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Efficient and accurate spectral method using generalized Jacobi functions for solving Riesz fractional differential equations

Zhiping Mao [a], Sheng Chen [a], Jie Shen [a,b,*]

[a] Fujian Provincial Key Laboratory on Mathematical Modeling & High Performance Scientific Computing and School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, PR China
[b] Department of Mathematics, Purdue University, West Lafayette, IN 47907-1957, USA

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ABSTRACT

We consider numerical approximation of the Riesz Fractional Differential Equations (FDEs), and construct a new set of generalized Jacobi functions, $\mathcal{J}_n^{-\alpha,-\alpha}(x)$, which are tailored to the Riesz fractional PDEs. We develop optimal approximation results in non-uniformly weighted Sobolev spaces, and construct spectral Petrov–Galerkin algorithms to solve the Riesz FDEs with two kinds of boundary conditions (BCs): (i) homogeneous Dirichlet boundary conditions, and (ii) Integral BCs. We provide rigorous error analysis for our spectral Petrov–Galerkin methods, which show that the errors decay exponentially fast as long as the data (right-hand side function) is smooth, despite that fact that the solution has singularities at the endpoints. We also present some numerical results to validate our error analysis.

1. Introduction

Fractional differential equations (FDEs) have attracted considerable attention recently due to their ability to model certain processes which can not be adequately described by usual partial differential equations. Among the many forms of FDEs, Riesz FDEs are closely related to the fractional power of Laplacian operator [5], and have been numerically studied extensively in the last decade. For example, Deng [7] developed a finite difference/predictor–corrector approximations for the Riesz fractional Fokker–Planck equation; Shen et al. [19] proposed a new weighted Riesz fractional finite-difference approximation scheme; Zhao et al. [27] presented a fourth order compact alternating direction implicit (ADI) scheme to solve the nonlinear Riesz fractional Schrödinger equation. On the other hand, finite element methods have been also developed for Riesz FDEs [4,29]. We observe that most of the numerical works for Riesz FDEs and more general FDEs are based on finite-element or finite difference methods. In particular, pioneer work with finite element analysis is carried out in [16] and [8], and fast algorithms for some FDEs have been developed in [22,23,14,10], among others.

Due to the non-local feature of the fractional derivatives, global methods such as spectral methods appear to be well-suited for FDEs. Indeed, spectral methods for some FDEs have been proposed in [11,12] where the well-posedness of some FDEs and their spectral approximations have been established; a Crank–Nicolson ADI Spectral method is presented in [26], a class of efficient spectral methods for solving multi-dimensional FDEs have been developed in [13]. However, the solution of

* Corresponding author at: Department of Mathematics, Purdue University, West Lafayette, IN 47907-1957, USA.
E-mail addresses: maozhiping87@gmail.com (Z. Mao), shengchen@stu.xmu.edu.cn (S. Chen), shen@math.purdue.edu (J. Shen).

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a FDE has low regularity in the usual Sobolev spaces even with a smooth data. It leads to slow convergence if a usual Spectral method is employed. Recently, some efficient spectral/spectral-element DG methods for a class of one-dimensional FDEs with one-sided fractional derivatives have been proposed in [24,25] by using eigenfunctions of fractional Sturm–Liouville problems as basis functions. Related spectral algorithms and their rigorous error analyses have been established in [6].

The main purpose of this paper is to extend most of the algorithms and analyses in [6] for one-sided FDEs to Riesz FDEs. To this end, we construct a new set of generalized Jacobi functions, $\mathcal{J}_n^{-\alpha,-\alpha}(x)$, which are tailored to the Riesz fractional PDEs, and stay their various properties including in particular optimal approximation results in non-uniformly weighted Sobolev spaces. We then construct efficient spectral Petrov–Galerkin algorithms to solve the Riesz FDEs with two kinds of boundary conditions (BCs): (i) homogeneous Dirichlet boundary conditions, and (ii) Integral BCs. We provide rigorous error analysis for our spectral Petrov–Galerkin methods, and show that the errors decay exponentially fast as long as the data (right-hand side function) is smooth, even though the solution may have singularities at the endpoints. We also present a number of numerical results to validate our error estimates.

The rest of the paper is organized as follows. We describe some basic notations and properties for fractional derivatives and Jacobi polynomials/functions in Section 2. We introduce a set of generalized Jacobi functions $\mathcal{J}_n^{-\alpha,-\alpha}(x)$ and discuss its properties in Section 3. Its approximation results are listed in Section 4. Then, in Section 5, we construct efficient Petrov–Galerkin method for two classes of Riesz FDEs, conduct error analysis and show a number of numerical results. Finally we give some concluding remarks.

2. Preliminaries

In this section, we review the fractional integrals and derivatives, and recall some crucial relationships between fractional derivative and Jacobi polynomials.

2.1. Fractional integrals and derivatives

Let $\mathbb{N}$ and $\mathbb{R}$ be respectively the set of positive integers and real numbers. We further denote

$$\mathbb{N}_0 := \{0\} \cup \mathbb{N}, \quad \mathbb{R}^+ := \{a \in \mathbb{R} : a > 0\}, \quad \mathbb{R}_0^+ := \{0\} \cup \mathbb{R}^+. \tag{2.1}$$

Recall the definitions of the fractional integrals and fractional derivatives in the sense of Riemann–Liouville (see e.g., [15]). In this paper, we fixed $\Lambda = [-1, 1]$. Let $\Gamma(\cdot)$ be the Gamma function.

Definition 1 *(Fractional integrals and derivatives).* For $\rho \in \mathbb{R}^+$, the left and right fractional integrals are respectively defined as

$$\begin{aligned} {}_{-1}I_x^{\rho} v(x) &:= \frac{1}{\Gamma(\rho)} \int_{-1}^{x} \frac{v(t)}{(x-t)^{1-\rho}} dt, \quad x > -1, \\ {}_{x}I_1^{\rho} v(x) &:= \frac{1}{\Gamma(\rho)} \int_{x}^{1} \frac{v(t)}{(t-x)^{1-\rho}} dt, \quad x < 1. \end{aligned} \tag{2.2}$$

For real $s \in [k-1, k)$ with $k \in \mathbb{N}$, the left-sided Riemann–Liouville fractional derivative (LRLFD) of order s is defined by

$$ {}_{-1}D_x^s v(x) = \frac{1}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_{-1}^{x} \frac{v(t)}{(x-t)^{s-k+1}} dt, \quad x \in \Lambda, \tag{2.3}$$

and the right-sided Riemann–Liouville fractional derivative (RRLFD) of order s is defined by

$$ {}_{x}D_1^s v(x) = \frac{(-1)^k}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_{x}^{1} \frac{v(t)}{(t-x)^{s-k+1}} dt, \quad x \in \Lambda. \tag{2.4}$$

And it is clear that for any $k \in \mathbb{N}_0$,

$$ {}_{-1}D_x^k = D^k, \quad {}_{x}D_1^k = (-1)^k D^k, \quad \text{where} \quad D^k := \frac{d^k}{dx^k}. \tag{2.5}$$

Then, we can express the RLFD as

$$\begin{aligned} {}_{-1}D_x^s v(x) &= D^k \{ {}_{-1}I_x^{k-s} v(x) \}, \\ {}_{x}D_1^s v(x) &= (-1)^k D^k \{ {}_{x}I_1^{k-s} v(x) \}. \end{aligned} \tag{2.6}$$

Definition 2 *(Riesz fractional integrals and derivatives).* (See [17], Equations (12.44) and (12.55).) For $\rho \in [0,1)$, the Riesz potential i.e. Riesz fractional integrals are respectively defined as

$$\begin{aligned} I_1^{\rho} v(x) &:= \frac{1}{2\Gamma(\rho)\sin(\pi\rho/2)} \int_{-1}^{1} \frac{\operatorname{sign}(x-t)}{|x-t|^{1-\rho}} v(t)dt = \frac{1}{2\sin(\pi\rho/2)} ({}_{-1}I_x^{\rho} - {}_xI_1^{\rho}) v(x), \quad x \in \Lambda, \\ I_2^{\rho} v(x) &:= \frac{1}{2\Gamma(\rho)\cos(\pi\rho/2)} \int_{-1}^{1} \frac{v(t)}{|x-t|^{1-\rho}} dt = \frac{1}{2\cos(\pi\rho/2)} ({}_{-1}I_x^{\rho} + {}_xI_1^{\rho}) v(x), \quad x \in \Lambda, \end{aligned} \tag{2.7}$$

where sign is the sign function.

Likewise, for real $s \in [k-1,k)$ with $k \in \mathbb{N}$, we define the Riesz fractional derivative (RFD) of order s:

$$D^s v(x) := \begin{cases} D^k I_1^{k-s} v(x), & k \text{ is odd}, \\ D^k I_2^{k-s} v(x), & k \text{ is even}, \end{cases} \quad x \in \Lambda. \tag{2.8}$$

We also define

$$D_1^s v(x) := D^k I_1^{k-s} v(x), \quad D_2^s v(x) := D^k I_2^{k-s} v(x), \quad x \in \Lambda, \tag{2.9}$$

Obviously,

$$\begin{aligned} D_1^s v(x) &:= D^s v(x), \quad \text{if } k \text{ is odd}, \\ D_2^s v(x) &:= D^s v(x), \quad \text{if } k \text{ is even}. \end{aligned} \tag{2.10}$$

2.2. Jacobi polynomials and its properties

Let us first recall the classical Jacobi polynomials. For $\alpha,\ \beta > -1$, let $P_n^{\alpha,\beta}(x)$ be the classical Jacobi polynomials which are orthogonal with respect to the weight function $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ over $(-1,1)$, i.e.

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx = \gamma_n^{\alpha,\beta} \delta_{mn}, \tag{2.11}$$

where

$$\gamma_n^{\alpha,\beta} = \| P_n^{\alpha,\beta}(x) \|^2_{\omega^{\alpha,\beta}(x)} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)} \tag{2.12}$$

and δ_{mn} is the Dirac Delta symbol. The Jacobi polynomials satisfy the three-term recurrence relation:

$$\begin{cases} P_0^{\alpha,\beta}(x) = 1, \\ P_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta), \\ P_{n+1}^{\alpha,\beta}(x) = (A_n^{\alpha,\beta} x - B_n^{\alpha,\beta}) P_n^{\alpha,\beta}(x) - C_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}(x), \ n \ge 1, \end{cases} \tag{2.13}$$

where

$$\begin{aligned} A_n^{\alpha,\beta} &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \\ B_n^{\alpha,\beta} &= \frac{(\alpha^2-\beta^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \\ C_n^{\alpha,\beta} &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \end{aligned} \tag{2.14}$$

And (cf. [3])

$$\omega^{\alpha,\beta}(x) P_n^{\alpha,\beta}(x) = \frac{(-1)^k (n-k)!}{2^k n!} \frac{d^k}{dx^k} \left\{ \omega^{\alpha+k,\beta+k}(x) P_{n-k}^{\alpha+k,\beta+k}(x) \right\}, \quad n \ge k \ge 0. \tag{2.15}$$

In our paper, we will make use of the Jacobi polynomials with real parameters $\alpha, \beta \in \mathbb{R}$, which are defined by (cf. [1,21]):

$$\begin{aligned} P_n^{\alpha,\beta}(x) &= \frac{(\alpha+1)_n}{n!} {}_2F_1\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\Big) \\ &= (-1)^n \frac{(\beta+1)_n}{n!} {}_2F_1\Big(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\Big) \end{aligned} \tag{2.16}$$

where

$$ {}_2F_1(a,b;c;x)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_jx^j}{(c)_j j!}, \quad |x|<1, \quad a,b,c\in\mathbb{R},\ -c\notin\mathbb{N}_0 \tag{2.17} $$

is hypergeometric function, and the rising factorial in the Pochhammer symbol for $a\in\mathbb{R},\ j\in\mathbb{N}_0$ is defined by

$$ (a)_0=1; \quad (a)_j:=a(a+1)\cdots(a+j-1)=\frac{\Gamma(a+j)}{\Gamma(a)},\ \text{for } j\geqslant 1. \tag{2.18} $$

It should be pointed out that the three-term recurrence relation (2.13) is also hold for the generalization of Jacobi polynomials with real parameter. However, the orthogonality does not carry over to the general case. From the Jacobi definition (2.16), we can derive the following property (cf. [1]).

Property 1. *If $-(n+\alpha+\beta+1)\notin\mathbb{N}_0$, we have*

$$ \frac{d^k}{dx^k}P_n^{\alpha,\beta}(x)=d_{n,k}^{\alpha,\beta}P_{n-k}^{\alpha+k,\beta+k},\ n\ge k, \tag{2.19} $$

where

$$ d_{n,k}^{\alpha,\beta}=\frac{\Gamma(n+k+\alpha+\beta+1)}{2^k\Gamma(n+\alpha+\beta+1)} \tag{2.20} $$

We also have following properties.

Property 2.

$$ P_n^{\alpha,\beta}(x)=(-1)^nP_n^{\beta,\alpha}(-x); \quad P_n^{\alpha,\beta}(1)=\frac{(\alpha+1)_n}{n!}. \tag{2.21} $$

2.3. Generalized Jacobi polynomials/functions

We start with the generalized Jacobi functions (GJFs) introduced in [9]:

$$ j_n^{\alpha,\beta}(x)=\begin{cases}(1-x)^{-\alpha}(1+x)^{-\beta}P_{\hat{n}}^{-\alpha,-\beta}, & (\alpha,\beta)\in\mathcal{N}_1,\ \hat{n}=n-[-\alpha]-[-\beta],\\ (1-x)^{-\alpha}P_{\hat{n}}^{-\alpha,\beta}, & (\alpha,\beta)\in\mathcal{N}_2,\ \hat{n}=n-[-\alpha],\\ (1+x)^{-\beta}P_{\hat{n}}^{\alpha,-\beta}, & (\alpha,\beta)\in\mathcal{N}_3,\ \hat{n}=n-[-\beta],\\ P_n^{\alpha,\beta}, & (\alpha,\beta)\in\mathcal{N}_4,\end{cases} \tag{2.22} $$

where [a] denotes the maximum integer $\le a$, and

$$ \mathcal{N}_1=\{(\alpha,\beta):\alpha,\beta\le-1\}, \quad \mathcal{N}_2=\{(\alpha,\beta):\alpha\le-1,\beta>-1\}, $$
$$ \mathcal{N}_3=\{(\alpha,\beta):\alpha>-1,\beta\le-1\}, \quad \mathcal{N}_4=\{(\alpha,\beta):\alpha,\beta>-1\}. $$

Let us recall some properties of GJFs which will be used in section 4. Here we just list the properties that will be used in this paper, for more information, please refer to [9]. The GJFs are mutually $L^2_{\omega^{\alpha,\beta}}$-orthogonal, i.e.,

$$ \int_{-1}^{1} j_m^{\alpha,\beta}(x)j_n^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)dx=\gamma_{\hat{n}}^{\bar{\alpha},\bar{\beta}}, \tag{2.23} $$

where $\hat{n}$ is defined in (2.22), $\gamma_{\hat{n}}^{\bar{\alpha},\bar{\beta}}$ is defined in (2.12), and

$$ \bar{\alpha}=\begin{cases}-\alpha, & \alpha\le-1\\ \alpha, & \alpha>-1\end{cases} \tag{2.24} $$

(likewise for $\bar{\beta}$). Let $k,l,m\in\mathbb{N}$, and if $m\le k,l$, then

$$ D^m j_n^{-k,-l}(x)=(-2)^m\frac{(n-k-l+m)!}{(n-k-l)!}j_{n-m}^{-k+m,-l+m}(x), \quad n\ge\max(k+l,m). \tag{2.25} $$

It also holds

$$ D^m j_n^{-k,-l}(1)=D^m j_n^{-l,-k}(-1)=0, \quad l,k>m. \tag{2.26} $$

2.4. The relationship of fractional derivatives and Jacobi polynomials

Before stating the relationship, for simplicity, we define

$$Q_n^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta} P_n^{\alpha,\beta}.$$

Then we have the following two lemmas.

Lemma 1 *(Spectral relationship for the classical Riesz potential). (See [15, page 184, Theorem 6.5].) If $0<\nu<1$ and r and k are two integers such that $r > -\frac{\nu+1}{2}$, $k > -\frac{\nu+3}{2}$, then for $x \in (-1,1)$ the following holds*

$$\int_{-1}^{1} \frac{Q_m^{\frac{\nu-1}{2}+r,\frac{\nu+1}{2}+k}(t)dt}{|x-t|^{\nu}} = \frac{\pi(-1)^r 2^{r+k+1}\Gamma(m+\nu)}{m!\Gamma(\nu)\cos\frac{\nu\pi}{2}} P_{m+r+k+1}^{\frac{\nu-1}{2}-r,\frac{\nu-3}{2}-k}(x),\ m+r+k+1 \geq 0, \tag{2.27}$$

Lemma 2. *(See [15, page 185, Theorem 6.7].) If $0<\nu<1$ and r and k are integer numbers such that $r, k > -\dfrac{\nu}{2} - 1$, then the following holds*

$$\int_{-1}^{1} \frac{\operatorname{sign}(x-t)}{|x-t|^{\nu}} Q_m^{\frac{\nu}{2}+r,\frac{\nu}{2}+k}(t)dt = \frac{(-1)^r \pi \Gamma(m+\nu)}{2^{-r-k-1} m!\Gamma(\nu)\sin(\frac{\pi\nu}{2})} P_{m+r+k+1}^{\frac{\nu}{2}-r-1,\frac{\nu}{2}-k-1}(x). \tag{2.28}$$

3. Generalized Jacobi function $\mathcal{J}_n^{-\alpha,-\alpha}(x)$ and its related properties

In this section, we will introduce a special set of generalized Jacobi functions $\mathcal{J}_n^{-\alpha,-\alpha}$ with $\alpha > -1$, $\alpha \notin \mathbb{N}_0$, $n = 0,1,2,\cdots$. As we will see in subsection 5.1, it can be used as basis functions for spectral methods to Riesz fractional differential equation.

3.1. Definition of $\mathcal{J}_n^{-\alpha,-\alpha}(x)$

First let us give its definition.

Definition 3. Define

$$\mathcal{J}_n^{-\alpha,-\alpha}(x) = Q_n^{\alpha,\alpha}(x) = (1-x^2)^{\alpha} P_n^{\alpha,\alpha}(x), \quad \text{for } \alpha > -1 \tag{3.1}$$

for all $x \in \Lambda$ and $n \in \mathbb{N}_0$.

From (2.22), we can see that

$$\mathcal{J}_n^{-\alpha,-\alpha}(x) = j_{n-2[-\alpha]}^{-\alpha,-\alpha}(x), \quad \text{if } \alpha > 1. \tag{3.2}$$

3.2. Properties of $\mathcal{J}_n^{-\alpha,-\alpha}(x)$

It can be readily verified from (2.21) that

$$\mathcal{J}_n^{-\alpha,-\alpha}(-x) = (-1)^n \mathcal{J}_n^{-\alpha,-\alpha}(x), \quad \alpha > -1, \tag{3.3}$$

and

$$D^k \mathcal{J}_n^{-\alpha,-\alpha}(\pm 1) = 0, \quad k = 0,1,\cdots,\lceil\alpha\rceil - 1, \tag{3.4}$$

where $\lceil a \rceil$ is the minimal integer that great or equals to a. It also satisfies the three-term recurrence relation:

$$\begin{cases} \mathcal{J}_0^{-\alpha,-\alpha}(x) = (1-x^2)^{\alpha}, \\ \mathcal{J}_1^{-\alpha,-\alpha}(x) = (\alpha+1)x(1-x^2)^{\alpha}, \\ \mathcal{J}_{n+1}^{-\alpha,-\alpha}(x) = A_n^{\alpha,\alpha} x \mathcal{J}_n^{-\alpha,-\alpha}(x) - C_n^{\alpha,\alpha}\mathcal{J}_{n-1}^{-\alpha,-\alpha}(x),\ n \geq 1, \end{cases} \tag{3.5}$$

where $A_n^{\alpha,\alpha}$, $C_n^{\alpha,\alpha}$ are defined in (2.14).

We derive from (2.11) that for $\alpha > -1$,

$$\int_{-1}^{1} \mathcal{J}_n^{-\alpha,-\alpha}(x)\mathcal{J}_m^{-\alpha,-\alpha}(x)\omega^{-\alpha,-\alpha}(x) = \gamma_n^{\alpha,\alpha}\delta_{mn}, \tag{3.6}$$

where $\gamma_n^{\alpha,\alpha}$ is defined in (2.12).

Next, let us study the integrals and derivatives of $\mathcal{J}_n^{-\alpha,-\alpha}(x)$. First by (2.15), we have the integer order derivatives of $\mathcal{J}_n^{-\alpha,-\alpha}(x)$, which is

$$\frac{d^k}{dx^k}\mathcal{J}_n^{-\alpha,-\alpha}(x) = \frac{(-1)^k 2^k (n+k)!}{n!}\mathcal{J}_{n+k}^{k-\alpha,k-\alpha}(x), \quad n,k \geq 0. \tag{3.7}$$

On the other hand, for the fractional integrals and derivatives of $\mathcal{J}_n^{-\alpha,-\alpha}(x)$, in view of (2.7), (2.8) and Lemma 1, 2, we have the following two results.

Theorem 1. *If $-1 < s < 0$, then*

$$I_2^{-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = \frac{\Gamma(m+1+s)}{m!}P_m^{\frac{s}{2},\frac{s}{2}}(x). \tag{3.8}$$

Proof. Let $r = 0$, $k = -1$ and $\nu = 1 + s$ in equation (2.27), then we have

$$\begin{aligned} I_2^{-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) &= \frac{1}{2\cos(\frac{-\pi s}{2})\Gamma(-s)}\int_{-1}^{1}\frac{\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(t)}{|x-t|^{1+s}}dt \\ &= \frac{1}{2\cos(\frac{\pi s}{2})\Gamma(-s)}\frac{\pi\Gamma(m+1+s)}{m!\Gamma(1+s)\cos\frac{(1+s)\pi}{2}}P_m^{\frac{s}{2},\frac{s}{2}}(t). \end{aligned}$$

Then we can derive (3.8) from above and the Euler's reflection formula (see e.g., [2]):

$$\Gamma(1-\tau)\Gamma(\tau) = \frac{\pi}{\sin(\pi\tau)} = \frac{\pi}{2\sin(\frac{\pi\tau}{2})\cos(\frac{\pi\tau}{2})}, \quad 0 < \tau < 1. \qquad \square \tag{3.9}$$

Theorem 2. *If $s \in (n-1, n)$ with $n \in \mathbb{N}$, then*

$$I_\nu^{n-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = C(n)\frac{\Gamma(m+s+1-n)}{2^{-n}m!}P_{m+n}^{\frac{s}{2}-n,\frac{s}{2}-n}(x), \tag{3.10}$$

and for $k = 0, 1, \cdots, n-1$,

$$D_\nu^{s-k}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = 2^k C(n)\frac{\Gamma(m-k+1+s)}{m!}P_{m+k}^{\frac{s}{2}-k,\frac{s}{2}-k}(x), \tag{3.11}$$

where

$$\nu = 1,\ C(n) = (-1)^{\frac{n-1}{2}}, \quad \text{if } n \text{ is odd;} \quad \nu = 2,\ C(n) = (-1)^{\frac{n}{2}}, \quad \text{if } n \text{ is even.} \tag{3.12}$$

Proof. We consider first the case when n is odd. Let $r = \frac{n-1}{2}$, $k = \frac{n-1}{2}$ and $\nu = s + 1 - n$ in equation (2.28), then we have

$$\begin{aligned} I_1^{n-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) &= \frac{1}{2\sin(\frac{\pi(n-s)}{2})\Gamma(n-s)}\int_{-1}^{1}\frac{\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(t)}{|x-t|^{1+s-n}}dt \\ &= \frac{1}{2\cos(\frac{\pi s}{2})\Gamma(n-s)}\frac{\pi\Gamma(m+s+1-n)}{2^{-n}m!\Gamma(s+1-n)\sin(\frac{\pi(s+1-n)}{2})}P_{m+n}^{\frac{s}{2}-n,\frac{s}{2}-n}(x) \end{aligned}$$

Using (3.9), the last equation becomes

$$I_1^{n-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = C(n)\frac{\Gamma(m+s+1-n)}{2^{-n}m!}P_{m+n}^{\frac{s}{2}-n,\frac{s}{2}-n}(x). \tag{3.13}$$

For $k = 0, 1, \cdots, n-1$, by the Riesz fractional derivative definition (2.9), we have

$$D_1^{s-k}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = D^{n-k}I_1^{n-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x). \tag{3.14}$$

Then by (3.13), we have

$$\begin{aligned} D_1^{s-k}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) &= C(n)\frac{\Gamma(m+s+1-n)}{2^{-n}m!}\frac{d^{n-k}}{dx^{n-k}}P_{m+n}^{\frac{s}{2}-n,\frac{s}{2}-n}(x) \\ &= C(n)\frac{\Gamma(m+s+1-n)}{2^{-n}m!}\frac{\Gamma(m-k+s+1)}{2^{n-k}\Gamma(m+s+1-n)}P_{m+k}^{\frac{s}{2}-k,\frac{s}{2}-k}(x) \\ &= 2^kC(n)\frac{\Gamma(m-k+1+s)}{m!}P_{m+k}^{\frac{s}{2}-k,\frac{s}{2}-k}(x). \end{aligned}$$

Now we consider the case when n is even. Let $r=\frac{n}{2}$, $k=\frac{n}{2}-1$ and $\nu=s+1-n$ in equation (2.27), then we have

$$\begin{aligned} I_2^{n-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) &= \frac{1}{2\cos(\frac{\pi(n-s)}{2})\Gamma(n-s)}\int_{-1}^{1}\frac{\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(t)}{|x-t|^{1+s-n}}dt \\ &= \frac{1}{2\cos(\frac{\pi s}{2})\Gamma(n-s)}\frac{\pi 2^n\Gamma(m+s+1-n)}{m!\Gamma(s+1-n)\cos\frac{(s+1-n)\pi}{2}}P_{m+n}^{\frac{s}{2}-n,\frac{s}{2}-n}(x) \end{aligned}$$

Using (3.9), the last equation becomes

$$I_2^{n-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = C(n)\frac{2^n\Gamma(m+s+1-n)}{m!}P_{m+n}^{\frac{s}{2}-n,\frac{s}{2}-n}(x). \tag{3.15}$$

For $k=0,1,\cdots,n-1$, by the Riesz fractional derivative definition (2.9), we have

$$D_2^{s-k}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = D^{n-k}I_2^{n-s}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x). \tag{3.16}$$

Then by (3.15), we have

$$\begin{aligned} D_2^{s-k}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) &= C(n)\frac{2^n\Gamma(m+s+1-n)}{m!}\frac{d^{n-k}}{dx^{n-k}}P_{m+n}^{\frac{s}{2}-n,\frac{s}{2}-n}(x) \\ &= C(n)\frac{2^n\Gamma(m+s+1-n)}{m!}\frac{\Gamma(m-k+s+1)}{2^{n-k}\Gamma(m+s+1-n)}P_{m+k}^{\frac{s}{2}-k,\frac{s}{2}-k}(x) \\ &= 2^kC(n)\frac{\Gamma(m-k+1+s)}{m!}P_{m+k}^{\frac{s}{2}-k,\frac{s}{2}-k}(x). \end{aligned}$$

Here we used the fact: $-(m+s+1-n)\notin\mathbb{N}_0$. This is because m, n are integers while s is a non-integer number. The proof is complete. □

Another useful property of $\mathcal{J}_n^{-\alpha,-\alpha}(x)$ is the orthogonality of its Riesz fractional derivatives. For $s>0$, $s\notin\mathbb{N}$, let $k=\lfloor\frac{s}{2}\rfloor+1$, where $\lfloor a\rfloor$ is the maximum integer that is less than or equal to a. Then for $0\le l\le m+k, n+k$, it hold

$$\int_{-1}^{1} D_\nu^{s-k+l}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x)D_\nu^{s-k+l}\mathcal{J}_n^{-\frac{s}{2},-\frac{s}{2}}(x)\omega^{\frac{s}{2}-k+l,\frac{s}{2}-k+l}(x)dx = h_{n,l}^{\frac{s}{2},\frac{s}{2}}\delta_{mn}, \tag{3.17}$$

where

$$\begin{aligned} h_{n,l}^{\frac{s}{2},\frac{s}{2}} &= 2^{2k}\frac{\Gamma^2(n-k+1+s)}{(n!)^2}(d_{n+k,l}^{\frac{s}{2}-k,\frac{s}{2}-k})^2\gamma_{n+k-l}^{\frac{s}{2}-k+l,\frac{s}{2}-k+l} \\ &= 2^{s+1}\frac{\Gamma^2(n+\frac{s}{2}+1)\Gamma(n-k+l+s+1)}{(2n+s+1)(n+k-l)!(n!)^2}, \end{aligned} \tag{3.18}$$

and ν is defined in (3.12). This equation can be derive from (2.11), (2.19) and (3.11). This property plays an essentially role in the analysis of the approximation of $\mathcal{J}_n^{-\frac{s}{2},-\frac{s}{2}}(x)$.

We point out in particular that with $k=0$ in (3.11), we have

Corollary 1. *If $s\in(n-1,n)$ with $n\in\mathbb{N}$, then*

$$D^s\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = C(n)\frac{\Gamma(m+1+s)}{m!}P_m^{\frac{s}{2},\frac{s}{2}}(x). \tag{3.19}$$

3.3. Sturm–Liouville problem

For $s \in (n-1, n)$, $n \in \mathbb{N}$, we define the Riesz fractional Sturm–Liouville type operator:

$$\mathcal{L}^s := -(1-x)^{\frac{s}{2}}(1+x)^{\frac{s}{2}} D^s (1-x)^{\frac{s}{2}}(1+x)^{\frac{s}{2}} D^s. \tag{3.20}$$

Thanks to (3.20), one can immediately derive that $\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x)$, $m = 0, 1, \cdots$, are eigenfunctions of Riesz fractional Sturm–Liouville type equations. Indeed, we have the following result:

Theorem 3. *Let $s \in (n-1, n)$, $n \in \mathbb{N}$ and $x \in \Lambda$, then*

$$\mathcal{L}^s \mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = \lambda_{m,s} \mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x), \quad m = 0, 1, \cdots \tag{3.21}$$

where

$$\lambda_{m,s} = -\frac{\Gamma^2(m+1+s)}{(m!)^2}. \tag{3.22}$$

Next, we derive an asymptotic estimate from $\lambda_{m,s}$ with fixed s.
From the following property of Gamma function:

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} \exp\Big(-x + \frac{\theta}{12x}\Big), \quad \forall x > 0,\ 0 < \theta < 1, \tag{3.23}$$

we can show that for any constant $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, $n + a > 1$ and $n + b > 1$ (cf. [28]),

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \le \nu_n^{a,b} n^{a-b}, \tag{3.24}$$

where

$$\nu_n^{a,b} = \exp\Big(\frac{a-b}{2(n+b-1)} + \frac{1}{12(n+a-1)} + \frac{(a-b)^2}{n}\Big). \tag{3.25}$$

Then we can immediately derive from (3.22) and the above that for fixed s, we have

$$\lambda_{m,s} = O(m^{2s}), \quad \text{for } m \gg 1. \tag{3.26}$$

4. Approximation by $\mathcal{J}_n^{-\alpha,-\alpha}(x)$

In view of (3.4), we know that $\mathcal{J}_n^{-\alpha,-\alpha}(x)$, $n = 0, 1, \cdots$ can be used as basis functions to deal with boundary value problem with boundary condition: $u^{(k)} = 0$, $k = 0, 1, \cdots, \lceil \alpha \rceil - 1$. Below, we establish their approximation properties.

We first introduce some notations about weighted Sobolev spaces. Let $\omega(x) > 0$ be a weight function, then the weighted Sobolev space is defined by:

$$L^2_\omega(\Lambda) = \Big\{u : \int_\Lambda u^2(x) w dx < +\infty\Big\}, \tag{4.1}$$

with norm

$$\|u\|_w = \Big(\int_\Lambda u^2(x) w dx\Big)^{\frac{1}{2}}.$$

Denote $L^2(\Lambda) = L^2_\omega(\Lambda)$ and $\|u\| = \|u\|_w$ if $w(x) = 1$.

Furthermore, we define the non-uniformly Jacobi weighted space involving Riesz fractional derivatives as follows: Let $k = \lfloor \frac{s}{2} \rfloor + 1$, $s \in (n-1, n)$ with $n \in \mathbb{N}$, and $l \in \mathbb{N}_0$, we denote

$$\mathcal{B}_s^{m,\nu} := \{u \in L^2_{\omega^{-\frac{s}{2},-\frac{s}{2}}}(\Lambda) : D_\nu^{s-k+l} u \in L^2_{\omega^{\frac{s}{2}-k+l,\frac{s}{2}-k+l}}(\Lambda), \text{ for } 0 \le l \le m\},\ m \in \mathbb{N}_0, \tag{4.2}$$

where ν is defined in (3.12).

Let $\mathbb{P}_N$ be the set of polynomials of degree at most N. For $\alpha > 0$, $\alpha \notin \mathbb{N}$, and any real number ν, μ, define the finite dimensional fractional-polynomial space

$$\mathbb{F}_N^{-\nu,-\mu}(\Lambda) := \big\{\phi = (1-x)^\nu (1+x)^\mu \varphi : \varphi \in \mathbb{P}_N\big\}. \tag{4.3}$$

In particular,

$$\mathbb{F}_N^{-\alpha,-\alpha}(\Lambda) = \text{span}\{\mathcal{J}_n^{-\alpha,-\alpha}(x),\ 0 \le n \le N\}.$$

We now show the completeness of $\{\mathcal{J}_n^{-\alpha,-\alpha}(x)\}$ in $L^2_{\omega^{-\alpha,-\alpha}}(\Lambda)$. Indeed, for any $u \in L^2_{\omega^{-\alpha,-\alpha}}(\Lambda)$, we have $(1-x^2)^{-\alpha}u \in L^2_{\omega^{\alpha,\alpha}}(\Lambda)$. For $\alpha > 0$, Jacobi polynomials $\{P_n^{\alpha,\alpha}\}_{n\ge 0}$ are mutually orthogonal and complete in $L^2_{\omega^{\alpha,\alpha}}(\Lambda)$, so we can uniquely expand

$$(1-x^2)^{-\alpha}u(x) = \sum_{n=0}^{\infty} \hat{u}_n^{\alpha,\alpha} P_n^{\alpha,\alpha}(x), \tag{4.4}$$

where

$$\begin{aligned}\hat{u}_n^{\alpha,\alpha} &= \frac{1}{\gamma_n^{\alpha,\alpha}} \int_{-1}^{1} (1-x^2)^{-\alpha} u(x) P_n^{\alpha,\alpha}(x)\omega^{\alpha,\alpha}(x)dx \\ &= \frac{1}{\gamma_n^{\alpha,\alpha}} \int_{-1}^{1} u(x)\mathcal{J}_n^{-\alpha,-\alpha}(x)\omega^{-\alpha,-\alpha}(x)dx.\end{aligned}$$

Multiplying both sides of (4.4) by $(1-x^2)^{-\alpha}$, we obtain owing to the orthogonality of (3.6), we can expand any $u \in L^2_{\omega^{-\alpha,-\alpha}}(\Lambda)$ as

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n^{-\alpha,-\alpha} \mathcal{J}_n^{-\alpha,-\alpha}(x). \tag{4.5}$$

And it holds the Parseval identity:

$$\|u\|^2_{\omega^{-\alpha,-\alpha}} = \sum_{n=0}^{\infty} \gamma_n^{\alpha,\alpha} |\hat{u}_n^{-\alpha,-\alpha}|^2. \tag{4.6}$$

Consider next the $L^2_{\omega^{-\alpha,-\alpha}}$-orthogonal projection on $\mathbb{F}_N^{-\alpha,-\alpha}(\Lambda)$ defined by

$$(\pi_N^{-\alpha,-\alpha}u - u, v_N)_{\omega^{-\alpha,-\alpha}} = 0, \quad \forall v_N \in \mathbb{F}_N^{-\alpha,-\alpha}(\Lambda). \tag{4.7}$$

By definition, we have

$$\pi_N^{-\alpha,-\alpha}u(x) = \sum_{n=0}^{N} \hat{u}_n^{-\alpha,-\alpha} \mathcal{J}_n^{-\alpha,-\alpha}(x). \tag{4.8}$$

Lemma 3. *Let $\alpha > 0$, $\alpha \ne n/2$ with $n \in \mathbb{N}$, if $k = \lfloor \alpha \rfloor + 1$, then $\forall l \in \mathbb{N}_0$, we have*

$$\left(D_\nu^{2\alpha-k+l}(\pi_N^{-\alpha,-\alpha}u - u), D^{l-k}w_N\right)_{\omega^{\alpha-k+l,\alpha-k+l}} = 0, \quad \forall w_N \in \mathbb{P}_N. \tag{4.9}$$

Moreover,

$$\|D_\nu^{2\alpha-k+l}u\|^2_{\omega^{\alpha-k+l,\alpha-k+l}} = \sum_{n=l}^{\infty} h_{n,l}^{\alpha,\alpha} |\hat{u}_n^{-\alpha,-\alpha}|^2, \tag{4.10}$$

where $h_{n,l}^{\alpha,\alpha}$ is same to (3.18) and

$$\nu = \begin{cases} 1, & \text{if } \lceil 2\alpha \rceil \text{ is odd,} \\ 2, & \text{if } \lceil 2\alpha \rceil \text{ is even.} \end{cases} \tag{4.11}$$

Proof. Since

$$(\pi_N^{-\alpha,-\alpha}u - u)(x) = \sum_{n=N+1}^{\infty} \hat{u}_n^{-\alpha,-\alpha} \mathcal{J}_n^{-\alpha,-\alpha}(x),$$

and $\mathbb{P}_N = \text{span}\{P_n^{\alpha,\alpha} : 0 \le n \le N\}$, by (2.11), (2.19) and (3.19), we can derive (4.9).

Then, (4.10) can be derived directly from (3.17) and (4.5). □

Now we are ready to show the main approximation results. In the sequel, we use c to denote a generic constant.

Theorem 4. *Assume $\alpha > 0$ and $\alpha \neq n/2$, $n \in \mathbb{N}$. Let $k = \lfloor \alpha \rfloor + 1$, $u \in \mathcal{B}^{m,\nu}_{2\alpha}$ with $m \in \mathbb{N}_0$, and ν be defined as in (4.11). Then for $0 \le l \le m \le N + k$,*

$$\|D_\nu^{2\alpha-k+l}(\pi_N^{-\alpha,-\alpha}u - u)\|_{\omega^{\alpha-k+l,\alpha-k+l}} \le cN^{\frac{l-m}{2}}\sqrt{\frac{(N+k-m+1)!}{(N+k-l+1)!}}\|D_\nu^{2\alpha-k+m}u\|_{\omega^{\alpha-k+m,\alpha-k+m}}. \tag{4.12}$$

In particular, if m is fixed, then

$$\|D_\nu^{2\alpha-k+l}(\pi_N^{-\alpha,-\alpha}u - u)\|_{\omega^{\alpha-k+l,\alpha-k+l}} \le cN^{l-m}\|D_\nu^{2\alpha-k+m}u\|_{\omega^{\alpha-k+m,\alpha-k+m}}. \tag{4.13}$$

For $0 \le m \le N + k$, we also have the $L^2_{\omega^{-\alpha,-\alpha}}$-estimates:

$$\|\pi_N^{-\alpha,-\alpha}u - u\|_{\omega^{-\alpha,-\alpha}} \le cN^{k-2\alpha}\sqrt{\frac{(N+k-m+1)!}{(N+k+m+1)!}}\|D_\nu^{2\alpha-k+m}u\|_{\omega^{\alpha-k+m,\alpha-k+m}}. \tag{4.14}$$

In particular, if m is fixed, then

$$\|\pi_N^{-\alpha,-\alpha}u - u\|_{\omega^{-\alpha,-\alpha}} \le cN^{k-(2\alpha+m)}\|D_\nu^{2\alpha-k+m}u\|_{\omega^{\alpha-k+m,\alpha-k+m}}. \tag{4.15}$$

Proof. We derive from (4.5), (4.7) and (4.9) that

$$\begin{aligned}\|D_\nu^{2\alpha-k+l}(\pi_N^{-\alpha,-\alpha}u - u)\|^2_{\omega^{\alpha-k+l,\alpha-k+l}} &= \sum_{n=N+1}^{\infty} h_{n,l}^{\alpha,\alpha}|\hat{u}_n^{-\alpha,-\alpha}|^2 = \sum_{n=N+1}^{\infty}\frac{h_{n,l}^{\alpha,\alpha}}{h_{n,m}^{\alpha,\alpha}}h_{n,m}^{\alpha,\alpha}|\hat{u}_n^{-\alpha,-\alpha}|^2 \\ &\le \frac{h_{N+1,l}^{\alpha,\alpha}}{h_{N+1,m}^{\alpha,\alpha}}\|D_\nu^{2\alpha-k+m}u\|^2_{\omega^{\alpha-k+m,\alpha-k+m}}.\end{aligned} \tag{4.16}$$

By (2.18), (3.18), we have

$$\begin{aligned}\frac{h_{N+1,l}^{\alpha,\alpha}}{h_{N+1,m}^{\alpha,\alpha}} &= \frac{\Gamma(N+l-k+2\alpha+2)}{\Gamma(N+m-k+2\alpha+2)}\frac{(N+k-m+1)!}{(N+k-l+1)!} \\ &= \frac{1}{(N+l-k+2\alpha+2)\cdots(N+m-k+2\alpha+1)}\frac{(N+k-m+1)!}{(N+k-l+1)!} \\ &\le N^{l-m}\frac{(N+k-m+1)!}{(N+k-l+1)!},\end{aligned} \tag{4.17}$$

which, together with (4.16), implies (4.12).

Using (3.24) and the fact that $\Gamma(n+1) = n!$, we have that for $m \le N$,

$$\frac{(N+k-m+1)!}{(N+k-l+1)!} \le N^{l-m}\nu_{N+k}^{2-m,2-l}N^{l-m}, \tag{4.18}$$

where $\nu_{N+k}^{2-m,2-l} \approx 1$ for fixed m and $N \gg 1$. Therefore, we can derive (4.15) from (4.12).

Using a similar argument, we can derive the $L^2_{\omega^{-\alpha,-\alpha}}$-estimates. Indeed, by (4.6) and (4.10),

$$\begin{aligned}\|\pi_N^{-\alpha,-\alpha}u - u\|^2_{\omega^{-\alpha,-\alpha}} &= \sum_{n=N+1}^{\infty}\gamma_n^{\alpha,\alpha}|\hat{u}_n^{-\alpha,-\alpha}|^2 = \sum_{n=N+1}^{\infty}\frac{\gamma_n^{\alpha,\alpha}}{h_{n,m}^{\alpha,\alpha}}h_{n,m}^{\alpha,\alpha}|\hat{u}_n^{-\alpha,-\alpha}|^2 \\ &\le \frac{\gamma_{N+1}^{\alpha,\alpha}}{h_{N+1,m}^{\alpha,\alpha}}\|D_\nu^{2\alpha-k+m}u\|^2_{\omega^{\alpha-k+m,\alpha-k+m}}.\end{aligned} \tag{4.19}$$

Similarly, by (2.12), (3.18) and (3.24) again, we obtain, for fixed m,

$$\begin{aligned}\frac{\gamma_{N+1}^{\alpha,\alpha}}{h_{N+1,m}^{\alpha,\alpha}} &= \frac{(N+1)!}{\Gamma(N+2\alpha+2)}\frac{(N+k+m+1)!}{\Gamma(N+2\alpha-k+m+2)}\frac{(N+k-m+1)!}{(N+k+m+1)!} \\ &\le \nu_N^{2,2+2\alpha}N^{-2\alpha}\nu_{N+m}^{2,2+2\alpha}(N+m)^{2k-2\alpha}\frac{(N-m+1)!}{(N+m+1)!} \\ &\le cN^{2k-(4\alpha+2m)}.\end{aligned}$$

This completes the proof. $\square$

5. Application to Riesz fractional differential equations

We consider in this section the application of $\{\mathcal{J}_n^{-\alpha,-\alpha}(x)\}$ to solve a class of fractional differential equations with Riesz derivatives with two kinds of boundary conditions: one is with Dirichlet boundary conditions, and the other is with integral BCs.

5.1. RFBVPs with homogeneous Dirichlet BCs

We consider the one-dimensional RFBVPs of order $2\alpha \in (2k-1, 2k)$ with $k \in \mathbb{N}$:

$$\begin{aligned} &(-1)^k D^{2\alpha} u(x) = f(x), \quad x \in \Lambda, \\ &\qquad u^{(l)}(\pm 1) = 0, \quad l = 0, 1, \cdots, k-1, \end{aligned} \tag{5.1}$$

where $f(x) \in L^2_{\omega^{\alpha,\alpha}}(\Lambda)$ is a given function.

For the above problem, we can determine its classical solution as follows. We expand:

$$f(x) = \sum_{m=0}^{\infty} f_m P_m^{\alpha,\alpha}(x), \tag{5.2}$$

and assume that the solution $u(x)$ of (5.1) takes the form:

$$u(x) = \sum_{n=0}^{\infty} \tilde{u}_n \mathcal{J}_n^{-\alpha,-\alpha}(x). \tag{5.3}$$

Obviously, $u(x)$ satisfies the boundary conditions, and by (3.19), we have

$$D^{2\alpha} u(x) = \sum_{n=0}^{\infty} \tilde{u}_n D^{2\alpha} \mathcal{J}_n^{-\alpha,-\alpha}(x) = (-1)^k \frac{\Gamma(n+1+2\alpha)}{n!} \sum_{n=0}^{\infty} \tilde{u}_n P_n^{\alpha,\alpha}(x). \tag{5.4}$$

Substituting (5.4) and (5.2) in (5.1), and thanks to the orthogonality of $P_n^{\alpha,\alpha}(x)$, we have

$$\tilde{u}_n = f_n \Big/ \Big(2\cos(\pi\alpha) \frac{\Gamma(n+1+2\alpha)}{n!} \Big). \tag{5.5}$$

Since the expansion of f is unique, so is the expansion (5.3).

Recall that $\mathbb{F}_N^{-\alpha,-\alpha} = \text{span}\{\mathcal{J}_n^{-\alpha,-\alpha}(x) : n = 0, 1, \cdots, N\}$. Then, a Petrov–Galerkin spectral method for (5.1) is: Find $u_N \in \mathbb{F}_N^{-\alpha,-\alpha}$ such that

$$(-1)^k (D^{2\alpha} u_N, v_N)_{\omega^{\alpha,\alpha}} = (f, v_N)_{\omega^{\alpha,\alpha}}, \quad \forall v_N \in P_N. \tag{5.6}$$

The solution to this discrete problem can be found directly as follows. Setting

$$u_N(x) = \sum_{n=0}^{N} \hat{u}_n \mathcal{J}_n^{-\alpha,-\alpha}(x), \tag{5.7}$$

and plugging the above and (5.2) in (5.6), using (3.19) and the orthogonality of $\{P_m^{\alpha,\alpha}\}$ in $L^2_{\omega^{\alpha,\alpha}}(\Lambda)$, we find

$$\hat{u}_n = \tilde{u}_n = f_n \Big/ \Big(2\cos(\pi\alpha) \frac{\Gamma(n+1+2\alpha)}{n!} \Big), \quad \forall 0 \le n \le N. \tag{5.8}$$

As for the error estimate, we have

Theorem 5. *Assuming $f^{(j)} \in L^2_{\omega^{\alpha+j,\alpha+j}}(\Lambda)$ for $0 \le j \le m$, we have*

$$\|u - u_N\|_{\omega^{-\alpha,-\alpha}} \le c N^{-2\alpha-m} \|f^{(m)}\|_{\omega^{\alpha+m,\alpha+m}}. \tag{5.9}$$

$$\|D^{2\alpha}(u - u_N)\|_{\omega^{\alpha,\alpha}} \le c N^{-m} \|f^{(m)}\|_{\omega^{\alpha+m,\alpha+m}}. \tag{5.10}$$

Proof. It is clear from (5.8) that $u_N = \Pi_N^{-\alpha,-\alpha} u$. Hence, by (4.15), we have

$$\|u - u_N\|_{\omega^{-\alpha,-\alpha}} = \|u - \Pi_N^{-\alpha,-\alpha} u\|_{\omega^{-\alpha,-\alpha}} \le c N^{-2\alpha-m} \|D^{2\alpha+m} u\|_{\omega^{\alpha+m,\alpha+m}}.$$

Since $D^{2\alpha+m} u = f^{(m)}$, we conclude (5.9).

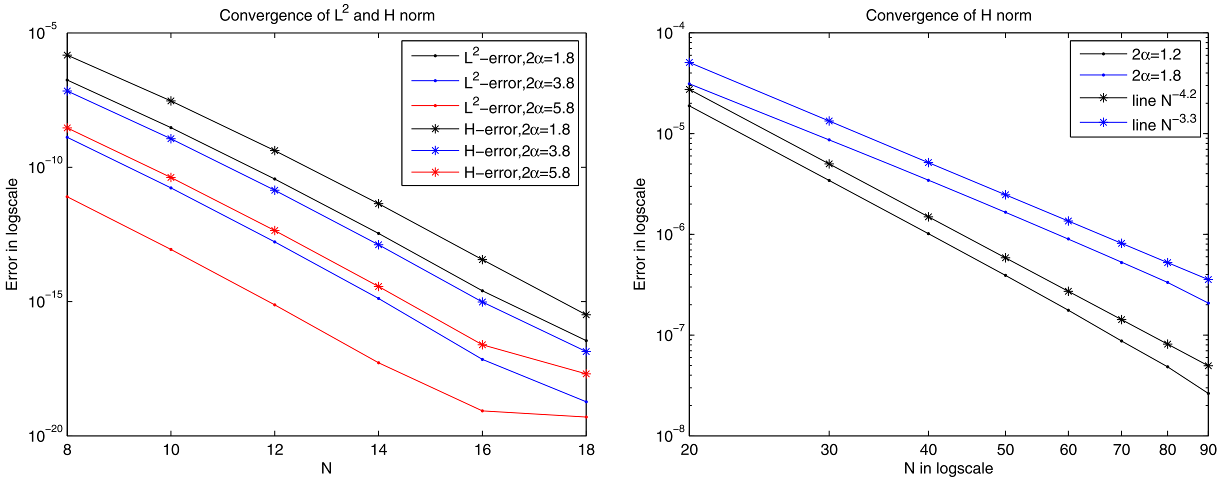

Fig. 1. Convergence results for problem (5.1) with Petrov–Galerkin method (left: $f(x) = \cos(\pi x)$, right: $u(x) = (1 - x^2)^2$).

On the other hand, it is easy to derive from (5.6) that $(-1)^k D^{2\alpha} u_N = \Pi_N^{\alpha,\alpha} f$, where $\Pi_N^{\alpha,\alpha}$ is the $L^2_{\omega^{\alpha,\alpha}}$-projector onto P_N. Hence, we have

$$(-1)^k D^{2\alpha}(u - u_N) = f - \Pi_N^{\alpha,\alpha} f.$$

Then, (5.10) is a direct consequence of the approximation result for $\Pi_N^{\alpha,\alpha}$ (cf. Thm. 3.35 in [20]). □

5.1.1. Numerical tests

Now let us present several numerical examples to illustrate our Petrov–Galerkin method and validate the theoretical results.

Example 1. (*Smooth right-hand side function*). We take $f(x) = \cos(\pi x)$ as an example. In this case, it is easy to see that the solution is not smooth in the usual Sobolev spaces as it exhibits singularities at the endpoints.

The results for $2\alpha = 1.8, 3.8, 5.8$ are shown in the left of Fig. 1 which is draw in semi-log scale. We observe that all errors decay exponentially which agrees with our error estimate (5.9)–(5.10).

Example 2. (*Smooth solution*). We take $u(x) = (1 - x^2)^2$ as an example. In this case, the right-hand side function $f(x)$ is non-smooth.

The results for $2\alpha = 1.2, 1.8$ are shown in the right of Fig. 1 which is draw in log–log scale. We observe that it only has algebraic convergence as expected.

5.2. RFBVPs with fractional integral BCs

Setting $r = 0$, $k = -1$ in (2.27), we have

$$\int_{-1}^{1} \frac{Q_m^{\frac{\nu-1}{2},\frac{\nu-1}{2}}(\tau)d\tau}{|t-\tau|^{\nu}} = \frac{\pi\Gamma(m+\nu)}{m!\Gamma(\nu)\cos\frac{\nu\pi}{2}} P_m^{\frac{\nu-1}{2},\frac{\nu-1}{2}}(t),\ m = 0, 1, \cdots. \tag{5.11}$$

By equation (5.11), one can readily derive that, for $s \in (n-1, n)$ with n is a even number, it holds

$$I_2^{n-s} Q_m^{\frac{s-n}{2},\frac{s-n}{2}} = \frac{\Gamma(m+s+1-n)}{m!} P_m^{\frac{s-n}{2},\frac{s-n}{2}}. \tag{5.12}$$

On the other hand, setting $r = 0$, $k = -1$ in (2.28), we have

$$\int_{-1}^{1} \frac{\text{sign}(x-t)}{|x-t|^{\nu}} Q_m^{\frac{\nu}{2},\frac{\nu}{2}-1}(t)dt = \frac{\pi\Gamma(m+\nu)}{m!\Gamma(\nu)\sin(\frac{\pi\nu}{2})} P_m^{\frac{\nu}{2}-1,\frac{\nu}{2}}(x), \tag{5.13}$$

which implies that, for $s \in (n-1, n)$ with n being a odd integer, it holds

$$I_1^{n-s} Q_m^{\frac{s+1-n}{2},\frac{s-1-n}{2}} = \frac{\Gamma(m+s+1-n)}{m!} P_m^{\frac{s-1-n}{2},\frac{s+1-n}{2}}. \tag{5.14}$$

Hence, (5.12) and (5.14) indicate that $Q_m^{\frac{s-n}{2},\frac{s-n}{2}}$ (resp. $Q_m^{\frac{s+1-n}{2},\frac{s-1-n}{2}}$) can be used as basis functions to RFBVPs of "even" (resp. "odd") order with fractional integral BCs. Here a FDE of order s is "even" (resp. "odd") means that $s \in (n-1, n)$ with n being an even (resp. odd) number.

To fixed the idea, we consider the RFBVPs of order $s \in (1,2)$ or $(2,3)$:

$$-D^s u(x) = f(x), \quad x \in \Lambda, \tag{5.15}$$

with boundary conditions:

$$I_2^{\mu} u(\pm 1) = 0,\ \mu = 2-s,\ \text{if}\ s \in (1,2); \tag{5.16}$$

or

$$I_1^{\mu} u(\pm 1) = 0,\ DI_1^{\mu} u(1) = D_1^{1-\mu} u(1) = 0,\ \mu = 3-s,\ \text{if}\ s \in (2,3). \tag{5.17}$$

Let us first consider the case $s \in (1,2)$. Take $v(x) := I_2^{\mu} u(x)$, then a (weighted) weak formulation to (5.15)–(5.16) is: for $f(x) \in H^{-1}_{\omega^{\alpha,\beta}}(\Lambda)$ with a given pair $-1 < \alpha, \beta < 1$, find $v \in H^1_{0,\omega^{\alpha,\beta}}(\Lambda)$, such that

$$a(v,w) = (f,w)_{\omega^{\alpha,\beta}}, \quad \forall\, w \in H^1_{0,\omega^{\alpha,\beta}}(\Lambda), \tag{5.18}$$

where

$$a(g,h) = \int_{\Lambda} Dg(x) D(h(x)\omega^{\alpha,\beta}(x))dx.$$

Here, $H^1_{\omega^{\alpha,\beta}}(\Lambda)$ is the usual weighted Sobolev space and $H^{-1}_{\omega^{\alpha,\beta}}(\Lambda)$ is its dual space, $H^1_{0,\omega^{\alpha,\beta}}(\Lambda) := \{v \in H^1_{\omega^{\alpha,\beta}}(\Lambda),\ v(\pm 1) = 0\}$. The continuity and coercivity of the bilinear form $a(\cdot,\cdot)$ is shown in [20] (Lemma 3.5). Hence, the above problem admits a unique solution $v \in H^1_{0,\omega^{\alpha,\beta}}$.

Let $\mathbb{P}_N^0 := \{u \in \mathbb{P}_N :\ u(\pm 1) = 0\}$. In order to derive an efficient method, we take $\alpha = \beta = s/2-1$. Then the Petrov–Galerkin method to (5.18) is: Find $v_N := I_2^{\mu} u_N \in \mathbb{P}_N^0$ such that

$$a(v_N, w_N) = (f, w_N)_{\omega^{s/2-1,s/2-1}}, \quad \forall\, w_N \in \mathbb{P}_N^0. \tag{5.19}$$

Since $\mathbb{P}_N^0$ is a subspace of $H^1_{0,\omega^{s/2-1,s/2-1}}(\Lambda)$, the problem (5.19) admits an unique solution.

Denote

$$\hat{P}_m^{s/2-1,s/2-1}(x) = \frac{m!\Gamma(s/2)}{\Gamma(m+s/2)} P_m^{s/2-1,s/2-1}(x),\ m \ge 0,$$

and set

$$\phi_k(x) = \hat{P}_k^{s/2-1,s/2-1}(x) - \hat{P}_{k+2}^{s/2-1,s/2-1}(x),\ k \ge 0.$$

By (2.21), we have $\phi_k(\pm 1) = 0,\ k \ge 0$. Hence, $\mathbb{P}_N^0 = \text{span}\{\phi_k(x) :\ k = 0, 1, \cdots, N-2\}$.

On the other hand, denote

$$\varphi_k(x) = \left(1-x^2\right)^{s/2-1}\left(a_k \hat{P}_k^{s/2-1,s/2-1}(x) - a_{k+2}\hat{P}_{k+2}^{s/2-1,s/2-1}(x)\right), \tag{5.20}$$

where $a_k = \frac{k!}{\Gamma(k+s-1)}$. Then by (5.12), we have

$$I_2^{\mu}\varphi_k(x) = \phi_k(x). \tag{5.21}$$

Therefore, setting

$$U_N := \text{span}\{\varphi_k(x) : \quad k = 0, 1, \cdots, N-2\}, \tag{5.22}$$

the Petrov–Galerkin approximation to problem (5.15)–(5.16) is: find $u_N \in U_N$, such that

$$(D_2^{1-\mu} u_N, D(w_N \omega^{s/2-1,s/2-1})) = (f, w_N)_{\omega^{s/2-1,s/2-1}}, \quad \forall\, w_N \in \mathbb{P}_N^0. \tag{5.23}$$

We now provide some implementation detail for (5.19) and (5.23). Writing $v_N = \sum_{k=0}^{N-2} v_k \phi_k(x)$ in (5.19) and taking $w_N = \phi_j(x)$ for $j = 0, 1, \cdot, N-2$, we arrive at the following linear system:

$$S\bar{v} = \bar{f}, \tag{5.24}$$

where $S=\{s_{i,j}\}_{i,j=0}^{N-2}$ with $s_{i,j}=a(\phi_j,\phi_i)$, $\bar{v}=[v_0,v_1,\cdots,v_{N-2}]^T$, and $\bar{f}=[f_0,f_1,\cdots,f_{N-2}]^T$ with $f_j=(f,w)_{\omega^{s/2-1,s/2-1}}$, $j=0,1,\cdots,N-2$. Therefore, thanks to (5.21), we can recover u_N by

$$u_N=\sum_{k=0}^{N-2} v_k\varphi_k(x), \tag{5.25}$$

where $\varphi_k(x)$ is defined in (5.20). It can be easily shown that S is a upper triangular matrix whose non-zero entries can be derived explicitly, so (5.24) can be easily solved.

Remark 1. One can also take $\alpha=\beta=0$ in (5.18) and the corresponding finite dimensional approximation. In this case, we can expand $v_N=\sum_{m=0}^{N-2}\tilde{v}_m P_m^{-1,-1}$ which leads to a diagonal system for $\{\tilde{v}_m\}$, but it is more difficult to compute $u_N=D_2^{\mu}v_N$. See the case $s\in(2,3)$ below for such an approach.

In order to describe the approximation error, we introduce the non-uniformly Jacobi–weighted Sobolev space $B^m_{\alpha,\beta}(\Lambda):=\{u: D^k u\in L^2_{\omega^{\alpha+k,\beta+k}}(\Lambda),\ 0\le k\le m\},\ k,m\in\mathbb{N}$.

Theorem 6. *Let u and u_N be the solution of (5.15)–(5.16) and (5.23), respectively. If $v=I_2^{\mu}u\in H^1_{0,\omega^{s/2-1,s/2-1}}(\Lambda)$ and $Dv=D_2^{1-\mu}u\in B^{m-1}_{s/2-1,s/2-1}(\Lambda)$ with $m\in\mathbb{N}$, then we have*

$$\|D_2^{1-\mu}(u-u_N)\|_{\omega^{s/2-1,s/2-1}}\lesssim N^{1-m}\|D_2^{m-\mu}u\|_{\omega^{s/2+m-2,s/2+m-2}}. \tag{5.26}$$

In particular, if $f^{(m-2)}\in L^2_{\omega^{s/2+m-2,s/2+m-2}}(\Lambda)$ with $m\ge 2$, then

$$\|D_2^{1-\mu}(u-u_N)\|_{\omega^{s/2-1,s/2-1}}\lesssim N^{1-m}\|f^{(m-2)}\|_{\omega^{s/2+m-2,s/2+m-2}}. \tag{5.27}$$

Proof. Using a standard argument, we derive immediately from (5.18) and (5.19) that

$$\|D(v-v_N)\|_{\omega^{s/2-1,s/2-1}}\lesssim \inf_{\hat{v}_N\in\mathbb{P}_N^0}\|D(v-\hat{v}_N)\|_{\omega^{s/2-1,s/2-1}}. \tag{5.28}$$

Let $\pi^{1,0}_{N,s/2-1,s/2-1}$ the orthogonal projection defined in [20] (cf. Equation (3.290)), and take $\hat{v}_N=\pi^{1,0}_{N,s/2-1,s/2-1}v$ in the last equation, we find

$$\begin{aligned}\|D(v-v_N)\|_{\omega^{s/2-1,s/2-1}}&\lesssim\|D(v-\pi^{1,0}_{N,s/2-1,s/2-1}v)\|_{\omega^{s/2-1,s/2-1}}\\&\lesssim N^{1-m}\|D^m v\|_{\omega^{s/2+m-2,s/2+m-2}}\ \text{(cf. [20] Thm. 3.39)},\end{aligned} \tag{5.29}$$

which, together with (5.15), implies (5.26), and (5.27), since $v=I_2^{\mu}u$ and $v_N=I_2^{\mu}u_N$. $\square$

We now present some numerical examples to illustrate our approximation results.

Example 3. (*With smooth $f(x)$*). We consider first (5.15) with a smooth $f(x)=\cos(\pi x)$.

The results are shown in the left of Fig. 2. We observe that, despite the fact that the solution is non-smooth at the endpoints, all errors decay exponentially. This is consistent with the error estimate (5.27).

Example 4. (*With non-smooth $f(x)$*). We also consider (5.15) with a non-smooth $f(x)=\cos(\pi x)(1+x)^{0.6}$.

The convergence results are shown in the right of Fig. 2. We observe that the error converges algebraically which is also consistent with (5.27).

Now let us turn to the case $s\in(2,3)$. Set $v(x):=I_1^{\mu}u(x)$, and define

$$\begin{aligned}W&=\{w(x)\in H^1_0(\Lambda): Dw\in L^2_{\omega^{-2,0}}(\Lambda)\},\\X&=\{w(x)\in W: D^2w\in L^2_{\omega^{0,2}}(\Lambda)\}.\end{aligned} \tag{5.30}$$

Then a weak formulation for (5.15) and (5.17) is: find $v\in W$, such that

$$(Dv,D^2w)=-(f,w),\quad \forall w\in X. \tag{5.31}$$

Now we apply the dual-Petrov–Galerkin approximation [18] for (5.31). We define a pair of dual approximation spaces

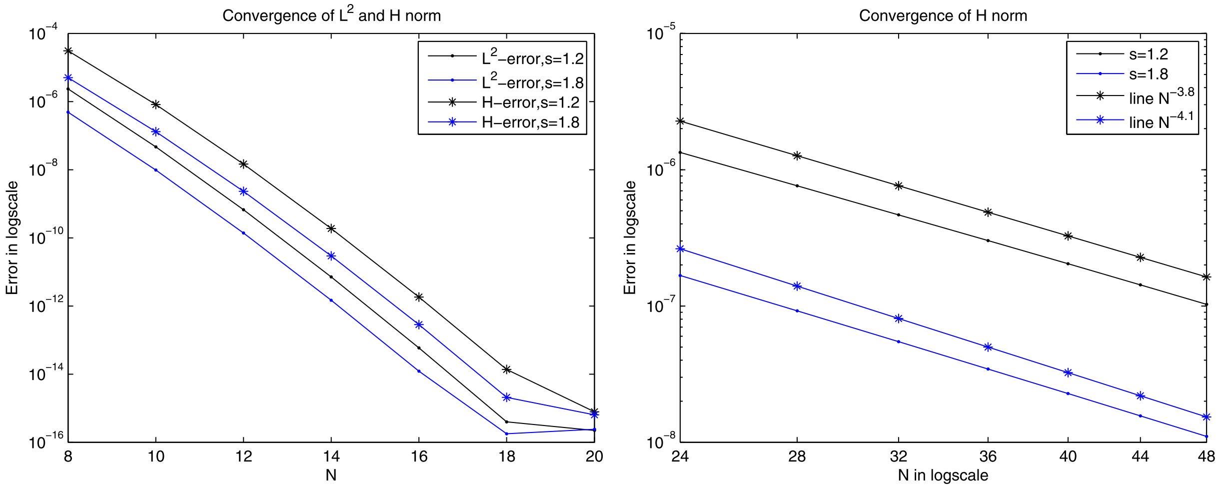

Fig. 2. Convergence results for problem (5.15)–(5.16) (left: $f(x) = \cos(\pi x)$, right: $f(x) = \cos(\pi x)(1+x)^{0.6}$).

$$\begin{aligned} W_N &= \{w(x) \in \mathbb{P}_N : w(\pm 1) = w'(1) = 0\}, \\ W_N^* &= \{w(x) \in \mathbb{P}_N : w(\pm 1) = w'(-1) = 0\}. \end{aligned} \tag{5.32}$$

Then the dual-Petrov–Galerkin approximation for (5.31) is: find $v \in W_N$, such that

$$(Dv_N, D^2 w_N) = -(f, w_N), \quad \forall w_N \in W_N^*. \tag{5.33}$$

The following result is shown in [18] (cf. also [20] Thm. 6.4).

Lemma 4. *Let $v = I_1^{\mu} u$ and $v_N = I_1^{\mu} u_N$ be the solution of (5.15)–(5.17) and (5.33), respectively. If $v \in W \cap B^m_{-2,-1}(\Lambda)$ with $m \ge 2$, then we have*

$$\|v - v_N\|_{\omega^{-1,1}} + N^{-1} \|D(v - v_N)\|_{\omega^{-1,0}} \lesssim N^{-m} \|D^m v\|_{\omega^{m-2,m-1}}. \tag{5.34}$$

We can then derive the following error estimate between u and u_N.

Theorem 7. *If $v \in W$ and $D_1^{1-\mu} u_N \in B^{m-1}_{-1,0}(\Lambda)$ with $m \ge 2$, then we have*

$$\|D_1^{1-\mu}(u - u_N)\|_{\omega^{-1,0}} \lesssim N^{1-m} \|D^{m-\mu} u\|_{\omega^{m-2,m-1}}. \tag{5.35}$$

In particular, if $f^{(m-3)} \in L^2_{\omega^{m-2,m-1}}$ with $m \ge 2$, then

$$\|D_1^{1-\mu}(u - u_N)\|_{\omega^{-1,0}} \lesssim N^{1-m} \|f^{(m-3)}\|_{\omega^{m-2,m-1}}. \tag{5.36}$$

We now provide some implementation detail for solving (5.33). Set

$$\tilde{\phi}_k(x) = j_{k+3}^{-2,-1}(x), \quad \bar{\phi}_k(x) = j_{k+3}^{-1,-2}(x).$$

It is clear from (2.26) that

$$W_N = \mathrm{span}\{\tilde{\phi}_0, \tilde{\phi}_1, \cdots, \tilde{\phi}_{N-3}\}, \quad W_N^* = \mathrm{span}\{\bar{\phi}_0, \bar{\phi}_1, \cdots, \bar{\phi}_{N-3}\}. \tag{5.37}$$

Thus, denote

$$\begin{aligned} v_N &= \sum_{k=0}^{N-3} \tilde{v}_k \tilde{\phi}_k(x), \quad \bar{v} = (\tilde{v}_0, \tilde{v}_1, \cdots, \tilde{v}_{N-3})^T; \\ f_k &= (I_N f, \bar{\phi}_k), \quad \bar{f} = (f_0, f_1, \cdots, f_{N-3})^T; \\ s_{i,j} &= (\tilde{\phi}_j, \bar{\phi}_i), \quad S = (s_{ij})_{i,j=0}^{N-3}, \end{aligned} \tag{5.38}$$

the linear system (5.33) becomes

$$S\bar{v} = \bar{f}. \tag{5.39}$$

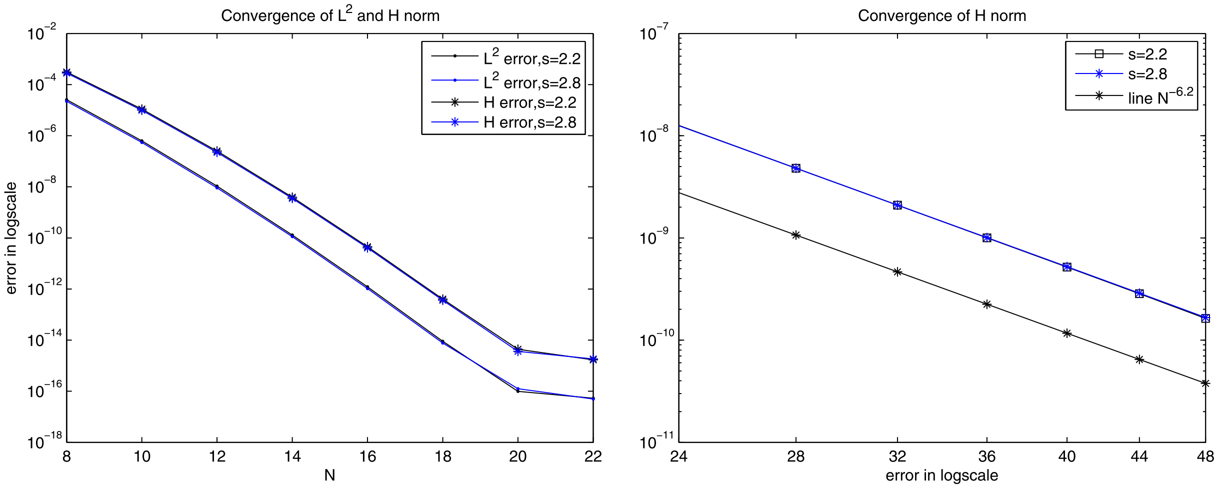

Fig. 3. Convergence results for problem (5.15)–(5.17). Left: $f(x) = \cos(\pi x)$; Right: $f(x) = \cos(\pi x)(1+x)^{0.6}$.

By the orthogonality (2.23), one can easily show that S is a diagonal matrix [18]. Since the basis function $\tilde{\phi}_k(x)$ is a linear combination of Legendre polynomials (see equation (6.9) in [20]), then we can rewrite $v_N(x)$ in the form

$$v_N(x) = \sum_{m=0}^{N} v_m P_m^{s/2-2,s/2-1}(x) \tag{5.40}$$

by Jacobi–Legendre transform. Then, we can recover u_N by (5.14) with $n = 3$:

$$u_N(x) = D_1^{3-s} v_N(x) = \sum_{m=0}^{N} \frac{v_m\, m!}{\Gamma(m+s-2)} Q_m^{s/2-2,s/2-1}(x). \tag{5.41}$$

Using the same data as in Examples 3 and 4, we plot the numerical result for (5.15)–(5.17) in Fig. 3. We observe that the results agree well with the error estimate (5.36).

Remark 2. For higher-order RFBVPs, one can apply a similar approach.

6. Concluding remarks

We developed in this paper efficient and accurate spectral Petrov–Galerkin methods for Riesz FPDEs with homogeneous Dirichlet BCs and fractional integral BCs. The methods are based on a new class of generalized Jacobi functions which are tailored to Riesz fractional derivatives. We derived useful properties of these generalized Jacobi functions, and in particular their optimal approximation results in non-uniformly weighted Sobolev spaces. By using various orthogonal properties of Jacobi polynomials and generalized Jacobi functions, we developed efficient Petrov–Galerkin methods for a class of Riesz FDEs, and derived rigorous error estimates. In particular, it is shown that the errors decay exponentially fast as long as the data (right-hand side function) is smooth, despite that fact that the solution has singularities at the endpoints. We also presented a number of numerical results to validate our error estimates.

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References

[1] A. Érdelyi, et al., Higher Transcendental Functions, vol. 2, California Institute of Technology, H. Bateman MS project, 1953.
[2] M. Abramovitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.
[3] Richard Askey, James Fitch, Integral representations for Jacobi polynomials and some applications, J. Math. Anal. Appl. 26 (2) (1969) 411–437.
[4] Weiping Bu, Yifa Tang, Jiye Yang, Galerkin finite element method for two-dimensional Riesz space fractional diffusion equations, J. Comput. Phys. 276 (2014) 26–38.
[5] Luis Caffarelli, Luis Silvestre, An extension problem related to the fractional Laplacian, Commun. Partial Differ. Equ. 32 (8) (2007) 1245–1260.
[6] Sheng Chen, Jie Shen, Li-Lian Wang, Generalized Jacobi functions and their applications to fractional differential equations, Math. Comput. 85 (300) (2016) 1603–1638.

[7] Kaiying Deng, Weihua Deng, Finite difference/predictor–corrector approximations for the space and time fractional Fokker–Planck equation, Appl. Math. Lett. 25 (11) (2012) 1815–1821.
[8] Vincent J. Ervin, Norbert Heuer, John Paul Roop, Numerical approximation of a time dependent, nonlinear, space-fractional diffusion equation, SIAM J. Numer. Anal. 45 (2) (2007) 572–591.
[9] Ben-Yu Guo, Jie Shen, Li-Lian Wang, Generalized Jacobi polynomials/functions and their applications, Appl. Numer. Math. 59 (5) (2009) 1011–1028.
[10] Siu-Long Lei, Hai-Wei Sun, A circulant preconditioner for fractional diffusion equations, J. Comput. Phys. 242 (2013) 715–725.
[11] X. Li, C. Xu, A space–time spectral method for the time fractional diffusion equation, SIAM J. Numer. Anal. 47 (3) (2009) 2108–2131.
[12] X. Li, C. Xu, Existence and uniqueness of the weak solution of the space–time fractional diffusion equation and a spectral method approximation, Commun. Comput. Phys. 8 (5) (2010) 1016.
[13] Zhiping Mao, Jie Shen, Efficient spectral–Galerkin methods for fractional partial differential equations with variable coefficients, J. Comput. Phys. 307 (2016) 243–261.
[14] Hong-Kui Pang, Hai-Wei Sun, Multigrid method for fractional diffusion equations, J. Comput. Phys. 231 (2) (2012) 693–703.
[15] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, San Diego, 1999.
[16] John Paul Roop, Computational aspects of FEM approximation of fractional advection dispersion equations on bounded domains in $\mathbb{R}^2$, J. Comput. Appl. Math. 193 (1) (2006) 243–268.
[17] S.G. Samko, A.A. Kilbas, O.I. Maričev, Fractional Integrals and Derivatives, Gordon and Breach Science Publ., 1993.
[18] Jie Shen, A new dual-Petrov–Galerkin method for third and higher odd-order differential equations: application to the KdV equation, SIAM J. Numer. Anal. 41 (2003) 1595–1619.
[19] S. Shen, F. Liu, V. Anh, I. Turner, J. Chen, A novel numerical approximation for the space fractional advection–dispersion equation, IMA J. Appl. Math. 79 (3) (2014) 431–444.
[20] J. Shen, T. Tang, L.L. Wang, Spectral Methods: Algorithms, Analysis and Applications, Springer, 2011.
[21] G. Szegö, Orthogonal Polynomials, Math. Soc. Colloq. Publ., Amer. Math. Soc., vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[22] Hong Wang, Ning Du, A superfast-preconditioned iterative method for steady-state space-fractional diffusion equations, J. Comput. Phys. 240 (2013) 49–57.
[23] Hong Wang, Ning Du, Fast solution methods for space-fractional diffusion equations, J. Comput. Appl. Math. 255 (2014) 376–383.
[24] M. Zayernouri, G.E. Karniadakis, Fractional Sturm–Liouville eigen-problems: theory and numerical approximation, J. Comput. Phys. 252 (2013) 495–517.
[25] Mohsen Zayernouri, George Em Karniadakis, Discontinuous spectral element methods for time- and space-fractional advection equations, SIAM J. Sci. Comput. 36 (4) (2014) B684–B707.
[26] Fanhai Zeng, Fawang Liu, Changpin Li, Kevin Burrage, Ian Turner, V. Anh, A Crank–Nicolson ADI spectral method for a two-dimensional Riesz space fractional nonlinear reaction–diffusion equation, SIAM J. Numer. Anal. 52 (6) (2014) 2599–2622.
[27] Xuan Zhao, Zhi-zhong Sun, Zhao-peng Hao, A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation, SIAM J. Sci. Comput. 36 (6) (2014) A2865–A2886.
[28] Xiaodan Zhao, Li-Lian Wang, Ziqing Xie, Sharp error bounds for Jacobi expansions and Gegenbauer–Gauss quadrature of analytic functions, SIAM J. Numer. Anal. 51 (3) (2013) 1443–1469.
[29] Jingjun Zhao, Jingyu Xiao, Yang Xu, A finite element method for the multiterm time–space Riesz fractional advection–diffusion equations in finite domain, Abstr. Appl. Anal. (2013) 868035, Hindawi Publishing Corporation, 2013.