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# On the Existence and Regularity of Solutions of a Quasilinear Mixed Equation of Leray–Lions Type

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**Abstract.** Our aim in this article is to study the existence and regularity of solutions of a quasilinear elliptic-hyperbolic equation. This equation appears in the design of blade cascade profiles. This leads to an inverse problem for designing two-dimensional channels with prescribed velocity distributions along channel walls. The governing equation is obtained by transformation of the physical domain to the plane defined by the streamlines and the potential lines of fluid. We establish an existence and regularity result of solutions for a more general framework which includes our physical problem as a specific example.

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**Key words.** Leray–Lions type operator, maximum principle, relative rearrangements, inverse method.

## 0. Introduction

The main object of this article is to study the existence and the regularity of solutions of a quasilinear equation. One specific application is the equation governing the flow of a perfect and isentropic fluid, obtained when solving the inverse problem of determination of transonic channels, with the deviation as well as the Mach number distributions prescribed along channel walls (see [3, 9, 10] and appendix).

This equation was established for a fluid verifying the exact isentropicity law: $p/\rho^\gamma = cst$ (where $\gamma$ is the ratio of specific heats ($\approx 1.4$), $p$ and $\rho$ are, respectively, the pressure and the density of fluid) after transformation of the physical domain to the plane defined by the streamlines and the potential lines of the fluid.

The unknowns of this equation are the velocity, the Mach number, and the density – the two last quantities are given as algebraic functions of the velocity by virtue of St. Venant's relations for isentropic fluids (see [4, 6, 10]).

The streamline curvatures in the physical domain can be determined by a function of these aerodynamic unknowns as well as the angle between the streamline tangent vector and the physical domain basis vector $\mathbf{i}$. The Cartesian coordinates of the channel walls are obtained by an integration of first-order differential equations; these equations are functions of angle and velocity along the streamlines defining the channel walls.

### 0.1. PHYSICAL PROBLEM MODEL

The considered domain is rectangular because being formed by the channel image in the plane defined by the streamlines ($\psi = cst$) and the potential lines ($\xi = cst$) (see Figures 1a and b).

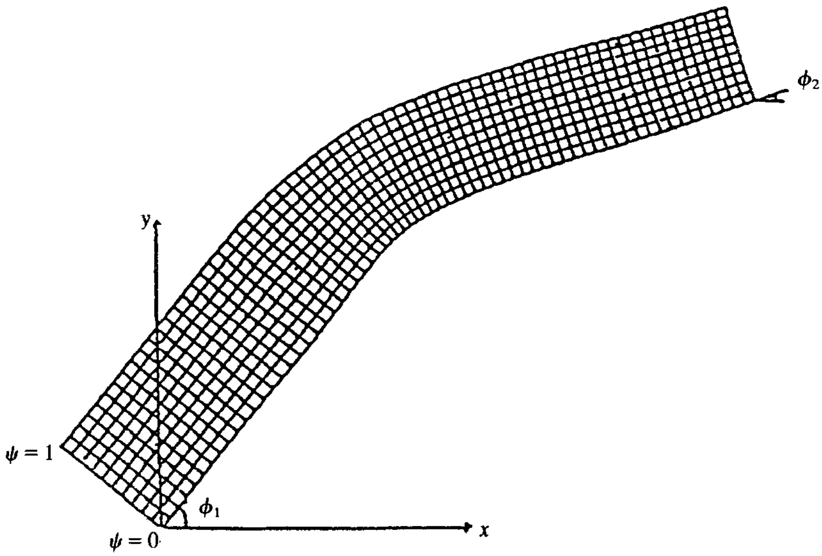


Fig. 1a. Physical domain.

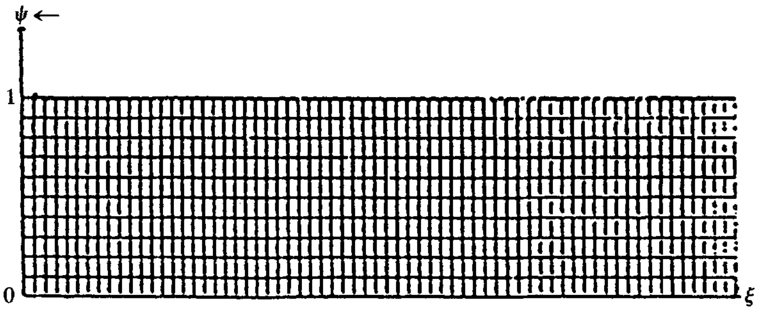


Fig. 1b. Computational domain.

The equations governing the fluid flow in this rectangular domain ($\mathscr{R}$) are as follows:

$$
\begin{aligned}
&-\frac{\partial^2 V}{\partial \psi^2} - \frac{1-M^2}{\rho^2}\frac{\partial^2 V}{\partial \xi^2} + \frac{1+M^2}{V}\left(\frac{\partial V}{\partial \psi}\right)^2 + \frac{1+\gamma M^4}{\rho^2 V}\left(\frac{\partial V}{\partial \xi}\right)^2 = 0 \quad \text{in } \mathscr{R},\\
&M = V\sqrt{\frac{2}{\gamma+1}\frac{1}{1-\frac{\gamma-1}{\gamma+1}V^2}} \quad \text{in } \bar{\mathscr{R}},\\
&\rho = \left(1 - \frac{\gamma-1}{\gamma+1}V^2\right)^{1/(\gamma-1)} \quad \text{in } \bar{\mathscr{R}},\\
&BD\colon\ V_{|\partial\mathscr{R}} = g,
\end{aligned}
\tag{0.1}
$$

where $V$ is the dimensionless velocity, $\rho$ is the dimensionless density and $g$ is obtained from the Mach number distributions prescribed on the channel walls (for more details, see the appendix).

In addition, we have the following expression for the streamline curvatures

$$
\chi = \rho\frac{\partial V}{\partial \psi} \quad \text{in } \bar{\mathscr{R}}. \tag{0.2}
$$

Finally, the deviation that generates the channel walls as well as their coordinates are obtained by integrations of the following equations in $\bar{\mathscr{R}}$:

$$
\begin{aligned}
&\frac{\partial \phi}{\partial \xi} = \frac{\chi}{V}, \qquad \frac{\partial x}{\partial \xi} = \frac{\cos\phi}{V}, \qquad \frac{\partial y}{\partial \xi} = \frac{\sin\phi}{V},\\
&BC\colon \quad \phi(0,\psi) = \phi_1, \qquad x(0,\psi) = x_1(\psi), \qquad y(0,\psi) = y_1(\psi),
\end{aligned}
\tag{0.3}
$$

where $\phi_1$, $x_1(\psi)$ and $y_1(\psi)$ are physical data. More details about the equations are given in the appendix (see also [10]).

### 0.2. MATHEMATICAL SETTING AND GENERAL FRAMEWORK

We analyze the equation (0.1) by setting it in the more general framework which follows:

($\mathscr{P}$) Find $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$
\begin{aligned}
&Au + F(u, \nabla u) = T \quad \text{in } \Omega,\\
&u - g \in W_0^{1,p}(\Omega).
\end{aligned}
$$

Here $\Omega$ denotes a bounded open set of $\mathbf{R}^N$, $Au = -(\partial/\partial x_i)\, a_i(x, u, \nabla u)^\star$, an elliptic-hyperbolic operator of Leray–Lions type which maps $W^{1,p}(\Omega) \cap L^\infty(\Omega)$

$^\star$ In the sequel, we will often omit the sum signs.

into $W^{-1,p'}(\Omega)$, $(1/p'+1/p=1)$ and such that

$$\begin{aligned}&\text{For almost every } x\in\Omega, \forall u\in I\subset\mathbb{R}, \forall\xi\in\mathbb{R}^N\\ &a_i(x,u,\xi)\xi_i\geqslant \nu(u)|\xi|^p. \qquad (0.4)\\ &g \text{ is given in } W^{1,p}(\Omega)\cap L^\infty(\Omega) \text{ and } F \text{ is in } W^{-1,p'}(\Omega).\end{aligned}$$

Some similar problems in the case of the homogeneous boundary condition (i.e. $g=0$) and in the case where the operator is *uniformly* elliptic on *all real axes* for $u$ (i.e. $\nu(u)=cst=\nu_0>0$, $\forall u\in\mathbb{R}$) were studied by several authors (see [1, 5, 7] and the references therein).

More recently, one of the authors found a situation in [17–19] where the operator can degenerate (i.e. $\nu(u)>0$ for $u\neq 0$ and $\nu(0)\geqslant 0$). In this last case, he proved a maximum principle which will be used in the present approach. In addition, three new approaches are presented in this article.

One can observe that one of the main differences between the papers quoted above [1, 5, 7] and ours is that the operator $A$ is not *elliptic* on all *the real axes* but only on some intervals. This is why we assume only the following condition on $\nu$: there exists an interval $(0,u_0]$, $u_0>0$ in the domain of the function $\nu$ on which the function $\nu$ is continuous and strictly positive. For example,

$$\nu(u)=1-|u|\sqrt{\frac{1}{1-a^2u^2}},\quad a>0.$$

The condition is related to the values of the Mach number (less than 1) for the physical model case which, in this case, represents a subsonic flow.

The second novelty appears when we use a principle of comparison to show that if there exists a constant $\delta>0$ such that $g\geqslant\delta$, then there exists a solution of the problem $(\mathcal{P})$ satisfying also $u\geqslant\delta$.

Also observe that here the boundary condition is nonhomogeneous on $\partial\Omega$. This is another case not treated in the papers [16, 17, 19]. Note that the one *side condition* on $F$ (see assumption $H_3(i)$) is not preserved by translation on $u$.

Finally, the third novelty resides in the fact that the domain of the function $u\mapsto F(x,u,\xi)$ as well as the function $u\mapsto a_i(x,u,\xi)$ are not necessary $\mathbb{R}$, for almost every $x$ and for all $\xi\in\mathbb{R}^N$. In [1, 5, 7, 19], $F$ is defined everywhere on $\Omega\times\mathbb{R}\times\mathbb{R}^N$. But in our case, it is only defined on a subset of this domain. Specifically, in our model case:

$$\operatorname{Dom} F(x,\cdot,\xi)=(0,a)\cup(-a,0),$$

where

$$a=\sqrt{\frac{\gamma+1}{\gamma-1}}\quad\text{with } \gamma\approx 1.4.$$

We can also treat some functions of the following form:

$$F(x, u, \xi) = \frac{-1}{\log u} |\xi|^p + \frac{\sqrt{1-u^2}}{u} |\xi|^{p-1}.$$

In [1], Boccardo *et al.* treated only the case where the operator is uniformly elliptic, and where the second member (i.e.: $T$) is a smooth function in $L^s$. Here, we do not need the notion of sub and sur-solution. The assumptions for the operator $A$ in [5] and [7] are the same as those in [1] and also, the growth of the function $F$ is less than $p$. Moreover, in [7] these operators are very smooth.

*Remark* 1. In addition, we will get a uniform upper bound for all solutions of $(\mathcal{P})$ by the method developed in [16–19] by one of the authors. To our knowledge, it is the first time that such a quasilinear equation with quadratic growth representing a physical problem has been so thoroughly studied.

The presentation of our work will be as follows:

In Section 1, we present the assumptions on the operator $A$, the function $F$ and the right-hand term $T$, and we will show how the physical model case can be represented by the general framework presented previously.

In Section 2, we introduce a family of modified problems $(\mathcal{P}_\epsilon)$ whose solution $u_\epsilon$ stays in a bounded domain of $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, we show that it possesses a principle of comparison.

We also prove that the sequence $u_\epsilon$ converges to a function $u$ strongly in $W^{1,p}(\Omega)$ and weak-star in $L^\infty(\Omega)$.

In Sections 3 to 5, we deduce that this function $u$ is, in fact, a weak solution of the problem $(\mathcal{P})$.

We establish in Section 6 the Hölder continuity for the solutions of this problem $(\mathcal{P})$.

We complete this article with an appendix which explains briefly how the problem (0.1) is established from the physical considerations on the inverse problem. Several numerical approaches of this type of *inverse problem* have been studied for a subsonic flow (see [20, 21]) as well as for a transonic flow (see [9, 10]). In [11], we propose a numerical scheme based on a finite difference method with various boundary conditions corresponding to different physical problems.

## 1. Hypotheses

Let $\Omega$ be a smooth bounded open set of $\mathbf{R}^N$, $(N \geq 1)$ and $p \in (1, +\infty)$. We propose to solve the following problem $(\mathcal{P})$:

$$(\mathcal{P}) \quad \begin{aligned} &\text{Find } u \in W^{1,p}(\Omega) \cap L^\infty(\Omega), \\ &Au + F(u, \nabla u) = T \quad \text{in } \Omega, \\ &u - g \in W_0^{1,p}(\Omega) \end{aligned}$$

under the following assumptions on $A$, $F$, $T$ and $g$:

(H$_1$) $T \in W^{-1,r}(\Omega)$, $r \geq p'$, $r > N/(p-1)$ and $T \geq 0$ in the sense of $W^{-1,p'}(\Omega)$ i.e. for all $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$, $\langle T, \varphi \rangle \geq 0$ where, $1/p + 1/p' = 1$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

(H$_2$) There exists a number $u_1 \in \bar{\mathbb{R}}_+^*$, (i.e., $u_1$ can be infinite but $u_1 > 0$), such that:

(i) The maps $a_i$ are Caratheodory functions from $\Omega \times (0, u_1) \times \mathbb{R}^N$ into $\mathbb{R}$: i.e.,

$\forall \eta \in ]0, u_1[, \forall \xi \in \mathbb{R}^N$,

$x \mapsto a_i(x, \eta, \xi)$ is measurable from $\Omega$ into $\mathbb{R}$.

For almost every $x \in \Omega$

$(\eta, \xi) \mapsto a_i(x, \eta, \xi)$ is continuous from $(0, u_1) \times \mathbb{R}^N$ into $\mathbb{R}$.

(ii) (growth) For almost every $x \in \Omega$, for all $\eta \in (0, u_1)$ and for all $\xi \in \mathbb{R}^N$,

$|a_i(x, \eta, \xi)| \leq a(\eta)(|\xi|^{p-1} + a_0(x))$,

where $a\colon (0, u_1) \mapsto \mathbb{R}_+$ is increasing

and $a_0 \in L^{p'}(\Omega)$.

(iii) (restricted coercivity) there exists a continuous function $\nu$ on its domain, such that:

$$(0, u_1) \subset \text{Domain of } \nu, \forall (\alpha, \beta) \in (0, u_1)^2, \min_{\alpha \leq \eta \leq \beta} \nu(\eta) > 0.$$

For almost every $x \in \Omega, \forall \eta \in (0, u_1), \forall \xi \in \mathbb{R}^N$,

$$\sum_{i=1}^{N} a_i(x, \eta, \xi)\xi_i \geq \nu(\eta)|\xi|^p.$$

(iv) (restricted monotony) For almost every $x \in \Omega$, $\forall \eta \in (0, u_1), \forall \xi \in \mathbb{R}^N$ and $\forall \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$:

$$\sum_{i=1}^{N} [a_i(x, \eta, \xi) - a_i(x, \eta, \xi')][\xi_i - \xi'_i] > 0.$$

(H$_3$)

(i) There exists a number $u_2 \in \bar{\mathbb{R}}_+^*$ such that the map $F$ is a Caratheodory function from $\Omega \times (0, u_2) \times \mathbb{R}^N$ into $\mathbb{R}^+$.

(ii) $\forall \epsilon > 0$, $\forall M \in (0, u_2)$, there exists a constant $C_\epsilon(M) > 0$ such that $\forall \eta \in (\epsilon, M)$, for almost every $x \in \Omega$, $\forall \xi \in \mathbb{R}^N$,

$|F(x, \eta, \xi)| \leq c_\epsilon(M)(|\xi|^p + f_0(x))$,

where $f_0 \in L_+^1(\Omega)$.

(iii) $F(x, \eta, 0) = 0$, for almost every $x \in \Omega$ and $\forall \eta \in (0, u_2)$.

($H_4$) $g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and there exists a constant $\delta \in (0, \min(u_1, u_2))$ such that $g \geqslant \delta$ in the sense of traces on $\partial\Omega$. (See Remark 2 below.)

SEVERAL EXAMPLES

We prove first that the model problem (0.1) satisfies the hypotheses ($H_1$) to ($H_4$). We transform (0.1) so that we can write it in the abstract form of problem ($\mathscr{P}$).

Next, let us consider (0.1), using the variables $x_1$ and $x_2$ instead of $\xi$ and $\psi$, we can write Equation (0.1) in the following form

$$-\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \sum_{i=1}^{2} a_{i,j}(V) \frac{\partial V}{\partial x_i} + F(V, \nabla V) = 0, \tag{1.1}$$

where

$$(a_{i,j}) = \begin{pmatrix} f(V) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad f(V) = \frac{1 - M^2(V)}{\rho^2(V)}. \tag{1.2}$$

We note that the functions $M$ and $\rho$ are continuous on their domain. We establish that (see appendix)

$$f'(V) = -(\gamma + 1)\frac{M^4}{\rho^2 V} \tag{1.3}$$

(see (0.1) for the definition of $M$ and $\rho$). Using (0.1), (1.2) and (1.3), we get

$$F(V, \nabla V) = \frac{(1 + M^2)}{V}\left[\left(\frac{\partial V}{\partial x_2}\right)^2 + f(V)\left(\frac{\partial V}{\partial x_1}\right)^2\right].$$

Hence, $\forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$,

$$a_i(x, \eta, \xi) = \sum_{j=1}^{2} a_{ij}(\eta)\xi_j.$$

The expression of the function $M$ (Mach number) implies that

$$u_1 < \sqrt{\frac{\gamma + 1}{\gamma - 1}}$$

and, moreover,

$$\nu(\eta) = \min(1, f(\eta)),$$

where

$$f(\eta) = \frac{1 - M^2(\eta)}{\rho^2(\eta)}.$$

Therefore, we have to choose $u_1 = 1$ so that $\nu$ satisfies $(H_2)$(ii). We also deduce that

$$F(\eta, \xi) = \frac{1 + M^2(\eta)}{\eta}[\xi_2^2 + f(\eta)\xi_1^2]$$

satisfies the required hypotheses if we choose $u_2 = 1$.

*Remark* 2. The condition on the boundary data ($g \geqslant \delta$) corresponds to the fact that the prescribed Mach number distributions on the channel walls do not vanish in the case of curve channels. Hence, these distributions are bounded from below by a strictly positive constant.

The hypothesis on $\nu$: $\nu(\eta) > 0$ corresponds to the case of a subsonic flow (i.e., the Mach number is less than 1).

*Mathematical examples*

*1st example*:

$$\begin{aligned}
&-\operatorname{div}(\nu(u)|\nabla u|^{p-2}\nabla u) - \frac{|\nabla u|^p}{\log u} + \frac{\sqrt{1-u^2}}{u}|\nabla u| a_0^2(x) = f(x),\\
&u - g \in W_0^{1,p}(\Omega),\\
&\nu(u) = \frac{1-u}{u(1+|u|^p)}, \ a_0(x) \in L^\infty(\Omega), \ f \in L_+^{N/p+\epsilon}(\Omega).
\end{aligned}$$

We can check that for this system, we have

$$\begin{aligned}
&u_1 = u_2 = 1,\\
&a_i(x, \eta, \xi) = \nu(\eta)|\xi|^{p-2}\xi_i,\\
&F(x, \eta, \xi) = -\frac{|\xi|^p}{\log \eta} + \frac{\sqrt{1-\eta^2}}{\eta}|\xi| a_0^2(x),\\
&g \in W^{1,p}(\Omega) \cap L^\infty(\Omega), \ g \geqslant \delta, \ \delta < 1.
\end{aligned}$$

*2nd example*:

$$\begin{aligned}
&-\operatorname{div}\left(\frac{P(u)}{1 + a_0^2(x)|u|^p}|\nabla u|^{p-2}\nabla u\right) + e^u|\xi|^q + a_1^2(x)|\xi|^p = T,\\
&u - g \in W_0^{1,p}(\Omega),
\end{aligned}$$

where $P$ is a polynomial whose real zeros are positive and $P(u) > 0$ for $u > 0$ in a neighborhood of 0. For example,

$$\begin{aligned}
&P(u) = u^m(2-u)^s, \ u_1 = 2,\\
&P(u) = u^2 + u + 1, \ u_1 = +\infty.
\end{aligned}$$

Hence, if $P$ does not admit any real zero or 0 is the unique real zero of $P$, then $u_1 = +\infty$. Otherwise, we take: $u_1 = \min\{t \in \mathbf{R}^+, P(t) = 0\}$ and in this case

$$u_2 = +\infty, \quad 0 \leqslant q \leqslant p,$$
$$a_i(x, \eta, \xi) = \frac{P(\eta)}{1 + a_0^2(x)|\eta|^p} |\xi|^{p-2}\xi_i,$$
$$F(x, \eta, \xi) = e^{\eta}|\xi|^q + a_1^2(x)|\xi|^p,$$
$$T \in W^{-1,r}(\Omega), \qquad r \geqslant N/(p-1), \quad r \geqslant p' = \frac{p}{p-1}, \ T \geqslant 0.$$

To simplify, we can take $a_i \in L^\infty(\Omega)$, $i = 0, 1$.

*Remark* 3. We note that the operator

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u)$$

is not necessary defined, even if we restrict ourselves to the class of functions of $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. This is one of the novelties of our work in relation to other treated cases (see [16, 19]).

In the present case we do not assume any differentiability hypotheses on the operators. We can imagine that the functions $a_i(x, \ldots, \xi)$ and $F(x, \ldots, \xi)$ are discontinuous outside the considered interval in the previous assumptions. For example:

$$-\frac{\partial}{\partial x_i}\left(h(u)\frac{\partial u}{\partial x_j}\right) + g(u)|\nabla u|^2 = f(x), \quad u - g \in W_0^{1,p}(\Omega)$$

where

$$h(\eta) = \begin{cases} \eta^\alpha, & 0 \leqslant \eta \leqslant 1, \\ 0, & \eta \leqslant 0, \\ 1/3, & \eta > 1, \ \eta \text{ is a rational number}, \\ 0, & \eta \geqslant 1, \ \eta \text{ is not a rational number}, \end{cases}$$

$$g(\eta) = \begin{cases} \dfrac{1}{\eta}, & 0 \leqslant \eta \leqslant 2, \\ 0, & \eta < 0, \\ \chi_Q(\eta), & \eta \geqslant 2, \end{cases}$$

where $\chi_Q$ is the characteristic function of $Q$ ($Q$ is the set of rational numbers)

$$g \geqslant \delta, \quad \delta < 1, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

In brief, to solve the problem, one has to check that $g$ is 'small' (i.e., $0 < \delta \leqslant g \leqslant m$) and that in a neighborhood of $g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, the considered operators $A$ and $F$ are well defined (i.e., $Aw \in W^{-1,p'}(\Omega)$, $F(\cdot, w, \nabla w) \in L^1_{loc}(\Omega)$ for $w$ in a neighborhood of $g$).

To give a meaning to the problem $(\mathcal{P})$, we introduce the following definition:

DEFINITION 1. We say that $u \in W^{1,p}(\Omega)$ is a weak solution of $(\mathcal{P})$ if

(a) $0 < \operatorname{ess\,inf}_\Omega u \leq \operatorname{ess\,sup}_\Omega u < \min(u_1, u_2)$.

(b) For all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\langle Au, v\rangle + \int_\Omega F(x, u(x), \nabla u(x)) v(x)\,\mathrm{d}x = \langle T, v\rangle .$$

(c) $u - g \in W_0^{1,p}(\Omega)$.

*Notation.* $\alpha_N =$ measure of the unit ball of $\mathbb{R}^N$,

$$F(u, \nabla u)(x) = F(x, u(x), \nabla u(x)),$$

$$\gamma = \left(\int_0^{|\Omega|} \sigma^{(1/N-1)((p-1)r/(p-1)r-1)}\,\mathrm{d}\sigma\right)^{1-1/(p-1)r}$$

Then, the main result of the problem $(\mathcal{P})$ is the following:

THEOREM 1. *Let* $u_0 \in (0, \min(u_1, u_2))$ *and* $\nu_0 = \min_{\delta \leq \eta \leq u_0} \nu(\eta)$ *such that*:

$$m = \operatorname{ess\,sup}_\Omega g(x) + \frac{\gamma}{\nu_0 N \alpha_N^{1/N}} \|T\|_{W^{-1,r}(\Omega)}^{p'/p} \leq u_0. \tag{1.4}$$

*Under the hypotheses* $(\mathrm{H}_1)$ *to* $(\mathrm{H}_4)$, *there exists at least a weak solution of* $(\mathcal{P})$ *in the sense of Definition* 1. *Moreover, all the weak solutions of* $\mathcal{P}$ *satisfy*:

$$\delta \leq u(x) \leq m. \tag{1.5}$$

*Remark* 4. The condition (1.4) can be explained in the model case (0.1) by the fact that on the boundary, the velocity $g$ stays small, i.e., the Mach number distributions on the channel walls are less than 1. The relation (1.5) then explains the fact that the Mach number also stays less than 1 inside the domain (subsonic case). These results are compatible with the numerical results (see [9, 11]).

## 2. A Family of Modified Problems

As in [16, 17] and [19], we introduce a family of modified problems arising from the problem $(\mathcal{P})$ for diverse reasons:

- We do not know *a priori* if the operators are well defined outside of the intervals $(0, u_i)$, $i = 1, 2$.
- We have to work within the interval $(0, u_0)$.
- The numerical results obtained for the model case confirm that the estimate (1.5) has to be satisfied by the solution. Therefore, we choose a modified problem whose solutions verify the estimate (1.5).

We define the function $a_i'$ as follows:

For almost every $x \in \Omega$, $\forall \eta \in \mathbf{R}$, $\forall \xi \in \mathbf{R}^N$,

$$a_i'(x, \eta, \xi) = \begin{cases} a_i(x, \eta, \xi), & \delta \leqslant \eta \leqslant u_0, \\ a_i(x, \delta, \xi), & \eta \leqslant \delta, \\ a_i(x, u_0, \xi), & u_0 \leqslant \eta. \end{cases} \tag{2.1}$$

We will use $C_i$ to denote different constants depending only on $\delta$, $g$, $T$, $u_0$ and $\Omega$ in the rest of this article.

*Properties of $a_i'$:*

($P_1$) (growth of $a_i'$) From the hypotheses ($H_2$)(i) and (ii) on the function $a_i$, we get

$$|a_i'(x, \eta, \xi)| \leqslant C_1(|\xi|^{p-1} + a_0(x))$$

for almost every $x \in \Omega$, $\forall \eta \in \mathbf{R}$, $\forall \xi \in \mathbf{R}^N$.

($P_2$) (monotonicity) For almost every $x \in \Omega$, $\forall \eta \in \mathbf{R}$, $\forall \xi \in \mathbf{R}^N$ and $\forall \xi' \in \mathbf{R}^N$, $\xi \neq \xi'$

$$\sum_{i=1}^N [a_i'(x, \eta, \xi) - a_i'(x, \eta, \xi')][\xi_i - \xi_i'] > 0$$

(the property ($P_2$) is a direct consequence of ($H_2$)(iv).)

($P_3$) (coercivity) let $\nu_0 = \min_{\delta \leqslant \eta \leqslant u_0} \nu(\eta) > 0$ and from ($H_2$)(iii) then for almost every $x \in \Omega$, $\forall \eta \in \mathbf{R}$, $\forall \xi \in \mathbf{R}^N$,

$$\sum_{i=1}^N a_i'(x, \eta, \xi)\xi_i \geqslant \nu_0 |\xi|^p$$

(the property ($P_3$) is a direct consequence of ($H_2$)(iii)). The map $a_i'$ are some Caratheodory functions on $\Omega \times \mathbf{R} \times \mathbf{R}^N$. Then, we define for $u \in W^{1,p}(\Omega)$, the operator $A'$ by

$$A'u = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i'(x, u, \nabla u). \tag{2.2}$$

Let $\epsilon > 0$, we define the real continuous function $h_\epsilon$ as follows

$$h_\epsilon(t) = \begin{cases} 1, & t \geqslant \epsilon, \\ 0, & t \leqslant 0, \\ \text{affine for} & 0 \leqslant t \leqslant \epsilon. \end{cases} \tag{2.3}$$

Then, we define

$$F'(x, \eta, \xi) = \begin{cases} F(x, \eta, \xi), & \delta \leqslant \eta \leqslant u_0, \\ F(x, \delta, \xi), & \eta \leqslant \delta, \\ F(x, u_0, \xi), & u_0 \leqslant \eta. \end{cases} \tag{2.4}$$

and

$$F_\epsilon(x, \eta, \xi) = h_\epsilon(\eta - \delta)\frac{F'(x, \eta, \xi)}{1 + \epsilon F'(x, \eta, \xi)}. \tag{2.5}$$

*Properties of $F_\epsilon$:*

($P_4$) $F_\epsilon$ is a Caratheodory function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}^+$.

($P_5$) $F_\epsilon \leq \dfrac{1}{\epsilon}$.

Then, we define the family of modified problems $(\mathcal{P}_\epsilon)$ as follows

$(\mathcal{P}_\epsilon)$ Find $u_\epsilon \in W^{1,p}(\Omega)$,

$$A'u_\epsilon + F_\epsilon(x, u_\epsilon, \nabla u_\epsilon) = T,$$

$$u_\epsilon - g \in W_0^{1,p}(\Omega).$$

LEMMA 1. *Under the hypotheses* ($H_1$) *to* ($H_4$), *the problem* $(\mathcal{P}_\epsilon)$ *admits at least one solution.*

*Proof.* We note $w_\epsilon = u_\epsilon - g$. The problem $(\mathcal{P}_\epsilon)$ then is equivalent to the following problem:

$(\mathcal{P}'_\epsilon)$ Find $w_\epsilon \in W_0^{1,p}(\Omega)$,

$$A''w_\epsilon + F''_\epsilon(x, w_\epsilon, \nabla w_\epsilon) = T,$$

where

$$A''v = -\frac{\partial}{\partial x_i} a'_i(x, g(x) + v, \nabla g(x) + \nabla v),$$

and if we note

$$a''_i(x, \eta, \xi) = a'_i(x, g(x) + \eta, \nabla g(x) + \xi),$$

$a''_i$ verifies the equivalent properties ($P_1$) and ($P_2$) for the functions $a'_i$.

Moreover,

$$\frac{\sum_{i=1}^N a''_i(x, \eta, \xi)\xi_i}{|\xi| + |\xi|^p} \mapsto_{|\xi| \mapsto +\infty} +\infty,$$

for all $|\eta|$ in a bounded set,

and if

$$F''_\epsilon(x, \eta, \xi) = F_\epsilon(x, g(x) + \eta, \nabla g(x) + \xi)$$

then, $F''_\epsilon$ is a Caratheodory function and also $F''_\epsilon \leq 1/\epsilon$.

Then, by applying to $(\mathcal{P}'_\epsilon)$ the theorem ([8], p. 183) of J. L. Lions, we conclude that there exists at least one solution of $(\mathcal{P}'_\epsilon)$ and, therefore, there also exists $u_\epsilon$ solutions of $(\mathcal{P}_\epsilon)$.

LEMMA 2. (*Principle of comparison*) *Under the hypotheses* ($H_1$) *to* ($H_4$), *all solutions* $u_\epsilon$ *of* $(\mathcal{P}_\epsilon)$ *verify*:

(a) $\delta \leq u_\epsilon(x)$, *almost everywhere in* $\Omega$,

(b) $u_\epsilon(x) \leq$ *ess* $\sup_\Omega g + \dfrac{\gamma}{\nu_0 N \alpha_N^{1/N}} \|T\|_{W^{-1,r}(\Omega)}^{p'/p} \leq u_0$.

*Proof*. Let $v = (u_\epsilon - \delta)_-$, since $g \geq \delta$, then $v \in W_0^{1,p}(\Omega)$ and

$$\langle A' u_\epsilon, v\rangle + \int_\Omega F_\epsilon(x, u_\epsilon, \nabla u_\epsilon) v \,\mathrm{d}x = \langle T, v\rangle \tag{2.6}$$

but

$$\begin{aligned}\langle A' u_\epsilon, v\rangle &= \int_\Omega a_i'(x, u_\epsilon, \nabla u_\epsilon)\frac{\partial v}{\partial x_i}\,\mathrm{d}x \\ &= -\int_{u_\epsilon \leq \delta} a_i'(x, u_\epsilon, \nabla u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}\,\mathrm{d}x \end{aligned}\tag{2.7}$$

and

$$\langle T, v\rangle \geq 0 \quad \text{(because } T \geq 0 \text{ in the sense of } W^{-1,p'}(\Omega)), \tag{2.8}$$

$$\int_\Omega F_\epsilon(x, u_\epsilon, \nabla u_\epsilon) v\,\mathrm{d}x = \int_{u_\epsilon \leq \delta} (u_\epsilon - \delta)_- h_\epsilon(u_\epsilon - \delta) F'(x, u_\epsilon, \nabla u_\epsilon)\,\mathrm{d}x = 0 \tag{2.9}$$

from the definition of $h_\epsilon$.

Then, we deduce from (2.6) to (2.9) that:

$$-\int_{u_\epsilon \leq \delta} a_i'(x, u_\epsilon, \nabla u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}\,\mathrm{d}x \geq 0 \tag{2.10}$$

and from ($P_3$) on the coercivity of $a_i'$

$$a_i'(x, u_\epsilon, \nabla u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i} \geq \nu_0 |\nabla u_\epsilon|^p. \tag{2.11}$$

Thus, from (2.10) and (2.11) we get:

$$\nu_0 \int_{u_\epsilon \leq \delta} |\nabla u_\epsilon|^p\,\mathrm{d}x \leq 0 \quad \text{i.e.,} \quad \int_\Omega |\nabla (u_\epsilon - \delta)_-|^p\,\mathrm{d}x \leq 0. \tag{2.12}$$

Let:

$$(u_\epsilon - \delta)_- = 0, \quad \text{i.e., } u_\epsilon \geq \delta \text{ a.e.} \tag{2.13}$$

The second part of the proof is issued from a technique developed by one of the authors (see [16, 18]). For the reader's convenience, we give here a sketch of the proof. For more details on the relative rearrangement used here, we refer to [13, 15] and [19].

Let $\theta > 0$, $h > 0$ be two fixed positive real numbers for which we associate two Lipchitz functions:

$$H_\theta(\tau)=0, \quad \tau\leq 0,$$
$$H_\theta(\tau)=\tau, \quad 0\leq\tau\leq\theta,$$
$$H_\theta(\tau)=\theta, \quad \theta\leq\tau,$$

$$S_{\theta,h}(\tau)=0, \qquad \tau\leq\theta,$$
$$S_{\theta,h}(\tau) \text{ is affine for } \quad \theta\leq\tau\leq\theta+h,$$
$$S_{\theta,h}(\tau)=1, \qquad \theta+h\leq\tau.$$

We note that $\theta_1=\theta+ess\sup_\Omega g$ and we define $v_\epsilon=S_{\theta_1,h}(u_\epsilon)H_\theta(u_\epsilon-g)$. With these definitions, we then get the following lemma which can be readily proved (see [19] for more details). The parts (a) and (b) derive directly from the definitions of $H_\theta$ and $S_{\theta,h}$.

LEMMA 3. *Let* $v_\epsilon=S_{\theta_1,h}(u_\epsilon)H_\theta(u_\epsilon-g)$, $\theta_1=\theta+ess\sup_\Omega g$, *then*

(a) $v_\epsilon\geq 0$,

(b) $v_\epsilon\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$,

(c) $$\frac{\partial v_\epsilon}{\partial x_i}=H_\theta(u_\epsilon-g)S'_{\theta_1,h}(u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}=\begin{cases}\dfrac{\theta}{h}\dfrac{\partial u_\epsilon}{\partial x_i}, & \theta_1<u_\epsilon\leq\theta_1+h,\\ 0, & \textit{otherwise}.\end{cases}$$

*End of Lemma* 2 *proof*:

Since $u_\epsilon$ is solution of the problem $(\mathscr{P}_\epsilon)$, we have

$$\langle A'u_\epsilon, v_\epsilon\rangle+\int_\Omega F_\epsilon(x,u_\epsilon,\nabla u_\epsilon)v_\epsilon\,\mathrm{d}x=\langle T,v_\epsilon\rangle \tag{2.14}$$

and since $v_\epsilon\geq 0$ and $F_\epsilon\geq 0$ from ($P_4$), we deduce

$$\langle A'u_\epsilon, v_\epsilon\rangle\leq\langle T,v_\epsilon\rangle. \tag{2.15}$$

But since $T\in W^{-1,r}(\Omega)$, let us consider $f_i\in L^r(\Omega)$, $(i=1,\dots,N)$, such that

$$T=-\sum_{i=1}^N\frac{\partial f_i}{\partial x_i}\quad\text{and}\quad\|T\|_{W^{-1,r}(\Omega)}=\sum_{i=1}^N\|f_i\|_{L^r(\Omega)}. \tag{2.16}$$

Using the expression of $A'u_\epsilon$. Equation (2.16) and part (c) of Lemma 3, we deduce

$$\langle T,v_\epsilon\rangle=\int_\Omega f_i\frac{\partial v_\epsilon}{\partial x_i}\,\mathrm{d}x=\frac{\theta}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h}f_i\frac{\partial u_\epsilon}{\partial x_i}\,\mathrm{d}x, \tag{2.17}$$

$$\langle A'u_\epsilon, v_\epsilon\rangle=\frac{\theta}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h}a'_i(x,u_\epsilon,\nabla u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}\,\mathrm{d}x. \tag{2.18}$$

Thus, from (2.15), (2.17) and (2.18), we find

$$\frac{1}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h}a'_i(x,u_\epsilon,\nabla u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}\,\mathrm{d}x\leq\frac{1}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h}f_i\frac{\partial u_\epsilon}{\partial x_i}\,\mathrm{d}x. \tag{2.19}$$

From the coercivity of $a_i'$, we deduce that

$$\frac{1}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h} a_i'(x, u_\epsilon, \nabla u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}\,\mathrm{d}x \geq \frac{\nu_0}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h} |\nabla u_\epsilon|^p\,\mathrm{d}x. \tag{2.20}$$

Letting

$$f=\left(\sum_{i=1}^N f_i^2\right)^{p'/2}$$

and using the Hölder inequality

$$\frac{1}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h} f_i\frac{\partial u_\epsilon}{\partial x_i}\,\mathrm{d}x$$
$$\leq\left(\frac{1}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h} f\,\mathrm{d}x\right)^{1/p'}\left(\frac{1}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h} |\nabla u_\epsilon|^p\,\mathrm{d}x\right)^{1/p}. \tag{2.21}$$

By (2.19) to (2.21), we deduce

$$\nu_0\left(\frac{1}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h} |\nabla u_\epsilon|^p\,\mathrm{d}x\right)\geq\left(\frac{1}{h}\int_{\theta_1<u_\epsilon\leq\theta_1+h} f\,\mathrm{d}x\right). \tag{2.22}$$

Let $\bar{u}_\epsilon=(u_\epsilon - \mathit{ess}\sup_\Omega g)_+$, then when $h$ goes to 0, (2.22) implies

$$\nu_0\left(-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\bar{u}_\epsilon>\theta} |\nabla \bar{u}_\epsilon|^p\,\mathrm{d}x\right)\leq -\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\bar{u}_\epsilon>\theta} f\,\mathrm{d}x. \tag{2.23}$$

If we note $f_{*\bar{u}_\epsilon}$ the relative rearrangement of $f$ with respect to $\bar{u}_\epsilon$, we deduce (see [16, 19])

$$-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\bar{u}_\epsilon>\theta} f\,\mathrm{d}x = -\mu'(\theta)f_{*\bar{u}_\epsilon}(\mu(\theta)) \quad \text{for almost every } \theta, \tag{2.24}$$

where $\mu(\theta)=|\bar{u}_\epsilon>\theta|$ and $\mu'$ is the derivative of $\mu$.

Since $\bar{u}_\epsilon\in W_0^{1,p}(\Omega)$, we can show that (see [16, 19]):

$$\left(-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\bar{u}_\epsilon>\theta} |\nabla \bar{u}_\epsilon|^p\,\mathrm{d}x\right)^{1/p}(-\mu'(\theta))^{1/p'}\geq N\alpha_N^{1/N}(\mu(\theta))^{1-1/N}. \tag{2.25}$$

From (2.23) to (2.25), we get

$$1\leq\frac{1}{\nu_0 N\alpha_N^{1/N}}(\mu(\theta))^{1/N-1}f_{*\bar{u}_\epsilon}^{1/p}(\mu(\theta))(-\mu'(\theta)). \tag{2.26}$$

Therefore, from (2.26) we derive (see [16, 18, 19]) that for all $s\in[0, |\Omega|]$:

$$\bar{u}_{\epsilon *}(s)\leq\frac{1}{\nu_0 N\alpha_N^{1/N}}\int_s^{|\Omega|}\sigma^{1/N-1}f_{*\bar{u}_\epsilon}^{1/p}(\sigma)\,\mathrm{d}\sigma, \tag{2.27}$$

where $\bar{u}_{\epsilon *}$ is the decreasing rearrangement of $\bar{u}_\epsilon$.

We deduce from the properties of the rearrangement and from the Hölder inequality that for all $s\in[0, |\Omega|]$

$$u_{\epsilon *}(s) - ess\sup_{\Omega} g \leqslant \frac{\gamma}{N\alpha_N^{1/N}} \|T\|_{W^{-1,r}}^{p'/p},$$

or

$$\gamma = \left( \int_s^{|\Omega|} \sigma^{(1/N-1)((p-1)r/(p-1)r-1)} \, d\sigma \right)^{1-(1/(p-1)r)}$$

In particular:

$$u_\epsilon(x) \leqslant \|u_\epsilon\|_\infty \leqslant ess\sup_{\Omega} g + \frac{\gamma}{\nu_0 N\alpha_N^{1/N}} \|T\|_{W^{-1,r}}^{p'/p} \leqslant u_0. \tag{2.28}$$

LEMMA 4. *There exists a constant $C_5$ independent of $\epsilon$ such that*

$$\|\nabla u_\epsilon\|_{L^p} \leqslant C_5.$$

In other words, the sequence $u_\epsilon$ stays in a bounded set of $W^{1,p}(\Omega)$. Let us recall the following lemma which is proved in [19].

LEMMA 5. *Let $\mu_1 > 0$ and $\mu_2 > 0$. There then exists a function $\sigma \in C^1(\mathbb{R})$ which is a solution of:*

$$\mu_1 \sigma'(t) - \mu_2 |\sigma(t)| = 1, \quad t \in \mathbb{R},$$

$$\sigma(0) = 0, \quad \sigma \text{ is odd}.$$

*Proof of Lemma* 4. Consider $w_\epsilon = \sigma(u_\epsilon - g)$ with $\mu_1 = \nu_0$ and $\mu_2 = C_8(u_0)$ (see $(H_3)$(ii)) then $w_\epsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\begin{aligned} &\int_\Omega a_i'(x, u_\epsilon, \nabla u_\epsilon) \sigma'(u_\epsilon - g) \frac{\partial u_\epsilon}{\partial x_i} \, dx \\ &\quad = \int_\Omega a_i'(x, u_\epsilon, \nabla u_\epsilon) \sigma'(u_\epsilon - g) \frac{\partial g}{\partial x_i} \, dx + \\ &\qquad + \int_\Omega F_\epsilon(x, u_\epsilon, \nabla u_\epsilon) w_\epsilon \, dx + \langle T, w_\epsilon \rangle. \end{aligned} \tag{2.29}$$

Since $F_\epsilon(x, u_\epsilon, \nabla u_\epsilon) \leqslant F(x, u_\epsilon, \nabla u_\epsilon)$, from $(H_3)$(ii) and Lemma 3, we deduce

$$F_\epsilon(x, u_\epsilon, \nabla u_\epsilon) \leqslant C_8(u_0)(|\nabla u_\epsilon|^p + f_0(x)) \tag{2.30}$$

and from the coercivity of $a_i'$ and since $\sigma' > 0$, we get

$$\int_\Omega a_i'(x, u_\epsilon, \nabla u_\epsilon) \frac{\partial u_\epsilon}{\partial x_i} \sigma'(u_\epsilon - g) \, dx \geqslant \nu_0 \int_\Omega \sigma'(u_\epsilon - g) |\nabla u_\epsilon|^p \, dx. \tag{2.31}$$

On the other hand, since $\sigma'(u_\epsilon - g) \leqslant C_6$ and from the growth property of $a_i'$, using the Hölder inequality, we deduce

$$\int_\Omega a_i'(x, u_\epsilon, \nabla u_\epsilon) \sigma'(u_\epsilon - g) \frac{\partial g}{\partial x_i} \, dx \leqslant C_7 \|\nabla u_\epsilon\|_{L^p(\Omega)}^{p/p'} + C_8 \tag{2.32}$$

and from (2.30), we get

$$\begin{aligned}\int_\Omega & F_\epsilon(x, u_\epsilon, \nabla u_\epsilon) w_\epsilon \, dx \\ &\leqslant \int_\Omega C_8(u_0)|\sigma(u_\epsilon - g)|\,|\nabla u_\epsilon|^p \, dx + C_9 \int_\Omega f_0(x)|\sigma| \, dx \\ &\leqslant \int_\Omega C_8(u_0)|\sigma(u_\epsilon - g)|\,|\nabla u_\epsilon|^p \, dx + C_{10} \end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\langle T, w_\epsilon\rangle &\leqslant \|T\|_{W^{-1,p'}(\Omega)} \|w_\epsilon\|_{W_0^{1,p}(\Omega)} \\ &\leqslant C_{11}\|\nabla u_\epsilon\|_{L^p(\Omega)} + C_{12}. \end{aligned} \tag{2.34}$$

From (2.29) to (2.34), we get

$$\begin{aligned}\int_\Omega & [\nu_0\sigma'(u_\epsilon - g) - C_8(u_0)|\sigma(u_\epsilon - g)|]|\nabla u_\epsilon|^p \, dx \\ &\leqslant C_7\|\nabla u_\epsilon\|_{L^p(\Omega)}^{p/p'} + c_{11}\|\nabla u_\epsilon\|_{L^p(\Omega)} + C_{13} \end{aligned} \tag{2.35}$$

and by the definition of $\sigma$, (2.35) becomes

$$\int_\Omega |\nabla u_\epsilon|^p \, dx \leqslant C_7\|\nabla u_\epsilon\|_{L^p(\Omega)}^{p/p'} + C_{11}\|\nabla u_\epsilon\|_{L^p(\Omega)} + C_{13}.$$

Finally, by Young's inequality, we deduce $\|\nabla u_\epsilon\|_{L^p(\Omega)} \leqslant C_{14}$.

We now assume that when $\epsilon \mapsto 0$,

$$\begin{aligned} &u_\epsilon \mapsto u \text{ weakly in } W^{1,p}(\Omega), \\ &u_\epsilon \mapsto u \text{ weakly-star in } L^\infty(\Omega), \\ &u_\epsilon \mapsto u \text{ almost everywhere in } \Omega. \end{aligned} \tag{2.36}$$

LEMMA 6. *The function $u$ defined by the relation* (2.36) *satisfies*

(a) $\delta \leqslant u(x) \leqslant \text{ess sup}_\Omega\, g + \dfrac{\gamma}{\nu_0 N \alpha_N^{1/N}} \|T\|_{W^{-1,r}(\Omega)}^{p'/p}$

*for almost every $x \in \Omega$*

(b) $u - g \in W_0^{1,p}(\Omega)$.

*Proof*. This lemma follows directly from Lemma 2, the definition of $u$ (2.36) *and the continuity of the trace function*.

*Remark* 5. Lemmas 2 and 6 ensure that almost every in $\Omega$,

$$\begin{aligned} &a_i'(x, u(x), \nabla u(x)) = a_i(x, u(x), \nabla u(x)), \\ &a_i(x, u_\epsilon(x), \nabla u_\epsilon(x)) = a_i'(x, u_\epsilon(x), \nabla u_\epsilon(x)). \end{aligned}$$

## 3. A Result of the Strong-Convergence in $W^{1,p}(\Omega)$

The purpose of this section is to pass to the limit in $(\mathcal{P}_\epsilon)$ as $\epsilon \mapsto 0$. For convenience, we denote by $\tilde{A}(u, \nabla u)$ the vector of $\mathbb{R}^N$ whose components are $a_i(x, u(x), \nabla u(x))$ and we write

$$\begin{aligned} &\sum_{i=1}^{N} a_i(x, u(x), \nabla u(x)) \frac{\partial u}{\partial x_i} = \tilde{A}(u, \nabla u)\nabla u, \\ &F(x, u, \nabla u) = F(u, \nabla u). \end{aligned} \tag{3.1}$$

LEMMA 7. *The sequence $u_\epsilon$ converges strongly to $u$ in $W^{1,p}(\Omega)$ as $\epsilon \mapsto 0$.*

*Proof.* We use essentially the property $(S_+)$ introduced by F. E. Browder (see [2]). The idea of the proof is partially due to Boccardo–Murat–Puel [1]. Since the operator $\tilde{A}$ satisfies the property $(S_+)$, it suffices to show that

$$\limsup_{\epsilon \mapsto 0} \int_\Omega [\tilde{A}(u_\epsilon, \nabla u_\epsilon) - \tilde{A}(u, \nabla u)]\nabla(u_\epsilon - u)\, dx \leq 0. \tag{3.2}$$

Let $\mu_1 = \nu_0$ and $\mu_2 = C_\delta(u_0)$, and $\sigma$ be the function associated to $\mu_1$ and $\mu_2$, according to Lemma 5. Then, $\forall \epsilon > 0$, $\forall \eta > 0$

$$\left.\begin{aligned} v_{\epsilon,\eta} &= \sigma(u_\epsilon - u_\eta) \\ v_{\eta,\epsilon} &= \sigma(u_\eta - u_\epsilon) \end{aligned}\right\} \in W_0^{1,p}(\Omega).$$

By replacing $v$ by $v_{\epsilon,\eta}$ (resp. $v_{\eta,\epsilon}$) in $(\mathcal{P}_\epsilon)$ (resp.$(\mathcal{P}_\eta)$), we get:

$$\langle Au_\epsilon, v_{\epsilon,\eta}\rangle + (F_\epsilon(u_\epsilon, \nabla u_\epsilon), v_{\epsilon,\eta}) = \langle T, v_{\epsilon,\eta}\rangle, \tag{3.3}$$

$$\langle Au_\eta, v_{\eta,\epsilon}\rangle + (F_\eta(u_\eta, \nabla u_\eta), v_{\eta,\epsilon}) = \langle T, v_{\eta,\epsilon}\rangle, \tag{3.4}$$

i.e.,

$$\langle Au_\epsilon, \sigma(u_\epsilon - u_\eta)\rangle + (F_\epsilon(u_\epsilon, \nabla u_\epsilon), \sigma(u_\epsilon - u_\eta)) = \langle T, \sigma(u_\epsilon - u_\eta)\rangle, \tag{3.5}$$

$$\langle Au_\eta, \sigma(u_\eta - u_\epsilon)\rangle + (F_\eta(u_\eta, \nabla u_\eta), \sigma(u_\eta - u_\epsilon)) = \langle T, \sigma(u_\eta - u_\epsilon)\rangle. \tag{3.6}$$

Since $\sigma$ is an odd function, we have

$$\sigma(u_\epsilon - u_\eta) = -\sigma(u_\eta - u_\epsilon).$$

Using this fact, we get by summing up relations (3.5) and (3.6)

$$\langle Au_\epsilon - Au_\eta, \sigma(u_\epsilon - u_\eta)\rangle + (F_\epsilon(u_\epsilon, \nabla u_\epsilon) - F_\eta(u_\eta, \nabla u_\eta), \sigma(u_\epsilon - u_\eta)) = 0$$

and we find

$$\begin{aligned} &\mu_1 \int_\Omega [\tilde{A}(u_\epsilon, \nabla u_\epsilon) - \tilde{A}(u_\eta, \nabla u_\eta)]\nabla(u_\epsilon - u_\eta)\sigma'(u_\epsilon - u_\eta)\, dx \\ &\quad \leq \mu_1 \int_\Omega [|F_\epsilon(u_\epsilon, \nabla u_\epsilon)| + |F_\eta(u_\eta, \nabla u_\eta)|]|\sigma(u_\epsilon - u_\eta)|\, dx. \end{aligned} \tag{3.7}$$

Using the assumption ($H_4$) on the growth of $F$, we obtain that

$$\int_\Omega [|F_\epsilon(u_\epsilon, \nabla u_\epsilon)| + |F_\eta(u_\eta, \nabla u_\eta)|]|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x$$
$$\leqslant \mu_2 \int_\Omega [|\nabla u_\epsilon|^p + |\nabla u_\eta|^p]|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x + 2\mu_2 \int_\Omega f_0(x)|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x. \tag{3.8}$$

From assumption ($H_3$), using the fact that $\|u_\epsilon\|_\infty \leqslant M$ and $\|u_\eta\|_\infty \leqslant M$, we get

$$\tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon \geqslant \mu_1|\nabla u_\epsilon|^p, \tag{3.9}$$
$$\tilde{A}(u_\eta, \nabla u_\eta)\nabla u_\eta \geqslant \mu_1|\nabla u_\eta|^p. \tag{3.10}$$

From (3.8), (3.9) and (3.10) we deduce that

$$\int_\Omega [|F_\epsilon(u_\epsilon, \nabla u_\epsilon)| + |F_\eta(u_\eta, \nabla u_\eta)|]|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x$$
$$\leqslant \mu_2 \int_\Omega \tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x +$$
$$+ \mu_2 \int_\Omega \tilde{A}(u_\eta, \nabla u_\eta)\nabla u_\eta|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x + 2\mu_1\mu_2 \int_\Omega f_0(x)|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x. \tag{3.11}$$

We now let $\epsilon$ go to 0 and then $\eta$ go to 0. Since $u_\epsilon \mapsto u$ almost everywhere in $\Omega$ as $\epsilon \mapsto 0$ (and also $u_\eta \mapsto u$ almost everywhere in $\Omega$ as $\eta \mapsto 0$), we deduce by Lebesgue's dominated convergence theorem, that

$$\lim_{\epsilon \mapsto 0} \int_\Omega f_0(x)|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x = \int_\Omega f_0(x)|\sigma(u - u_\eta)|\,\mathrm{d}x$$

and (3.12)

$$\lim_{\eta \mapsto 0} \int_\Omega f_0(x)|\sigma(u - u_\eta)|\,\mathrm{d}x = 0.$$

From (3.7) and (3.11) we get

$$\mu_1 \int_\Omega [\tilde{A}(u_\epsilon, \nabla u_\epsilon) - \tilde{A}(u_\eta, \nabla u_\eta)]\nabla(u_\epsilon - u_\eta)\sigma'(u_\epsilon - u_\eta)\,\mathrm{d}x$$
$$\leqslant \mu_2 \int_\Omega \tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x +$$
$$+ \mu_2 \int_\Omega \tilde{A}(u_\eta, \nabla u_\eta)\nabla u_\eta|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x + 2\mu_1\mu_2 \int_\Omega f_0(x)|\sigma(u_\epsilon - u_\eta)|\,\mathrm{d}x. \tag{3.13}$$

Let us write:

$$\tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon = \tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla(u_\epsilon - u_\eta) + \tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\eta, \tag{3.14}$$

$$\tilde{A}(u_\eta, \nabla u_\eta)\nabla u_\eta = -\tilde{A}(u_\eta, \nabla u_\eta)\nabla(u_\epsilon - u_\eta) + \tilde{A}(u_\eta, \nabla u_\eta)\nabla u_\epsilon. \tag{3.15}$$

Hence, from (3.13), (3.14), (3.15) and the following relation

$$\mu_1\sigma'(u_\epsilon - u_\eta) - \mu_2|\sigma(u_\epsilon - u_\eta)| = 1$$

we obtain

$$\begin{aligned}
&\int_\Omega [\tilde{A}(u_\epsilon, \nabla u_\epsilon) - \tilde{A}(u_\eta, \nabla u_\eta)]\nabla(u_\epsilon - u_\eta)\sigma'(u_\epsilon - u_\eta)\,dx \\
&\quad\leqslant \mu_2 \int_\Omega \tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\eta|\sigma(u_\epsilon - u_\eta)|\,dx + \\
&\quad\quad + \mu_2 \int_\Omega \tilde{A}(u_\eta, \nabla u_\eta)\nabla u_\epsilon|\sigma(u_\epsilon - u_\eta)|\,dx + 2\mu_1\mu_2 \int_\Omega f_0(x)|\sigma(u_\epsilon - u_\eta)|\,dx,
\end{aligned} \tag{3.16}$$

since $\tilde{A}(u_\epsilon, \nabla u_\epsilon)$ is in a bounded set of $(L^{p'}(\Omega))^N$, we can subtract a subsequence still denoted by $\epsilon$ such that

$$\tilde{A}(u_\epsilon, \nabla u_\epsilon) \mapsto U \quad \text{weakly in } (L^{p'}(\Omega))^N \quad \text{as } \epsilon \mapsto 0.$$

For a fixed $\eta$, we take lim sup as $\epsilon \mapsto 0$ in (3.16). Using Lebesgue's dominated convergence theorem, we obtain:

$$\begin{aligned}
&\limsup_{\epsilon\mapsto 0} \int_\Omega \tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon\,dx - \int_\Omega \tilde{A}(u_\eta, \nabla u_\eta)\nabla u\,dx + \\
&\quad + \int_\Omega \tilde{A}(u_\eta, \nabla u_\eta)\nabla u_\eta\,dx \\
&\quad\leqslant \mu_2 \int_\Omega U\nabla u_\eta|\sigma(u - u_\eta)|\,dx + \mu_2 \int_\Omega \tilde{A}(u_\eta, \nabla u_\eta)\nabla u|\sigma(u - u_\eta)|\,dx + \\
&\quad\quad + 2\mu_1\mu_2 \int_\Omega f_0(x)|\sigma(u - u_\eta)|\,dx.
\end{aligned} \tag{3.17}$$

When $\eta \mapsto 0$ in (3.17), we find that

$$\limsup_{\epsilon\mapsto 0} \int_\Omega \tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon\,dx \leqslant \int_\Omega U\nabla u\,dx \tag{3.18}$$

we conclude that

$$\begin{aligned}
&\limsup_{\epsilon\mapsto 0} \int_\Omega [\tilde{A}(u_\epsilon, \nabla u_\epsilon) - \tilde{A}(u, \nabla u)]\nabla(u_\epsilon - u)\,dx \\
&\quad\leqslant \limsup_{\epsilon\mapsto 0} \int_\Omega \tilde{A}(u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon\,dx - \int_\Omega U\nabla u\,dx - \\
&\quad\quad - \int_\Omega \tilde{A}(u, \nabla u)\nabla u\,dx + \int_\Omega \tilde{A}(u, \nabla u)\nabla u\,dx.
\end{aligned} \tag{3.19}$$

From (3.18), we see that the second member of inequality (3.19) is less than or equal to zero. This proves the desired result (3.2).

## 4. Existence of Solution for the Problem $(\mathscr{P})$

We will now prove the existence of a solution for the problem $(\mathscr{P})$ by passing to the limit in $(\mathscr{P}_\epsilon)$. Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we can write

$$\langle Au_\epsilon, v\rangle = \int_\Omega [a_i(u_\epsilon, \nabla u_\epsilon) - a_i(u, \nabla u)] \frac{\partial v}{\partial x_i} \, dx + \\ + \int_\Omega a_i(u, \nabla u) \frac{\partial v}{\partial x_i} \, dx.$$

By Vitali's theorem, we deduce by using Lemma 7 that

$$a_i(u_\epsilon, \nabla u_\epsilon) - a_i(u, \nabla u) \mapsto 0 \quad \text{in } L^{p'}(\Omega) \text{ as } \epsilon \mapsto 0.$$

Hence,

$$\lim_{\epsilon \mapsto 0} \langle Au_\epsilon, v\rangle = \int_\Omega a_i(u, \nabla u) \frac{\partial v}{\partial x_i} \, dx. \\ = \langle Au, v\rangle$$

In addition, we deduce from Lemmas 6 and 7 and assumption $(H_3)$(iii), that for almost every $x \in \Omega$:

$$\lim_{\epsilon \mapsto 0} h_\epsilon(u_\epsilon - \delta) \frac{F'(x, u_\epsilon, \nabla u_\epsilon)}{1 + \epsilon F'(x, u_\epsilon, \nabla u_\epsilon)} = F'(x, u(x), \nabla u(x)) \\ = F(x, u(x), \nabla u(x)).$$

Using Vitali's theorem, we have

$$F_\epsilon(u_\epsilon, \nabla u_\epsilon) \mapsto F(u, \nabla u) \quad \text{in } L^1(\Omega)\text{-strong as } \epsilon \mapsto 0.$$

Hence,

$$\int_\Omega F_\epsilon(u_\epsilon, \nabla u_\epsilon) v \, dx \\ = \int_\Omega [F(u_\epsilon, \nabla u_\epsilon) - F(u, \nabla u)] v \, dx + \\ + \int_\Omega F(u, \nabla u) v \, dx \mapsto_{\epsilon \mapsto 0} \int_\Omega F(u, \nabla u) v \, dx$$

and we conclude from these convergence results that

$$\forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

$$\langle Au, v\rangle + \int_\Omega F(u, \nabla u) v \, dx = \langle T, v\rangle.$$

Lemma 6 and this last relation ensure that $u$ is a weak solution of $(\mathscr{P})$ in the sense of Definition 1.

## 5. A Priori Estimate for the Weak Solution of $(\mathscr{P})$. (End of Theorem 1 proof)

LEMMA 8. *All the weak solutions of* $(\mathscr{P})$ *satisfy*

$$u(x) \leqslant ess \sup_{\Omega} g + \frac{\gamma}{\bar{\nu}_0 N \alpha_N^{1/N}} \|T\|_{W^{-1,r}(\Omega)}^{p'/p} \tag{5.1}$$

*where*

$$\bar{\nu}_0 = \min_{\{ess \inf_\Omega u \leqslant \eta \leqslant ess \sup_\Omega u\}} \nu(\eta) > 0.$$

*Proof.* Since $u$ is a weak solution, by definition

$$0 < ess \inf_{\Omega} u \leqslant ess \sup_{\Omega} u < \min(u_1, u_2).$$

This relation implies that for almost every $x \in \Omega$,

$$\sum_{i=1}^{N} a_i(x, u(x), \nabla u(x)) \frac{\partial u}{\partial x_i} \geqslant \bar{\nu}_0 |\nabla u(x)|^p, \tag{5.2}$$

where $\bar{\nu}_0$ is given by (2.1) and $\bar{\nu}_0 > 0$ (see $(H_2)$(ii)).

We deduce also that for a weak solution of $\mathscr{P}$, we have the following growth property

$$\begin{gathered} \text{for almost every } x \in \Omega \\ 0 \leqslant F(x, u(x), \nabla u(x)) \leqslant C_{is}(|\nabla u(x)|^p + f_0(x)), \end{gathered} \tag{5.3}$$

where $C_{is}$ is a constant dependent on $ess \inf_\Omega u$ and $ess \sup_\Omega u$.

From relations (5.2) and (5.3), the proof of Lemma 8 is exactly the same as for $u_\epsilon$ if we replace $\nu_0$ by $\bar{\nu}_0$.

This completes the proof of Theorem 1.

## 6. Regularity of Solution of $(\mathscr{P})$

THEOREM 6.1. *Assume* $(H_1)$ *to* $(H_4)$ *and in addition, that* $a_0 \in L^r(\Omega)$, $f_0 \in L^{r/p'}(\Omega)$. *Then every weak solution of* $(\mathscr{P})$ *satisfies the* $\alpha$*-Hölder condition in* $\Omega$.

*Proof.* Let us denote

$$\alpha = ess \inf_{\Omega} u, \qquad \beta = ess \sup_{\Omega} u.$$

Since $u$ is a weak solution of $(\mathscr{P})$, then

$$0<\alpha \leqslant \beta < \min(u_1, u_2).$$

From this relation, we deduce the following growth properties for $A$ and $F$:

$(\mathrm{Q}_1)$ $\sum_{i=1}^N a_i(x, u(x), \nabla u(x))\partial u/\partial x_i \geqslant \bar{\nu}_0|\nabla u(x)|^p$,
$\bar{\nu}_0 = \min_{\alpha\leqslant\eta\leqslant\beta}\nu(\eta) > 0$.

$(\mathrm{Q}_2)$ $|a_i(x, u(x), \nabla u(x))| \leqslant C(\beta)(|\nabla u(x)|^p + a_0(x))$,
$C(\beta)$ is a constant dependent only on $\beta$.

$(\mathrm{Q}_3)$ $u(x) > 0, \qquad F(x, u(x), \nabla u(x)) \geqslant 0$.

$(\mathrm{Q}_4)$ $F(x, u(x), \nabla u(x)) \leqslant C_\alpha(\beta)(|\nabla u(x)|^p + f_0(x))$,
$T \in W^{-1,r}(\Omega)$, $r > N/(p-1)$.

The relations $(\mathrm{Q}_1)$ to $(\mathrm{Q}_4)$ imply that the assumptions of the theorem proved in [16, 19] for *local* Hölder continuity are satisfied. We can conclude that $u$ is Hölder continuous inside of $\Omega$.

*Remark*: In [16, 19] the boundary condition is homogeneous, but the proof does not change in the general case, since we show a local result.

In addition, the condition $(\mathrm{Q}_3)$ ensures that the solution of $(\mathscr{P})$ is also a solution of a variational inequality with the constraint set

$$K = \{v \in W_0^{1,p}(\Omega) + g,\ v \geqslant 0\}.$$

### 6.1. CASE WHERE $T=0$

The model case given in (0.1) corresponds to the case $T=0$. In this particular case, many precise results are given by many authors (see [7, 12, 22]). In the present case, we will use the results of N. S. Trudinger [22]. For the reader's convenience, we recall these results.

### 6.2. TRUDINGER'S RESULTS

Let $u \in W^{1,p}(\Omega)$ be the solution of

$(\mathrm{E}_1)$ $\operatorname{div}(A(x, u(x), \nabla u(x))) + B(x, u(x), \nabla u(x)) = 0$.

We assume that $A$ and $B$ satisfy

$(\mathrm{I}_0)$ $|A(x, u, \xi)| \leqslant C_{15}(|\xi|^{p-1} + a_0(x))$,
$\xi A(x, u, \xi) \geqslant C_{16}|\xi|^p$,
$|B(x, u, \xi)| \leqslant C_{17}(|\xi|^p + a_1(x))$,
$C_{15}, C_{16}, C_{17} > 0$ and $a_i \in L^\infty(\Omega)$, $\|a_i\|_\infty \leqslant \mu$.

Then, $u$ is a solution of ($E_1$) if

$$(E_2)\quad \forall \phi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),$$
$$\int_\Omega (\nabla\phi A(x, u(x), \nabla u(x)) - \phi B(x, u(x), \nabla u(x)))\,dx = 0.$$

6.3. HARNACK'S INEQUALITIES

Let $u$ be a solution of ($E_2$) such that $0 \leqslant u(x) \leqslant M$. Then for all cube $K_\rho$ of edge $\rho$ in $\Omega$:

$$(I_1)\quad \max_{K_\rho} u(x) \leqslant C \min_{K_\rho} u(x),$$

where $C$ depends only on $N$, $p$, $\mu$, $M$, $C_{15}$, $C_{16}$ and $C_{17}$ and

$$(I_2)\quad \rho^{-N/\gamma}\|u\|_{L^\gamma(K_{2\rho})} \leqslant C \min_{K_\rho} u(x).$$

For all $\gamma$ such that

$$\gamma < \frac{N(p-1)}{N-p}, \quad p<N,$$
$$\gamma \leqslant \infty, \quad p>N,$$

$$(I_3)\quad K_{2\rho}\subset\Omega,$$
$$\max_{K_\rho} u(x) \leqslant C\rho^{-N/q}\|u\|_{L^q(K_{2\rho})}, \quad \text{for all } q>p-1$$

$$(I_4)\quad \rho^{N/(1-p)}\|u\|_{L^{p-1}(K_{2\rho})} \leqslant C(\min_{K_\rho} u + m(\rho)),$$
where $m(\rho) = \mu\rho + (\mu\rho)^{p/(p-1)}, \quad C = C(p, N, a_i, M),$
$$\sup_{K_\rho} u \leqslant C\left(\frac{\rho}{\rho_0}\right)^\delta (M + m(\rho)),$$
for all $\rho \leqslant \rho_0$, where $\delta > 0$ and $C = C(p, N, a_i, C_{15}, C_{16}, C_{17}, M)$.

THEOREM 6.2 (*Trudinger*). *We assume that* $\partial\Omega$ *is of class* $C^1$ *and* $g$ *is continuous on* $\Omega$. *We also assume* ($I_0$). *Then, any solution of*

$$\operatorname{div}(A(x, u(x), \nabla u(x)) + B(x, u(x), \nabla u(x)) = 0, \quad \text{in } \Omega,$$
$$u = g, \quad \text{on } \partial\Omega,$$

*is continuous in* $\Omega$. *Moreover, if* $g$ *satisfies the* $\alpha$-*Hölder condition in* $\bar\Omega$, *then the solution* $u$ *satisfies also the* $\alpha$-*Hölder condition in* $\bar\Omega$.

COROLLARY. *We assume that* $T = 0$, $g$ *is* $\alpha$*-Hölder continuous in* $\bar{\Omega}$, $a_0 \in L^\infty(\Omega)$, $f_0 \in L^\infty(\Omega)$. *Then, every weak solution* $u$ *of* $\mathcal{P}$ *is* $\alpha$*-Hölder continuous in* $\bar{\Omega}$ *and satisfies Harnack's inequalities* ($\mathrm{I}_1$) *to* ($\mathrm{I}_4$).

*Proof*. From the inequality

$$0 < ess \min_{\Omega} u(x) \leq ess \sup_{\Omega} u(x) < \min(u_1, u_2)$$

we deduce that the assumptions ($I_0$) on the operators are well satisfied.

## 7. Appendix

### 7.1. AERODYNAMIC HYPOTHESES

- Plane flow (two-dimensional),
- Isentropic flow (nonviscous flow, no shock),
- Perfect fluid,
- Uniform upstream data (and also the downstream conditions).

We deduce from these hypotheses that:

(a) The dynamic equation is reduced to: curl $\mathbf{V} = \mathbf{0}$.
(b) The density $\rho$ and the pressure $p$ are linked by the following equation (isentropic law) $p/\rho^\gamma = cst$.

### 7.2. GENERALITIES OF THERMODYNAMICS

We recall here some differential relations that we will use later. From the first principle of thermodynamics, we write

$$\mathrm{d}H_i = 0 = \mathrm{d}h + V\,\mathrm{d}V = \frac{\mathrm{d}p}{\rho} + V\,\mathrm{d}V.$$

Since the enthalpy is constant, then for an isentropic flow, we have

$$\frac{\mathrm{d}p}{\rho} = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_S \frac{\mathrm{d}\rho}{\rho} = a^2 \frac{\mathrm{d}\rho}{\rho},$$

which implies

$$a^2 \frac{\mathrm{d}\rho}{\rho} + V\,\mathrm{d}V = 0,$$

therefore

$$\mathrm{d}(\rho V) = (1 - M^2)\rho\,\mathrm{d}V, \quad \mathrm{d}\rho = -\rho M^2 \frac{\mathrm{d}V}{V}. \tag{7.1}$$

Similarly, we establish the following relations from the same thermodynamic considerations (see [3, 4, 6, 10])

$$\mathrm{d}a = -\frac{\gamma-1}{2} M \,\mathrm{d}V, \qquad \mathrm{d}M = M\left(1+\frac{\gamma-1}{2} M^2\right)\frac{\mathrm{d}V}{V} \tag{7.2}$$

*Relations of St. Venant*

Let $p_i$, $T_i$, $\rho_i$, etc. be the characteristic of the steady state of the flow (generator state). If we consider an isentropic fluid, we also have from the first principle of thermodynamics, the following equations of St. Venant (see [4, 10]):

$$1+\frac{\gamma-1}{2} M^2 = \frac{T_i}{T} = \left(\frac{p_i}{p}\right)^{(\gamma-1/\gamma)} = \left(\frac{\rho_i}{\rho}\right)^{\gamma-1} = \left(\frac{a_i}{a}\right)^2 \tag{7.3}$$

and we deduce from these relations that

$$1+\frac{\gamma-1}{2} M^2 = \frac{1}{1-\dfrac{\gamma-1}{\gamma+1}\tilde{V}^2},$$

$$M = \tilde{V}\sqrt{\frac{2}{\gamma+1}\frac{1}{1-\dfrac{\gamma-1}{\gamma+1}\tilde{V}^2}},$$

$$\tilde{\rho} = \left(1-\frac{\gamma-1}{\gamma+1}\tilde{V}^2\right)^{(1/\gamma-1)}, \tag{7.4}$$

where $\tilde{V} = V/a_c$, $\tilde{\rho} = \rho/\rho_i$ and the characteristics of the critical state of fluid are given by

$$\frac{T_c}{T_i} = \left(\frac{p_c}{p_i}\right)^{(\gamma-1/\gamma)} = \left(\frac{\rho_c}{\rho_i}\right)^{\gamma-1} = \left(\frac{a_c}{a_i}\right)^2 = \frac{2}{\gamma+1}, \tag{7.5}$$

where $c$ denotes the critical state of fluid.

### 7.3. VELOCITY EQUATION IN FRENET'S COORDINATES

By introducing the continuity equation, we get the equations which represent the flow of a 2-D perfect fluid:

$$\operatorname{curl} \mathbf{V} = \mathbf{0}, \qquad \operatorname{div} \rho\mathbf{V} = 0. \tag{7.6}$$

Let $(\mathbf{t}, \mathbf{n})$ be Frenet's vectors associated to a streamline of fluid, $s$ and $n$ be the coordinates along the streamlines and the orthogonal lines of the streamlines ($\mathbf{V} = V\mathbf{t}$) and $\mathbf{B}$ be the normal vector of the physical plane.

From (7.6), (7.1) and (7.2) we get (see [3, 9, 10])

$$\mathbf{grad}\,\mathbf{V} = -\left(\frac{V}{1-M^2}\operatorname{div}\mathbf{t}\right)\mathbf{t} + V(\mathbf{B}\cdot\operatorname{curl}\mathbf{t})\mathbf{n}. \tag{7.7}$$

On the other hand, we also have (see [3, 10])

$$\mathbf{B}\cdot\operatorname{curl}\mathbf{t} = \chi = \frac{\partial\phi}{\partial s}, \qquad \operatorname{div}\mathbf{t} = \chi' = \frac{\partial\phi}{\partial n}, \tag{7.8}$$

where $\chi$ (resp. $\chi'$) is the curvature of the streamlines (resp. of the orthogonal lines of the streamlines).

Therefore, we deduce the following relations between angles and velocity.

$$\chi = \frac{\partial\phi}{\partial s} = \frac{1}{V}\frac{\partial V}{\partial n},$$

$$\chi' = \frac{\partial\phi}{\partial n} = -\frac{1-M^2}{V}\frac{\partial V}{\partial s}. \tag{7.9}$$

It is clear that curl $\mathbf{grad}\,\phi = \mathbf{0}$. Taking then the scalar product of this vector with $\mathbf{B}$, we get from (7.1), (7.2) and (7.9) the velocity equation in the Frenet coordinates

$$\frac{\partial^2 V}{\partial n^2} + (1-M^2)\frac{\partial^2 V}{\partial s^2} - \frac{2}{V}\left(\frac{\partial V}{\partial n}\right)^2 - \frac{2-M^2+\gamma M^4}{V}\left(\frac{\partial V}{\partial s}\right)^2 = 0. \tag{7.10}$$

*Computational domain.* Introduction of the parameters $\xi$ and $\psi$.

From the continuity equation div $\rho\mathbf{V} = 0$, we deduce that there exists a stream function $\psi$ such that $\mathrm{d}\psi = \rho V\,\mathrm{d}n$ and from the dynamic equation curl $\mathbf{V} = \mathbf{0}$, we deduce that there exists a potential function $\xi$ such that $\mathrm{d}\xi = V\,\mathrm{d}s$.

The potential lines and the streamlines define a mesh of orthogonal lines in the physical domain.

We use the following change of variables $(\xi, \psi) \mapsto (\tilde{\xi}, \tilde{\psi})$:

$$\mathrm{d}\tilde{\xi} = \frac{\tilde{V}}{D}\,\mathrm{d}s, \qquad \mathrm{d}\tilde{\psi} = \frac{\tilde{\rho}\tilde{V}}{D}\,\mathrm{d}n, \tag{7.11}$$

where $D$ is the channel flow. Noting by the sub-index 1 the upstream conditions and $h_1$ the upstream channel spacing, we then have $D = h_1\tilde{\rho}_1\tilde{V}_1\cos\phi_1$.

Therefore, the computational domain is rectangular because being defined by the transformation of the physical domain to the plane defined by the streamlines and the potential lines of fluid (see Figure 1).

From (7.11) and Equations (7.9) and (7.10), we get the following equations.

*Velocity equation*:

$$-\frac{\partial^2\tilde{V}}{\partial\tilde{\psi}^2} - \frac{1-M^2}{\tilde{\rho}^2}\frac{\partial^2\tilde{V}}{\partial\tilde{\xi}^2} + \frac{1+M^2}{\tilde{V}}\left(\frac{\partial\tilde{V}}{\partial\tilde{\psi}}\right)^2 + \frac{1+\gamma M^4}{\tilde{\rho}^2\tilde{V}}\left(\frac{\partial\tilde{V}}{\partial\tilde{\xi}}\right)^2 = 0. \tag{7.12}$$

*Relations between angles and velocity*:

$$\chi = \tilde{\rho}\frac{\partial \tilde{V}}{\partial \tilde{\psi}}, \qquad \chi' = -(1-M^2)\frac{\partial \tilde{V}}{\partial \tilde{\xi}},$$

$$\frac{\partial \phi}{\partial \tilde{\xi}} = \frac{\chi}{\tilde{V}}, \qquad \frac{\partial \phi}{\partial \tilde{\psi}} = \frac{\chi'}{\tilde{\rho}\tilde{V}}. \tag{7.13}$$

Similarly, from the differential relations between the Cartesian coordinates of the physical plane and the Frenet coordinates

$$\mathrm{d}x = \cos\phi\,\mathrm{d}s + \sin\phi\,\mathrm{d}n, \qquad \mathrm{d}y = \sin\phi\,\mathrm{d}s - \cos\phi\,\mathrm{d}n$$

we get in the computational domain

$$\frac{\partial x}{\partial \tilde{\xi}} = \frac{\cos\phi}{\tilde{V}}, \qquad \frac{\partial x}{\partial \tilde{\psi}} = \frac{\sin\phi}{\tilde{\rho}\tilde{V}},$$

$$\frac{\partial y}{\partial \tilde{\xi}} = \frac{\sin\phi}{\tilde{V}}, \qquad \frac{\partial y}{\partial \tilde{\psi}} = -\frac{\cos\phi}{\tilde{\rho}\tilde{V}}. \tag{7.14}$$

Let us establish now the relation (1.3). We have

$$f(\tilde{V}) = \frac{1-M^2(\tilde{V})}{\tilde{\rho}^2(\tilde{V})}$$

and from (7.1), (7.2) and (7.4), one can readily check that

$$f'(\tilde{V}) = -(\gamma+1)\frac{M^4}{\tilde{\rho}^2\tilde{V}}.$$

### 7.4. BOUNDARY CONDITIONS AND APPLICATION

The physical data for the inverse problem are the inlet and outlet angles and the Mach number distribution on each channel wall. These distributions are the same at the upstream and downstream points of the channel walls because we assume that the flow's upstream and downstream conditions are uniform.

We compute first the potential difference $\Delta\xi'_{\text{low}}$ on the lower channel wall on which the length is equal to 1

$$\Delta\xi'_{\text{low}} = \frac{1}{D}\int_0^1 \tilde{V}_{\text{low}}(\tau)\,\mathrm{d}\tau.$$

and we multiply it by a constant $L_{\text{low}}$ to get

$$\Delta\tilde{\xi}_{\text{low}} = \Delta\xi'_{\text{low}}L_{\text{low}}.$$

The potential difference on the upper channel wall has to be equal to one on the lower channel wall (see Figure 1) $\Delta\tilde{\xi}_{\text{upp}} = \Delta\tilde{\xi}_{\text{low}}$.

In the same way, we have

$$\Delta\xi'_{\mathrm{upp}} = \frac{1}{D}\int_0^1 \tilde{V}_{\mathrm{upp}}(\tau)\,\mathrm{d}\tau$$

which gives us a second constant $L_{\mathrm{upp}}$

$$L_{\mathrm{upp}} = \frac{\Delta\tilde{\xi}_{\mathrm{upp}}}{\Delta\xi'_{\mathrm{upp}}}.$$

The constants $L_{\mathrm{low}}$ and $L_{\mathrm{upp}}$ are respectively the length of the lower and upper channel wall over the upstream channel spacing $h_1$. Therefore, from the same Mach number distribution $M = F(\tau)$, $\tau \in [0, 1]$, there are as many velocity distributions $V = g(\tilde{\xi})$ on the channel walls (boundary condition on the velocity) as the pair $(L_{\mathrm{inf}}, L_{\mathrm{sup}})$ because multiplying by $L_{\mathrm{low}}$ and $L_{\mathrm{upp}}$ is the same as dilating the $\tilde{\xi}$-axis. There exists, moreover a particular pair of values $(L_{\mathrm{inf}}, L_{\mathrm{sup}})$ which gives exactly the total deviation $\Delta\phi$ so desired. Hence, we are sure to get the deviation. However, the upstream channel spacing is totally defined and it can be modified only by changing the initial Mach number distributions.

If we prefer to have the upstream channel spacing as data, we will fix it and one of the inlet and outlet angles. In this case, the deviation will be obtained from the computation as in the previous case for the upstream channel spacing.

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