q^{n+1/2}} \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}, \mathbf{u}^{n+1/2} \right) \\ &+ \frac{1}{2\sqrt{E(\tilde{\mathbf{u}}^{n+1/2}) + \delta}} \left((\tilde{\mathbf{u}}^{n+1/2} \cdot \nabla \tilde{\mathbf{u}}^{n+1/2}, \mathbf{u}^{n+1/2}) - \int_{\partial\Omega} (\mathbf{n} \cdot \mathbf{g}^{n+1/2}) \cdot \frac{1}{2} |\mathbf{g}^{n+1/2}|^2 ds \right). \end{aligned} \tag{2.11}$$

The above scheme enjoys the following stability result.

THEOREM 2.1. *Let* $\boldsymbol{f} \equiv 0$. *The scheme* (2.7)–(2.8) *is unconditionally energy stable in the sense that*

$$|q^{n+1}|^2 - |q^n|^2 = -\Delta t \nu \|\nabla \boldsymbol{u}^{n+1/2}\|_{L^2}^2.$$

Proof. We recall that for $\mathbf{u} \in H := \{\mathbf{u} \in L^2(\Omega) : \nabla \cdot \mathbf{u} = 0,\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0\}$, we have the identity

$$(\mathbf{u} \cdot \mathbf{v}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in H^1(\Omega). \tag{2.12}$$

Taking the inner products of (2.7) and (2.8) with $\mathbf{u}^{n+1/2}$ and $2q^{n+1/2}$, respectively, and summing up the results and using the above identity, we obtain immediately the desired result. ∎

We now describe how to solve the semidiscrete-in-time scheme (2.7)–(2.9) efficiently. Inspired by the work in [12], we denote

$$S^{n+1} = \frac{q^{n+1/2}}{\sqrt{E(\tilde{\mathbf{u}}^{n+1/2})+\delta}}, \quad \mathbf{u}^{n+1} = \hat{\mathbf{u}}^{n+1} + S^{n+1}\check{\mathbf{u}}^{n+1}, \quad p^{n+1} = \hat{p}^{n+1} - S^{n+1}\check{p}^{n+1}. \tag{2.13}$$

Plugging the above into (2.7) and (2.9), we find that

$$\begin{cases} \dfrac{\hat{\mathbf{u}}^{n+1}}{\Delta t} - \dfrac{\nu}{2}\Delta\hat{\mathbf{u}}^{n+1} + \nabla\hat{p}^{n+1/2} = \mathbf{f}^{n+1/2} + \dfrac{\mathbf{u}^n}{\Delta t} + \dfrac{\nu}{2}\Delta\mathbf{u}^n, & (2.14)\\ \nabla\cdot\hat{\mathbf{u}}^{n+1} = 0, & (2.15)\end{cases}$$

$$\begin{cases} \dfrac{\check{\mathbf{u}}^{n+1}}{\Delta t} - \dfrac{\nu}{2}\Delta\check{\mathbf{u}}^{n+1} - \nabla\check{p}^{n+1/2} = -\tilde{\mathbf{u}}^{n+1/2}\cdot\nabla\tilde{\mathbf{u}}^{n+1/2}, & (2.16)\\ \nabla\cdot\check{\mathbf{u}}^{n+1} = 0, & (2.17)\end{cases}$$

which are linear systems that can be solved independently of S^{n+1}!

It remains to determine S^{n+1}. Taking the inner product of (2.7) with $\mathbf{u}^{n+1/2}$, we have

$$\begin{aligned}\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}, \mathbf{u}^{n+1/2}\right) + \nu\|\nabla\mathbf{u}^{n+1/2}\|^2 + S^{n+1}(\tilde{\mathbf{u}}^{n+1/2}\cdot\nabla\tilde{\mathbf{u}}^{n+1/2}, \mathbf{u}^{n+1/2})&\\ = (\mathbf{f}^{n+1/2}, \mathbf{u}^{n+1/2}).&\end{aligned} \tag{2.18}$$

Taking the inner product of (2.8) with $2q^{n+1/2}$ leads to

$$\frac{(q^{n+1})^2-(q^n)^2}{\Delta t} = \left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}, \mathbf{u}^{n+1/2}\right) + S^{n+1}(\tilde{\mathbf{u}}^{n+1/2}\cdot\nabla\tilde{\mathbf{u}}^{n+1/2}, \mathbf{u}^{n+1/2}). \tag{2.19}$$

Combining (2.18) with (2.19) results in

$$\frac{(q^{n+1})^2-(q^n)^2}{\Delta t} + \nu\|\nabla\mathbf{u}^{n+1/2}\|^2 = (\mathbf{f}^{n+1/2}, \mathbf{u}^{n+1/2}). \tag{2.20}$$

Recalling (2.13), we find that

$$X_{1,n+1}(S^{n+1})^2 + X_{2,n+1}S^{n+1} + X_{3,n+1} = 0, \tag{2.21}$$

where

$$\begin{aligned} X_{1,n+1} &= \frac{4}{\Delta t}(E(\tilde{\mathbf{u}}^{n+1/2})+\delta) + \frac{\nu}{4}\|\nabla\check{\mathbf{u}}^{n+1}\|^2,\\ X_{2,n+1} &= \frac{\nu}{2}(\nabla(\hat{\mathbf{u}}^{n+1}+\mathbf{u}^n), \nabla\check{\mathbf{u}}^{n+1}) - \frac{4q^n}{\Delta t}\sqrt{E(\tilde{\mathbf{u}}^{n+1/2})+\delta} - \frac{1}{2}(\mathbf{f}^{n+1/2}, \check{\mathbf{u}}^{n+1}),\\ X_{3,n+1} &= \frac{\nu}{4}\|\nabla(\hat{\mathbf{u}}^{n+1}+\mathbf{u}^n)\|^2 - \frac{1}{2}(\mathbf{f}^{n+1/2}, \mathbf{u}^n + \hat{\mathbf{u}}^{n+1}).\end{aligned}$$

Note that (2.21) is a quadratic equation for S^{n+1} which can be solved directly by using the quadratic formula. Once S^{n+1} is known, we can obtain $\mathbf{u}^{n+1}$ and $p^{n+1/2}$ through (2.13).

Remark 2.3. The nonlinear quadratic equation (2.21) has two solutions. Since the exact solution is 1, we should choose the root which is closer to 1. In fact, to make sure that (2.8) makes sense, i.e., $q^{n+1/2} \neq 0$, we need to fix a constant $\kappa \in (0,1)$ and choose a root satisfying $q^{n+1/2} \geq \kappa$.

2.2. Fully discrete case. We describe below the finite difference method on the staggered grids, i.e., the MAC scheme, for the spatial discretization of (2.7)–(2.9). To fix the idea, we consider a two-dimensional rectangular domain in $\mathbb{R}^2$, i.e., $\Omega = (L_{lx}, L_{rx}) \times (L_{ly}, L_{ry})$. We refer the reader to the appendix for detailed notation about the finite difference method on the staggered grids.

Given $\{\mathbf{U}^k, P^k, Q^k\}_{k=0}^n$ approximations to $\{\mathbf{u}^k, p^k, q^k\}_{k=0}^n$, we find $\{\mathbf{U}^{n+1}, P^{n+1}, Q^{n+1}\}$ such that

$$\begin{aligned} &d_t U_1^{n+1} + \frac{Q^{n+1/2}}{B^{n+1/2}} \left(\tilde{U}_1^{n+1/2} D_x(\mathcal{P}_h \tilde{U}_1^{n+1/2}) + \mathcal{P}_h \tilde{U}_2^{n+1/2} d_y(\mathcal{P}_h \tilde{U}_1^{n+1/2}) \right) \\ &\qquad -\nu D_x(d_x U_1)^{n+1/2} - \nu d_y(D_y U_1)^{n+1/2} + [D_x P]^{n+1/2} = f_1^{n+1/2}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} &d_t U_2^{n+1} + \frac{Q^{n+1/2}}{B^{n+1/2}} \left(\mathcal{P}_h \tilde{U}_1^{n+1/2} d_x(\mathcal{P}_h \tilde{U}_2^{n+1/2}) + \tilde{U}_2^{n+1/2} D_y(\mathcal{P}_h \tilde{U}_2^{n+1/2}) \right) \\ &\qquad -\nu D_y(d_y U_2)^{n+1/2} - \nu d_x(D_x U_2)^{n+1/2} + [D_y P]^{n+1/2} = f_2^{n+1/2}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} d_t Q^{n+1} = &\frac{1}{2B^{n+1/2}} (\mathcal{P}_h \tilde{\mathbf{U}}^{n+1/2} \cdot \nabla_h (\mathcal{P}_h \tilde{\mathbf{U}}^{n+1/2}), \mathbf{U}^{n+1/2})_{l^2} \\ &+ \frac{1}{2Q^{n+1/2}} (d_t \mathbf{U}^{n+1}, \mathbf{U}^{n+1/2})_{l^2}, \end{aligned} \tag{2.24}$$

$$d_x U_1^{n+1} + d_y U_2^{n+1} = 0, \tag{2.25}$$

where $B^{n+1/2} = \sqrt{E_h(\tilde{\mathbf{U}}^{n+1/2}) + \delta}$ with $E_h(\tilde{\mathbf{U}}^{n+1/2}) = \frac{1}{2} \|\tilde{\mathbf{U}}^{n+1/2}\|_{l^2}^2$, and

$$\begin{aligned} &(\mathcal{P}_h \tilde{\mathbf{U}}^{n+1/2} \cdot \nabla_h (\mathcal{P}_h \tilde{\mathbf{U}}^{n+1/2}), \mathbf{U}^{n+1/2})_{l^2} \\ &\quad = \left(\tilde{U}_1^{n+1/2} D_x(\mathcal{P}_h \tilde{U}_1^{n+1/2}) + \mathcal{P}_h \tilde{U}_2^{n+1/2} d_y(\mathcal{P}_h \tilde{U}_1^{n+1/2}), U_1^{n+1/2} \right)_{l^2,T,M} \\ &\qquad + \left(\mathcal{P}_h \tilde{U}_1^{n+1/2} d_x(\mathcal{P}_h \tilde{U}_2^{n+1/2}) + \tilde{U}_2^{n+1/2} D_y(\mathcal{P}_h \tilde{U}_2^{n+1/2}), U_2^{n+1/2} \right)_{l^2,M,T}; \end{aligned}$$

here $\mathcal{P}_h$ is the bilinear interpolation operator.

The boundary and initial conditions are

$$\begin{cases} U_{1,0,j+1/2}^n = U_{1,N_x,j+1/2}^n = 0, & 0 \le j \le N_y - 1, \\ U_{1,i,0}^n = U_{1,i,N_y}^n = 0, & 0 \le i \le N_x, \\ U_{2,0,j}^n = U_{2,N_x,j}^n = 0, & 0 \le j \le N_y, \\ U_{2,i+1/2,0}^n = U_{2,i+1/2,N_y}^n = 0, & 0 \le i \le N_x - 1, \\ U_{1,i,j+1/2}^0 = u_{1,i,j+1/2}^0, & 0 \le i \le N_x, 0 \le j \le N_y, \\ U_{2,i+1/2,j}^0 = u_{2,i+1/2,j}^0, & 0 \le i \le N_x, 0 \le j \le N_y, \end{cases} \tag{2.26}$$

where $\mathbf{u}^0 = (u_1^0, u_2^0)$ is the initial condition.

Note that the above fully discretized scheme can be efficiently solved using exactly the same procedure as in the semidiscrete case for (2.7)–(2.9).

For the reader's convenience, we still give the implementation of the fully discrete scheme (2.22)–(2.25). Denote

$$K^{n+1} = \frac{Q^{n+1/2}}{B^{n+1/2}}, \quad \mathbf{U}^{n+1} = \hat{\mathbf{U}}^{n+1} + K^{n+1} \check{\mathbf{U}}^{n+1}, \quad P^{n+1} = \hat{P}^{n+1} - K^{n+1} \check{P}^{n+1}. \tag{2.27}$$

The discrete scheme (2.22)–(2.25) can be recast as

$$
\begin{cases}
\dfrac{\hat U_1^{n+1}}{\Delta t} - \dfrac{\nu}{2} D_x(d_x \hat U_1)^{n+1} - \dfrac{\nu}{2} d_y(D_y \hat U_1)^{n+1} + [D_x \hat P]^{n+1/2} \\
= f_1^{n+1/2} + \dfrac{U_1^n}{\Delta t} + \dfrac{\nu}{2} D_x(d_x U_1)^n + \dfrac{\nu}{2} d_y(D_y U_1)^n, & (2.28)\\
\dfrac{\hat U_2^{n+1}}{\Delta t} - \dfrac{\nu}{2} D_y(d_y \hat U_2)^{n+1} - d_x(D_x \hat U_2)^{n+1} + [D_y \hat P]^{n+1/2} \\
= f_2^{n+1/2} + \dfrac{U_2^n}{\Delta t} + \dfrac{\nu}{2} D_y(d_y U_2)^n + \dfrac{\nu}{2} d_x(D_x U_2)^n, & (2.29)\\
d_x \hat U_1^{n+1} + d_y \hat U_2^{n+1} = 0 & (2.30)
\end{cases}
$$

and

$$
\begin{cases}
\dfrac{\check U_1^{n+1}}{\Delta t} - \dfrac{\nu}{2} D_x(d_x \check U_1)^{n+1} - \dfrac{\nu}{2} d_y(D_y \check U_1)^{n+1} - [D_x \check P]^{n+1/2} \\
= -\left(\tilde U_1^{n+1/2} D_x(\mathcal{P}_h \tilde U_1^{n+1/2}) + \mathcal{P}_h \tilde U_2^{n+1/2} d_y(\mathcal{P}_h \tilde U_1^{n+1/2})\right), & (2.31)\\
\dfrac{\check U_2^{n+1}}{\Delta t} - \dfrac{\nu}{2} D_y(d_y \check U_2)^{n+1} - d_x(D_x \check U_2)^{n+1} - [D_y \check P]^{n+1/2} \\
= -\left(\mathcal{P}_h \tilde U_1^{n+1/2} d_x(\mathcal{P}_h \tilde U_2^{n+1/2}) + \tilde U_2^{n+1/2} D_y(\mathcal{P}_h \tilde U_2^{n+1/2})\right), & (2.32)\\
d_x \check U_1^{n+1} + d_y \check U_2^{n+1} = 0. & (2.33)
\end{cases}
$$

The above two discrete generalized Stokes systems can be efficiently solved thanks to the structure of the MAC scheme [17]. Next we can determine K^{n+1} from (2.24) by solving a quadratic algebraic equation. Finally, we obtain (U^{n+1}, P^{n+1}) from (2.27).

As in the semidiscrete case in Remark 2.3, we should only be concerned with the roots satisfying

$$
|Q^{n+1/2}| > \kappa \tag{2.34}
$$

for a given $\kappa \in (0,1)$ and choose the root which is closer to the exact solution 1.

2.3. Energy stability. In this section, we will demonstrate that the second-order fully discrete scheme (2.22)–(2.25) is unconditionally energy stable. The energy stability of the semi-discrete scheme (2.7)–(2.9) can be established similarly.

THEOREM 2.2. *In the absence of the external force* $\boldsymbol{f}$*, the scheme* (2.22)–(2.25) *is unconditionally stable, and the following discrete energy law holds for any* Δt*:*

$$
|Q^{n+1}|^2 - |Q^n|^2 = -\nu \Delta t \|D\,\boldsymbol{U}^{n+1/2}\|^2 \quad \forall n \geq 0. \tag{2.35}
$$

Proof. Multiplying (2.22) by $U_{1,i,j+1/2}^{n+1/2} hk$, making summation on i, j for $1 \leq i \leq N_x - 1,\ 0 \leq j \leq N_y - 1$, and recalling Lemma A.1, we have

$$
\begin{aligned}
&(d_t U_1^{n+1}, U_1^{n+1/2})_{l^2,T,M} + \nu \|d_x U_1^{n+1/2}\|_{l^2,M}^2 + \nu \|D_y U_1^{n+1/2}\|_{l^2,T_y}^2 \\
&+ \frac{Q^{n+1/2}}{B^{n+1/2}} \left(\tilde U_1^{n+1/2} D_x(\mathcal{P}_h \tilde U_1^{n+1/2}) + \mathcal{P}_h \tilde U_2^{n+1/2} d_y(\mathcal{P}_h \tilde U_1^{n+1/2}), U_1^{n+1/2}\right)_{l^2,T,M} \\
&- (P^{n+1/2}, d_x U_1^{n+1/2})_{l^2,M} = (f_1^{n+1/2}, U_1^{n+1/2})_{l^2,T,M}.
\end{aligned} \tag{2.36}
$$

Similarly, multiplying (2.23) by $U^{n+1/2}_{2,i+1/2,j}hk$, and making summation on i,j for $0\le i\le N_x-1,\ 1\le j\le N_y-1$, we can obtain

(2.37)
$$\begin{aligned}&(d_tU_2^{n+1},U_2^{n+1/2})_{l^2,M,T}+\nu\|d_yU_2^{n+1/2}\|^2_{l^2,M}+\nu\|D_xU_2^{n+1/2}\|^2_{l^2,T_x}\\&+\frac{Q^{n+1/2}}{B^{n+1/2}}\left(\mathcal{P}_h\tilde{U}_1^{n+1/2}d_x(\mathcal{P}_h\tilde{U}_2^{n+1/2})+\tilde{U}_2^{n+1/2}D_y(\mathcal{P}_h\tilde{U}_2^{n+1/2}),U_2^{n+1/2}\right)_{l^2,M,T}\\&-(P^{n+1/2},d_yU_2^{n+1/2})_{l^2,M}=(f_2^{n+1/2},U_2^{n+1/2})_{l^2,M,T}.\end{aligned}$$

Multiplying (2.24) by $2Q^{n+1/2}$ yields

$$\begin{aligned}\frac{1}{\Delta t}(|Q^{n+1}|^2-|Q^n|^2)=&\frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2},\mathbf{U}^{n+1/2})_{l^2}\\&+(d_t\mathbf{U}^{n+1},\mathbf{U}^{n+1/2})_{l^2}.\end{aligned}\tag{2.38}$$

Combining (2.38) with (2.36) and (2.37) and noting (2.25) lead to

$$\begin{aligned}&|Q^{n+1}|^2-|Q^n|^2+\nu\Delta t\|D\mathbf{U}^{n+1/2}\|^2\\&=\Delta t(f_1^{n+1/2},U_1^{n+1/2})_{l^2,T,M}+\Delta t(f_2^{n+1/2},U_2^{n+1/2})_{l^2,M,T},\end{aligned}\tag{2.39}$$

which implies the desired result (2.35). □

3. Error estimates. In this section we carry out a rigorous error analysis for the fully discrete scheme (2.22)–(2.25). More precisely, we shall prove the following main result: In what follows, $(\mathbf{u}^n,p^n,q^n)$ represents the exact solution of (2.3)–(2.5) at time t^n.

THEOREM 3.1. *Assume that the exact solution* $(\boldsymbol{u},p)$ *of* (2.3)–(2.5) *is sufficiently smooth such that* $\boldsymbol{u}\in W^3_\infty(J;W^4_\infty(\Omega))^2$, $p\in W^3_\infty(J;W^3_\infty(\Omega))$. *Denote* $(\boldsymbol{u}^n,p^n,q^n)=(\boldsymbol{u}(t^n),p(t^n),q(t^n))$, *where* q *is defined as in* (2.1). *Then for the fully discrete scheme* (2.22)–(2.25) *satisfying* (2.34) *for given* $\kappa\in(0,1)$, *there exists* $C_*>0$ *such that for* $\hat{h}=\min(h,k)$ *sufficiently small with* $\Delta t\le C_*\hat{h}$, *we have the following error estimates:*

$$\|d_x(U_1^m-u_1^m)\|_{l^2,M}+\|d_y(U_2^m-u_2^m)\|_{l^2,M}\le C(\Delta t^2+h^2+k^2),\quad m\le N,\tag{3.1}$$

$$\begin{aligned}&\|\boldsymbol{U}^m-\boldsymbol{u}^m\|_{l^2}+\left(\sum_{l=1}^m\Delta t\|P^{l-1/2}-p^{l-1/2}\|^2_{l^2,M}\right)^{1/2}+|Q^m-q^m|\\&\le C(\Delta t^2+h^2+k^2),\quad m\le N,\end{aligned}\tag{3.2}$$

$$\|D_y(U_1^m-u_1^m)\|_{l^2,T_y}\le C(\Delta t^2+h^2+k^{3/2}),\quad m\le N,\tag{3.3}$$

$$\|D_x(U_2^m-u_2^m)\|_{l^2,T_x}\le C(\Delta t^2+h^{3/2}+k^2),\quad m\le N,\tag{3.4}$$

where the positive constant C *is independent of* h, k, *and* Δt.

Remark 3.1. The above error estimates show, in particular, that at least one root of the nonlinear algebraic equation (2.21) will converge to the exact solution $\frac{q(t)}{\sqrt{E(\mathbf{u})+\delta}}\equiv 1$. The numerical result presented in Figure 1 clearly verifies this assertion.

The proof of Theorem 3.1 involves several major steps. First, we shall define an auxiliary problem in the next subsection and recall an existing result in [18, 9] for the part of the error corresponding to the time-dependent Stokes problem. Then, we shall derive error estimates in section 3.3 depending on the bound

$$L_m = \max_{n=0,\dots,m} \|\mathbf{U}^n\|_{L^\infty}. \tag{3.5}$$

Finally, we show in section 3.4 that L_m can be uniformly bounded to complete the proof of Theorem 3.1.

3.1. An auxiliary problem. We consider first an auxiliary problem which will be used in what follows.

Set $\mathbf{g} = \mathbf{f} - \mathbf{u}\cdot\nabla\mathbf{u}$. We recast (1.1) as

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{g} \quad \text{in } \Omega\times J, \tag{3.6a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times J \tag{3.6b}$$

and consider its approximation by the MAC scheme: For each $n = 0,\dots,N-1$, let $\{W^{n+1}_{1,i,j+1/2}\}$, $\{W^{n+1}_{2,i+1/2,j}\}$, and $\{H^{n+1}_{i+1/2,j+1/2}\}$ be such that

$$\begin{aligned} &d_t W^{n+1}_{1,i,j+1/2} - \nu D_x(d_x W_1)^{n+1/2}_{i,j+1/2} - \nu d_y(D_y W_1)^{n+1/2}_{i,j+1/2} + [D_x H]^{n+1/2}_{i,j+1/2} \\ &\qquad = g^{n+1/2}_{1,i,j+1/2}, \quad 1\le i\le N_x-1, \quad 0\le j\le N_y-1, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &d_t W^{n+1}_{2,i+1/2,j} - \nu D_y(d_y W_2)^{n+1/2}_{i+1/2,j} - \nu d_x(D_x W_2)^{n+1/2}_{i+1/2,j} + D_y H^{n+1/2}_{i+1/2,j} \\ &\qquad = g^{n+1/2}_{2,i+1/2,j}, \quad 0\le i\le N_x-1, \quad 1\le j\le N_y-1, \end{aligned} \tag{3.8}$$

$$d_x W^{n+1/2}_{1,i+1/2,j+1/2} + d_y W^{n+1/2}_{2,i+1/2,j+1/2} = 0, \quad 0\le i\le N_x-1, \quad 0\le j\le N_y-1, \tag{3.9}$$

where the boundary and initial approximations are the same as in (2.26).

By following closely the same arguments as in [18, 9], we can prove the following.

LEMMA 3.2. *Assuming that $\boldsymbol{u}\in W^3_\infty(J;W^4_\infty(\Omega))^2$, $p\in W^3_\infty(J;W^3_\infty(\Omega))$, we have the following results:*

$$\|d_x(W_1^{n+1}-u_1^{n+1})\|_{l^2,M} + \|d_y(W_2^{n+1}-u_2^{n+1})\|_{l^2,M} \le O(\Delta t^2 + h^2 + k^2), \tag{3.10}$$

$$\left(\sum_{l=0}^{n}\Delta t\|d_t(\boldsymbol{W}^{l+1}-\boldsymbol{u}^{l+1})\|^2_{l^2}\right)^{1/2} + \|\boldsymbol{W}^{n+1}-\boldsymbol{u}^{n+1}\|_{l^2} \le O(\Delta t^2 + h^2 + k^2), \tag{3.11}$$

$$\|D_y(W_1^{n+1}-u_1^{n+1})\|_{l^2,T_y} \le O(\Delta t^2 + h^2 + k^{3/2}), \tag{3.12}$$

$$\|D_x(W_2^{n+1}-u_2^{n+1})\|_{l^2,T_x} \le O(\Delta t^2 + h^{3/2} + k^2), \tag{3.13}$$

$$\left(\sum_{l=1}^{N}\Delta t\|(H-p)^{l-1/2}\|^2_{l^2,M}\right)^{1/2} \le O(\Delta t^2 + h^2 + k^2). \tag{3.14}$$

3.2. Discrete LBB condition. In order to carry out our error analysis, we need the discrete LBB condition.

Here we use the same notation and results as Rui and Li [18, Lemma 3.3]. Let

$$b(\mathbf{v}, q) = -\int_\Omega q\,\nabla\cdot\mathbf{v}dx,\ \ \mathbf{v}\in\mathbf{V},\ \ q\in W,$$

where

$$\mathbf{V} = H_0^1(\Omega)\times H_0^1(\Omega),\quad W = \left\{q\in L^2(\Omega):\int_\Omega q dx = 0\right\}.$$

We construct the finite-dimensional subspaces of W and $\mathbf{V}$ by introducing three different partitions $\mathcal{T}_h, \mathcal{T}_h^1, \mathcal{T}_h^2$ of Ω. The original partition $\delta_x\times\delta_y$ is denoted by $\mathcal{T}_h$. The partition $\mathcal{T}_h^1$ is generated by connecting all the midpoints of the vertical sides of $\Omega_{i+1/2,j+1/2}$ and extending the resulting mesh to the boundary Γ. Similarly, for all $\Omega_{i+1/2,j+1/2}\in\mathcal{T}_h$ we connect all the midpoints of the horizontal sides of $\Omega_{i+1/2,j+1/2}$ and extend the resulting mesh to the boundary Γ; then we obtain the third partition, which is denoted by $\mathcal{T}_h^2$.

Corresponding to the quadrangulation $\mathcal{T}_h$, define W_h, a subspace of W,

$$W_h = \left\{q_h:\ q_h|_T = \text{constant}\ \forall T\in\mathcal{T}_h \text{ and} \int_\Omega q dx = 0\right\}.$$

Furthermore, let $\mathbf{V}_h$ be a subspace of $\mathbf{V}$ such that $\mathbf{V}_h{=}S_h^1\times S_h^2$, where

$$S_h^l = \left\{g\in C^{(0)}(\overline{\Omega}):\ g|_{T^l}\in Q_1(T^l)\ \forall T^l\in\mathcal{T}_h^l \textit{ and } g|_\Gamma = 0\right\},\ l=1,2,$$

and Q_1 denotes the space of all polynomials of degree ≤ 1 with respect to each of the two variables x and y.

We introduce the bilinear forms

$$b_h(\mathbf{v}_h, q_h) = -\sum_{\Omega_{i+1/2,j+1/2}\in\mathcal{T}_h}\int_{\Omega_{i+1/2,j+1/2}} q_h\Pi_h(\nabla\cdot\mathbf{v}_h)dx,\ \ \mathbf{v}_h\in\mathbf{V}_h,\ \ q_h\in W_h,$$

where

$$\begin{aligned}\Pi_h:\ & C^{(0)}(\overline{\Omega}_{i+1/2,j+1/2})\to Q_0(\Omega_{i+1/2,j+1/2}) \text{ such that}\\ & (\Pi_h\varphi)_{i+1/2,j+1/2} = \varphi_{i+1/2,j+1/2}\quad\forall\ \Omega_{i+1/2,j+1/2}\in\mathcal{T}_h.\end{aligned}$$

Then, we have the following result [18].

LEMMA 3.3. *There is a constant $\beta>0$ independent of h and k such that*

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{b_h(\boldsymbol{v}_h,q_h)}{\|D\boldsymbol{v}_h\|}\ge\beta\|q_h\|_{l^2,M}\quad\forall q_h\in W_h. \tag{3.15}$$

We also define the operator $\mathbf{I}_h:\ \mathbf{V}\to\mathbf{V}_h$, such that

$$(\nabla\cdot\mathbf{I}_h\mathbf{v}, w) = (\nabla\cdot\mathbf{v}, w)\ \forall w\in W_h, \tag{3.16}$$

with the following approximation properties [2]:

$$\|\mathbf{v}-\mathbf{I}_h\mathbf{v}\|\le C\|\mathbf{v}\|_{W_2^1(\Omega)}\hat{h}, \tag{3.17}$$

$$\|\nabla\cdot(\mathbf{v}-\mathbf{I}_h\mathbf{v})\|\le C\|\nabla\cdot\mathbf{v}\|_{W_2^1(\Omega)}\hat{h}, \tag{3.18}$$

and the inverse inequality

$$\|\mathbf{U}^k-\mathbf{I}_h\mathbf{u}^k\|_\infty \le C\hat{h}^{-1}\|\mathbf{U}^k-\mathbf{I}_h\mathbf{u}^k\|_{l^2} \quad \forall\, 1\le k\le N, \tag{3.19}$$

where $\hat{h}=\max\{h,k\}$, and the positive constant C is independent of $\hat{h}$.

In addition, by the definition of $\mathbf{I}_h\mathbf{v}$ and the midpoint rule of integration, the L^∞ norm of the projection is obtained by

$$\|\mathbf{v}-\mathbf{I}_h\mathbf{v}\|_\infty \le C\|\mathbf{v}\|_{W^2_\infty(\Omega)}\hat{h}. \tag{3.20}$$

Furthermore, we have the following estimate [3]:

$$\|\mathbf{v}-\mathbf{I}_h\mathbf{v}\|_{l^2} \le C\hat{h}^2. \tag{3.21}$$

3.3. A first error estimate with bound depending on L_m. For simplicity, we set

$$\begin{aligned} e^n_{\mathbf{u}} &= (\mathbf{U}^n-\mathbf{W}^n)+(\mathbf{W}^n-\mathbf{u}^n) := \boldsymbol{\xi}^n+\boldsymbol{\gamma}^n,\\ e^n_p &= (P^n-H^n)+(H^n-p^n) := \eta^n+\zeta^n,\\ e^n_q &= Q^n-q^n. \end{aligned} \tag{3.22}$$

The main result of this subsection is as follows.

PROPOSITION 3.1. *Assuming $\boldsymbol{u}\in W^3_\infty(J;W^4_\infty(\Omega))^2$, $p\in W^3_\infty(J;W^3_\infty(\Omega))$, we have*

$$\begin{aligned} &\|\boldsymbol{\xi}^{m+1}\|^2_{l^2}+\sum_{n=0}^m\Delta t\|d_t\boldsymbol{\xi}^{n+1}\|^2_{l^2}+\|D\boldsymbol{\xi}^{m+1}\|^2+\sum_{n=0}^m\Delta t\|\eta^{n+1/2}\|^2_{l^2,M}+|e^{m+1}_q|^2\\ &\quad\le C(L_m)(\Delta t^4+h^4+k^4), \end{aligned} \tag{3.23}$$

where η^k, $\boldsymbol{\xi}^k$, and e^k_q are defined as in (3.22)*, and the positive constant $C(L_m)$ is independent of h, k, and Δt but dependent on L_m.*

We shall prove Proposition 3.1 through a sequence of lemmas below.

First we prove the boundedness of the discrete velocity in the discrete L^2 norm by using the energy stability.

LEMMA 3.4. *Let $\{U^k\}$ be the solution of* (2.22)–(2.25)*. We have*

$$\|\boldsymbol{U}^{m+1}\|_{l^2}\le C(L_m), \tag{3.24}$$

where $C(L_m)$ is independent of h, k, and Δt but dependent on L_m.

Proof. Multiplying (2.22) by $d_tU^{n+1}_{1,i,j+1/2}hk$, making summation on i,j for $1\le i\le N_x-1,\ 0\le j\le N_y-1$, and recalling Lemma A.1, we have

(3.25)

$$\begin{aligned} &\|d_tU_1^{n+1}\|^2_{l^2,T,M}+\frac{\nu}{2\Delta t}(\|d_xU_1^{n+1}\|^2_{l^2,M}-\|d_xU_1^n\|^2_{l^2,M}+\|D_yU_1^{n+1}\|^2_{l^2,T_y}-\|D_yU_1^n\|^2_{l^2,T_y})\\ &+\frac{Q^{n+1/2}}{B^{n+1/2}}\left(\tilde{U}_1^{n+1/2}D_x(\mathcal{P}_h\tilde{U}_1^{n+1/2})+\mathcal{P}_h\tilde{U}_2^{n+1/2}d_y(\mathcal{P}_h\tilde{U}_1^{n+1/2}),d_tU_1^{n+1}\right)_{l^2,T,M}\\ &-(P^{n+1/2},d_xd_tU_1^{n+1})_{l^2,M}=(f_1^{n+1/2},d_tU_1^{n+1/2})_{l^2,T,M}. \end{aligned}$$

Similarly, multiplying (2.23) by $d_tU_{2,i+1/2,j}^{n+1}hk$ and making summation on i,j for $0\le i\le N_x-1,\ 1\le j\le N_y-1$, we can obtain

$$\begin{aligned}&\|d_tU_2^{n+1}\|_{l^2,M,T}^2+\frac{\nu}{2\Delta t}(\|d_yU_2^{n+1}\|_{l^2,M}^2-\|d_yU_2^n\|_{l^2,M}^2+\|D_xU_2^{n+1}\|_{l^2,T_x}^2-\|D_xU_2^n\|_{l^2,T_x}^2)\\&+\frac{Q^{n+1/2}}{B^{n+1/2}}\left(\mathcal{P}_h\tilde{U}_1^{n+1/2}d_x(\mathcal{P}_h\tilde{U}_2^{n+1/2})+\tilde{U}_2^{n+1/2}D_y(\mathcal{P}_h\tilde{U}_2^{n+1/2}),d_tU_2^{n+1}\right)_{l^2,M,T}\\&-(P^{n+1/2},d_yd_tU_2^{n+1})_{l^2,M}=(f_2^{n+1/2},d_tU_2^{n+1})_{l^2,M,T}.\end{aligned}\tag{3.26}$$

Combining (3.25) with (3.26) results in

$$\begin{aligned}&\|d_t\mathbf{U}^{n+1}\|_{l^2}^2+\frac{\nu}{2\Delta t}(\|D\mathbf{U}^{n+1}\|^2-\|D\mathbf{U}^n\|^2)\\&\quad=(\mathbf{f}^{n+1/2},d_t\mathbf{U}^{n+1})_{l^2}-\frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}),d_t\mathbf{U}^{n+1})_{l^2}.\end{aligned}\tag{3.27}$$

Recalling (2.39) and using the Cauchy–Schwarz and Poincaré inequalities, we obtain

$$\begin{aligned}|Q^{n+1}|^2-|Q^0|^2+\nu\sum_{k=0}^n\Delta t\|D\mathbf{U}^{k+1/2}\|^2&=\sum_{k=0}^n\Delta t(\mathbf{f}^{k+1/2},\mathbf{U}^{k+1/2})\\&\le\frac{\nu}{2}\sum_{k=0}^n\Delta t\|D\mathbf{U}^{k+1/2}\|^2+C\sum_{k=0}^n\Delta t\|\mathbf{f}^{k+1/2}\|_{l^2}^2,\end{aligned}\tag{3.28}$$

which implies

$$|Q^{n+1}|\le C.\tag{3.29}$$

Using (3.29), the last term on the right-hand side of (3.27) can be estimated by

$$\begin{aligned}&\left|-\frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}),d_t\mathbf{U}^{n+1})_{l^2}\right|\\&\qquad\le C(L_n)(\|D\mathbf{U}^n\|^2+\|D\mathbf{U}^{n-1}\|^2)+\frac{1}{4}\|d_t\mathbf{U}^{n+1}\|_{l^2}^2.\end{aligned}\tag{3.30}$$

Combining (3.27) with (3.30) and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\|d_t\mathbf{U}^{n+1}\|_{l^2}^2+\frac{\nu}{2\Delta t}(\|D\mathbf{U}^{n+1}\|^2-\|D\mathbf{U}^n\|^2)\\&\qquad\le C(L_n)(\|D\mathbf{U}^n\|^2+\|D\mathbf{U}^{n-1}\|^2)+\frac{1}{2}\|d_t\mathbf{U}^{n+1}\|_{l^2}^2+\frac{1}{2}\|\mathbf{f}^{n+1/2}\|_{l^2}^2.\end{aligned}\tag{3.31}$$

Multiplying (3.31) by $2\Delta t$, summing over n from 0 to m, and applying the Gronwall inequality give that

$$\|D\mathbf{U}^{m+1}\|^2\le C(L_m)\sum_{n=0}^m\Delta t\|\mathbf{f}^{n+1/2}\|_{l^2}^2.\tag{3.32}$$

Thus, we get the desired result (3.24) by applying the discrete Poincaré inequality. □

LEMMA 3.5. *Assuming* $\boldsymbol{u} \in W^3_\infty(J; W^4_\infty(\Omega))^2$, $p \in W^3_\infty(J; W^3_\infty(\Omega))$, *we have*

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{\xi}^{m+1}\|^2_{l^2} + \frac{\nu}{2}\sum_{n=0}^{m}\Delta t\|D\boldsymbol{\xi}^{n+1/2}\|^2 + |e_q^{m+1}|^2 \\
&\qquad \leq C(L_m)\sum_{n=0}^{m}\Delta t\|\boldsymbol{\xi}^n\|^2_{l^2} + \frac{1}{2}\sum_{n=0}^{m}\Delta t\|d_t\boldsymbol{\xi}^{n+1}\|^2_{l^2} \\
&\qquad + \frac{1}{\kappa}C(L_m)\sum_{n=0}^{m}\Delta t|e_q^{n+1}|^2 + \frac{1}{\kappa}C(L_m)(\Delta t^4 + h^4 + k^4),
\end{aligned} \tag{3.33}$$

where $\boldsymbol{\xi}^k$ *and* e_q^k *are defined in* (3.22), κ *is the constant in* (2.34), *and the positive constant* $C(L_m)$ *is independent of* h, k, *and* Δt *but dependent on* L_m.

Proof. Subtracting (3.7) from (2.22), we obtain

$$\begin{aligned}
&d_t\xi^{n+1}_{1,i,j+1/2} - \nu D_x(d_x\xi_1)^{n+1/2}_{i,j+1/2} - \nu d_y(D_y\xi_1)^{n+1/2}_{i,j+1/2} \\
&\qquad + [D_x\eta]^{n+1/2}_{i,j+1/2} = T^{n+1/2}_{1,i,j+1/2},
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}
T_1^{n+1/2} =& -\frac{Q^{n+1/2}}{B^{n+1/2}}\left(\tilde{U}_1^{n+1/2}D_x(\mathcal{P}_h\tilde{U}_1^{n+1/2}) + \mathcal{P}_h\tilde{U}_2^{n+1/2}d_y(\mathcal{P}_h\tilde{U}_1^{n+1/2})\right) \\
&+ \frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}\left(u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y}\right)^{n+1/2}.
\end{aligned} \tag{3.35}$$

Subtracting (3.8) from (2.23), we obtain

$$\begin{aligned}
&d_t\xi^{n+1}_{2,i+1/2,j} - \nu D_y(d_y\xi_2)^{n+1/2}_{i+1/2,j} - \nu d_x(D_x\xi_2)^{n+1/2}_{i+1/2,j} \\
&\qquad + [D_y\eta]^{n+1/2}_{i+1/2,j} = T^{n+1/2}_{2,i+1/2,j},
\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}
T_2^{n+1/2} =& -\frac{Q^{n+1/2}}{B^{n+1/2}}\left(\mathcal{P}_h\tilde{U}_1^{n+1/2}d_x(\mathcal{P}_h\tilde{U}_2^{n+1/2}) + \tilde{U}_2^{n+1/2}D_y(\mathcal{P}_h\tilde{U}_2^{n+1/2})\right) \\
&+ \frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}\left(u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y}\right)^{n+1/2}.
\end{aligned} \tag{3.37}$$

Subtracting (2.4) from (2.24), we obtain

$$d_te_q^{n+1} = \frac{1}{2B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}), \boldsymbol{\xi}^{n+1/2})_{l^2} + \sum_{k=1}^{3}S_k^{n+1/2}, \tag{3.38}$$

where

$$\begin{aligned}
S_1^{n+1/2} &= \frac{\mathrm{dq}^{n+1/2}}{\mathrm{dt}} - d_tq^{n+1}, \\
S_2^{n+1/2} &= \frac{1}{2B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}), \mathbf{W}^{n+1/2})_{l^2} \\
&\quad - \frac{1}{2\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}\int_\Omega \mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2}\cdot\mathbf{u}^{n+1/2}d\mathbf{x}, \\
S_3^{n+1/2} &= \frac{1}{2Q^{n+1/2}}(d_t\mathbf{U}^{n+1}, \mathbf{U}^{n+1/2})_{l^2} - \frac{1}{2q^{n+1/2}}\int_\Omega \frac{\partial\mathbf{u}^{n+1/2}}{\partial t}\cdot\mathbf{u}^{n+1/2}d\mathbf{x}.
\end{aligned}$$

Multiplying (3.34) by $\xi_{1,i,j+1/2}^{n+1/2}hk$, making summation on i,j for $1 \le i \le N_x - 1,\ 0 \le j \le N_y - 1$, and applying Lemma A.1, we have

$$(d_t\xi_1^{n+1}, \xi_1^{n+1/2})_{l^2,T,M} + \nu\|d_x\xi_1^{n+1/2}\|_{l^2,M}^2 + \nu\|D_y\xi_1^{n+1/2}\|_{l^2,T_y}^2 - (\eta^{n+1/2}, d_x\xi_1^{n+1/2})_{l^2,M} = (T_1^{n+1/2}, \xi_1^{n+1/2})_{l^2,T,M}. \tag{3.39}$$

Multiplying (3.36) by $\xi_{2,i+1/2,j}^{n+1/2}hk$, making summation on i,j for $0 \le i \le N_x - 1,\ 1 \le j \le N_y - 1$, and applying Lemma A.1 lead to

$$(d_t\xi_2^{n+1}, \xi_2^{n+1/2})_{l^2,M,T} + \nu\|d_y\xi_2^{n+1/2}\|_{l^2,M}^2 + \nu\|D_x\xi_2^{n+1/2}\|_{l^2,T_x}^2 - (\eta^{n+1/2}, d_y\xi_2^{n+1/2})_{l^2,M} = (T_2^{n+1/2}, \xi_2^{n+1/2})_{l^2,M,T}. \tag{3.40}$$

Multiplying (3.38) by $(e_q^{n+1} + e_q^n)$ leads to

$$\frac{(e_q^{n+1})^2 - (e_q^n)^2}{\Delta t} = \frac{e_q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2} \cdot \nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}), \boldsymbol{\xi}^{n+1/2})_{l^2} + 2\sum_{k=1}^{3} S_k^{n+1/2} e_q^{n+1/2}. \tag{3.41}$$

Combining (3.39) with (3.40), we have

$$(d_t\boldsymbol{\xi}^{n+1}, \boldsymbol{\xi}^{n+1/2})_{l^2} + \nu\|D\boldsymbol{\xi}^{n+1/2}\|^2 - (\eta^{n+1/2}, d_x\xi_1^{n+1/2} + d_y\xi_2^{n+1/2})_{l^2,M} = (\mathbf{T}^{n+1/2}, \boldsymbol{\xi}^{n+1/2})_{l^2}, \tag{3.42}$$

where $\mathbf{T} = (T_1, T_2)$. Subtracting (3.9) from (2.25), we obtain

$$d_x\xi_1^{n+1} + d_y\xi_2^{n+1} = 0, \quad 0 \le i \le N_x - 1, \quad 0 \le j \le N_y - 1. \tag{3.43}$$

Thus we have

$$(\eta^{n+1/2}, d_x\xi_1^{n+1/2} + d_y\xi_2^{n+1/2})_{l^2,M} = 0. \tag{3.44}$$

The term on the right-hand side of (3.42) can be recast as

$$\begin{aligned}(\mathbf{T}^{n+1/2}, \boldsymbol{\xi}^{n+1/2})_{l^2} = &- \frac{e_q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2} \cdot \nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}), \boldsymbol{\xi}^{n+1/2})_{l^2}\\ &- \frac{q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2} \cdot \nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}), \boldsymbol{\xi}^{n+1/2})_{l^2}\\ &+ \frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2}) + \delta}}(\mathbf{u}^{n+1/2} \cdot \nabla\mathbf{u}^{n+1/2}, \boldsymbol{\xi}^{n+1/2})_{l^2}.\end{aligned} \tag{3.45}$$

The last two terms on the right-hand side of (3.45) can be transformed into the following:

$$
\begin{aligned}
&\frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}(\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2},\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&-\frac{q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&\le(\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2},\boldsymbol{\xi}^{n+1/2})_{l^2}\left(\frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}-\frac{q^{n+1/2}}{B^{n+1/2}}\right)\\
&-\frac{q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2})-\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2},\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&-\frac{q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&-\frac{q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\boldsymbol{\xi}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}.
\end{aligned}
\tag{3.46}
$$

Recalling the midpoint approximation property of the rectangle quadrature formula and using the Cauchy–Schwarz and Poincaré inequalities, the first term on the right-hand side of (3.46) can be estimated as

$$
\begin{aligned}
&(\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2},\boldsymbol{\xi}^{n+1/2})_{l^2}\left(\frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}-\frac{q^{n+1/2}}{B^{n+1/2}}\right)\\
&\quad\le Cq^{n+1/2}\|\boldsymbol{\xi}^{n+1/2}\|_{l^2}|E_h(\tilde{\mathbf{U}}^{n+1/2})-E(\mathbf{u}^{n+1/2})|\\
&\quad\le C|q^{n+1/2}|\|\boldsymbol{\xi}^{n+1/2}\|_{l^2}\|\tilde{\mathbf{U}}^{n+1/2}+\mathbf{u}^{n+1/2}\|_{l^2}\|\tilde{\mathbf{U}}^{n+1/2}-\mathbf{u}^{n+1/2}\|_{l^2}+C(h^4+k^4)\\
&\quad\le\frac{\nu}{8}\|D\boldsymbol{\xi}^{n+1/2}\|^2+C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)\\
&\quad+C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2}^2+\|\boldsymbol{\gamma}^{n-1}\|_{l^2}^2)+C(L_n)(\Delta t^4+h^4+k^4).
\end{aligned}
\tag{3.47}
$$

Using the Cauchy–Schwarz and Poincaré inequalities, we see that the second term on the right-hand side of (3.46) can be estimated as

$$
\begin{aligned}
&-\frac{q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2})-\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2},\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&\quad=-\frac{q^{n+1/2}}{B^{n+1/2}}\left((\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}-\mathbf{u}^{n+1/2})\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2}\right)_{l^2}\\
&\quad-\frac{q^{n+1/2}}{B^{n+1/2}}(\mathbf{u}^{n+1/2}\cdot(\nabla_h\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2}-\nabla\mathbf{u}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&\quad\le\frac{\nu}{8}\|D\boldsymbol{\xi}^{n+1/2}\|^2+C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)\\
&\quad+C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2}^2+\|\boldsymbol{\gamma}^{n-1}\|_{l^2}^2)+C(L_n)\|\nabla_h\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2}-\nabla\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2}\|^2\\
&\quad+C(L_n)\|\nabla\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2}-\nabla\tilde{\mathbf{u}}^{n+1/2}\|^2+C(L_n)\|\nabla\tilde{\mathbf{u}}^{n+1/2}-\nabla\mathbf{u}^{n+1/2}\|^2\\
&\quad\le\frac{\nu}{8}\|D\boldsymbol{\xi}^{n+1/2}\|^2+C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)\\
&\quad+C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2}^2+\|\boldsymbol{\gamma}^{n-1}\|_{l^2}^2)+C(L_n)(\Delta t^4+h^4+k^4).
\end{aligned}
\tag{3.48}
$$

Recalling Lemma A.1 and using the Cauchy–Schwarz inequality, we see that the third term on the right-hand side of (3.46) can be controlled by

$$\begin{aligned}
-\frac{q^{n+1/2}}{B^{n+1/2}}&(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&\le C(L_n)|(\nabla_h(\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}|\\
&\le\frac{\nu}{8}\|D\boldsymbol{\xi}^{n+1/2}\|^2+C(L_n)(\Delta t^4+h^4+k^4).
\end{aligned}\tag{3.49}$$

Using Lemma A.1 and the Cauchy–Schwarz inequality, the last term on the right-hand side of (3.46) can be bounded by

$$\begin{aligned}
-\frac{q^{n+1/2}}{B^{n+1/2}}&(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\boldsymbol{\xi}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&\le\frac{\nu}{8}\|D\boldsymbol{\xi}^{n+1/2}\|^2+C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2).
\end{aligned}\tag{3.50}$$

Combining (3.42) with (3.43)–(3.50) results in

$$\begin{aligned}
&\frac{\|\boldsymbol{\xi}^{n+1}\|_{l^2}^2-\|\boldsymbol{\xi}^n\|_{l^2}^2}{2\Delta t}+\nu\|D\boldsymbol{\xi}^{n+1/2}\|^2+\frac{e_q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&\qquad\le C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)+C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2}^2+\|\boldsymbol{\gamma}^{n-1}\|_{l^2}^2)\\
&\qquad+\frac{\nu}{2}\|D\boldsymbol{\xi}^{n+1/2}\|^2+C(L_n)(\Delta t^4+h^4+k^4).
\end{aligned}\tag{3.51}$$

Next we estimate the last term, which is a sum of three terms, on the right-hand side of (3.41):

$$2S_1^{n+1/2}e_q^{n+1/2}\le C(|e_q^{n+1}|^2+|e_q^n|^2)+C\|q\|_{W_\infty^3(J)}^2\Delta t^4,\tag{3.52}$$

$$\begin{aligned}
2S_2^{n+1/2}e_q^{n+1/2}=&\frac{e_q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h(\tilde{\boldsymbol{\xi}}^{n+1/2}+\tilde{\boldsymbol{\gamma}}^{n+1/2})),\mathbf{W}^{n+1/2})_{l^2}\\
&+\frac{e_q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2}),\mathbf{W}^{n+1/2})_{l^2}\\
&-\frac{e_q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}\int_\Omega\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2}\cdot\mathbf{u}^{n+1/2}d\mathbf{x}.
\end{aligned}\tag{3.53}$$

The analysis of the first term on the right-hand side of (3.53) can be carried out with the help of Lemmas A.1 and 3.2:

$$\begin{aligned}
\frac{e_q^{n+1/2}}{B^{n+1/2}}&(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h(\tilde{\boldsymbol{\xi}}^{n+1/2}+\tilde{\boldsymbol{\gamma}}^{n+1/2})),\mathbf{W}^{n+1/2})_{l^2}\\
&\le C(L_n)|e_q^{n+1/2}||(\nabla_h(\mathcal{P}_h(\tilde{\boldsymbol{\xi}}^{n+1/2}+\tilde{\boldsymbol{\gamma}}^{n+1/2})),\mathbf{W}^{n+1/2})_{l^2}|\\
&\le C(L_n)|e_q^{n+1/2}|\|\tilde{\boldsymbol{\xi}}^{n+1/2}+\tilde{\boldsymbol{\gamma}}^{n+1/2}\|_{l^2}\|D\mathbf{W}^{n+1/2}\|\\
&\le C|e_q^{n+1/2}|^2+C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)\\
&+C(L_n)(\Delta t^4+h^4+k^4),
\end{aligned}\tag{3.54}$$

where, thanks to Lemma 3.2, we used the fact that $\|D\mathbf{W}^{n+1/2}\|\le C$.

The last two terms on the right-hand side of (3.53) can be handled similarly to (3.47):

$$
\begin{aligned}
&\frac{e_q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2}),\mathbf{W}^{n+1/2})_{l^2}\\
&\qquad -\frac{e_q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}\int_\Omega \mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2}\cdot\mathbf{u}^{n+1/2}d\mathbf{x}\\
&\qquad \le C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)+C|e_q^{n+1/2}|^2\\
&\qquad + C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2}^2+\|\boldsymbol{\gamma}^{n-1}\|_{l^2}^2)\\
&\qquad + C(L_n)(\Delta t^4+h^4+k^4),
\end{aligned}\tag{3.55}
$$

where, with the aid of Lemma 3.2, we use the fact that $\|\mathbf{W}^{n+1/2}\|_{l^2}\le C$. Recalling Lemma 3.4 and using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
2S_3^{n+1/2}e_q^{n+1/2} =& \frac{e_q^{n+1/2}}{Q^{n+1/2}}(d_t\mathbf{U}^{n+1},\mathbf{U}^{n+1/2})_{l^2}-\frac{e_q^{n+1/2}}{q^{n+1/2}}\int_\Omega\frac{\partial\mathbf{u}^{n+1/2}}{\partial t}\cdot\mathbf{u}^{n+1/2}d\mathbf{x}\\
=& \frac{e_q^{n+1/2}}{Q^{n+1/2}}(d_t(\boldsymbol{\xi}^{n+1}+\boldsymbol{\gamma}^{n+1}),\mathbf{U}^{n+1/2})_{l^2}+\frac{e_q^{n+1/2}}{Q^{n+1/2}}(d_t\mathbf{u}^{n+1},\boldsymbol{\xi}^{n+1/2}+\boldsymbol{\gamma}^{n+1/2})_{l^2}\\
&+\frac{e_q^{n+1/2}}{Q^{n+1/2}}(d_t\mathbf{u}^{n+1},\mathbf{u}^{n+1/2})_{l^2}-\frac{e_q^{n+1/2}}{q^{n+1/2}}\int_\Omega\frac{\partial\mathbf{u}^{n+1/2}}{\partial t}\cdot\mathbf{u}^{n+1/2}d\mathbf{x}\\
\le& \frac{1}{\kappa}C(L_n)|e_q^{n+1/2}|^2+\frac{1}{2}\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2+\frac{1}{\kappa}C(L_n)\|d_t\boldsymbol{\gamma}^{n+1}\|_{l^2}^2\\
&+C\|\boldsymbol{\xi}^{n+1}\|_{l^2}^2+C\|\boldsymbol{\gamma}^{n+1}\|_{l^2}^2+\frac{1}{\kappa}C(L_n)(\Delta t^4+h^4+k^4).
\end{aligned}\tag{3.56}
$$

Combining (3.41) with (3.52)–(3.56) leads to

$$
\begin{aligned}
\frac{(e_q^{n+1})^2-(e_q^n)^2}{\Delta t}\le& \frac{e_q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}),\boldsymbol{\xi}^{n+1/2})_{l^2}\\
&+C(L_n)(|e_q^{n+1}|^2+|e_q^n|^2)+\frac{1}{2}\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2\\
&+C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)\\
&+C(L_n)(\Delta t^4+h^4+k^4).
\end{aligned}\tag{3.57}
$$

Then by combining (3.51) with (3.57), we can obtain

$$
\begin{aligned}
&\frac{\|\boldsymbol{\xi}^{n+1}\|_{l^2}^2-\|\boldsymbol{\xi}^n\|_{l^2}^2}{2\Delta t}+\frac{\nu}{2}\|D\boldsymbol{\xi}^{n+1/2}\|^2+\frac{(e_q^{n+1})^2-(e_q^n)^2}{\Delta t}\\
&\qquad\le C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)+\frac{1}{2}\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2\\
&\qquad + C(L_n)(|e_q^{n+1}|^2+|e_q^n|^2)+C(L_n)(\Delta t^4+h^4+k^4).
\end{aligned}\tag{3.58}
$$

Then we can obtain the desired result (3.33) by multiplying (3.58) by Δt and summing over n from 0 to m. ∎

LEMMA 3.6. *Assuming* $\boldsymbol{u} \in W^3_\infty(J; W^4_\infty(\Omega))^2$, $p \in W^3_\infty(J; W^3_\infty(\Omega))$, *we then have*

(3.59)
$$\sum_{n=0}^{m} \Delta t \|d_t \boldsymbol{\xi}^{n+1}\|^2_{l^2} + \frac{\nu}{2}\|D\boldsymbol{\xi}^{m+1}\|^2 \leq C(L_m)\sum_{n=0}^{m}\Delta t\|\boldsymbol{\xi}^n\|^2_{l^2} + C(L_m)\sum_{n=0}^{m}\Delta t|e_q^{n+1/2}|^2 + C(L_m)\sum_{n=0}^{m}\Delta t\|D\boldsymbol{\xi}^n\|^2 + C(L_m)(\Delta t^4 + h^4 + k^4),$$

where $\boldsymbol{\xi}^k$ *and* e_q^k *are defined as in* (3.22), *and the positive constant* $C(L_m)$ *is independent of* h, k, *and* Δt *but dependent on* L_m.

Proof. Multiplying (3.34) by $d_t\xi^{n+1}_{1,i,j+1/2}hk$, making summation on i, j for $1 \leq i \leq N_x - 1$, $0 \leq j \leq N_y - 1$, and applying Lemma A.1, we have

(3.60)
$$\|d_t\xi_1^{n+1}\|^2_{l^2,T,M} + \frac{\nu}{2}\frac{\|d_x\xi_1^{n+1}\|^2_{l^2,M} - \|d_x\xi_1^n\|^2_{l^2,M}}{\Delta t} + \frac{\nu}{2}\frac{\|D_y\xi_1^{n+1}\|^2_{l^2,T_y} - \|D_y\xi_1^n\|^2_{l^2,T_y}}{\Delta t} - (\eta^{n+1/2}, d_x d_t \xi_1^{n+1})_{l^2,M} = (T_1^{n+1/2}, d_t\xi_1^{n+1})_{l^2,T,M}.$$

Multiplying (3.36) by $d_t\xi^{n+1}_{2,i+1/2,j}hk$, making summation on i, j for $0 \leq i \leq N_x - 1$, $1 \leq j \leq N_y - 1$ and applying Lemma A.1 lead to

(3.61)
$$\|d_t\xi_2^{n+1}\|^2_{l^2,M,T} + \frac{\nu}{2}\frac{\|d_y\xi_2^{n+1}\|^2_{l^2,M} - \|d_y\xi_2^n\|^2_{l^2,M}}{\Delta t} + \frac{\nu}{2}\frac{\|D_x\xi_2^{n+1}\|^2_{l^2,T_x} - \|D_x\xi_2^n\|^2_{l^2,T_x}}{\Delta t} - (\eta^{n+1/2}, d_y d_t \xi_2^{n+1})_{l^2,M} = (T_2^{n+1/2}, d_t\xi_2^{n+1})_{l^2,M,T}.$$

Combining (3.60) with (3.61), we have

$$\|d_t\boldsymbol{\xi}^{n+1}\|^2_{l^2} + \frac{\nu}{2}\frac{\|D\boldsymbol{\xi}^{n+1}\|^2 - \|D\boldsymbol{\xi}^n\|^2}{\Delta t} = (\mathbf{T}^{n+1/2}, d_t\boldsymbol{\xi}^{n+1})_{l^2}. \tag{3.62}$$

The right-hand side of (3.62) can be estimated as

(3.63)
$$\begin{aligned}(\mathbf{T}^{n+1/2}, d_t\boldsymbol{\xi}^{n+1})_{l^2} &= \left(\frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}} - \frac{Q^{n+1/2}}{B^{n+1/2}}\right)(\mathbf{u}^{n+1/2}\cdot\nabla_h\mathbf{u}^{n+1/2}, d_t\boldsymbol{\xi}^{n+1})_{l^2} \\ &\quad - \frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2} - \mathbf{u}^{n+1/2}\cdot\nabla_h\mathbf{u}^{n+1/2}, d_t\boldsymbol{\xi}^{n+1})_{l^2} \\ &\quad - \frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\boldsymbol{\xi}}^{n+1/2}, d_t\boldsymbol{\xi}^{n+1})_{l^2} \\ &\quad - \frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2}, d_t\boldsymbol{\xi}^{n+1})_{l^2}.\end{aligned}$$

The first term on the right-hand side of (3.63) can be handled similarly to (3.47):

$$\begin{aligned}(3.64)\quad &\left(\frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}-\frac{Q^{n+1/2}}{B^{n+1/2}}\right)(\mathbf{u}^{n+1/2}\cdot\nabla_h\mathbf{u}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}\\
&=\left(\frac{q^{n+1/2}}{\sqrt{E(\mathbf{u}^{n+1/2})+\delta}}-\frac{q^{n+1/2}}{B^{n+1/2}}\right)(\mathbf{u}^{n+1/2}\cdot\nabla_h\mathbf{u}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}\\
&-\frac{e_q^{n+1/2}}{B^{n+1/2}}(\mathbf{u}^{n+1/2}\cdot\nabla_h\mathbf{u}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}\\
&\le\frac{1}{6}\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2+C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)\\
&+C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2}^2+\|\boldsymbol{\gamma}^{n-1}\|_{l^2}^2)+C(L_n)|e_q^{n+1/2}|^2\\
&+C(L_n)(\Delta t^4+h^4+k^4).\end{aligned}$$

Using (3.29) and the definition of $\mathcal{P}_h$, we can estimate the second term on the right-hand side of (3.63) as

$$\begin{aligned}&-\frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\mathbf{u}}^{n+1/2}-\mathbf{u}^{n+1/2}\cdot\nabla_h\mathbf{u}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}\\
(3.65)\quad&\le C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)+\frac{1}{6}\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2\\
&+C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2}^2+\|\boldsymbol{\gamma}^{n-1}\|_{l^2}^2)+C(L_n)(\Delta t^4+h^4+k^4).\end{aligned}$$

Applying the Cauchy–Schwarz inequality, the third term on the right-hand side of (3.63) can be controlled by

$$\begin{aligned}&-\frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\boldsymbol{\xi}}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}\\
(3.66)\quad&\le C(L_n)(\|D\boldsymbol{\xi}^n\|^2+\|D\boldsymbol{\xi}^{n-1}\|^2)+\frac{1}{6}\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2\\
&+C(L_n)(h^4+k^4).\end{aligned}$$

Combining (3.62) with (3.63)-(3.66) yields

$$\begin{aligned}&\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2+\frac{\nu}{2}\frac{\|D\boldsymbol{\xi}^{n+1}\|^2-\|D\boldsymbol{\xi}^n\|^2}{\Delta t}\\
&\le-\frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}\\
(3.67)\quad&+\frac{1}{2}\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2+C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2}^2+\|\boldsymbol{\xi}^{n-1}\|_{l^2}^2)\\
&+C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2}^2+\|\boldsymbol{\gamma}^{n-1}\|_{l^2}^2)+C(L_n)|e_q^{n+1/2}|^2\\
&+C(L_n)(\|D\boldsymbol{\xi}^n\|^2+\|D\boldsymbol{\xi}^{n-1}\|^2)\\
&+C(L_n)(\Delta t^4+h^4+k^4).\end{aligned}$$

Multiplying (3.67) by $2\Delta t$ and summing over n from 0 to m, we have

$$\begin{aligned}&\sum_{n=0}^{m}\Delta t\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2+\nu\|D\boldsymbol{\xi}^{m+1}\|^2\\&\quad\le-2\sum_{n=0}^{m}\Delta t\frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}\\&\qquad+C(L_m)\sum_{n=0}^{m}\Delta t\|\boldsymbol{\xi}^n\|_{l^2}^2+C(L_m)\sum_{n=0}^{m}\Delta t|e_q^{n+1/2}|^2\\&\qquad+C(L_m)\sum_{n=0}^{m}\Delta t\|D\boldsymbol{\xi}^n\|^2+C(L_m)(\Delta t^4+h^4+k^4).\end{aligned}\tag{3.68}$$

From the discrete-integration-by-parts, the first term on the right-hand side of (3.68) can be transformed into

$$\begin{aligned}&-2\sum_{n=0}^{m}\Delta t\frac{Q^{n+1/2}}{B^{n+1/2}}(\mathcal{P}_h\tilde{\mathbf{U}}^{n+1/2}\cdot\nabla_h\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}\\&\quad\le C(L_m)|\sum_{n=0}^{m}\Delta t(\nabla_h\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2},d_t\boldsymbol{\xi}^{n+1})_{l^2}|\\&\quad\le C(L_m)|(\nabla_h\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{m+1/2},\boldsymbol{\xi}^{m+1})_{l^2}-\sum_{n=1}^{m}\Delta t(\nabla_h d_t\mathcal{P}_h\tilde{\boldsymbol{\gamma}}^{n+1/2},\boldsymbol{\xi}^n)_{l^2}|\\&\quad\le C(L_m)\sum_{n=1}^{m}\Delta t\|d_t\tilde{\boldsymbol{\gamma}}^{n+1/2}\|_{l^2}^2+C(L_m)\sum_{n=1}^{m}\Delta t\|D\boldsymbol{\xi}^n\|^2\\&\quad+\frac{\nu}{2}\|D\boldsymbol{\xi}^{m+1}\|_{l^2}^2+C(L_m)(\Delta t^4+h^4+k^4).\end{aligned}\tag{3.69}$$

Substituting (3.69) into (3.68) leads to

$$\begin{aligned}&\sum_{n=0}^{m}\Delta t\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2+\frac{\nu}{2}\|D\boldsymbol{\xi}^{m+1}\|^2\\&\quad\le C(L_m)\sum_{n=0}^{m}\Delta t\|\boldsymbol{\xi}^n\|_{l^2}^2+C(L_m)\sum_{n=0}^{m}\Delta t|e_q^{n+1/2}|^2\\&\quad+C(L_m)\sum_{n=0}^{m}\Delta t\|D\boldsymbol{\xi}^n\|^2+C(L_m)(\Delta t^4+h^4+k^4).\end{aligned}\tag{3.70}$$

□

LEMMA 3.7. *Assuming* $\boldsymbol{u}\in W_\infty^3(J;W_\infty^4(\Omega))^2$ *and* $p\in W_\infty^3(J;W_\infty^3(\Omega))$, *we have*

$$\begin{aligned}\sum_{n=0}^{m}\Delta t\|\eta^{n+1/2}\|_{l^2,M}^2&\le C(L_m)\sum_{n=0}^{m}\Delta t\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2\\&\quad+C(L_m)\sum_{n=0}^{m}\Delta t\|D\boldsymbol{\xi}^{n+1/2}\|^2+C(L_m)\sum_{n=0}^{m}\Delta t\|\boldsymbol{\xi}^n\|_{l^2}^2\\&\quad+C(L_m)\sum_{n=0}^{m}\Delta t|e_q^{n+1/2}|^2+C(L_m)(\Delta t^4+h^4+k^4),\end{aligned}\tag{3.71}$$

where η^k, $\boldsymbol{\xi}^k$, *and* e_q^k *are defined as in* (3.22), *and the positive constant* $C(L_m)$ *is independent of* h, k, *and* Δt *but dependent on* L_m.

Proof. For a discrete function $\{v_{1,i,j+1/2}^{n+1/2}\}$ such that $v_{1,i,j+1/2}^{n+1/2}|_{\partial\Omega} = 0$, multiplying (3.34) by $v_{1,i,j+1/2}^{n+1/2}hk$ and making summation for i, j with $i = 1, \ldots, N_x - 1,\ j = 0, \ldots, N_y - 1$, and recalling Lemma A.1 lead to

$$\begin{aligned}&(d_t\xi_1^{n+1}, v_1^{n+1/2})_{l^2,T,M} + \nu(d_x\xi_1^{n+1/2}, d_xv_1^{n+1/2})_{l^2,M} + \nu(D_y\xi_1^{n+1/2}, D_yv_1^{n+1/2})_{l^2,T_y}\\&- (\eta^{n+1/2}, d_xv_1^{n+1/2})_{l^2,M} = (T_1^{n+1/2}, v_1^{n+1/2})_{l^2,T,M}.\end{aligned} \tag{3.72}$$

Similarly, in the y direction we can obtain

$$\begin{aligned}&(d_t\xi_2^{n+1}, v_2^{n+1/2})_{l^2,M,T} + \nu(d_y\xi_2^{n+1/2}, d_yv_2^{n+1/2})_{l^2,M} + \nu(D_x\xi_2^{n+1/2}, D_xv_2^{n+1/2})_{l^2,T_x}\\&- (\eta^{n+1/2}, d_yv_2^{n+1/2})_{l^2,M} = (T_2^{n+1/2}, v_2^{n+1/2})_{l^2,M,T}.\end{aligned} \tag{3.73}$$

Combining (3.72) with (3.73) results in

$$\begin{aligned}&(d_t\boldsymbol{\xi}^{n+1}, \mathbf{v}^{n+1/2})_{l^2} + \nu(D\boldsymbol{\xi}^{n+1/2}, D\mathbf{v}^{n+1/2})\\&- (\eta^{n+1/2}, d_xv_1^{n+1/2} + d_yv_2^{n+1/2})_{l^2,M}\\&= (\mathbf{T}^{n+1/2}, \mathbf{v}^{n+1/2})_{l^2}.\end{aligned} \tag{3.74}$$

Using Lemma 3.3, (3.45), and the discrete Poincaré inequality, we can obtain

$$\begin{aligned}\beta\|\eta^{n+1/2}\|_{l^2,M} &\le \sup_{\mathbf{v}\in\mathbf{V}_h} \frac{(\eta^{n+1/2}, d_xv_1^{n+1/2} + d_yv_2^{n+1/2})_{l^2,M}}{\|D\mathbf{v}^{n+1/2}\|}\\&\le C\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2} + C\|D\boldsymbol{\xi}^{n+1/2}\| + C(L_n)(\|\boldsymbol{\xi}^n\|_{l^2} + \|\boldsymbol{\xi}^{n-1}\|_{l^2})\\&+ C(L_n)|e_q^{n+1/2}| + C(L_n)(\|\boldsymbol{\gamma}^n\|_{l^2} + \|\boldsymbol{\gamma}^{n-1}\|_{l^2})\\&+ C(L_n)(\Delta t^2 + h^2 + k^2).\end{aligned} \tag{3.75}$$

Then we can obtain the desired result (3.71). ∎

We are now in position to prove Proposition 3.1.

Proof of Proposition 3.1. Combining the above results, we obtain the following under the $l_m^\infty(L^\infty)$ bound assumption: Combining (3.33) with (3.59), we have

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{\xi}^{m+1}\|_{l^2}^2 + \frac{1}{2}\sum_{n=0}^{m}\Delta t\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2 + \frac{\nu}{2}\|D\boldsymbol{\xi}^{m+1}\|^2 + |e_q^{m+1}|^2\\&\le\ C(L_m)\sum_{n=0}^{m}\Delta t\|\boldsymbol{\xi}^n\|_{l^2}^2 + C(L_m)\sum_{n=0}^{m}\Delta t|e_q^{n+1}|^2\\&+ C(L_m)\sum_{n=0}^{m}\Delta t\|D\boldsymbol{\xi}^n\|^2 + C(L_m)(\Delta t^4 + h^4 + k^4).\end{aligned} \tag{3.76}$$

Then applying the discrete Gronwall inequality, we arrive at

$$\|\boldsymbol{\xi}^{m+1}\|_{l^2}^2 + \sum_{n=0}^{m}\Delta t\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2 + \|D\boldsymbol{\xi}^{m+1}\|^2 + |e_q^{m+1}|^2 \le C(L_m)(\Delta t^4 + h^4 + k^4). \tag{3.77}$$

Recalling (3.71), we have

(3.78)
$$\begin{aligned}\sum_{n=0}^{m}\Delta t\|\eta^{n+1/2}\|_{l^2,M}^2 &\leq C(L_m)\sum_{n=0}^{m}\Delta t\|d_t\boldsymbol{\xi}^{n+1}\|_{l^2}^2+C(L_m)\sum_{n=0}^{m}\Delta t\|\boldsymbol{\xi}^n\|_{l^2}^2\\ &+C(L_m)\sum_{n=0}^{m}\Delta t|e_q^{n+1}|^2+C(L_m)\sum_{n=0}^{m}\Delta t\|D\boldsymbol{\xi}^{n+1/2}\|^2+C(L_m)(\Delta t^4+h^4+k^4)\\ &\leq C(L_m)(\Delta t^4+h^4+k^4),\end{aligned}$$

which, together with (3.77), completes the proof of Proposition 3.1. ∎

3.4. Uniform bound on L_m. It remains to show that L_m in (3.5) can be uniformly bounded.

LEMMA 3.8. *Assume that the assumptions of Theorem* 3.1 *hold and suppose that $\hat{h}$ is sufficiently small. Then there exists a positive constant C_* such that $\Delta t \leq C_*\hat{h}$, and we have*

$$\|\boldsymbol{U}^m\|_\infty \leq C_1 \quad \forall\ 0\leq m\leq N=T/\Delta t, \tag{3.79}$$

where $\hat{h}=\max\{h,k\}$, and C_1 is a positive constant independent of h, k and Δt.

Proof. We proceed with the following two steps using a bootstrap argument.

Step 1 (definition of C_1). Using the scheme (2.22)–(2.25) for $n=0$, Proposition 3.1, the properties of the operator $\mathbf{I}_h$, and the inverse inequality (3.19), we can get the approximation $\mathbf{U}^1$ and the following property:

$$\begin{aligned}\|\mathbf{U}^1\|_\infty &= \|\mathbf{U}^1-\mathbf{I}_h\mathbf{u}^1\|_\infty+\|\mathbf{I}_h\mathbf{u}^1-\mathbf{u}^1\|_\infty+\|\mathbf{u}^1\|_\infty\\ &\leq C\hat{h}^{-1}\|\mathbf{U}^1-\mathbf{I}_h\mathbf{u}^1\|_{l^2}+\|\mathbf{I}_h\mathbf{u}^1-\mathbf{u}^1\|_\infty+\|\mathbf{u}^1\|_\infty\\ &\leq C\hat{h}^{-1}(\|\boldsymbol{\xi}^1+\boldsymbol{\gamma}^1\|_{l^2}+\|\mathbf{I}_h\mathbf{u}^1-\mathbf{u}^1\|_{l^2})+\|\mathbf{I}_h\mathbf{u}^1-\mathbf{u}^1\|_\infty+\|\mathbf{u}^1\|_\infty\\ &\leq C\hat{h}^{-1}(\Delta t^2+\hat{h}^2)+\|\mathbf{u}^1\|_\infty\leq C,\end{aligned}$$

where $\hat{h}$ and Δt are selected such that $\hat{h}^{-1}\Delta t^2$ is sufficiently small.

Thus define the positive constant C_1 independent of $\hat{h}$ and Δt such that

$$C_1\geq \max\{\|\mathbf{U}^1\|_\infty, 2\|\mathbf{u}\|_{L^\infty(L^\infty)}\}.$$

Step 2 (induction). We can easily obtain that hypothesis (3.79) holds true for $m=1$ by the definition of C_1. Assuming that $\|\mathbf{U}^m\|_\infty\leq C_1$ holds true for an integer $m=1,\dots,N-1$ and using Proposition 3.1, we obtain

$$\|\boldsymbol{\xi}^{m+1}\|_{l^2}\leq C(L_m)(\Delta t^2+\hat{h}^2).$$

Next, we prove that $\|\mathbf{U}^{m+1}\|_\infty\leq C_1$ holds true since

$$\begin{aligned}\|\mathbf{U}^{m+1}\|_\infty &= \|\mathbf{U}^{m+1}-\mathbf{I}_h\mathbf{u}^{m+1}\|_\infty+\|\mathbf{I}_h\mathbf{u}^{m+1}-\mathbf{u}^{m+1}\|_\infty+\|\mathbf{u}^{m+1}\|_\infty\\ &\leq C\hat{h}^{-1}(\|\boldsymbol{\xi}^{m+1}+\boldsymbol{\gamma}^{m+1}\|_{l^2}+\|\mathbf{I}_h\mathbf{u}^{m+1}-\mathbf{u}^{m+1}\|_{l^2})\\ &+\|\mathbf{I}_h\mathbf{u}^{m+1}-\mathbf{u}^{m+1}\|_\infty+\|\mathbf{u}^{m+1}\|_\infty\\ &\leq C_2\hat{h}^{-1}(\Delta t^2+\hat{h}^2)+\|\mathbf{u}^{m+1}\|_\infty.\end{aligned}\tag{3.80}$$

Let $\Delta t \le C_* \hat{h}$, and let a positive constant $\hat{h}_*$ be small enough to satisfy

$$C_2(1+C_*^2)\hat{h}_* \le \frac{C_1}{2}.$$

Then for $\hat{h} \in (0, \hat{h}_*]$, (3.80) can be controlled by

$$\begin{aligned} \|\mathbf{U}^{m+1}\|_\infty &\le C_2 \hat{h}^{-1}(\Delta t^2 + \hat{h}^2) + \|\mathbf{u}^{m+1}\|_\infty \\ &\le C_2(1+C_*^2)\hat{h}_* + \frac{C_1}{2} \le C_1. \end{aligned} \tag{3.81}$$

Then the induction hypothesis (3.79) is proved. ∎

Proof of Theorem 3.1. Combining Proposition 3.1 and Lemma 3.8 leads to

$$\begin{aligned} &\|\boldsymbol{\xi}^{m+1}\|_{l^2}^2 + \sum_{n=0}^{m} \Delta t \|d_t \boldsymbol{\xi}^{n+1}\|_{l^2}^2 + \|D\boldsymbol{\xi}^{m+1}\|^2 + \sum_{n=0}^{m} \Delta t \|\eta^{n+1/2}\|_{l^2,M}^2 + |e_q^{m+1}|^2 \\ &\le C(\Delta t^4 + h^4 + k^4). \end{aligned} \tag{3.82}$$

Recalling Lemma 3.2 and (3.22), we arrive at the conclusions of Theorem 3.1. ∎

4. Numerical experiments. In this section, we provide some numerical results to verify the accuracy of the proposed numerical scheme.

We take $\Omega = (0,1) \times (0,1)$, $T = 1$, $\nu = 1$, and $\delta = 0.1$, and set $\Delta t = h = k$. We denote

$$\begin{cases} \|e_X\|_{\infty,2} = \max\limits_{0 \le n \le m} \|e_X^n\|, \\ \|e_p\|_{2,2} = \left(\sum\limits_{n=0}^{m} \Delta t \left\| P^{n+1/2} - p^{n+1/2} \right\|_{l^2,M}^2 \right)^{1/2}, \\ \|e_q\|_\infty = \max\limits_{0 \le n \le m} |Q^n - q^n|, \end{cases}$$

where $X = \mathbf{u}, d_x u_1, D_y u_1$.

Example 1. The right-hand sides of the equations are computed according to the analytic solution given as

$$\begin{cases} p(x,y,t) = \exp(t)(x^3 - 1/4), \\ u_1(x,y,t) = -\exp(t) x^2 (x-1)^2 y(y-1)(2y-1)/256, \\ u_2(x,y,t) = \exp(t) x(x-1)(2x-1) y^2 (y-1)^2/256. \end{cases}$$

The numerical results for Example 1 are presented in Tables 1 and 2. We observe that the results are consistent with the error estimates in Theorem 3.1.

Example 2. The right-hand sides of the equations are computed according to the analytic solution given as

$$\begin{cases} p(x,y,t) = \exp(t)(\sin(\pi y) - 2/\pi), \\ u_1(x,y,t) = \exp(t)\sin^2(\pi x)\sin(2\pi y), \\ u_2(x,y,t) = -\exp(t)\sin(2\pi x)\sin^2(\pi y). \end{cases}$$

The numerical results for Example 2 are presented in Tables 3 and 4. We observe uniform second-order convergence for all quantities, including $D_y u_1$ for which Theorem 3.1 predicts only 3/2-order convergence. This is due to the fact that for this

TABLE 1
Convergence rates of the velocity for Example 1.

$N_x \times N_y$	$\|e_{\mathbf{u}}\|_{\infty,2}$	Rate	$\|e_{d_x u_1}\|_{\infty,2}$	Rate	$\|e_{D_y u_1}\|_{\infty,2}$	Rate
$2^4 \times 2^4$	1.05E-6	—	2.78E-6	—	8.71E-6	—
$2^5 \times 2^5$	2.59E-7	2.02	6.82E-7	2.03	3.21E-6	1.44
$2^6 \times 2^6$	6.41E-8	2.01	1.65E-7	2.04	1.16E-6	1.47
$2^7 \times 2^7$	1.59E-8	2.01	4.01E-8	2.05	4.16E-7	1.48

TABLE 2
Convergence rates of the pressure and auxiliary variable for Example 1.

$N_x \times N_y$	$\|e_p\|_{2,2}$	Rate	$\|e_q\|_{\infty}$	Rate
$2^4 \times 2^4$	1.01E-3	—	5.10E-11	—
$2^5 \times 2^5$	2.52E-4	2.00	1.36E-11	1.90
$2^6 \times 2^6$	6.30E-5	2.00	3.44E-12	1.99
$2^7 \times 2^7$	1.57E-5	2.00	8.57E-13	2.00

particular exact solution, we have $\frac{\partial^2 u^x}{\partial y^2} = 0$ for $y = 0$ and $y = 1$ and $\frac{\partial^2 u^y}{\partial x^2} = 0$ for $x = 0$ and $x = 1$, which lead to a super-convergence for $D_y u_1$ (see related results in [18, 10]).

Note that we only presented the results for u_1 in both examples since the results for u_2 are similar.

Example 3. We take the initial condition to be $u_1^0(x, y) = \sin^2(\pi x)\sin(2\pi y)$, $u_2^0(x, y) = \sin(2\pi x)\sin^2(\pi y)$, and $\mathbf{f} = 0$.

In Figure 1 we present the time evolutions of the two approximate solutions of (2.21) for Example 3 as $\Delta t = 1/N \to 0$ in (2.21). We clearly observe that one solution of (2.21) converges to the exact solution 1, while the other solution converges to zero.

Appendix A. Finite difference discretization on the staggered grids. To fix the idea, we consider $\Omega = (L_{lx}, L_{rx}) \times (L_{ly}, L_{ry})$. Three-dimensional rectangular domains can be dealt with similarly.

The two-dimensional domain Ω is partitioned by $\Omega_x \times \Omega_y$, where

$$\Omega_x : L_{lx} = x_0 < x_1 < \cdots < x_{N_x-1} < x_{N_x} = L_{rx},$$
$$\Omega_y : L_{ly} = y_0 < y_1 < \cdots < y_{N_y-1} < y_{N_y} = L_{ry}.$$

For simplicity we also use the following notation:

$$\begin{cases} x_{-1/2} = x_0 = L_{lx}, \quad x_{N_x+1/2} = x_{N_x} = L_{rx}, \\ y_{-1/2} = y_0 = L_{ly}, \quad y_{N_y+1/2} = y_{N_y} = L_{ry}. \end{cases} \tag{A.1}$$

For possible integers i, j, $0 \le i \le N_x$, $0 \le j \le N_y$, define

$$x_{i+1/2} = \frac{x_i + x_{i+1}}{2}, \quad h_{i+1/2} = x_{i+1} - x_i, \quad h = \max_i h_{i+1/2},$$
$$h_i = x_{i+1/2} - x_{i-1/2} = \frac{h_{i+1/2} + h_{i-1/2}}{2},$$
$$y_{j+1/2} = \frac{y_j + y_{j+1}}{2}, \quad k_{j+1/2} = y_{j+1} - y_j, \quad k = \max_j k_{j+1/2},$$
$$k_j = y_{j+1/2} - y_{j-1/2} = \frac{k_{j+1/2} + k_{j-1/2}}{2},$$
$$\Omega_{i+1/2,j+1/2} = (x_i, x_{i+1}) \times (y_j, y_{j+1}).$$

TABLE 3
Convergence rates of the velocity for Example 2.

$N_x \times N_y$	$\|e_{\mathbf{u}}\|_{\infty,2}$	Rate	$\|e_{d_x u_1}\|_{\infty,2}$	Rate	$\|e_{D_y u_1}\|_{\infty,2}$	Rate
$2^4 \times 2^4$	2.15E-2	—	4.94E-2	—	9.53E-2	—
$2^5 \times 2^5$	5.21E-3	2.05	1.28E-2	1.94	2.31E-2	2.04
$2^6 \times 2^6$	1.28E-3	2.02	3.29E-3	1.96	5.70E-3	2.02
$2^7 \times 2^7$	3.18E-4	2.01	8.20E-4	2.01	1.41E-3	2.01

TABLE 4
Convergence rates of the pressure and auxiliary variable for Example 2.

$N_x \times N_y$	$\|e_p\|_{2,2}$	Rate	$\|e_q\|_{\infty}$	Rate
$2^4 \times 2^4$	6.38E-2	—	1.35E-2	—
$2^5 \times 2^5$	1.42E-2	2.17	3.49E-3	1.95
$2^6 \times 2^6$	3.27E-3	2.12	8.72E-4	2.00
$2^7 \times 2^7$	7.97E-4	2.04	2.17E-4	2.01

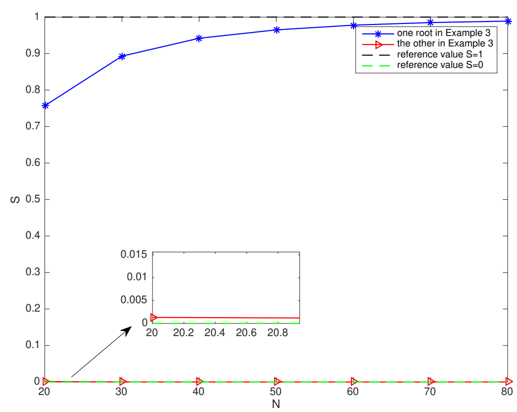

FIG. 1. *Time evolutions of the two approximate solutions of* (2.21) *as* $\Delta t \to 0$ *for Example* 3.

It is clear that

$$h_0 = \frac{h_{1/2}}{2}, \quad h_{N_x} = \frac{h_{N_x-1/2}}{2}, \quad k_0 = \frac{k_{1/2}}{2}, \quad k_{N_y} = \frac{k_{N_y-1/2}}{2}.$$

For a function $f(x, y)$, let $f_{l,m}$ denote $f(x_l, y_m)$, where l may take values $i,\ i + 1/2$ for integer i, and m may take values $j,\ j + 1/2$ for integer j. For discrete functions with values at proper nodal-points, define

$$\text{(A.2)} \quad \begin{cases} [d_x f]_{i+1/2,m} = \dfrac{f_{i+1,m} - f_{i,m}}{h_{i+1/2}}, & [D_y f]_{l,j+1} = \dfrac{f_{l,j+3/2} - f_{l,j+1/2}}{k_{j+1}}, \\ [D_x f]_{i+1,m} = \dfrac{f_{i+3/2,m} - f_{i+1/2,m}}{h_{i+1}}, & [d_y f]_{l,j+1/2} = \dfrac{f_{l,j+1} - f_{l,j}}{k_{j+1/2}}. \end{cases}$$

For functions f and g, define some discrete l^2 inner products and norms as follows:

$$\text{(A.3)} \qquad (f, g)_{l^2,M} \equiv \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} h_{i+1/2} k_{j+1/2} f_{i+1/2,j+1/2} g_{i+1/2,j+1/2},$$

$$(f,g)_{l^2,T_x} \equiv \sum_{i=0}^{N_x}\sum_{j=1}^{N_y-1} h_i k_j f_{i,j} g_{i,j}, \tag{A.4}$$

$$(f,g)_{l^2,T_y} \equiv \sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y} h_i k_j f_{i,j} g_{i,j}, \tag{A.5}$$

$$\|f\|^2_{l^2,\xi} \equiv (f,f)_{l^2,\xi}, \qquad \xi = M,\ T_x,\ T_y. \tag{A.6}$$

Further define discrete l^2 inner products and norms as follows:

$$(f,g)_{l^2,T,M} \equiv \sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y-1} h_i k_{j+1/2} f_{i,j+1/2} g_{i,j+1/2}, \tag{A.7}$$

$$(f,g)_{l^2,M,T} \equiv \sum_{i=0}^{N_x-1}\sum_{j=1}^{N_y-1} h_{i+1/2} k_j f_{i+1/2,j} g_{i+1/2,j}, \tag{A.8}$$

$$\|f\|^2_{l^2,T,M} \equiv (f,f)_{l^2,T,M}, \quad \|f\|^2_{l^2,M,T} \equiv (f,f)_{l^2,M,T}. \tag{A.9}$$

For vector-valued functions $\mathbf{u} = (u_1, u_2)$, it is clear that

$$\|d_x u_1\|^2_{l^2,M} \equiv \sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1} h_{i+1/2} k_{j+1/2} |d_x u_{1,i+1/2,j+1/2}|^2, \tag{A.10}$$

$$\|D_y u_1\|^2_{l^2,T_y} \equiv \sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y} h_i k_j |D_y u_{1,i,j}|^2, \tag{A.11}$$

and $\|d_y u_2\|_{l^2,M}$, $\|D_x u_2\|_{l^2,T_x}$ can be represented similarly. Finally, define the discrete H^1 norm and discrete l^2 norm of a vector-valued function $\mathbf{u}$ as

$$\|D\mathbf{u}\|^2 \equiv \|d_x u_1\|^2_{l^2,M} + \|D_y u_1\|^2_{l^2,T_y} + \|D_x u_2\|^2_{l^2,T_x} + \|d_y u_2\|^2_{l^2,M}, \tag{A.12}$$

$$\|\mathbf{u}\|^2_{l^2} \equiv \|u_1\|^2_{l^2,T,M} + \|u_2\|^2_{l^2,M,T}. \tag{A.13}$$

For simplicity we only consider the case when uniform meshes are used in both the x- and y-directions with all $h_{i+1/2} = h$ and $k_{j+1/2} = k$.

Finally we present the following useful lemma.

LEMMA A.1 ([25]). *Let* $\{V_{1,i,j+1/2}\}$, $\{V_{2,i+1/2,j}\}$ *and* $\{q_{1,i+1/2,j+1/2}\}$, $\{q_{2,i+1/2,j+1/2}\}$ *be discrete functions with* $V_{1,0,j+1/2} = V_{1,N_x,j+1/2} = V_{2,i+1/2,0} = V_{2,i+1/2,N_y} = 0$, *with proper integers* i *and* j. *Then there holds*

$$\begin{cases} (D_x q_1, V_1)_{l^2,T,M} = -(q_1, d_x V_1)_{l^2,M}, \\ (D_y q_2, V_2)_{l^2,M,T} = -(q_2, d_y V_2)_{l^2,M}. \end{cases} \tag{A.14}$$

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