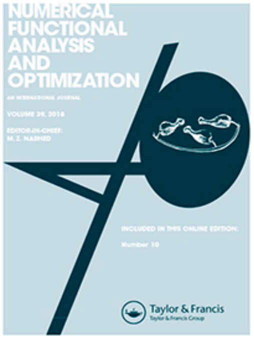

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Convergence of approximate attractors for a fully discrete system for Reaction-Diffusion Equations

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ABSTRACT

The reaction-diffusion equations are approximated by a fully discrete system: a Legendre-Galerkin approximation for the space variables and a semi-implicit scheme for the time integration. The stability and the convergence of the fully discrete system are established. It is also shown that, under a restriction on the space dimension and the growth rate of the nonlinear term, the approximate attractors of the discrete finite dimensional dynamical systems converge to the attractor of the original infinite dimensional dynamical systems. An error estimate of optimal order is derived as well without any further regularity assumption.

1 Introduction

It is well known that the permanent regime of a large class of dissipative dynamical systems can be represented by a finite number of determining modes, e.g. by the universal (global) attractors or the inertial manifolds (if they exist). We refer to Temam [13] for an extensive review on this subject. In particular, for the reaction-diffusion equations, the existence of a maximal attractor $\mathcal{A}$ and the estimates of its Hausdorff and fractal dimensions have been given in Babin-Vishik [1] and Marion [8].

Since the attractors or the inertial manifolds play an important role in the understanding of the long time behavior of the solutions of certain dynamical systems, it is then worthwhile to consider approximations for the attractors or the inertial manifolds. The problem has been approched in various directions, among them are: approximations to the attractor of the Navier-Stokes equations by a simpler infinite dimensional dynamical system (see for instance Brefort et al. [2]) or by a semi-discrete scheme based on the eigenvectors of the Stokes operator (see Sec. 4.7 of [13]); approximations to the inertial manifolds (see among others Foias et al. [5], Marion [9]). The aim of this paper is to study a fully discrete scheme, which is readily implementable in computers, for the reaction-diffusion equations. The discrete scheme generates a discrete dynamical system, and our goal is to show how the attractors of the discrete dynamical systems converge to the attractor of the original dynamical system. A necessary step for this purpose is to derive a uniform bound, which is independent of time and space mesh sizes, in the strong topology $H_0^1(\Omega)$ for the approximate solutions given by the discrete scheme.

The upper bound for approximate solutions to evolutionary partial differential equations, given by a large number of existing stability analyses, often increases indefinitely when the time interval [0,T] goes to infinity. Such a stability result is certainly irrelevant for the long time integrations. In a previous paper [11], the author proved the $L^\infty(\mathbb{R}^+; H^\alpha)$ $(\alpha = 0, 1)$ stability for fully discrete nonlinear Galerkin method for the Navier-Stokes equations. However the space discretization used there was based on the eigenvectors of the Stokes operator which are in general not readily available. We consider in this paper a Legendre-Galerkin approximation for the space variables while using a first order semi-implicit scheme for the time integration. The techniques we use here are similar to those used in [11], but differ from those in the choice of an appropriate test function which enables us to obtain the strong stability. Although only the Legendre-Galerkin approximation is considered, the techniques apply also to a fairly large class of spatial approximations.

The strong stability of the approximate solutions and its convergence towards the exact solution are established. Under a restriction on the space dimension and the growth rate of the nonlinear term, we are able to prove the convergence of the approximate attractors of the fully discrete systems to the attractor of the reaction-diffusion equations. In addition, we are able to obtain an error estimate of optimal order without assuming further regularity of the exact solution. Our results are not complete in the high space dimensional cases due to the lack of the uniform $L^\infty(\mathbb{R}^+ \times \Omega)$ stability for the approximate solutions.

The paper is organized as follows. In the next section, we introduce the discrete scheme which is to be studied and we recall two discrete Gronwall lemmas and several classical results about the exact solution. In section 3, we derive the uniform bounds for the approximate solutions in $L^2(\Omega)$ as well as in $H_0^1(\Omega)$, which ensure that the approximate attractors are uniformly bounded in $H_0^1(\Omega)$. Then we prove that the approximate solutions converge to the exact solution in various topologies in section 4. Finally, in section 5, we establish the convergence of the approximate attractors towards the attractor of the original dynamical system.

2 Notations and Some Preliminary Results

We consider the following reaction-diffusion equation

$$\frac{\partial u}{\partial t} - d\Delta u + g(u) = 0, \tag{1}$$

$$u(0) = u_0 \tag{2}$$

subjected to the homogeneous Dirichlet boundary conditions.
$d > 0$ is given, Ω is an open bounded set in $\mathbb{R}^I$ with sufficient smooth boundary, and g is a polynomial of odd degree with a positive leading coefficient

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j \ , \ b_{2p-1} > 0. \tag{3}$$

We will deal with the nonlinear cases: $p > 1$.

Let us first derive some inequalities on g which will be used frequently in this paper. Since $b_{2p-1} > 0$, by using the Young's inequality we derive from (3) that there exist $c_1, c_2 > 0$ such that

$$|\sum_{j=0}^{2p-2} b_j s^{j+1}| \leq \frac{1}{2} b_{2p-1} s^{2p} + c_1 \ , \ \forall\ s, \tag{4}$$

$$\begin{aligned} \frac{2p-1}{2} b_{2p-1} s^{2p-2} - c_2 \ &\leq \ g'(s) = \sum_{j=1}^{j=2p-1} j b_j s^{j-1} \\ &\leq \ \frac{3}{2}(2p-1) b_{2p-1} s^{2p-2} + c_2 \ , \ \forall\ s. \end{aligned} \tag{5}$$

Also we infer from (4) that

$$\frac{1}{2} b_{2p-1} s^{2p} - c_1 \leq g(s)s \leq \frac{3}{2} b_{2p-1} s^{2p} + c_1 \ , \ \forall\ s. \tag{6}$$

In addition, since

$$g(s)^2 = (\sum_{j=0}^{2p-1} b_j s^j)^2 = b_{2p-1}^2 s^{4p-2} + \sum_{j=0}^{4p-4} \gamma_j s^j,$$

we can derive by using the Young's inequality that

$$g(s)^2 \leq 2 b_{2p-1}^2 s^{4p-2} + c_3 \ , \ \forall\ s. \tag{7}$$

Let us denote

$$V = H_0^1(\Omega) \ , \ H = L^2(\Omega).$$

The norm in H and V will be denoted respectively by

$$|u| = \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \text{ and } ||u|| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

The corresponding inner product in H and V will be denoted respectively by $(\cdot,\cdot)$ and $(\nabla\cdot,\nabla\cdot)$. We also denote the norm in $H^s(\Omega)$ by $||\cdot||_s$.

The following classical existence and uniqueness results are well known (see for instance Lions [6]).

Theorem 1 *For $u_0 \in H$, the system (1)-(2) admits a unique solution u satisfying $u \in C(\mathbb{R}^+; H)$ and $u \in L^2(0,T;V) \cap L^{2p}((0,T)\times\Omega)$, $\forall T > 0$.*

Moreover, if $u_0 \in V$, then $u \in C(\mathbb{R}^+; V) \cap L^2(0,T; H^2(\Omega)\cap V)$, $\forall T > 0$.

Further regularity results are also available. For example, we can prove (see for instance Temam [13] and Marion [8]) the following results.

Theorem 2 *For $u_0 \in H$ and $0 < t_0 < T < \infty$, there exist $d_1 = d_1(d,\Omega,t_0,|u_0|)$, $d_2 = d_2(d,\Omega,t_0,T,|u_0|)$ such that the unique solution u of (1)-(2) satisfies*

$$||u||_{L^\infty(t_0,\infty;H^2(\Omega))} \le d_1 \text{ and } ||u'||_{L^2(t_0,T;H^1(\Omega))} \le d_2.$$

We now introduce some notations for the spectral-Legendre approximation. We define

- $\Omega = (-1,1)^I$;
- S_N: the set of polynomials such that the order of each variable is less or equal than N;
- P_N: the orthonormal projection operator from H onto S_N;
- $V_N = S_N \cap V$;
- $a(u,v) = (\nabla u, \nabla v)$, $\forall\ u,v \in V$;
- Π_N: the orthogonal projection operator form V onto V_N defined by

$$a(u - \Pi_N u, v) = 0 \ , \ \forall\ v \in V_N. \tag{8}$$

We refer to the recent book by Canuto et al. [3] and the references therein for further properties and applications of the spectral-Legendre approximations.

We consider in this paper the following scheme which uses Legendre-Galerkin approximation for the space variables while time discretization is made by a first order semi-implicit scheme.

Find $\{u^n\} = \{u^n_{k,N}\}_1^\infty$ satisfying

$$(\frac{u^{n+1}-u^n}{k}, v) + da(u^{n+1}, v) = (-g(u^n), v) \ , \ \forall \ v \in V_N, n \geq 0 \tag{9}$$

with

$$u^0 = P_N u_0 \ \text{ or } \ u^0 = \Pi_N u_0 \ \text{ if } \ u_0 \in V. \tag{10}$$

For u^n given in S_N, the existence and the uniqueness of the $u^{n+1} \in V_N$ satisfying (9) is clear thanks to the classical Lax-Milgram theorem. We can then define a map

$$s(N,k): \ u^{n-1} \in S_N \to u^n \in S_N$$

associated to (9)-(10). It is straightforward that $\{s(N,k)^n\}$ satisfies the discrete semi-group properties

$$s(N,k)^0 = \text{Identity} \ \text{ and } \ s(N,k)^{n+m} = s(N,k)^n \circ s(N,k)^m \ , \ \forall \ n,m \geq 0$$

and of course $s(N,k)^n u^0 = u^n$.

Before all, let us recall two time discrete analogs of Gronwall lemmas which are essential to obtain the long time stabilities of the discrete scheme.

Lemma 1 *Let a^n, b^n be two series satisfying*

$$\frac{a^{n+1}-a^n}{k} + \lambda a^{n+1} \leq b^n \text{ and } \mathrm{b}^n \leq \mathrm{b}, \ \forall \ \mathrm{n} \geq 0.$$

Then

$$a^n \leq \frac{1}{(1+k\lambda)^n} a^0 + \frac{1+k\lambda}{\lambda}(1 - \frac{1}{(1+k\lambda)^{n+1}})b \ , \ \forall \ n \geq 0$$

provided $k, 1+k\lambda > 0$.

The second one is a time discrete counterpart of the uniform Gronwall lemma (see shen [11]).

Lemma 2 *Let d^n, g^n, h^n be three series satisfying*

$$\frac{d^{n+1}-d^n}{k} \leq g^n d^n + h^n \ , \ \forall \ n \geq n_0,$$

and

$$\begin{cases} k\sum_{n=k_0}^{N+k_0} g^n \leq a_1 \\ k\sum_{n=k_0}^{N+k_0} h^n \leq a_2 & , \ \forall \ k_0 \geq n_0 \\ k\sum_{n=k_0}^{N+k_0} d^n \leq a_3 \end{cases}$$

with $kN = r$. Then

$$d^n \leq (a_2 + \frac{a_3}{r}) exp(a_1) \ , \ \forall \ n \geq n_0 + N.$$

3 Absorbing Sets and Attractors

From now on, we will use c_i, c_i' to denote positive constants only depending on Ω; b_i, B_i and G_i to denote positive constants depending on some data: $b_i = b_i(d, \Omega, |u_0|)$, $B_i = B_i(d, \Omega, |u_0|, n)$, $G_i = G_i(d, \Omega, |u_0|, n, r)$. We will assume hereafter $k \leq K_0$ (for some $K_0 > 0$ fixed).

3.1 Absorbing Set in H

Let us denote

$$b_0 = |u_0|^2 + \frac{1 + K_0 d\delta}{d\delta}(c_1 + c_3 K_0)|\Omega|. \tag{11}$$

We are going to prove

Theorem 3 *We assume k and N are such that*

$$\begin{cases} c_8 k^2 N^{4I(p-1)} b_0^{2p-2} \leq \frac{1}{2} - \delta \\ c_8 d^{-1} k N^{4I(p-1)-4} b_0^{2p-2} \leq \frac{3}{2} - \delta \end{cases} \tag{12}$$

where I is the space dimension and δ is given in $(0, \frac{1}{2})$. Then, we have

$$|u^n|^2 \leq \frac{1}{(1 + k d\delta c_9^2)^n}|u^0|^2 + \frac{1 + K_0 d\delta c_9^2}{d\delta c_9^2}(c_1 + c_3 K_0)|\Omega| = B_0(n) \leq b_0 \ , \ \forall\ n \geq 0.$$

$$\begin{aligned} \sum_{n=n_0}^{n_0+\frac{r}{k}-1} \{\delta|u^{n+1} - u^n|^2 \ &+\ k d\delta||u^{n+1}||^2 + \frac{b_{2p-1}}{2}\int_\Omega (u^n)^{2p} dx\} \\ &\leq\ B_0(n_0) + r(c_1 + c_3 K_0)|\Omega| = G_0(n_0, r)\ ,\ \forall\ r > 0, n_0 \geq 0, \end{aligned}$$

where c_8, c_9 is to be given in the process of the proof.

Remark 1 *Let $\rho_0 = \frac{1+K_0 d\delta c_9^2}{d\delta c_9^2}(c_1 + c_3 K_0)|\Omega|$. Then $\forall \rho_0' > \rho_0$, we infer form this theorem that for n sufficiently large, we have $|u^n|^2 \leq \rho_0'$. In other word, $B_0 = B_H(0, \rho_0')$ is an absorbing set of the discrete semi-group $\{s(N, k)^n\}$ in H.*

PROOF: Taking the inner product of (9) with $2ku^{n+1}$, by using the identity

$$(a - b, 2a) = |a|^2 - |b|^2 + |a - b|^2, \tag{13}$$

we obtain

$$\begin{aligned} |u^{n+1}|^2 \ &-\ |u^n|^2 + |u^{n+1} - u^n|^2 + 2kd||u^{n+1}||^2 = -2k(g(u^n), u^{n+1}) \\ &=\ -2k(g(u^n), u^n) - 2k(g(u^n), u^{n+1} - u^n). \end{aligned} \tag{14}$$

By using (4) and the Schwarz inequality, we obtain

$$\begin{aligned}|\ u^{n+1}\ |^2 - |u^n|^2 + |u^{n+1} - u^n|^2 + 2kd||u^{n+1}||^2 \\ \leq\quad -\frac{k}{2}b_{2p-1}\int_\Omega (u^n)^{2p}dx + c_1k|\Omega| + \frac{1}{2}|u^{n+1} - u^n|^2 + 2k^2\int_\Omega g(u^n)^2dx.\end{aligned} \tag{15}$$

We infer from (15) and (7) that

$$\begin{aligned}|u^{n+1}|^2 \quad - \quad & |u^n|^2 + \frac{1}{2}|u^{n+1} - u^n|^2 + 2kd||u^{n+1}||^2 + \frac{k}{2}b_{2p-1}\int_\Omega (u^n)^{2p}dx \\ \leq \quad & c_1k|\Omega| + c_3k^2|\Omega| + 2b_{2p-1}^2k^2\int_\Omega (u^n)^{4p-2}dx.\end{aligned} \tag{16}$$

Thanks to the Sobolev embedding theorem, we have

$$H^\alpha \subset L^{4p-2} \text{ with } \alpha = \frac{\mathrm{I(p-1)}}{2\mathrm{p}-1}. \tag{17}$$

- case $\alpha < 1$: We recall that (see for instance Lions & Magenes [7])

$$||u||_\alpha \leq c(\alpha,\Omega)|u|^{1-\alpha}||u||^\alpha \ ,\ \forall\ 0 < \alpha < 1. \tag{18}$$

Hence by using (17) and (18)

$$\begin{aligned}\int_\Omega (u^n)^{4p-2}dx \quad &= \quad (|u^n|_{L^{4p-2}})^{4p-2} \leq c_5||u^n||_\alpha^{4p-2} \\ &\leq \quad c_6|u^n|^{(1-\alpha)(4p-2)}||u^n||^{(4p-2)\alpha}.\end{aligned} \tag{19}$$

By using the following inverse inequality (see Canuto & Quarteroni [4])

$$||u||_t \leq c_4N^{2(t-s)}||u||_s \ ,\ \forall\ t > s \geq 0\ ,\ \forall\ u \in S_N, \tag{20}$$

we derive that

$$\int_\Omega (u^n)^{4p-2}dx \leq c_7N^{4I(p-1)-4}|u^n|^{4p-4}||u^n||^2. \tag{21}$$

- case $\alpha \geq 1$: Using (17) and (20), we derive

$$\begin{aligned}\int_\Omega (u^n)^{4p-2}dx \quad &\leq \quad c_5||u^n||_\alpha^{4p-4}||u^n||_\alpha^2 \\ &\leq \quad c_6'N^{8\alpha(p-1)}|u^n|^{4p-4}N^{4(\alpha-1)}||u^n||^2 \\ &\leq \quad c_7'N^{4I(p-1)-4}|u^n|^{4p-4}||u^n||^2.\end{aligned}$$

We infer from (20) that

$$\begin{aligned}||u^n||^2 \quad &\leq \quad 2||u^{n+1} - u^n||^2 + 2||u^{n+1}||^2 \\ &\leq \quad 2c_4^2N^4|u^{n+1} - u^n|^2 + 2||u^{n+1}||^2.\end{aligned} \tag{22}$$

Therefore

$$\int_\Omega (u^n)^{4p-2}dx \le c_7'' N^{4I(p-1)-4}|u^n|^{4(p-1)}(2c_4^2N^4|u^{n+1}-u^n|^2+2||u^{n+1}||^2). \quad (23)$$

Putting (23) into (16), we derive

$$\begin{aligned}|u^{n+1}|^2 &- |u^n|^2+(\frac{1}{2}-c_8k^2N^{4I(p-1)}|u^n|^{4(p-1)})|u^{n+1}-u^n|^2\\ &+ kd(2-c_8d^{-1}kN^{4I(p-1)-4}|u^n|^{4(p-1)})||u^{n+1}||^2\\ &+ \frac{k}{2}b_{2p-1}\int_\Omega (u^n)^{2p}dx \le k(c_1+c_3K_0)|\Omega|. \quad (24)\end{aligned}$$

Assuming that k and N verify the hypothesis (12), we will prove by induction that

$$|u^q|^2 \le B_0(q) \le b_0\ ,\ \forall\ q \ge 0. \quad (25)$$

- (25) at $q=0$ is obvious;
- assuming (25) is true up to $q=n$, then by using (12), the inequality (24) becomes

$$|u^{n+1}|^2-|u^n|^2+\delta|u^{n+1}-u^n|^2+kd\delta||u^{n+1}||^2 \le k(c_1+c_3K_0)|\Omega|. \quad (26)$$

Therefore by using the Poincaré inequality

$$||u|| \ge c_9|u|\ ,\ \forall\ u\in V. \quad (27)$$

After dropping some unnecessary terms, we arrive to

$$\frac{|u^{n+1}|^2-|u^n|^2}{k}+d\delta c_9^2|u^{n+1}|^2 \le (c_1+c_3K_0)|\Omega|.$$

We can now apply Lemma 1 to this last inequality with $a^n=|u^n|^2$, $b^n=(c_1+c_3K_0)|\Omega|$ and $\lambda=d\delta c_9^2$, from which we derive

$$\begin{aligned}|u^{n+1}|^2 &\le \frac{1}{(1+kd\delta c_9^2)^{n+1}}|u^0|^2+\frac{1+K_0d\delta c_9^2}{d\delta c_9^2}(c_1+c_3K_0)|\Omega|\\ &= B_0(n+1) \le |u^0|^2+\frac{1+K_0d\delta c_9^2}{d\delta c_9^2}(c_1+c_3K_0)|\Omega| = b_0. \quad (28)\end{aligned}$$

The proof of (25) is complete.

In order to prove the last inequality of Theorem 3, we take the sum of (26) for n from n_0 to $n_0+\frac{r}{k}-1$, which lead to

$$\begin{aligned}|u^{n_0+\frac{r}{k}}|^2 &+ \sum_{n=n_0}^{n_0+\frac{r}{k}-1}\left\{\delta|u^{n+1}-u^n|^2+kd\delta||u^{n+1}||^2+\frac{b_{2p-1}}{2}\int_\Omega (u^n)^{2p}dx\right\}\\ &\le B_0(n_0)+r(c_1+c_3K_0)|\Omega| = G_0(n_0,r)\ ,\ \forall\ r>0, n_0\ge 0. \quad (29)\end{aligned}$$

This completes the proof of Theorem 3. ¶

3.2 Absorbing Set in V

Let us denote

$$G(s) = \int_0^s g(t)dt = \sum_{j=0}^{2p-1} \frac{b_j}{j+1} s^{j+1}.$$

We prove first

Lemma 3 *There exists $c_1' > 0$ such that $\forall\ u, v \in H$, we have*

$$\begin{aligned}(u - v, g(v)) \ &\geq \ \int_\Omega [G(u) - G(v)]dx - c_1'|u - v|^2 \\ &- \ \frac{3b_{2p-1}}{4p} \int_\Omega (u - v)^2 (u^{2p-2} + v^{2p-2}) dx. \end{aligned} \tag{30}$$

PROOF: By the definitions of G and g, we find

$$G(u) - G(v) = \sum_{j=0}^{2p-1} \frac{b_j}{j+1}(u^{j+1} - v^{j+1}) = \sum_{j=0}^{2p-1} \frac{b_j}{j+1}(u - v)(\sum_{k=0}^{j} u^k v^{j-k}).$$

Since

$$\begin{aligned}\sum_{k=0}^{j} u^k v^{j-k} \ &= \ v^j + \sum_{k=1}^{j}(u^k - v^k + v^k)v^{j-k} \\ &= \ (j+1)v^j + \sum_{k=1}^{j}(u^k - v^k)v^{j-k} \\ &= \ (j+1)v^j + (u - v)f_j(u, v),\end{aligned}$$

where

$$f_j(u, v) = \sum_{k=1}^{j} v^{j-k}(\sum_{m=1}^{k} u^m v^{k-1-m}) \ , \ \forall\ j$$

We then derive that

$$\begin{aligned}G(u) - G(v) \ &= \ (u - v)\left\{\sum_{j=0}^{2p-1} b_j v^j + (u - v)\sum_{j=0}^{2p-1} \frac{b_j}{j+1} f_j(u, v)\right\} \\ &= \ (u - v)g(v) + (u - v)^2 \sum_{j=0}^{2p-1} \frac{b_j}{j+1} f_j(u, v).\end{aligned} \tag{31}$$

Since $\frac{b_{2p-1}}{2p} > 0$, by using Young's inequality, we can find $c_1' > 0$ such that

$$\frac{b_{2p-1}}{4p}(u^{2p-2} + v^{2p-2}) - c_1' \leq \sum_{j=0}^{2p-1} \frac{b_j}{j+1} f_j(u, v) \leq \frac{3b_{2p-1}}{4p}(u^{2p-2} + v^{2p-2}) + c_1'. \tag{32}$$

Therefore, integrating (31) over Ω, taking into account (32), we recover (30). ¶

Now we are in position to prove

Theorem 4 *There exist two constants $b_3, b_4 > 0$ only depending on d, Ω and $|u_0|$ such that if k and N satisfy*

$$\left\{ \begin{array}{l} k^2 N^{4I(p-1)} \leq b_3 \\ k N^{4I(p-1)-4} \leq b_4 \end{array} \right. \tag{33}$$

then we have

- *for u_0 in H, and $\forall\, r > 0, n_0 \geq 0$,*

$$||u^n||^2+\frac{b_{2p-1}}{(2p-1)}\int_\Omega (u^n)^{2p}dx \leq \frac{G_0(n_0,r)}{r}(\frac{1}{d\delta}+2c_{12})+4c_{10}|\Omega| = G_1(n_0,r)\ ,\ \forall\ n \geq \frac{r}{k}+n_0,$$

$$\sum_{n=m}^{m+\frac{r}{k}-1}[\frac{1}{k}|u^{n+1}-u^n|^2+||u^{n+1}-u^n||^2] \leq G_1(m,r)\ ,\ \forall\ m \geq \frac{r}{k}+n_0.$$

- *Moreover, if $u_0 \in V \cap L^{2p}$,*

$$||u^n||^2+\frac{b_{2p-1}}{2(2p-1)}\int_\Omega (u^n)^{2p}dx \leq ||u^0||^2+\frac{3b_{2p-1}}{(2p-1)}\int_\Omega |u^0|^{2p}dx+4c_{10}|\Omega| = b_1\ ,\ \forall\ n \geq 0,$$

$$\sum_{n=n_0}^{n_0+\frac{r}{k}-1}[\frac{1}{k}|u^{n+1}-u^n|^2+||u^{n+1}-u^n||^2] \leq b_1\ ,\ \forall\ n_0 \geq 0, r > 0,$$

where b_3, b_4, c_{10} and c_{12} are to be estimated in the process of the proof.

Remark 2 *Let $\rho_1 = \frac{1}{r}(c_1+c_3K_0)|\Omega|(\frac{1}{d\delta}+2c_{12})+4c_{10}|\Omega|$. Then $\forall \rho_1' > \rho_1$, we infer form this theorem that for n sufficiently large, we have*

$$||u^n||^2+\frac{b_{2p-1}}{2(2p-1)}\int_\Omega (u^n)^{2p}dx \leq \rho_1'.$$

In other word, $B_1 = B_{V\cap L^{2p}}(0,\rho_1')$ is an absorbing set of the discrete semi-group $\{s(N,k)^n\}$ in $V \cap L^{2p}$.

Proof: We take the inner product of (9) with $2\frac{u^{n+1}-u^n}{k}$,

$$\frac{2}{k^2}|u^{n+1}-u^n|^2+\frac{1}{k}(||u^{n+1}||^2-||u^n||^2+||u^{n+1}-u^n||^2)+\frac{2}{k}(g(u^n),u^{n+1}-u^n) = 0. \tag{34}$$

By using lemma 3, we derive

$$\begin{aligned} \frac{2}{k^2}|u^{n+1}-u^n|^2 &+ \frac{1}{k}(||u^{n+1}||^2-||u^n||^2+||u^{n+1}-u^n||^2) \\ &+ \frac{2}{k}(\int_\Omega G(u^{n+1})dx - \int_\Omega G(u^n)dx) \leq \frac{2c_1'}{k}|u^{n+1}-u^n|^2 \\ &+ \frac{c_3'}{k}\int_\Omega (u^{n+1}-u^n)^2[(u^{n+1})^{2p-2}+(u^n)^{2p-2}]dx. \end{aligned} \tag{35}$$

Let us first majorize $\int_\Omega u^2v^{2p-2}dx$ for $u, v \in S_N$. We apply Hölder inequality with $\frac{1}{2}+\frac{1}{4p-2}+\frac{p-1}{2p-1}=1$, using $H^\alpha \subset L^{4p-2}$ (see Thm. 3) and the inverse inequality, we derive

$$\begin{aligned}
(u^2, v^{2p-2}) &\leq c_4'|u|_{L^{4p-2}}|u|\cdot|v^{2p-2}|_{L^{\frac{2p-1}{p-1}}} \\
&\leq c_5'||u||_\alpha\cdot|u|\cdot|v|^{2p-2}_{L^{4p-2}} \\
&\leq \frac{1}{4c_3'k}|u|^2+c_6'k||v||_\alpha^{4p-4}||u||^2_\alpha \\
&\leq \frac{1}{4c_3'k}|u|^2+c_7'kN^{4I(p-1)}|v|^{4p-4}|u|^2.
\end{aligned} \tag{36}$$

Since $|u^n|^2 \leq b_0$, $\forall\ n$ (see Thm. 3), we have

$$\begin{aligned}
\frac{c_3'}{k}\int_\Omega(u^{n+1} &- u^n)^2[(u^{n+1})^{2p-2}+(u^n)^{2p-2}]dx \\
&\leq \frac{1}{4k^2}|u^{n+1}-u^n|^2+\frac{c_8'k^2N^{4I(p-1)}b_0^2}{k^2}|u^{n+1}-u^n|^2.
\end{aligned} \tag{37}$$

By choosing b_3, b_4 sufficiently small in (33), we can derive from (35) and (37) that

$$\begin{aligned}
\frac{1}{k^2}|u^{n+1}-u^n|^2 &+ \frac{1}{k}(||u^{n+1}||^2-||u^n||^2+||u^{n+1}-u^n||^2) \\
&+ \frac{2}{k}\Big(\int_\Omega G(u^{n+1})dx-\int_\Omega G(u^n)dx\Big)\leq 0.
\end{aligned} \tag{38}$$

- For $u_0 \in H$, by dropping some unnecessary terms in (38), we obtain

$$\frac{1}{k}\Big[\Big(||u^{n+1}||^2+2\int_\Omega G(u^{n+1})dx\Big)-\Big(||u^n||^2+2\int_\Omega G(u^n)dx\Big)\Big]\leq 0. \tag{39}$$

We intend to apply Lemma 2 (the discrete uniform Gronwall lemma) to this inequality with $d^n=||u^n||^2+2\int_\Omega G(u^n)dx$, $h^n=g^n=0$.

By Young's inequality, there exists $c_{10}>0$ such that

$$\frac{b_{2p-1}}{2(2p-1)}s^{2p}-c_{10}\leq G(s)\leq\frac{3b_{2p-1}}{2(2p-1)}s^{2p}+c_{10}. \tag{40}$$

We infer from Theorem 3 and (40) that $\forall r>0$, we have

$$k\sum_{n=n_0}^{n_0+\frac{r}{k}}||u^{n+1}||^2\leq\frac{G_0(n_0,r)}{d\delta}\ ,\ \forall\ n_0\geq 0$$

$$\begin{aligned}
k\sum_{n=n_0}^{n_0+\frac{r}{k}}\int_\Omega G(u^{n+1})dx &\leq c_{10}r|\Omega|+c_{11}k\sum_{n=n_0}^{n_0+\frac{r}{k}}\int_\Omega(u^{n+1})^{2p}dx \\
&\leq c_{10}r|\Omega|+c_{12}G_0(n_0,r)\ ,\ \forall\ n_0\geq 0.
\end{aligned}$$

Hence

$$k\sum_{n=n_0}^{n_0+\frac{r}{k}} d^n \le \frac{G_0(n_0,r)}{d\delta} + 2(c_{10}r|\Omega| + c_{12}G_0(n_0,r)) = G_2(n_0,r)\ ,\ \forall\ n_0 \ge 0.$$

We then derive from Lemma 2 that

$$||u^{n+1}||^2 + 2\int_\Omega G(u^{n+1})dx \le \frac{G_2(n_0,r)}{r}\ ,\ \forall\ n \ge \frac{r}{k} + n_0. \tag{41}$$

Then by using (40), we find

$$||u^{n+1}||^2 + \frac{b_{2p-1}}{(2p-1)}\int_\Omega (u^{n+1})^{2p}dx \le \frac{G_2(n_0,r)}{r} + 2c_{10}|\Omega| = G_1(n_0,r)\ ,\ \forall\ n \ge \frac{r}{k} + n_0.$$

We now take the sum of (38) from m to $m+\frac{r}{k}-1$, by using (41), we derive

$$\begin{aligned} 2\int_\Omega G(u^{m+\frac{r}{k}})dx \quad &+ \sum_{n=m}^{m+\frac{r}{k}-1}[\frac{1}{k}|u^{n+1}-u^n|^2 + ||u^{n+1}-u^n||^2] \\ &\le \quad ||u^m||^2 + 2\int_\Omega G(u^m)dx \le \frac{G_2(m,r)}{r}\ ,\ \forall\ r>0, m \ge \frac{r}{k}+n_0. \end{aligned}$$

Therefore, it follows from (40) that

$$\sum_{n=m}^{m+\frac{r}{k}-1}[\frac{1}{k}|u^{n+1}-u^n|^2 + ||u^{n+1}-u^n||^2] \le G_1(m,r)\ ,\ \forall\ r>0, m \ge \frac{r}{k}+n_0.$$

• If $u_0 \in V \cap L^{2p}$, we derive from (39) that

$$||u^n||^2 + 2\int_\Omega G(u^n)dx \le ||u^0||^2 + 2\int_\Omega G(u^0)dx\ ,\ \forall\ n \ge 0, \tag{42}$$

$$\sum_{n=n_0}^{n_0+\frac{r}{k}-1}[\frac{1}{k}|u^{n+1}-u^n|^2 + ||u^{n+1}-u^n||^2] + 2\int_\Omega G(u^{\frac{T}{k}})dx \le ||u^0||^2 + 2\int_\Omega G(u^0)dx. \tag{43}$$

We infer from (40) and (43) that

$$||u^n||^2 + \frac{b_{2p-1}}{(2p-1)}\int_\Omega |u^n|^{2p}dx \le ||u^0||^2 + \frac{3b_{2p-1}}{(2p-1)}\int_\Omega |u^0|^{2p}dx + 4c_{10}|\Omega| = b_1\ ,\ \forall\ n \ge 0, \tag{44}$$

$$\sum_{n=n_0}^{n_0+\frac{r}{k}-1}[\frac{1}{k}|u^{n+1}-u^n|^2 + ||u^{n+1}-u^n||^2] \le b_1\ ,\ \forall\ r>0,\ n_0 \ge 0. \tag{45}$$

This ends the proof of theorem 4. ¶

3.3 Lipschitz Continuity of the Discrete Semi-group

Lemma 4 *We assume (33) is satisfied with b_3, b_4 sufficiently small. Then the discrete semi-group $\{s(N,k)^n\}$ are Lipschitz continuous from S_N (associated with the norm in H) to itself.*

PROOF: Let $u^0, v^0 \in S_N$ and $\{u^n\}, \{v^n\}$ denote the corresponding solutions of (9)-(10). Then $e^n = u^n - v^n$ satisfies

$$\begin{aligned}(\frac{e^{n+1}-e^n}{k}, v) + da(e^{n+1}, v) &= -(g(u^n) - g(v^n), v) \\ &= -(e^n f(u^n, v^n), v)\ ,\ \forall\ v \in V_N, \qquad (46)\end{aligned}$$

where

$$f(a,b) = b_1 + \sum_{j=2}^{2p-1} b_j \sum_{k=0}^{j-1} a^k b^{j-1-k}. \qquad (47)$$

Taking the inner product of (46) with $2ke^{n+1}$, we obtain

$$\begin{aligned}|e^{n+1}|^2 &- |e^n|^2 + |e^{n+1} - e^n|^2 + 2kd||e^{n+1}||^2 = -2k(e^n f(u^n, v^n), e^{n+1}) \\ &= -2k(e^n f(u^n, v^n), e^n) - 2k(e^n f(u^n, v^n), e^{n+1} - e^n). \qquad (48)\end{aligned}$$

Since $b_{2p-1} > 0$, by using Young's inequality, we can find $c_{14} > 0$ such that

$$\frac{b_{2p-1}}{2}(a^{2p-2} + b^{2p-2}) - c_{14} \leq f(a,b) \leq \frac{3b_{2p-1}}{2}(a^{2p-2} + b^{2p-2}) + c_{14}. \qquad (49)$$

By using (49), we have

$$-2k(e^n f(u^n, v^n), e^n) \leq -c_{15}k \int_\Omega (a^{2p-2} + b^{2p-2} - 1)(e^n)^2 dx. \qquad (50)$$

Using Hölder inequality with $\frac{1}{2} + \frac{p-1}{2p-1} + \frac{1}{4p-2} = 1$, since $H^\alpha \subset L^{4p-2}$, we derive

$$\begin{aligned}-2k(e^n f(u^n, v^n) &,\ e^{n+1} - e^n) \leq c_{16}k|e^n|_{L^{4p-2}}|f(u^n, v^n)|_{L^{\frac{2p-1}{p-1}}}|e^{n+1} - e^n| \\ &\leq c_{17}k||e^n||_\alpha |f(u^n, v^n)|_{L^{\frac{2p-1}{p-1}}}|e^{n+1} - e^n| \\ &\leq \frac{1}{2}|e^{n+1} - e^n|^2 + c_{18}k^2|f(u^n, v^n)|^2_{L^{\frac{2p-1}{p-1}}}||e^n||^2_\alpha. \qquad (51)\end{aligned}$$

We infer from (49) that

$$|f(u^n, v^n)|^2_{L^{\frac{2p-1}{p-1}}} \leq c_{20}\{|u^n|^{4p-4}_{L^{4p-2}} + |v^n|^{4p-4}_{L^{4p-2}} + 1\}.$$

Let $\beta = max(0, \alpha - 1)$, By using the inverse inequality (20) and the results of Theorem 3, we find

$$\begin{aligned}||e^n||^2_\alpha |f(u^n, v^n)|^2_{L^{\frac{2p-1}{p-1}}} &\leq c_{21}||e^n||^2_\alpha[(||u^n||^{4p-4}_\alpha + ||v^n||^{4p-4}_\alpha) + 1] \\ &\leq c_{22}N^{4\beta}||e^n||^2[N^{8\alpha(p-1)}(|u^n|^{4p-4} + |v^n|^{4p-4}) + 1] \\ &\leq c_{23}N^{4I(p-1)-4}||e^n||^2. \qquad (52)\end{aligned}$$

Putting these inequalities into (50), and using the relation

$$||e^n||^2 \leq 2c_4^2 N^4 |e^{n+1} - e^n|^2 + 2||e^{n+1}||^2,$$

we arrive to

$$|e^{n+1}|^2 \quad - \quad |e^n|^2 + (\frac{1}{2} - c_{31}k^2 N^{4I(p-1)})|e^{n+1} - e^n|^2$$

$$
\begin{aligned}
&+ \quad kd(2 - c_{31}d^{-1}kN^{4I(p-1)-4})\|e^{n+1}\|^2 \\
&+ \quad ck\int_\Omega((u^n)^{2p-2} + (v^n)^{2p-2})(e^n)^2dx \le c_{32}k|e^n|^2. \qquad (53)
\end{aligned}
$$

If we chose b_3, b_4 in (33) small enough, we can derive from (53) that

$$|e^{n+1}|^2 \le (1 + c_{32}k)|e^n|^2 \ , \ \forall\ n \ge 0.$$

Therefore

$$|e^n|^2 \le (1 + c_{32}k)^n|e^0|^2 \le exp(c_{32}kn)|e^0|^2$$

which means the semi-group is Lipschitz continuous from S_N to itself. ¶

We have then proven that the discrete semi-groups $\{s(N,k)^n\}$ are continuous from S_N to itself and they possess absorbing sets in H as well as in V (see Remark 1 & 2). Therefore by applying the abstract existence theorem (Thm. 1.1 of [13]) in the discrete case, we have

Theorem 5 *The discrete semi-group $\{s(N,k)^n\}$ possesses a maximal attractor $\mathcal{A}_{N,k}$ in S_N. Moreover, $\{\mathcal{A}_{N,k}\}$ are uniformly bounded (independently of N and k) in V.*

More precisely we have

$$\mathcal{A}_{N,k} = \bigcap_{n\ge0}\overline{\bigcup_{m\ge n} s(N,k)^m(B_1)} \qquad (54)$$

where B_1 is given in the remark 2.

4 Convergence of the Approximate Solutions

Let us first introduce some approximate functions of $u(t)$.

DEFINITION:

- $u_1(t) = u_1^{(k,N)}(t) : \mathbb{R}^+ \to H$, is the piecewise constant function which equals to u^n on $[nk,(n+1)k)$;
- $u_2(t) = u_2^{(k,N)}(t) : \mathbb{R}^+ \to H$, is the piecewise constant function which equals to u^n on $[nk,(n+1)k)$;
- $u_3(t) = u_3^{(k,N)}(t) : \mathbb{R}^+ \to H$, is the continuous function which is linear on $[nk,(n+1)k)$ and $u_3(nk) = u^n$, $u_3((n+1)k) = u^{n+1}$.

As in [11], we can prove

Theorem 6 *Under the hypothesis (33), for $u_0 \in H$, we have*

- $u_i^{(k,N)} \to u$ *(as $k, N^{-1} \to 0$) $i = 1,2,3$, in $L^\infty(t_0,\infty;L^{2p}(\Omega)\cap V)$ weak-star,* $\forall t_0 > 0$;

- $u_i^{(k,N)} \to u$ *(as* $k, N^{-1} \to 0)$ $i = 1,2,3,$ *in* $L^2(0,T;V) \cap L^q(0,T;H)$ *strongly,* $\forall\, T > 0,\ 1 \le q < +\infty$, *provided* $kN^4 \to 0$ *in case of i=1,3.*

SKETCH OF THE PROOF: By the definitions of u_i, we can reformulate the scheme (9) as

$$(\frac{\partial u_3(t)}{\partial t}, v) + da(u_2(t), v) + (g(u_1(t)), v) = 0\ ,\ \forall\ v \in V_N. \tag{55}$$

From the definitions of u_i, One can readily check that (see for instance [11])

$$\int_0^T ||u_3(t)||^2 dt = \frac{k}{2}\sum_{n=0}^{\frac{T}{k}-1}(||u^{n+1}||^2 + ||u^n||^2) \le G_0(0,T),$$

$$\int_0^T |u_1(t) - u_3(t)|^2 dt = \frac{k}{3}\sum_{n=0}^{\frac{T}{k}-1}|u^{n+1} - u^n|^2 \le \frac{k}{3\delta}G_0(n_0,T). \tag{56}$$

Similarly,

$$\int_0^T |u_2(t) - u_3(t)|^2 dt \le \frac{k}{3\delta}G_0(n_0,T). \tag{57}$$

Then Theorem 3 and Theorem 4 (under $u_0 \in H$) can be reinterpreted as

- $u_i^{(k,N)}(t)(i = 1,2,3)$ are bounded independently of k, N in $L^\infty(t_0,\infty; L^{2p}(\Omega) \cap V)$ and $L^2(0,T;V)$, $\forall\ t_0, T > 0$;
- $\frac{\partial}{\partial t}u_3^{(k,N)}(t)$ is bounded independently of k, N in $L^2(0,T;V')$;
- $g(u_1^{(k,N)}(t))$ is bounded independently of k, N in $L^q((0,T)\times\Omega)$.

where q is given by the relation $\frac{1}{q} + \frac{1}{2p} = 1$.

Hence there exist $U_i \in L^\infty(t_0,\infty; L^{2p}(\Omega) \cap V) \cap L^2(0,T;V)$, $\forall\ t_0, T > 0$ and a subsequence (k', N') such that

$$\begin{cases} u_i = u_i^{(k',N')} \to U_i\ (\text{as k}, \mathrm{N}^{-1} \to 0) \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{V}) \text{ weakly },\mathrm{i} = 1,2,3 \\ g(u_1^{(k,N)}) \to g(U_1)\ (\text{as k}, \mathrm{N}^{-1} \to 0) \text{ in } \mathrm{L}^q((0,\mathrm{T})\times\Omega) \\ \frac{\partial u_3(t)}{\partial t} = \frac{\partial u_3^{(k',N')}(t)}{\partial t} \to \frac{\partial U_3(t)}{\partial t}\ (\text{as k}, \mathrm{m}^{-1} \to 0) \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{V}') \text{ weakly} \end{cases} \tag{58}$$

where $V' = H^{-1}$ is the dual space of V.

In virtue of a classical compactness theorem (see for example Lions [6]), we derive from (58) that

$$u_3 = u_3^{(k',N')} \to U_3 \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{H}) \text{ strongly.}$$

We infer from (56)-(57)that $U_1 = U_2 = U_3 = u^*$ and

$$u_i = u_i^{(k',N')} \to u^* \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{H}) \text{ strongly, } \mathrm{i} = 1,2.$$

With these strong convergence results, the passage to the limit in (55) is standard (see for instance Temam [12] for more details) and we find out that u^* is indeed the solution of (1)-(2).

The strong convergence in $L^2(0,T;V)$ can be proven by the following arguments similar as those used in [12].

Let us define

$$\begin{aligned}
X &= X^{(k,N)} = |u_3(T) - u(T)|^2 \\
&+ \sum_{n=0}^{\frac{T}{k}-1} |u^{n+1} - u^n|^2 + 2d\int_0^T ||u_2(t) - u(t)||^2 dt \\
&= \left\{ |u_3(T)|^2 + \sum_{n=0}^{\frac{T}{k}-1} |u^{n+1} - u^n|^2 + 2d\int_0^T ||u_2(t)||^2 dt \right\} \\
&+ \left\{ |u(T)|^2 + 2d\int_0^T ||u(t)||^2 dt \right\} \\
&+ \left\{ -2(u_3(T), u(T)) - 4d\int_0^T (\nabla u_2(t), \nabla u(t)) dt \right\} \\
&= X_1^{(k,N)} + X_2^{(k,N)} + X_3^{(k,N)}.
\end{aligned}$$

From (58), we derive that

$$X_3^{(k,N)} \to -2|u(T)|^2 - 4d\int_0^T ||u(t)||^2 dt.$$

Taking the sum of (14) for n from 0 to $\frac{T}{k} - 1$, we obtain

$$\begin{aligned}
|u_3(T)|^2 &+ \sum_{n=0}^{\frac{T}{k}-1} |u^{n+1} - u^n|^2 + 2d\int_0^T ||u_2(t)||^2 dt \\
&= |u_3(0)|^2 - 2\int_0^T (g(u_1(t)), u_2(t)) dt.
\end{aligned}$$

Let $k, N^{-1} \to 0$ in the last relation, we derive from (58) that

$$X_1^{(k,N)} \to |u(0)|^2 - 2\int_0^T (g(u(t)), u(t)) dt.$$

By taking the inner product of (1) with u, we have

$$\frac{\partial |u(t)|^2}{\partial t} + 2d||u(t)||^2 + 2(g(u(t)), u(t)) = 0.$$

The integration of which over [0,T] implies

$$|u(T)|^2 + 2d\int_0^T ||u(t)||^2 dt + 2\int_0^T (g(u(t), u(t)) dt = |u(0)|^2.$$

Combining all these relations, we derive

$$X^{(k,N)} = X_1^{(k,N)} + X_2^{(k,N)} + X_3^{(k,N)} \to 0 \ \ (\text{as } k, N^{-1} \to 0).$$

This implies

$$u_2 \to u \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{V}) \text{ strongly.}$$

Finally, by the definition of $u_1(t), u_2(t)$, we have

$$\int_0^T ||u_1(t)-u_2(t)||^2 dt = k\sum_{n=0}^{\frac{T}{k}-1} ||u^{n+1}-u^n||^2 \le kN^4 \sum_{n=0}^{\frac{T}{k}-1} |u^{n+1}-u^n|^2 \le \frac{kN^4}{\delta} G_0(n_0,T).$$

Similarly as in (56)

$$\int_0^T ||u_3(t)-u_2(t)||^2 dt \le \frac{k}{3}\sum_{n=0}^{\frac{T}{k}-1} ||u^{n+1}-u^n||^2 \le \frac{k}{3}N^4 \sum_{n=0}^{\frac{T}{k}-1} |u^{n+1}-u^n|^2 \le \frac{kN^4}{3\delta} G_0(n_0,T).$$

Therefore

$$u_1, u_3 \to u \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{V}) \text{ strongly provided } \mathrm{kN}^4 \to 0.¶$$

5 Convergence of the Attractors

For some technical reasons, more precisely due to the lack of the uniform $L^\infty(\mathbb{R}^+ \times \Omega)$ stability for the approximate solutions, we will restrict ourselves in this section to the cases

$$I \le 2 \text{ or } \mathrm{I}=3, \mathrm{p}=2 \tag{59}$$

which ensure the following continuous embedding

$$H^1(\Omega) \subset L^{4p-2}(\Omega). \tag{60}$$

Let us first establish the following

Lemma 5 *We assume (33) and (59). Then $\forall\ 0 < t_0 < T < \infty$, $s, \sigma \ge 1$, we have*

- *for $u_0 \in H$,*

$$\begin{aligned}|u^n - u(nk)|^2 \quad \le \quad & d_3 \exp(d_4(nk-t_0))\Big\{k^2 + |u_3(t_0)-u(t_0)|^2 + N^{-2s}||u||^2_{L^\infty(t_0,T;H^s)} \\ & + N^{-2(\sigma+1)}||u'||^2_{L^2(t_0,T;H^\sigma)}\Big\}\ ,\ \forall\ t_0 \le nk \le T. \end{aligned} \tag{61}$$

- *for $u_0 \in V \cap L^{2p}$,*

$$\begin{aligned}|u^n \quad - \quad u(nk)|^2 \le & F_3 \exp(F_4 nk)\Big\{k^2 + N^{-2s}||u||^2_{L^2(0,T;H^s)} \\ & + N^{-2s}||u(nk)||_s^2 + N^{-2(\sigma+1)}||u'||^2_{L^2(0,T;H^\sigma)}\Big\}\ ,\ \forall\ 0 \le nk \le T, \end{aligned} \tag{62}$$

where $d_3, d_4 > 0$ depends on d, Ω, t_0, T and $|u_0|$; F_3, F_4 depends on d, Ω, T and $||u_0||$.

PROOF: Let us denote

$$e(t) = \Pi_N u(t) - u_3(t),\ \bar{e}(t) = u(t) - \Pi_N u(t)\ ,\ \forall\ t > 0$$

and $e^0 = 0$.

Subtracting (9) from (1), by using (8), we find

$$(e' + \bar{e}', v) + da(e, v) = (g(u_1) - g(u), v) \ , \ \forall \ v \in V_N.$$

Hence

$$(e', v) + da(e, v) = (g(u_1) - g(u), v) - (\bar{e}', v) \ , \ \forall \ v \in V_N. \tag{63}$$

Since $e \in V_N$, we can take $v = 2e$ in (63) which gives

$$\frac{d}{dt}|e(t)|^2 + 2d||e(t)||^2 \le d||e(t)||^2 + \frac{1}{d}||\bar{e}(t)'||_{-1}^2 + 2(g(u_1(t)) - g(u(t)), e(t)). \tag{64}$$

We infer from Hölder inequality that

$$\begin{aligned} 2(g(u_1) - g(u), e) &= 2((u_1 - u)f(u_1, u), e) \\ &\le 2|u_1 - u||f(u_1, u)|_{L^{\frac{2p-1}{p-1}}} |e|_{L^{4p-2}} \end{aligned} \tag{65}$$

where f is the function given in (47).

By using (49), we derive

$$\begin{aligned} |f(u_1, u)|_{L^{\frac{2p-1}{p-1}}} &\le c_{44}\Big(\int_\Omega (u_1^{2p-2} + u^{2p-2} + 1)^{\frac{2p-1}{p-1}} dx\Big)^{\frac{p-1}{2p-1}} \\ &\le c_{45}(|u_1|_{L^{4p-2}}^{2p-2} + |u|_{L^{4p-2}}^{2p-2} + 1). \end{aligned}$$

It follows from (60) and the results of Theorem 4 that

$$|f(u_1, u)|_{L^{\frac{2p-1}{p-1}}} \le c_{46}(||u_1||^{2p-2} + ||u||^{2p-2} + 1) \le G_3(0, r) \ , \ \forall \ t \ge r$$

Therefore

$$\begin{aligned} 2(g(u_1) - g(u), e) &\le \frac{d}{2}||e||^2 + c_{47}G_3(n_0, r)^2|u_1 - u|^2 \\ &\le \frac{d}{2}||e||^2 + G_4(|u_1 - u_3|^2 + |e|^2 + |\bar{e}|^2) \ , \ \forall \ t \ge r \end{aligned} \tag{66}$$

where we have set $G_4 = G_4(0, r) = c_{47}G_3(0, r)^2$.

For t_0 given in $(0, T)$, we can choose r, n_0 such that $r + n_0 k \le t_0$ for k sufficiently small, it follows from (64) and (66) that

$$\frac{d}{dt}|e(t)|^2 \le G_4|e|^2 + [\frac{1}{d}||e'||_{-1}^2 + G_4(|u_1 - u_3|^2 + |\bar{e}|^2)] \ , \ \forall \ t \ge t_0. \tag{67}$$

By using the Gronwall lemma, we derive that $\forall \ t \in (t_0, T)$, we have

$$|e(t)|^2 \le exp(G_4(t - t_0))\{|e(t_0)|^2 + \int_{t_0}^T [\frac{1}{d}||e'||_{-1}^2 + G_4(|u_1 - u_3|^2 + |\bar{e}|^2)]dt\}. \tag{68}$$

We infer from Theorem 4 that

$$\int_{t_0}^{T} |u_1 - u_3|^2 dx = \frac{k}{3} \sum_{n=\frac{t_0}{k}}^{\frac{T}{k}} |u^{n+1} - u^n|^2 \leq \frac{k^2}{3} G_1(\frac{t_0}{k}, r).$$

We recall that (see for instance Canuto et al. [3])

$$||u - \Pi_N u|| \leq c_{48} N^{1-s} ||u||_s \ , \ \forall \ s \geq 1, u \in H^s(\Omega) \cap H_0^1(\Omega), \tag{69}$$

$$|u - \Pi_N u| \leq c_{49} N^{-s} ||u||_s \ , \ \forall \ s \geq 1, u \in H^s(\Omega) \cap H_0^1(\Omega). \tag{70}$$

Therefore

$$\int_{t_0}^{T} |\bar{e}|^2 dt \leq c_{50} N^{-2s} \int_{t_0}^{T} ||u||_s^2 dt = c_{50} N^{-2s} ||u||^2_{L^2(t_0,T;H^s)},$$

and by using Lemma 6 (see below)

$$\int_{t_0}^{T} ||e'||_{-1}^2 dt \leq c_{51} N^{-2(1+\sigma)} \int_{t_0}^{T} ||u'||_s^\sigma dt = c N^{-2(1+\sigma)} ||u'||^2_{L^2(t_0,T;H^\sigma)}.$$

Combining these results into (68), we derive

$$\begin{aligned} |e(t)|^2 \quad &\leq \quad G_5 exp(G_4(t-t_0))(|e(t_0)|^2 + N^{-2s}||u||^2_{L^2(t_0,T;H^s)} \\ &\qquad + N^{-2(1+\sigma)}||u'||^2_{L^2(t_0,T;H^\sigma)} + k^2) \ , \ \forall \ t \in (t_0, T). \end{aligned}$$

Hence

$$\begin{aligned} |(u \quad - \quad u_3)(t)|^2 &\leq 2(|e(t)|^2 + |u(t) - \Pi_N u(t)|^2) \\ &\leq G_6 exp(G_4(t-t_0)) \Big\{ k^2 + |e(t_0)|^2 + N^{-2s}||u||^2_{L^2(t_0,T;H^s)} \\ &\quad + N^{-2s}||u(t)||_s^2 + N^{-2(1+\sigma)}||u'||^2_{L^2(t_0,T;H^\sigma)} \Big\} \ , \ \forall \ t \in (t_0, T). \end{aligned} \tag{71}$$

Therefore by taking $t = nk$ in (71), we conclude that (61) is true with some constants d_3, d_4 only depending on $d, \Omega, t_0, T, |u_0|$.

For $u_0 \in V \cap L^{2p}$, we proceed exactly as above with t_0 replaced by 0, by using the results of Theorem 4 and $e(0) = 0$, we recover (62). ¶

Remark 3 *For $u_0 \in V \cap L^{2p}$, this theorem provides an optimal uniform error estimate in any inteval $[t_0, T]$ ($t_0 > 0$) without further regularity assumptions on the solution.*

It remains to prove

Lemma 6

$$||u - \Pi_N u||_{-1} \leq c_{55} N^{-(s+1)} ||u||_s \ , \ \forall \ s \geq 1, u \in H^s(\Omega) \cap H_0^1(\Omega). \tag{72}$$

PROOF: Let us define an operator T by

$$a(Tg, v) = < g, v >_{(H^{-1}, H_0^1)} . \tag{73}$$

One can readily check that T is continuous from H^s into $H^{s+2} \cap H_0^1$ for all $s \geq 0$ (see Maday & Quarteroni [10]).

Therefore, $\forall\, u \in H_0^1 \cap H^s$, we infer from (73) and (8) that

$$\begin{aligned} ||u - \Pi_N u||_{-1} &= \sup_{\phi \in H_0^1} \frac{< u - \Pi_N u, T >}{||\phi||} \\ &= \sup_{\phi \in H_0^1} \frac{a(u - \Pi_N u, T\phi - \Pi_N T\phi)}{||\phi||} \\ &\leq c_{56} ||u - \Pi_N u|| \sup_{\phi \in H_0^1} \frac{||T\phi - \Pi_N \phi||}{||\phi||}. \end{aligned}$$

We then derive from (69) that

$$||u - \Pi_N u||_{-1} \leq c_{57} N^{-2} ||u - \Pi_N u|| \sup_{\phi \in H_0^1} \frac{||T\phi||_3}{||\phi||} \leq c_{55} N^{-(s+1)} ||u - \Pi_N u||_s . \P$$

Thanks to Theorem 2, we can apply Lemma 5 with $s = 2$ and $\sigma = 1$, which gives

Corollary 1 *For $|u_0| \leq R_0$, there exist two constants d_5, d_6 only depending on d, Ω, t_0, T, R_0 such that*

$$|u^n - u(nk)|^2 \leq d_5 exp(d_6(nk - t_0)) \left\{ |u^{\frac{t_0}{k}} - u(t_0)|^2 + N^{-4} + k^2 \right\} \ ,\ \forall\, t_0 \leq nk \leq T. \tag{74}$$

We are now in position to state our main theorem.

Theorem 7 *Under the assumption (33), $\mathcal{A}_{N,k}$ converges to $\mathcal{A}$ in the sense of semi-distance*

$$d(\mathcal{A}_{N,k}, \mathcal{A}) \to 0 \ \text{as}\ k, N^{-1} \to 0.$$

Equivalently, this means that for any neighborhood $\mathcal{V}$ of $\mathcal{A}$ in H, there exists $N_0(\mathcal{V})$ and $k_0(\mathcal{V})$ such that

$$\mathcal{A}_{N,k} \subset \mathcal{V} \ ,\ \forall\, N \geq N_0 \ \text{and}\ k \leq k_0.$$

PROOF: In order to apply the general convergence theorem (Thm. 1.2 of [13]) in the discrete case, the only condition to be verified is that

($\star$) u^n converges to $u(nk)$ uniformly for u_0 in a bounded set of H and for nk in any compact set of $(0, \infty)$.

In virtue of Corollary 1, we see that ($\star$) is true if we can show for $t_0 > 0$ fixed and $nk = t_0$, we have

$$|u^n - u(t_0)| \to 0 \ \text{uniformly for}\ u_0 \ \text{in a bounded set of}\ H. \tag{75}$$

We will prove (75) by a contradiction argument.

If (75) were not true, we could find $R_1, \delta > 0$, subsequences $k_j, N_j^{-1} \to 0$ and $u_{0,j}$ satisfying $|u_{0,j}| \leq R_1$, $\forall\, j$ such that

$$|u_j^n - u_j(t_0)| \geq \delta > 0 \ , \ \forall\, j \tag{76}$$

where u_j is the solution of (1)-(2) with initial value $u_{0,j}$; and $u_j^n = u_j^n(k_j, N_j)$ is the corresponding approximate solution of (9)-(10).

By extracting a subsequence, still denote by $\{j\}$, we can assume

$$u_{0,j} \to v_0 \ \text{weakly in H as} \ j \to \infty.$$

Let $v(t)$ denote the solution of (1)-(2) with initial value v_0, the *a priori* estimates we obtained in Thm. 4 for u^n and the analog estimates for $v(t)$ enable us to prove, in a standard manner, the following convergence results

$$u_j(t_0) \to v(t_0) \ \text{weakly in V as} \ j \to \infty,$$

$$u_j^n \to v(t_0) \ \text{weakly in V as} \ j \to \infty.$$

Hence

$$u_j^n - u_j(t_0) \to 0 \ \text{weakly in V as} \ j \to \infty.$$

Since the embedding of V in H is compact, the above convergence holds strongly in H, in contradiction with (76). ¶

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