ELSEVIER

Applied Numerical Mathematics 21 (1996) 71–90

APPLIED NUMERICAL MATHEMATICS

On a new pseudocompressibility method for the incompressible Navier–Stokes equations ☆

Jie Shen

Department of Mathematics, Penn State University, University Park, PA 16802, USA

Abstract

We propose and analyze a new pseudocompressibility method which is obtained by introducing a pressure stabilizing/regularizing term in the equation of mass conservation. The perturbed system can be viewed as an approximation to the incompressible Navier–Stokes equations, and its discretization can lead to efficient and accurate numerical schemes for the Navier–Stokes equations. An error analysis for a related artificial compressibility method for the Navier–Stokes equations is also carried out.

Keywords: Artificial compressibility; Navier–Stokes equations; Perturbation; Pressure stabilization; Projection method; Pseudocompressibility

1. Introduction

Let $\Omega \in \mathbb{R}^d$ (with $d = 2$ or 3) be an open bounded set with a sufficiently smooth boundary. We consider the unsteady incompressible Navier–Stokes equations:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega\times[0,T], \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\times[0,T], \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \tag{1.2}$$

subject to, for the sake of simplicity, the homogeneous Dirichlet boundary condition for the velocity $\boldsymbol{u}$, i.e., $\boldsymbol{u}|_{\partial\Omega} = 0$, $\forall t \in [0,T]$. The purpose of this work is to propose and study a new pseudocompressibility approximation for the system (1.1)–(1.2).

On of the main difficulties in a numerical procedure for approximating the solution of the Navier–Stokes equations is introduced by the incompressibility constraint "$\operatorname{div}\boldsymbol{u} = 0$", which not only couples the velocity $\boldsymbol{u}$ and the pressure p, but also requires that the approximation spaces for the velocity and the pressure satisfy the so called Babuška–Brezzi inf–sup condition. There exists a vast literature on numerical approximations of the Stokes equations and the incompressible Navier–Stokes equations in

☆ This work is partially supported by NSF grant DMS-9205300.

SSDI 0168-9274(95)00132-8

primitive-variable formulation. These numerical methods can be classified in three categories according to how the incompressibility constraint is treated:

(i) Using a divergence-free subspace for the velocity approximation (see, for instance, [17,19]): the pressure is then eliminated from the system, resulting in a well-behaved discrete system with a significant smaller number of unknowns. However, the divergence-free subspaces are usually not easy to construct and they involve in general tedious programming.

(ii) Using a pair of compatible subspaces which satisfy the Babuška–Brezzi inf–sup condition for the velocity and the pressure (cf. [3,9] for finite element methods and cf. [5] for spectral methods): the resulting discrete system is coupled and unfortunately indefinite.

(iii) Relaxing the incompressibility constraint in an appropriate way: this leads to a class of pseudocompressibility methods, among which are: the *penalty method* (cf. [12,24,26]) and the *artificial compressibility method* in which the pressure can be eliminated but the resulting discrete system becomes ill-conditioned when the perturbation parameter $\varepsilon \ll 1$ (cf. [6,28]); the *pressure stabilization method* (cf. [2,4,11] for the steady case and cf. [18,21] for the unsteady case) which results in a coupled discrete system which is positive definite, though in general nonsymmetric; and the *projection method* which leads to a cascade of decoupled discrete Helmholtz equations for the velocity and for the pressure (see, for instance, [7,8,20,27]). The projection method is perhaps the most efficient and the easiest to implement for solving the unsteady Navier–Stokes equations.

In this work, we introduce a new pseudocompressibility method, very similar to the pressure-stabilization method (see [18,21]) in the sense that the artificial compressibility method is similar to the penalty method. The new pseudocompressibility method we propose is to approximate the solution $(\boldsymbol{u}, p)$ of the Navier–Stokes equations (1.1)–(1.2) by $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ satisfying the following perturbed system:

$$\boldsymbol{u}_t^\varepsilon - \nu\Delta\boldsymbol{u}^\varepsilon + \widetilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) + \nabla p^\varepsilon = \boldsymbol{f}, \tag{1.3}$$

$$\operatorname{div}\boldsymbol{u}^\varepsilon - \varepsilon\Delta p_t^\varepsilon = 0, \qquad \frac{\partial p_t^\varepsilon}{\partial \boldsymbol{n}}\Big|_{\partial\Omega} = 0, \tag{1.4}$$

with $\boldsymbol{u}^\varepsilon|_{t=0} = \boldsymbol{u}_0^\varepsilon$, $p^\varepsilon|_{t=0} = p_0^\varepsilon$. We note that $\widetilde{B}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \frac{1}{2}(\nabla\cdot\boldsymbol{u})\boldsymbol{v}$ is the modified bilinear term, introduced in [26] to ensure the dissipativity of the velocity.

The perturbed system, similar to the system in artificial compressibility method, should be viewed as an approximation to the system (1.1)–(1.2). We emphasize that efficient and accurate numerical schemes for the Navier–Stokes equations can be constructed by discretizing (1.3)–(1.4). For instance, we consider the following second-order scheme for (1.3)–(1.4) (corresponding to $\varepsilon = \beta\Delta t^2$):

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\Delta(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n) + \nabla p^{n+1/2} = \boldsymbol{f}(t_{n+1/2}) - \mathrm{NLT}, \\ \nabla\cdot\boldsymbol{u}^{n+1} - \beta\Delta t\Delta(p^{n+1} - p^n) = 0, \qquad \dfrac{\partial p^{n+1}}{\partial\boldsymbol{n}}\Big|_{\partial\Omega} = \dfrac{\partial p^n}{\partial\boldsymbol{n}}\Big|_{\partial\Omega}, \end{cases} \tag{1.5}$$

where β is an appropriate constant, NLT is a certain second-order approximation to $B(\boldsymbol{u}(t_{n+1/2}), \boldsymbol{u}(t_{n+1/2}))$. We shall consider two different choices for $p^{n+1/2}$:

(i) $p^{n+1/2} = (1/2)(p^{n+1} + p^n)$. Then (1.5) is a *coupled* positive definite, though nonsymmetric, system for $(\boldsymbol{u}^{n+1}, p^{n+1})$.

(ii) $p^{n+1/2} = (1/2)(3p^n - p^{n-1})$. In this case, $\boldsymbol{u}^{n+1}$ and p^{n+1} in (1.5) are totally *decoupled* and can be obtained very efficiently by solving a vector Helmholtz equation (for $\boldsymbol{u}^{n+1}$) and a scalar Poisson equation (for p^{n+1}).

Furthermore, the scheme (1.5) is very flexible in the sense that one can use virtually any discretization pair for the velocity and the pressure, in particular, finite elements of equal order which are otherwise unstable.

Moreover, the projection methods proposed in [13,31] can also be interpreted as time discretizations of the perturbed system (1.3)–(1.4) (cf. [23,25]). Indeed, let us consider the following projection scheme considered in [22]:

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\Delta\left(\widetilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n\right) + \nabla p^n = \boldsymbol{f}(t_{n+1/2}) - \text{NLT}, \\ \left(\widetilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n\right)\big|_{\partial\Omega} = 0, \end{cases} \tag{1.6}$$

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\Delta t} - \dfrac{1}{2}\nabla\left(p^{n+1} - p^n\right) = 0, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n}|_{\partial\Omega} = 0. \end{cases} \tag{1.7}$$

Let P be the projector in $\boldsymbol{L}^2(\Omega)$ onto the divergence free subspace

$$\boldsymbol{H} = \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\colon\ \nabla \cdot \boldsymbol{v} \in L^2(\Omega),\ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\right\}. \tag{1.8}$$

Then we have $\boldsymbol{u}^{n+1} = P\widetilde{\boldsymbol{u}}^{n+1}$, which explains why we refer (1.6)–(1.7) as a projection scheme. Note that $\boldsymbol{u}^{n+1}$ can be eliminated from (1.6)–(1.7). Taking the sum of (1.6) at step n and (1.7) at step $n-1$, and applying the divergence operator to (1.7), we obtain

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}}^{n+1} - \widetilde{\boldsymbol{u}}^n}{\Delta t} - \dfrac{\nu}{2}\Delta\left(\widetilde{\boldsymbol{u}}^{n+1} + P\widetilde{\boldsymbol{u}}^n\right) + \dfrac{1}{2}\nabla\left(3p^n - p^{n-1}\right) = \boldsymbol{f}(t_{n+1/2}) - \text{NLT}, \\ \left(\widetilde{\boldsymbol{u}}^{n+1} + P\widetilde{\boldsymbol{u}}^n\right)\big|_{\partial\Omega} = 0. \end{cases} \tag{1.9}$$

$$\nabla \cdot \widetilde{\boldsymbol{u}}^{n+1} - \frac{1}{2}\Delta t\Delta\left(p^{n+1} - p^n\right) = 0, \qquad \frac{\partial p^{n+1}}{\partial \boldsymbol{n}}\bigg|_{\partial\Omega} = \frac{\partial p^n}{\partial \boldsymbol{n}}\bigg|_{\partial\Omega}. \tag{1.10}$$

Thus, (1.9)–(1.10) is a second-order time discretization of the perturbed system (1.3)–(1.4) with $\varepsilon = (1/2)\Delta t^2$.

Therefore, the error behaviors of the scheme (1.5) and the projection method (1.6)–(1.7) are intimately related to the approximation error of (1.3)–(1.4) with respect to (1.1)–(1.2). The goal of this work is to study the perturbed system (1.3)–(1.4) and carry out an error analysis in term of the perturbation parameter ε. The results here may serve, in particular, as the guideline for obtaining second-order error estimates for the scheme (1.5) and the projection methods proposed in [31] and [13]. Moreover, it is hoped that the system (1.3)–(1.4) would lead to new efficient schemes for approximating the incompressible Navier–Stokes equations.

One of the main difficulties in analyzing (1.3)–(1.4) is that it lacks a dissipative mechanism for the pressure and does not possess the smoothing property enjoyed by (1.1)–(1.2). In fact, certain compatibility conditions for the initial data are needed for the solution of (1.3)–(1.4) to be more regular or to have ε-independent *a priori* estimates which are essential to the error analysis. These compatibility conditions are conveniently satisfied by using the solution of (1.1)–(1.2) at a positive time t_0 as the initial data for (1.3)–(1.4). The above comments apply as well to the artificial compressibility method introduced in [6,28]. For the sake of comparison, we will also derive error estimates for an artificial compressibility method. The results should be interesting for its own sake, although convergence rate of some slightly compressible flows to the incompresible flows in different contexts and by totally different techniques are also available (cf. [14,15]).

The rest of the paper is organized as follows. In the next section, we study the existence, uniqueness and regularity of the perturbed system (1.3)–(1.4). In Section 3, we perform an error analysis for the perturbed system with respect to (1.1)–(1.2). Then in Section 4, we derive error estimates for an artificial compressibility method with respect to (1.3)–(1.4).

2. Study of the perturbed problem

2.1. Preliminaries

We describe below some of the notations and results which will be frequently used in this paper.

We will use the standard notations $L^2(\Omega)$, $H^k(\Omega)$ and $H_0^k(\Omega)$ to denote the usual Sobolev spaces over Ω. The norm corresponding to $H^k(\Omega)$ will be denoted by $\|\cdot\|_k$. In particular, we will use $\|\cdot\|$ to denote the norm in $L^2(\Omega)$ and $(\cdot,\cdot)$ to denote the scalar product in $L^2(\Omega)$. As usual, the dual space of $H_0^1(\Omega)$ will be denoted by $H^{-1}(\Omega)$ and the duality between them will be denoted by $\langle\cdot,\cdot\rangle$. The vector functions and vector spaces will be denoted by bold face letters. To simplify the notation, we shall omit the space variables from the notation, i.e., $\boldsymbol{v}(t)$ should be considered as a function of t with value in a Sobolev space. We will use C to denote a generic constant which may depend on the data Ω, ν, $\boldsymbol{f}, \ldots,$ but will be *independent* of the perturbation parameter ε. Since we are only interested in the case $\varepsilon \ll 1$, we will assume throughout the paper that $0 < \varepsilon \leqslant 1$.

We now introduce some operators usually associated with the Navier–Stokes equations and its approximations.

$$B(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \qquad \widetilde{B}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \frac{1}{2}(\nabla\cdot\boldsymbol{u})\boldsymbol{v},$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \big(B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\big), \qquad \tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \big(\widetilde{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\big).$$

We note that

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \quad \forall \boldsymbol{u}\in\boldsymbol{V},\ \forall \boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega), \tag{2.1}$$

where $\boldsymbol{V} = \{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\colon\ \nabla\cdot\boldsymbol{v} = 0\}$. One can also easily check with integration by part that

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \frac{1}{2}\big\{b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) - b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\big\}, \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega). \tag{2.2}$$

Therefore, we have

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \quad \forall \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega). \tag{2.3}$$

The inequality (cf. [1]) is useful for dealing with the trilinear form $\tilde{b}$:

$$\|v\|_{L^\infty(\Omega)} \leqslant C \begin{cases} \|v\|^{1/2}\|v\|_2^{1/2} & \text{if } d=2, \\ \|v\|_1^{1/2}\|v\|_2^{1/2} & \text{if } d=3, \end{cases} \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.4}$$

The following three inequalities will be used repeatedly in the sequel.

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leqslant C\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1^{1/2}\|\boldsymbol{v}\|_2^{1/2}\|\boldsymbol{w}\|, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}_0^1(\Omega),\ \boldsymbol{u},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \tag{2.5}$$

The above inequality is valid for $d \leqslant 3$. Occasionally, we will use the following inequality which is only valid for $d = 2$:

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leqslant C\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|_1^{1/2}\big(\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|^{1/2}\|\boldsymbol{w}\|_1^{1/2} + \|\boldsymbol{w}\|_1\|\boldsymbol{v}\|^{1/2}\|\boldsymbol{v}\|_1^{1/2}\big),$$
$$\forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \tag{2.6}$$

In most cases, the following inequality, which is valid for $d \leqslant 4$, is sufficient for our purposes:

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leqslant \begin{cases} \|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, & \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \\ \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|\|\boldsymbol{w}\|_1, & \forall \boldsymbol{u} \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega),\ \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \\ \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|, & \forall \boldsymbol{u} \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega),\ \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \\ \|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|, & \forall \boldsymbol{v} \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega),\ \boldsymbol{u},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \end{cases} \tag{2.7}$$

These inequalities can be proved by using (2.2), (2.4), Holder's inequality and Sobolev inequalities (see, for instance, [29, Lemma 2.1]).

We shall frequently use, without mentioning, the following norm equivalences:

$$\|v\|_1 \sim \|\nabla_v\|,\ \forall v \in H_0^1(\Omega) \text{ or } H^1(\Omega)/\mathbb{R}, \qquad \|v\|_2 \sim \|\Delta_v\|,\ \forall v \in H^2(\Omega)\cap H_0^1(\Omega).$$

The following lemma of Gronwall type will be repeatedly used.

Lemma 2.1 (Gronwall lemma). *Let $y(t)$, $h(t)$, $g(t)$, $f(t)$ be nonnegative functions such that $\int_0^T g(t)\,\mathrm{d}t \leqslant M$ and either*

$$y(t) + \int_0^t h(s)\,\mathrm{d}s \leqslant y(0) + \int_0^t \big(g(s)y(s) + f(s)\big)\,\mathrm{d}s, \quad \forall\, 0 \leqslant t \leqslant T,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + h(t) \leqslant g(t)y(t) + f(t), \quad \forall\, 0 \leqslant t \leqslant T.$$

Then

$$y(t) + \int_0^t h(s)\,\mathrm{d}s \leqslant \exp(M)\left(y(0) + \int_0^t f(s)\,\mathrm{d}s\right), \quad \forall\, 0 < t \leqslant T.$$

2.2. Existence, uniqueness and regularity for the perturbed problem

Let us first establish the following results.

Theorem 2.1. *We assume that*

$$\boldsymbol{u}_0^\varepsilon \in \boldsymbol{H}_0^1(\Omega), \qquad p_-^\varepsilon \in H^1(\Omega)/\mathbb{R}, \qquad \boldsymbol{f} \in L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big).$$

Then there exists $K_0 = K_0(\nu, \boldsymbol{f}, T, \Omega, \boldsymbol{u}_0^\varepsilon, p_0^\varepsilon)$ *such that for*

$$T_0 = T_0(\varepsilon) = \begin{cases} T, & d = 2, \\ \min\{T, \varepsilon^2/K_0\}, & d = 3, \end{cases}$$

the problem (1.3)–(1.4) *admits a unique solution* $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ *in* $[0, T_0]$ *satisfying*

$$\boldsymbol{u}^\varepsilon \in L^2\big(0,T_0;\boldsymbol{H}^2(\Omega)\big) \cap L^\infty\big(0,T_0;\boldsymbol{H}_0^1(\Omega)\big), \quad p^\varepsilon \in L^\infty\big(0,T_0;\boldsymbol{H}^1(\Omega)/\mathbb{R}\big).$$

Proof (*Outline*). We use c_i to denote constants only depending on Ω and use K_i to denote constants depending on ν, $\boldsymbol{f}$, T, Ω, $\boldsymbol{u}_0^\varepsilon$ and p_0^ε.

The proof is based on a sequence of *a priori* estimates. We assume first that $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ is a sufficiently smooth solution of the problem (1.3)–(1.4). Taking the inner product of (1.3) with $\boldsymbol{u}^\varepsilon$ and of (1.4) with p^ε, summing up the two relations and using (2.3), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^\varepsilon\|^2 + \nu\|\nabla\boldsymbol{u}^\varepsilon\|^2 + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla p^\varepsilon\|^2 = \langle\boldsymbol{f}, \boldsymbol{u}^\varepsilon\rangle \leqslant \frac{\nu}{2}\|\nabla\boldsymbol{u}^\varepsilon\|^2 + \frac{1}{2\nu}\|\boldsymbol{f}\|_{-1}^2.$$

We then integrate the above inequality to get

$$\big\|\boldsymbol{u}^\varepsilon(t)\big\|^2 + \nu\int_0^t \big\|\nabla\boldsymbol{u}^\varepsilon(s)\big\|^2\mathrm{d}s + \varepsilon\big\|\nabla p^\varepsilon(t)\big\|^2 \leqslant K_1, \quad \forall t \in [0,T], \tag{2.8}$$

with

$$K_1 = \frac{1}{\nu}\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{H}^{-1})}^2 + \|\boldsymbol{u}_0^\varepsilon\|^2 + \|\nabla p_0^\varepsilon\|^2.$$

To obtain further *a priori* estimates, we rewrite (1.3) as an equation for $\boldsymbol{u}^\varepsilon$ only, considering ∇p^ε as a source function:

$$\begin{aligned} &\boldsymbol{u}_t^\varepsilon - \nu\Delta\boldsymbol{u}^\varepsilon + \widetilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) = \boldsymbol{f} - \nabla p^\varepsilon, \\ &\boldsymbol{u}^\varepsilon(0) = \boldsymbol{u}_0^\varepsilon. \end{aligned} \tag{2.9}$$

Taking the inner product of (2.9) with $-\Delta\boldsymbol{u}^\varepsilon$, we derive

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}^\varepsilon\|^2 + \nu\|\Delta\boldsymbol{u}^\varepsilon\|^2 &= -(\boldsymbol{f} - \nabla p^\varepsilon, \Delta\boldsymbol{u}^\varepsilon) + \tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon, \Delta\boldsymbol{u}^\varepsilon) \\ &\leqslant \frac{\nu}{4}\|\Delta\boldsymbol{u}^\varepsilon\|^2 + \frac{2}{\nu}\big(\|\boldsymbol{f}\|^2 + \|\nabla p^\varepsilon\|^2\big) + \tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon, \Delta\boldsymbol{u}^\varepsilon). \end{aligned} \tag{2.10}$$

Now the computation is different according to the space dimension.

(i) For $d=2$, we derive from the Sobolev inequality (2.4) that

$$\begin{aligned}\tilde{b}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}^\varepsilon,\Delta\boldsymbol{u}^\varepsilon) &\leqslant c_1\|\boldsymbol{u}^\varepsilon\|_{\boldsymbol{L}^\infty(\Omega)}\|\boldsymbol{u}^\varepsilon\|_1\|\boldsymbol{u}^\varepsilon\|_2\\ &\leqslant c_2\|\boldsymbol{u}^\varepsilon\|^{1/2}\|\boldsymbol{u}^\varepsilon\|_1\|\boldsymbol{u}^\varepsilon\|_2^{3/2}\leqslant \frac{\nu}{4}\|\Delta\boldsymbol{u}^\varepsilon\|^2+\frac{c_3}{\nu}\|\boldsymbol{u}^\varepsilon\|^2\|\Delta\boldsymbol{u}^\varepsilon\|^4.\end{aligned}$$

Thanks to the estimate (2.8), we have

$$\frac{2c_3}{\nu}\int_0^T\|\boldsymbol{u}^\varepsilon\|^2\|\Delta\boldsymbol{u}^\varepsilon\|^2\mathrm{d}s\leqslant\frac{2c_3}{\nu}\frac{K_1^2}{\nu}=K_2.$$

Hence, integrating (2.10) and applying Lemma 2.1, we obtain

$$\begin{aligned}\left\|\nabla\boldsymbol{u}^\varepsilon(t)\right\|^2+\nu\int_0^t\left\|\nabla\boldsymbol{u}^\varepsilon(s)\right\|^2\mathrm{d}s &\leqslant\exp(K_2)\left(\|\nabla\boldsymbol{u}_0^\varepsilon\|^2+\int_0^t\left(\left\|\boldsymbol{f}(s)\right\|^2+\left\|\nabla p^\varepsilon(s)\right\|^2\right)\mathrm{d}s\right)\\ &\leqslant\frac{K_3}{\varepsilon},\quad \forall t\in[0,T].\end{aligned}\tag{2.11}$$

(ii) For $d=3$, we use (2.4) and Young's inequality to derive

$$\begin{aligned}\tilde{b}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}^\varepsilon,\Delta\boldsymbol{u}^\varepsilon) &\leqslant c_4\|\boldsymbol{u}^\varepsilon\|_{\boldsymbol{L}^\infty(\Omega)}\|\boldsymbol{u}^\varepsilon\|_1\|\boldsymbol{u}^\varepsilon\|_2\\ &\leqslant c_5\|\nabla\boldsymbol{u}^\varepsilon\|^{3/2}\|\Delta\boldsymbol{u}^\varepsilon\|^{3/2}\leqslant\frac{\nu}{4}\|\Delta\boldsymbol{u}^\varepsilon\|^2+\frac{c_6}{\nu}\|\nabla\boldsymbol{u}^\varepsilon\|^6.\end{aligned}$$

Thanks to (2.8) and the last inequality, we derive from (2.10) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}^\varepsilon\|^2+\nu\|\Delta\boldsymbol{u}^\varepsilon\|^2\leqslant\frac{K_4}{\varepsilon}+K_5\|\nabla\boldsymbol{u}^\varepsilon\|^6.\tag{2.12}$$

Setting $y(t)=K_4/\varepsilon+\|\nabla\boldsymbol{u}^\varepsilon(t)\|^2$, since we can always assume $K_5\geqslant 1$, we infer from (2.12) that y satisfies the differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}y\leqslant K_5y^3,\qquad y(0)=\frac{K_4}{\varepsilon}+\|\nabla\boldsymbol{u}_0^\varepsilon\|^2.$$

Solving the above inequality, we get

$$y(t)\leqslant\frac{y(0)}{\sqrt{1-2K_5y(0)^2t}},\quad \text{for } t\leqslant\frac{1}{2K_5y(0)^2},$$

which also implies

$$y(t)\leqslant 2y(0),\quad \text{for } t\leqslant\frac{1}{4K_5y(0)^2}.$$

Since there exists $K_0>0$ such that

$$\frac{1}{4K_5y(0)^2}=\frac{1}{4K_5(K_4/\varepsilon+\|\nabla\boldsymbol{u}_0^\varepsilon\|^2)^2}\leqslant\frac{\varepsilon^2}{K_0},$$

we conclude from the last inequality that

$$\left\|\nabla\boldsymbol{u}^\varepsilon(t)\right\|^2\leqslant 2y(0)=2\left(\frac{K_4}{\varepsilon}+\|\nabla\boldsymbol{u}_0^\varepsilon\|^2\right)\leqslant\frac{K_6}{\varepsilon},\quad\forall 0\leqslant t\leqslant T_0(\varepsilon)=\min\left\{T,\frac{\varepsilon^2}{K_0}\right\}.\tag{2.13}$$

Finally, we derive from the last estimate, (2.8) and (2.12) that

$$\int_0^t \|\nabla u^\varepsilon(s)\|^2 \mathrm{d}s \leqslant \frac{K_7}{\varepsilon^2}, \quad \forall t \in [0, T_0]. \tag{2.14}$$

Now we have derived all the necessary *a priori* estimates. The remainder of the proof is then standard with an implementation of the Galerkin method and an utilization of a compactness theorem. We refer to [16] and [30] for more details on this matter. □

By using the ε-independent *a priori* estimate (2.8) for u^ε, one can prove by standard technique (see, for instance, [30, Chapter 3]) the existence of a weak solution $(u^\varepsilon, p^\varepsilon) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \times H^{-1}(Q)$ in the three dimensional case. One can also establish the following convergence results.

Proposition 2.1. *Let (u, p) be the unique strong solution of* (1.1)–(1.2) *in $Q = [0, T_1] \times \Omega$ ($T_1 = T$ if $d = 2$, and $T_1 \leqslant T$ is a constant depending on the data if $d = 3$). Then under the assumption of Theorem* 2.1 *and as $\varepsilon \to 0$, we have:*

$$u^\varepsilon \to u \quad \text{in } L^2(0, T_1; H_0^1(\Omega)) \text{ and in } L^q(0, T_1; L^2(\Omega)) \text{ for all } 1 \leqslant q < \infty,$$

$$\nabla p^\varepsilon \to \nabla p \quad \text{in } H^{-1}(Q),$$

where $(u^\varepsilon, p^\varepsilon)$ is the strong solution of (1.3)–(1.4) *when $d = 2$, and $(u^\varepsilon, p^\varepsilon)$ is any weak solution of* (1.3)–(1.4) *when $d = 3$.*

We note that the convergence of ∇p^ε is too weak to have any practical significance, owing to the lack of ε-independent estimate of $\|\nabla p^\varepsilon\|$. In the next section we shall derive, by using a special initial data $(u_0^\varepsilon, p_0^\varepsilon)$, ε-independent estimates on T_0 and improved convergence results for the velocity and the pressure.

The solution of (1.3)–(1.4) possesses further regularities provided that the data is more regular. For instance, we have the following regularity results.

Theorem 2.2. *We assume that*

$$u_0^\varepsilon \in H^2(\Omega), \qquad p_0^\varepsilon \in H^1(\Omega)/\mathbb{R}, \qquad f \in L^\infty(0,T;L^2(\Omega)), \qquad f_t \in L^2(0,T;H^{-1}(\Omega)).$$

Then the solution $(u^\varepsilon, p^\varepsilon)$ of (1.3)–(1.4) *satisfies*

$$u^\varepsilon \in L^\infty(0,T_0;H^2(\Omega) \cap H_0^1(\Omega)), \qquad u_t^\varepsilon \in L^\infty(0,T_0;L^2(\Omega)) \cap L^2(0,T_0;H_0^1(\Omega)),$$

$$p^\varepsilon, p_t^\varepsilon \in L^\infty(0,T_0;H^1(\Omega)/\mathbb{R}).$$

Proof. Taking the time derivative of (1.3)–(1.4), we obtain

$$u_{tt}^\varepsilon - \nu\Delta u_t^\varepsilon + \widetilde{B}(u_t^\varepsilon, u^\varepsilon) + \widetilde{B}(u^\varepsilon, u_t^\varepsilon) + \nabla p_t^\varepsilon = f_t, \tag{2.15}$$

$$\operatorname{div} u_t^\varepsilon - \varepsilon\Delta p_{tt}^\varepsilon = 0, \qquad \left.\frac{\partial p_{tt}^\varepsilon}{\partial n}\right|_{\partial\Omega} = 0. \tag{2.16}$$

We complete the system (2.15)–(2.16) with the initial condition $(\boldsymbol{u}_t^\varepsilon(0), p_t^\varepsilon(0))$ defined by

$$\boldsymbol{u}_t^\varepsilon(0) = \boldsymbol{f}(0) + \nu\Delta\boldsymbol{u}_0^\varepsilon - \widetilde{B}(\boldsymbol{u}_0^\varepsilon, \boldsymbol{u}_0^\varepsilon) - \nabla p_0^\varepsilon, \tag{2.17}$$

$$\varepsilon\Delta p_t^\varepsilon(0) = \nabla\cdot\boldsymbol{u}_0^\varepsilon, \qquad \frac{\partial p_t^\varepsilon(0)}{\partial \boldsymbol{n}}\bigg|_{\partial\Omega} = 0. \tag{2.18}$$

The assumption on $\boldsymbol{f}$, $\boldsymbol{u}_0^\varepsilon$ and p_0^ε implies that

$$\left\|\boldsymbol{u}_t^\varepsilon(0)\right\| \leqslant C, \qquad \left\|\nabla p_t^\varepsilon(0)\right\| \leqslant \frac{C}{\varepsilon}\|\boldsymbol{u}_0^\varepsilon\| \leqslant \frac{C}{\varepsilon}. \tag{2.19}$$

We note that (2.17)–(2.18) can be rigorously justified by using the *a priori* estimates below and by implementing a Galerkin method.

Taking the inner product of (2.15) with $\boldsymbol{u}_t^\varepsilon$ and of (2.16) with p_t^ε, in view of (2.3), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_t^\varepsilon\|^2 + \nu\|\nabla\boldsymbol{u}_t^\varepsilon\|^2 + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla p_t^\varepsilon\|^2 &= \langle \boldsymbol{f}_t, \boldsymbol{u}_t^\varepsilon\rangle - \tilde{b}(\boldsymbol{u}_t^\varepsilon, \boldsymbol{u}^\varepsilon, \boldsymbol{u}_t^\varepsilon)\\ &\leqslant \frac{\nu}{4}\|\nabla\boldsymbol{u}_t^\varepsilon\|^2 + C\|\boldsymbol{f}_t\|_{-1}^2 - \tilde{b}(\boldsymbol{u}_t^\varepsilon, \boldsymbol{u}^\varepsilon, \boldsymbol{u}_t^\varepsilon).\end{aligned} \tag{2.20}$$

We now bound the nonlinear term as follows according to the space dimension.

(i) For $d = 2$, we use (2.6) and Young's inequality to derive

$$\begin{aligned}\tilde{b}(\boldsymbol{u}_t^\varepsilon, \boldsymbol{u}^\varepsilon, \boldsymbol{u}_t^\varepsilon) &\leqslant C\|\boldsymbol{u}_t^\varepsilon\|\,\|\boldsymbol{u}_t^\varepsilon\|_1\|\boldsymbol{u}^\varepsilon\|_1 + C\|\boldsymbol{u}_t^\varepsilon\|^{1/2}\|\boldsymbol{u}_t^\varepsilon\|_1^{3/2}\|\boldsymbol{u}^\varepsilon\|^{1/2}\|\boldsymbol{u}^\varepsilon\|_1^{1/2}\\ &\leqslant \frac{\nu}{4}\|\nabla\boldsymbol{u}_t^\varepsilon\|^2 + C\big(\|\boldsymbol{u}^\varepsilon\|^2+1\big)\|\boldsymbol{u}^\varepsilon\|_1^2\|\boldsymbol{u}_t^\varepsilon\|^2.\end{aligned}$$

Using (2.8), (2.19) and applying Lemma 2.1 to (2.20), we get

$$\left\|\boldsymbol{u}_t^\varepsilon(t)\right\|^2 + \int_0^t \left\|\nabla\boldsymbol{u}_t^\varepsilon(s)\right\|^2\mathrm{d}s + \varepsilon\|\nabla p_t^\varepsilon\|^2 \leqslant \frac{C}{\varepsilon}, \quad \forall t\in[0,T].$$

(ii) For $d = 3$, we use (2.5) and Young's inequality to obtain

$$\tilde{b}(\boldsymbol{u}_t^\varepsilon, \boldsymbol{u}^\varepsilon, \boldsymbol{u}_t^\varepsilon) \leqslant C\|\boldsymbol{u}_t^\varepsilon\|\,\|\boldsymbol{u}_t^\varepsilon\|_1\|\boldsymbol{u}^\varepsilon\|_2^{1/2}\|\boldsymbol{u}^\varepsilon\|_1^{1/2} \leqslant \frac{\nu}{4}\|\nabla\boldsymbol{u}_t^\varepsilon\|^2 + C\|\boldsymbol{u}^\varepsilon\|_1\|\boldsymbol{u}^\varepsilon\|_2\|\boldsymbol{u}_t^\varepsilon\|^2. \tag{2.21}$$

By using (2.13), (2.14) and Schwartz inequality,

$$\int_0^{T_0} \left\|\boldsymbol{u}^\varepsilon(s)\right\|_1\left\|\boldsymbol{u}^\varepsilon(s)\right\|_2\mathrm{d}s \leqslant C\varepsilon^{-1/2}\left(\int_0^{T_0}\|\boldsymbol{u}^\varepsilon\|_2^2\,\mathrm{d}s\right)^{1/2} \leqslant \frac{C}{\varepsilon^{3/2}},$$

then application of Lemma 2.1 to (2.20) with (2.21) implies that

$$\left\|\boldsymbol{u}_t^\varepsilon(t)\right\|^2 + \int_0^t \left\|\nabla\boldsymbol{u}_t^\varepsilon(s)\right\|^2\mathrm{d}s + \varepsilon\left\|\nabla p_t^\varepsilon(t)\right\|^2 \leqslant \frac{C}{\varepsilon}\exp\left(\frac{C}{\varepsilon^{3/2}}\right), \quad \forall t\in[0,T_0].$$

We now rewrite (1.3), for each t, as an elliptic equation:

$$-\nu\Delta\boldsymbol{u}^\varepsilon(t) + \widetilde{B}\big(\boldsymbol{u}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t)\big) = \boldsymbol{f}(t) - \boldsymbol{u}_t^\varepsilon(t) - \nabla p^\varepsilon(t),$$

with $\boldsymbol{u}^\varepsilon(t)|_{\partial\Omega} = 0$. Taking the inner product of the last relation with $-\Delta\boldsymbol{u}^\varepsilon(t)$, using (2.5) and previous estimates, we derive readily that

$$\left\|\boldsymbol{u}^\varepsilon(t)\right\|_2 \leqslant M(\varepsilon), \quad \forall t \in (0, T_0],$$

where $M(\varepsilon)$ is some function $\to +\infty$ as $\varepsilon \to 0$. The proof of the theorem is then complete. □

Remark 2.1. We note that further regularity for the solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ is only possible if the initial data $(\boldsymbol{u}_0^\varepsilon, p_0^\varepsilon)$ satisfies certain compatibility conditions. In fact, in order to have $\boldsymbol{u}_t^\varepsilon \in L^\infty(0, T_0; \boldsymbol{H}_0^1(\Omega))$, we must have $\boldsymbol{u}_t^\varepsilon(0)$, as defined in (2.17), in $\boldsymbol{H}_0^1(\Omega)$. In other wards, the data $(\boldsymbol{u}_0^\varepsilon, p_0^\varepsilon)$ must satisfy the following compatibility condition

$$\boldsymbol{f}(0) + \nu\Delta\boldsymbol{u}_0^\varepsilon - \widetilde{B}(\boldsymbol{u}_0^\varepsilon, \boldsymbol{u}_0^\varepsilon) - \nabla p_0^\varepsilon \in \boldsymbol{H}_0^1(\Omega). \tag{2.22}$$

The above condition is reminiscent to the first compatibility condition for the Navier–Stokes equations (see, for instance, [10,29]), but there is an essential difference. In the Navier–Stokes equations, $p|_{t=0}$ is determined by $(\boldsymbol{u}_0, \boldsymbol{f}, \Omega)$ so that the compatibility conditions are of nonlocal nature and in general cannot be satisfied no matter how smooth are the data $(\boldsymbol{u}_0, \boldsymbol{f}, \Omega)$. On the other hand, the compatibility condition (2.22) is not of nonlocal nature since p_0^ε is of our choice and for sufficiently smooth $(\boldsymbol{u}_0^\varepsilon, \boldsymbol{f})$ one can always choose p_0^ε to satisfy (2.22) and even further compatibility conditions.

3. Error estimates

In the last section, we have derived some *a priori* estimates for the solution of (1.3)–(1.4). However, most of these estimates are ε-dependent. Therefore, they are not suitable for the error analysis with respect to ε. Here we shall first derive some ε-independent *a priori* estimates by using $(\boldsymbol{u}(t_0), p(t_0))$, the solution of (1.1)–(1.2) at $t = t_0$, as the initial data for the system (1.3)–(1.4). We then use these ε-independent estimates to derive the desired error estimates.

Let us recall first some regularity results for the Navier–Stokes equations (1.1)–(1.2). It is well known that (see, for instance, [10, Theorem 2.4]) for

$$\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}, \qquad \boldsymbol{f} \in C\big([0,T]; \boldsymbol{L}^2(\Omega)\big), \tag{3.1}$$

there exists $T_1 \leqslant T$ ($T_1 = T$ if $d = 2$) such that the solution of (1.1)–(1.2) satisfies

$$\left\|\boldsymbol{u}(t)\right\|_2 + \left\|\boldsymbol{u}_t(t)\right\| + \left\|p(t)\right\|_1 \leqslant C, \quad \forall t \in [0, T_1]. \tag{3.2}$$

Higher regularity for the solution at $t = 0$ requires that the data $\boldsymbol{u}_0$ and $\boldsymbol{f}(0)$ satisfy certain nonlocal compatibility conditions. However, thanks to the smoothing property of the Navier–Stokes equations, the solution becomes as smooth as the data allows for $t > 0$. We now state a regularity result which is sufficient for our error analysis (see, for instance, [10]).

Proposition 3.1. *In addition to* (3.1), *we assume that*

$$\boldsymbol{f}_t, \boldsymbol{f}_{tt} \in C\big([0, T_1]; \boldsymbol{L}^2(\Omega)\big). \tag{3.3}$$

Then for any $t_0 \in (0, T_1)$, *the solution of* (1.1)–(1.2) *satisfies*

$$\left\|\boldsymbol{u}_t(t)\right\|_2^2 + \left\|p_t(t)\right\|_1^2 + \int_{t_0}^{t} \left(\left\|\boldsymbol{u}_{tt}(s)\right\|_2^2 + \left\|p_{tt}(s)\right\|_1^2\right) \mathrm{d}s \leqslant C, \quad \forall t \in [t_0, T_1]. \tag{3.4}$$

Due to the lack of dissipative mechanism in the pressure variable, the system (1.3)–(1.4) does not possess a similar smoothing property. In fact, for the solution of (1.3)–(1.4) to be more regular or to have ε-independent *a priori* estimates, the initial data must satisfy certain compatibility conditions. In the rest of this section, we fix $t_0 > 0$ and consider the system (1.3)–(1.4) for $t \in [t_0, T_1]$ with the initial condition at t_0 given by

$$\boldsymbol{u}^\varepsilon(t_0) = \boldsymbol{u}(t_0), \qquad p^\varepsilon(t_0) = p(t_0). \tag{3.5}$$

We note that the initial data in (3.5) satisfies in particular the compatibility condition (2.22). Moreover, it allows us to obtain further ε-independent *a priori* estimates required by the error analysis.

The main results are collected in the following theorem.

Theorem 3.1. *We assume* (3.1) *and* (3.3). *Then for* t_0 *sufficiently small, there exists* ε*-independent* $T_0 \in (t_0, T_1]$ ($T_0 = T$ *if* $d = 2$) *and* $C > 0$ *such that* $\forall t \in [t_0, T_0]$, *we have*

$$\int_{t_0}^{t} \left\|\boldsymbol{u}(s) - \boldsymbol{u}^\varepsilon(s)\right\|^2 \mathrm{d}s + \varepsilon^{1/2}\left\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\right\|^2 + \varepsilon\left(\left\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\right\|_1^2 + \left\|p(t) - p^\varepsilon(t)\right\|^2\right) \leqslant C\varepsilon^2.$$

Remark 3.1. It appears that the results in Theorem 3.1 are comparable to those for the pressure stabilization method in term of ε (see [18,21]). However, as pointed out in the introduction, the projection schemes based on the pressure stabilization method are stable only if $\varepsilon \sim \Delta t$, while the projection schemes based on the system (1.3)–(1.4) are stable for $\varepsilon \sim (\Delta t)^2$. Therefore, Theorem 3.1 suggests that the schemes (1.5) and (1.6)–(1.7) are second-order accurate for the velocity.

The remainder of this section is devoted to the proof of this theorem. We will first derive some ε-independent *a priori* estimates. Then we will split the errors into two parts and treat them separately. The first part, which is the dominating part of the error, is the error introduced by perturbing the linear operator. The second part, which is relatively smaller and easier to handle, is associated with the nonlinear term.

3.1. ε-independent a priori estimates

In order to prove Theorem 3.1, we need some ε-independent *a priori* estimates which become available thanks to the special initial data given in (3.5).

Lemma 3.1. *We assume* (3.1) *and* (3.3). *Then for* t_0 *sufficiently small, there exists* ε*-independent* $T_0 \in (t_0, T_0]$ ($T_0 = T$ *if* $d = 2$) *such that*

$$\left\|\boldsymbol{u}^\varepsilon(t)\right\|_2^2 + \left\|p^\varepsilon(t)\right\|_1^2 \leqslant C, \quad \forall t \in [t_0, T_0], \tag{3.6}$$

$$\int_{t_0}^{t} \left\|\boldsymbol{u}_t^\varepsilon(s)\right\|_2^2 \mathrm{d}s + \left\|\boldsymbol{u}_t^\varepsilon(t)\right\|_1^2 + \left\|p_t^\varepsilon(t)\right\|_1^2 \leqslant C, \quad \forall t \in [t_0, T_0]. \tag{3.7}$$

Furthermore, we have the following error estimates:

$$\left\|\boldsymbol{u}_t^\varepsilon(t) - \boldsymbol{u}_t(t)\right\|^2 \leqslant C\varepsilon, \quad \forall t \in [t_0, T_0]. \tag{3.8}$$

Proof. Let us denote $e = u - u^\varepsilon$, $q = p - p^\varepsilon$. Subtracting (1.3)–(1.4) from (1.1)–(1.2), we obtain the error equation for $(u^\varepsilon, p^\varepsilon)$:

$$e_t - \nu\Delta e + \widetilde{B}(u^\varepsilon, e) + \widetilde{B}(e, u) + \nabla q = 0, \tag{3.9}$$

$$(\nabla\cdot e, \gamma) + \varepsilon(\nabla q_t, \nabla\gamma) = \varepsilon(\nabla p_t, \nabla\gamma), \quad \forall\gamma\in H^1(\Omega)/\mathbb{R}, \tag{3.10}$$

with $e(t_0) = 0$ and $q(t_0) = 0$.

Taking the inner product of (3.9) with e and setting $\gamma = q$ in (3.10), thanks to (2.3) and (2.5), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 + \nu\|\nabla e\|^2 + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q\|^2 &= \varepsilon(\nabla p_t, \nabla q) - \tilde{b}(e, u, e)\\ &\leqslant \varepsilon(\nabla p_t, \nabla q) + C\|e\|\,\|e\|_1\|u\|_1^{1/2}\|u\|_2^{1/2}\\ &\leqslant \frac{\varepsilon}{2}\|\nabla p_t\|^2 + \frac{\varepsilon}{2}\|\nabla q\|^2 + \frac{\nu}{2}\|\nabla e\|^2 + C\|u\|_1\|u\|_2\|e\|^2.\end{aligned}$$

Applying Lemma 2.1 to the above inequality and using (3.2), we obtain

$$\big\|e(t)\big\|^2 + \nu\int_{t_0}^{t}\|\nabla e(s)\|^2\,\mathrm{d}s + \varepsilon\|\nabla q\|^2 \leqslant C\varepsilon\int_{t_0}^{t}\big\|\nabla p_t(s)\big\|^2\,\mathrm{d}s \leqslant C\varepsilon, \quad \forall t\in[t_0, T]. \tag{3.11}$$

We now take the inner product of (3.9) with $-\Delta e$, using (2.5) and (2.7), we derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla e\|^2 + \nu\|\Delta e\|^2 &= (\nabla q, \nabla e) + \tilde{b}(u^\varepsilon, e, \Delta e) + \tilde{b}(e, u, \Delta e)\\ &= (\nabla q, \Delta e) + \tilde{b}(u - e, e, \Delta e) + \tilde{b}(e, u, \Delta e)\\ &\leqslant (\nabla q, \Delta e) + C\|\nabla e\|^{3/2}\|\Delta e\|^{3/2} + C\|u\|_2\|\nabla e\|\,\|\Delta e\|\\ &\leqslant \frac{\nu}{2}\|\Delta e\|^2 + \frac{1}{\nu}\|\nabla q\|^2 + C\big(\|u\|_2^2 + \|\nabla e\|^4\big)\|\nabla e\|^2.\end{aligned} \tag{3.12}$$

Thanks to (3.2) and (3.11), we infer from the above inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(C_1 + \|\nabla e\|^2\big) \leqslant C_2\big(C_1 + \|\nabla e\|^2\big)^3.$$

As in the proof of Theorem 2.1, we can derive from above that

$$\big\|\nabla e(t)\big\|^2 \leqslant 2C_1, \quad \forall t_0 \leqslant t \leqslant T_0 = \min\left\{T_1, \frac{1}{4C_2C_1^2}\right\}. \tag{3.13}$$

Using the above result and (3.12), we also get

$$\int_{t_0}^{t}\big\|\nabla e(s)\big\|^2\,\mathrm{d}s \leqslant C, \quad \forall t_0 \leqslant t \leqslant T_0. \tag{3.14}$$

We note that if $d = 2$, by using (2.6) on the nonlinear terms in (3.12), we have $T_0 = T$.

We now take the time derivative of (3.9)–(3.10) to obtain

$$e_{tt} - \nu\Delta e_t + \widetilde{B}(u_t^\varepsilon, e) + \widetilde{B}(u^\varepsilon, e_t) + \widetilde{B}(e_t, u) + \widetilde{B}(e, u_t) + \nabla q_t = 0, \tag{3.15}$$

$$(\nabla\cdot \boldsymbol{e}_t,\gamma)+\varepsilon(\nabla q_{tt},\nabla\gamma)=\varepsilon(\nabla p_{tt},\nabla\gamma),\quad \forall\gamma\in H^1(\Omega)/\mathbb{R}, \tag{3.16}$$

with

$$\boldsymbol{e}_t(t_0)=0,\qquad q_t(t_0)=p_t(t_0), \tag{3.17}$$

as determined by (3.5) and (3.9)–(3.10) at $t=t_0$.

Taking the inner product of (3.15) with $\boldsymbol{e}_t$ and setting $\gamma=q_t$ in (3.16), summing up the two relations, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}_t\|^2+\nu\|\nabla\boldsymbol{e}_t\|^2+\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q_t\|^2\\
&\quad=\varepsilon(\nabla p_{tt},\nabla q_t)-\tilde b(\boldsymbol{u}_t^\varepsilon,\boldsymbol{e},\boldsymbol{e}_t)-\tilde b(\boldsymbol{e}_t,\boldsymbol{u},\boldsymbol{e}_t)-\tilde b(\boldsymbol{e},\boldsymbol{u}_t,\boldsymbol{e}_t)\\
&\quad\leqslant\frac{\varepsilon}{2}\|\nabla q_t\|^2+\frac{\varepsilon}{2}\|\nabla p_{tt}\|^2-\tilde b(\boldsymbol{u}_t^\varepsilon,\boldsymbol{e},\boldsymbol{e}_t)-\tilde b(\boldsymbol{e}_t,\boldsymbol{u},\boldsymbol{e}_t)-\tilde b(\boldsymbol{e},\boldsymbol{u}_t,\boldsymbol{e}_t).
\end{aligned} \tag{3.18}$$

We shall use (2.5), (2.7) and Young's inequality to bound the nonlinear terms on the right-hand side as follows:

$$\begin{aligned}
\tilde b(\boldsymbol{u}_t^\varepsilon,\boldsymbol{e},\boldsymbol{e}_t)&=\tilde b(\boldsymbol{u}_t-\boldsymbol{e}_t,\boldsymbol{e},\boldsymbol{e}_t)\\
&\leqslant C\|\boldsymbol{u}_t\|_1\|\boldsymbol{e}\|_1\|\nabla\boldsymbol{e}_t\|+C\|\boldsymbol{e}_t\|\,\|\nabla\boldsymbol{e}_t\|\,\|\boldsymbol{e}\|_1^{1/2}\|\boldsymbol{e}\|_2^{1/2}\\
&\leqslant\frac{\nu}{8}\|\nabla\boldsymbol{e}_t\|^2+C\|\boldsymbol{u}_t\|_1^2\|\boldsymbol{e}\|_1^2+C\|\boldsymbol{e}\|_1\|\boldsymbol{e}\|_2\|\boldsymbol{e}_t\|^2;\\
\tilde b(\boldsymbol{e}_t,\boldsymbol{u},\boldsymbol{e}_t)&\leqslant\|\boldsymbol{e}_t\|\,\|\nabla\boldsymbol{e}_t\|\,\|\boldsymbol{u}\|_2\leqslant\frac{\nu}{8}\|\nabla\boldsymbol{e}_t\|^2+C\|\boldsymbol{u}\|_2\|\boldsymbol{e}_t\|^2;\\
\tilde b(\boldsymbol{e},\boldsymbol{u}_t,\boldsymbol{e}_t)&\leqslant C\|\boldsymbol{e}\|_1\|\boldsymbol{u}_t\|_1\|\nabla\boldsymbol{e}_t\|\leqslant\frac{\nu}{4}\|\nabla\boldsymbol{e}_t\|^2+C\|\boldsymbol{u}_t\|_1^2\|\boldsymbol{e}\|_1^2.
\end{aligned}$$

Thanks to (3.13)–(3.14), we have

$$\int_{t_0}^{T_0}\|\boldsymbol{e}(s)\|_1\|\boldsymbol{e}(s)\|_2\,\mathrm{d}s\leqslant C\int_{t_0}^{T_0}\|\boldsymbol{e}(s)\|_2\,\mathrm{d}s\leqslant C\left(\int_{t_0}^{T_0}\|\boldsymbol{e}(s)\|_2^2\,\mathrm{d}s\right)^{1/2}\leqslant C.$$

Collecting the above inequalities into (3.18), thanks to (3.2), (3.4), (3.11) and (3.17), we derive by using Lemma 2.1 with $y(t)=\|\boldsymbol{e}(t)\|^2+\varepsilon\|\nabla q_t(t)\|^2$ that

$$\|\boldsymbol{e}_t(t)\|^2+\nu\int_{t_0}^{t}\|\nabla\boldsymbol{e}_t(s)\|^2\,\mathrm{d}s+\varepsilon\|\nabla q_t(t)\|^2\leqslant C\varepsilon,\quad \forall t\in[t_0,T_0].$$

Rewriting (3.9), for each t, in the following form:

$$-\nu\Delta\boldsymbol{e}(t)+\widetilde B(\boldsymbol{u}^\varepsilon(t),\boldsymbol{e}(t))+\widetilde B(\boldsymbol{e}(t),\boldsymbol{u}(t))=-\boldsymbol{e}_t(t)-\nabla q(t),$$

taking the inner product of the above equation with $-\Delta\boldsymbol{e}(t)$ and using the available estimates on $\|\nabla q(t)\|$, $\|\boldsymbol{e}(t)\|_1$ and $\|\boldsymbol{e}_t(t)\|$, we conclude that

$$\|\boldsymbol{e}(t)\|_2\leqslant C,\quad \forall t\in[t_0,T_0].$$

In order to get the remaining estimates in (3.7), we only have to consider ∇q_t in (3.15) as a source term and apply the standard procedure. We leave the details to the interested reader. □

Remark 3.2. We note that further differentiation in time to (3.15)–(3.16) would not lead to further ε-independent estimates. Indeed, we derive from (3.15)–(3.17) that $e_{tt}(t_0) = -\nabla q_t(t_0) = -\nabla p_t(t_0) \neq 0$ which prevents us from obtaining

$$\|\nabla p_{tt}^{\varepsilon}(t)\| \leqslant C, \quad \forall t \in [t_0, T_0]. \tag{3.19}$$

3.2. Error estimates for a linearly perturbed problem

Let $(\boldsymbol{u}, p)$ be the solution of the Navier–Stokes equations (1.1)–(1.2), we consider the linearly perturbed problem:

$$\boldsymbol{v}_t^{\varepsilon} - \nu\Delta\boldsymbol{v}^{\varepsilon} + \nabla r^{\varepsilon} = \boldsymbol{f} - B(\boldsymbol{u}, \boldsymbol{u}), \tag{3.20}$$

$$\nabla\cdot\boldsymbol{v}^{\varepsilon} - \varepsilon\Delta r_t^{\varepsilon} = 0, \qquad \frac{\partial r_t^{\varepsilon}}{\partial \boldsymbol{n}}\Big|_{\partial\Omega} = 0, \tag{3.21}$$

where $\boldsymbol{u}(t)$ is the solution of the Navier–Stokes equations (1.1)–(1.2) and $\boldsymbol{v}^{\varepsilon}(t_0) = \boldsymbol{u}(t_0)$, $r^{\varepsilon}(t_0) = p(t_0)$.

Denoting $\boldsymbol{\xi} = \boldsymbol{u} - \boldsymbol{v}^{\varepsilon}$ and $\psi = p - r^{\varepsilon}$, subtracting (3.20)–(3.21) from (1.1)–(1.2), we obtain:

$$\boldsymbol{\xi}_t - \nu\Delta\boldsymbol{\xi} + \nabla\psi = 0, \tag{3.22}$$

$$(\nabla\cdot\boldsymbol{\xi}, \gamma) + \varepsilon(\nabla\psi_t, \nabla\gamma) = \varepsilon(\nabla p_t, \nabla\gamma), \quad \forall\gamma \in H^1(\Omega)/\mathbb{R}, \tag{3.23}$$

with $\boldsymbol{\xi}(t_0) = 0$ and $\psi(t_0) = 0$.

It is obvious that the results in Lemma 3.1 are also valid for the linear case. In particular, we have

$$\|r^{\varepsilon}(t)\|_1 + \|r_t^{\varepsilon}(t)\|_1 + \|\boldsymbol{v}^{\varepsilon}\|_2 \leqslant C, \qquad \|\boldsymbol{\xi}_t(t)\|^2 \leqslant C\varepsilon, \quad \forall t \in [t_0, T_0]. \tag{3.24}$$

Lemma 3.2. *Assuming* (3.1) *and* (3.3), *we have*

$$\int_{t_0}^{t} \|\boldsymbol{\xi}(s)\|^2\,\mathrm{d}s + \varepsilon^{1/2}\|\boldsymbol{\xi}(t)\|^2 + \varepsilon\big(\|\boldsymbol{\xi}(t)\|_1^2 + \|\psi(t)\|^2\big) \leqslant C\varepsilon^2, \quad \forall t \in [t_0, T_0].$$

Proof. We begin by using a parabolic duality argument. Given $t \in [t_0, T_0]$, let $(\boldsymbol{w}, q)$ be defined by the dual problem:

$$\begin{aligned} &\boldsymbol{w}_s + \nu\Delta\boldsymbol{w} + \nabla q = \boldsymbol{\xi}(s), \quad s \in [t_0, t], \\ &\nabla\cdot\boldsymbol{w} = 0, \quad s \in [t_0, t], \\ &\boldsymbol{w}|_{\partial\Omega} = 0, \qquad \boldsymbol{w}(t) = 0. \end{aligned} \tag{3.25}$$

It is standard to show that

$$\int_{t_0}^{t} \big(\|\Delta\boldsymbol{w}(s)\|^2 + \|\nabla q(s)\|^2\big)\,\mathrm{d}s \leqslant C\int_{t_0}^{t} \|\boldsymbol{\xi}(s)\|^2\,\mathrm{d}s. \tag{3.26}$$

Taking the inner product of the first equation of (3.25) with $\boldsymbol{\xi}(s)$, using the error equations (3.22)–(3.23) and the fact that "$\nabla\cdot\boldsymbol{w} = 0$", we get

$$\begin{aligned}\|\boldsymbol{\xi}(s)\|^2 &= (\boldsymbol{\xi}, \boldsymbol{w}_s) + \nu(\boldsymbol{\xi}, \Delta\boldsymbol{w}) - (q, \nabla\cdot\boldsymbol{\xi})\\ &= \frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{\xi}, \boldsymbol{w}) - (\boldsymbol{\xi}_s, \boldsymbol{w}) + \nu(\Delta\boldsymbol{\xi}, \boldsymbol{w}) - \varepsilon(\nabla q, \nabla r_s^\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{\xi}, \boldsymbol{w}) - \varepsilon(\nabla q, \nabla r_s^\varepsilon).\end{aligned}$$

Integrating the above relation in s from t_0 to t, since $\boldsymbol{\xi}(t_0) = 0$ and $\boldsymbol{w}(t) = 0$, we derive that for any $\delta > 0$,

$$\begin{aligned}\int_{t_0}^{t} \|\boldsymbol{\xi}(s)\|^2 \,\mathrm{d}s &\leqslant \varepsilon \int_{t_0}^{t} \|\nabla q(s)\|\,\|\nabla r_s^\varepsilon(s)\|\,\mathrm{d}s\\ &\leqslant \delta \int_{t_0}^{t} \|\nabla q(s)\|^2 \,\mathrm{d}s + \varepsilon^2 C(\delta) \int_{t_0}^{t} \|\nabla r_s^\varepsilon(s)\|^2 \,\mathrm{d}s.\end{aligned}$$

For δ sufficiently small, we derive from (3.26) and (3.24) that

$$\int_{t_0}^{t} \|\boldsymbol{\xi}(s)\|^2 \,\mathrm{d}s \leqslant C\varepsilon^2, \quad \forall t \in [t_0, T_0]. \tag{3.27}$$

We now consider another dual problem: Given $t \in [t_0, T_0]$, we define $(\boldsymbol{w}, q)$ by

$$\begin{aligned}&\boldsymbol{w}_s + \nu\Delta\boldsymbol{w} + \nabla q = \boldsymbol{\xi}_s(s), \quad s \in [t_0, t],\\ &\nabla\cdot\boldsymbol{w} = 0, \quad s \in [t_0, t],\\ &\boldsymbol{w}|_{\partial\Omega} = 0, \qquad \boldsymbol{w}(t) = 0.\end{aligned} \tag{3.28}$$

Again, the solution $(\boldsymbol{w}, q)$ of (3.28) satisfies

$$\int_{t_0}^{t} \left(\|\Delta\boldsymbol{w}(s)\|^2 + \|\nabla q(s)\|^2\right) \mathrm{d}s \leqslant C \int_{t_0}^{t} \|\boldsymbol{\xi}_s(s)\|^2 \,\mathrm{d}s. \tag{3.29}$$

Taking the time derivative in (3.23), we get

$$(\nabla\cdot\boldsymbol{\xi}_t, \gamma) + \varepsilon(\nabla\psi_{tt}, \nabla\gamma) = \varepsilon(\nabla p_{tt}, \nabla\gamma), \quad \forall\gamma \in H^1(\Omega)/\mathbb{R}. \tag{3.30}$$

Taking the inner product of (3.22) with $\boldsymbol{\xi}_t$ and using (3.30), we obtain

$$\begin{aligned}\|\boldsymbol{\xi}_t\|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{\xi}\|^2 &= (\psi, \nabla\cdot\boldsymbol{\xi}_t) = \varepsilon(\nabla\psi, \nabla p_{tt}) - \varepsilon(\nabla\psi, \nabla\psi_{tt})\\ &= \varepsilon(\nabla\psi, \nabla p_{tt}) - \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\psi, \nabla\psi_t) + \varepsilon\|\nabla\psi_t\|^2.\end{aligned}$$

Integrating the above equation and dropping some unnecessary terms, since $\psi(t_0) = 0$, we get

$$\begin{aligned}\int_{t_0}^{t} \|\boldsymbol{\xi}_t\|^2 \,\mathrm{d}s + \nu\|\nabla\boldsymbol{\xi}(t)\|^2 &\leqslant C\varepsilon \int_{t_0}^{t} \left(\|p_{tt}(s)\|_1^2 + \|\psi(s)\|_1^2\right) \mathrm{d}s\\ &\quad + C\varepsilon\|\psi_t(t)\|_1\|\psi(t)\|_1 + C\varepsilon \int_{t_0}^{t} \|\psi_t(s)\|_1^2 \,\mathrm{d}s.\end{aligned} \tag{3.31}$$

By using the estimate (3.24), we have

$$\|\psi_t(t)\|_i \leqslant C, \quad \forall t \in [t_0, T_0], \ i = 0, 1.$$

We then conclude from (3.4) and (3.31) that

$$\int_{t_0}^{t} \|\boldsymbol{\xi}_t\|^2 \,\mathrm{d}s + \|\nabla \boldsymbol{\xi}(t)\|^2 \leqslant C\varepsilon, \quad \forall t \in [t_0, T_0].$$

Now, taking the inner product of the first equation of (3.28) with $\boldsymbol{\xi}(s)$, we derive as above that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\boldsymbol{\xi}(s)\|^2 = \frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{\xi}, \boldsymbol{w}) - \varepsilon(\nabla q, \nabla r_s^\varepsilon).$$

Integrating the above relation in s from t_0 to t, using (3.29) and (3.24), we obtain

$$\begin{aligned}\|\boldsymbol{\xi}(t)\|^2 &\leqslant \varepsilon \int_{t_0}^{t} \|\nabla q(s)\| \, \|\nabla r_s^\varepsilon(s)\| \,\mathrm{d}s \leqslant \varepsilon^{1/2} \int_{t_0}^{t} \|\nabla q(s)\|^2 \,\mathrm{d}s + \varepsilon^{3/2} \int_{t_0}^{t} \|\nabla r_s^\varepsilon(s)\|^2 \,\mathrm{d}s \\ &\leqslant C\varepsilon^{1/2} \int_{t_0}^{t} \|\boldsymbol{\xi}_s(s)\|^2 \,\mathrm{d}s + C\varepsilon^{3/2} \leqslant C\varepsilon^{3/2}, \quad \forall t \in [t_0, T_0].\end{aligned}$$

Finally, we derive from (3.22), (3.24) and the previous estimates that

$$\begin{aligned}\|\psi(t)\| &\leqslant C \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{(\psi(t), \nabla \cdot \boldsymbol{v})}{\|\boldsymbol{v}\|_1} = C \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{(\boldsymbol{\xi}_t(t), \boldsymbol{v}) + (\nabla \boldsymbol{\xi}, \nabla \boldsymbol{v})}{\|\boldsymbol{v}\|_1} \\ &\leqslant C(\|\boldsymbol{\xi}_t(t)\| + \|\nabla \boldsymbol{\xi}(t)\|) \leqslant C\varepsilon^{1/2}, \quad \forall t \in [t_0, T_0]. \qquad \square\end{aligned}$$

3.3. *Error estimates for the nonlinear problem*

Let us denote $\boldsymbol{\eta} = \boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon$ and $\phi = r^\varepsilon - p^\varepsilon$. Subtracting (1.3)–(1.4) from (3.20)–(3.21), we obtain

$$\boldsymbol{\eta}_t - \nu \Delta \boldsymbol{\eta} + \nabla \phi = \widetilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) - \widetilde{B}(\boldsymbol{u}, \boldsymbol{u}), \tag{3.32}$$

$$(\nabla \cdot \boldsymbol{\eta}, \gamma) + \varepsilon(\nabla \phi_t, \nabla \gamma) = 0, \quad \forall \gamma \in H^1(\Omega)/\mathbb{R}, \tag{3.33}$$

with $\boldsymbol{\eta}(t_0) = 0$, $\phi(t_0) = 0$.

Lemma 3.3. *Under the assumption* (3.1) *and* (3.3), *we have*

$$\|\boldsymbol{\eta}(t)\|^2 + \int_{t_0}^{t} \|\nabla \boldsymbol{\eta}(s)\|^2 \,\mathrm{d}s + \varepsilon \|\nabla \phi(t)\|^2 \leqslant C\varepsilon^2, \quad t \in [t_0, T_0].$$

Proof. We recall that $\boldsymbol{u} - \boldsymbol{u}^\varepsilon = \boldsymbol{u} - \boldsymbol{v}^\varepsilon + \boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon = \boldsymbol{\xi} + \boldsymbol{\eta}$, hence

$$\tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon, \boldsymbol{\eta}) - \tilde{b}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\eta}) = -\tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{\xi} + \boldsymbol{\eta}, \boldsymbol{\eta}) - \tilde{b}(\boldsymbol{\xi} + \boldsymbol{\eta}, \boldsymbol{u}, \boldsymbol{\eta}). \tag{3.34}$$

Taking the inner product of (3.32) with η and setting $\gamma = \phi$ in (3.33), using (3.34) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2 + \nu\|\nabla\eta\|^2 + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|^2 = -\tilde{b}(u^\varepsilon, \xi, \eta) - \tilde{b}(\xi + \eta, u, \eta). \tag{3.35}$$

The nonlinear terms can be bounded as follows:

$$\tilde{b}(u^\varepsilon, \xi, \eta) \leqslant C\|u^\varepsilon\|_2\|\eta\|_1\|\xi\| \leqslant \frac{\nu}{4}\|\nabla\eta\|^2 + C\|u^\varepsilon\|_2^2\|\xi\|^2,$$

$$\tilde{b}(\xi + \eta, u, \eta) \leqslant C\big(\|\xi\| + \|\eta\|\big)\|u\|_2\|\eta\|_1 \leqslant \frac{\nu}{4}\|\nabla\eta\|^2 + C\|u\|_2^2\big(\|\xi\|^2 + \|\eta\|^2\big).$$

Collecting the above inequalities into (3.35), thanks to Lemmas 2.1, 3.1 and 3.2, we get

$$\|\eta(t)\|^2 + \nu\int_{t_0}^{t}\|\nabla\eta(s)\|^2\,\mathrm{d}s + \varepsilon\|\nabla\phi(t)\|^2 \leqslant C\varepsilon^2, \quad \forall t \in [t_0, T_0]. \qquad \square$$

Proof of Theorem 3.1. We note that $u - u^\varepsilon = \xi + \eta$ and $p - p^\varepsilon = \psi + \phi$. Taking the inner product of (3.32) with $-\Delta\eta$, in light of the results in Lemmas 3.1–3.3, it is an easy matter to get the crude estimate: $\|\eta\|_1^2 \leqslant c\varepsilon$, $\forall t \in [t_0, T_0]$. Then the results in Theorem 3.1 are direct consequences of this inequality and Lemmas 3.2 and 3.3. $\square$

4. Error estimates of an artificial compressibility method

As mentioned in the introduction, the new method (1.3)–(1.4) is similar to the artificial compressibility method in the sense that the pressure stabilization method in similar to the penalty method. Hence, it is interesting and informative to compare the approximation properties of the new method (1.3)–(1.4) with the artificial compressibility method. We consider here the following artificial compressibility method:

$$u_t^\varepsilon - \nu\Delta u^\varepsilon + \widetilde{B}(u^\varepsilon, u^\varepsilon) + \nabla p^\varepsilon = f \quad \text{in } \Omega \times [t_0, T], \tag{4.1}$$

$$\nabla \cdot u^\varepsilon + \varepsilon p_t^\varepsilon = 0 \quad \text{in } \Omega \times [t_0, T]; \qquad u^\varepsilon|_{\partial\Omega} = 0, \tag{4.2}$$

subjected to the initial conditions: $u^\varepsilon(t_0) = u(t_0)$ and $p^\varepsilon(t_0) = p(t_0)$.

Similar to (1.3)–(1.4), the main value of (4.1)–(4.2) is that its time discretizations are easier to solve than direct time discretizations of (1.1)–(1.2). Let us consider for instance the following second-order scheme for (4.1)–(4.2) with $\varepsilon = \beta\Delta t^2$:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \dfrac{\nu}{2}\Delta\big(u^{n+1} + u^n\big) + \nabla p^{n+1/2} = f(t_{n+1/2}) - \mathrm{NLT}, \\ \nabla \cdot u^{n+1} + \beta\Delta t\big(p^{n+1} - p^n\big) = 0. \end{cases} \tag{4.3}$$

As in (1.5), β is an appropriate constant and NLT is a certain second-order approximation to $B(u(t_{n+1/2}), u(t_{n+1/2}))$. If we choose $p^{n+1/2} = (1/2)(p^{n+1} + p^n)$, then we can either view (4.3) is a *coupled* positive definite nonsymmetric system for (u^{n+1}, p^{n+1}), or eliminate p^{n+1} from (4.3) to obtain a positive definite symmetric, though ill-conditioned for $\Delta t \ll 1$, system for u^{n+1} only.

Hence, (4.3) is easier to solve than a direct discretization of the Navier–Stokes equations (1.1)–(1.2). However, the scheme (4.3) becomes unstable if we choose $p^{n+1/2} = (1/2)(3p^n - p^{n-1})$.

Theorem 4.1. *We assume* (3.1), (3.3) *and let* $t_0 \in (0, T_1)$. *Then for* ε *sufficient small, we have*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\|^2 + \int_{t_0}^{t} \|\boldsymbol{u}(s) - \boldsymbol{u}^\varepsilon(s)\|_1^2 \,\mathrm{d}s + \varepsilon\|p(t) - p^\varepsilon(t)\|^2 \leqslant C\varepsilon^2, \quad \forall t \in [t_0, T_1],$$

where T_1 *is the same as in the Proposition* 3.1.

Proof. Let us denote $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}^\varepsilon$ and $q = p - p^\varepsilon$, subtracting (4.1)–(4.2) from (1.1)–(1.2), we obtain

$$\boldsymbol{e}_t - \nu\Delta\boldsymbol{e} + \nabla q = -\widetilde{B}(\boldsymbol{e}, \boldsymbol{u}) - \widetilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{e}), \tag{4.4}$$

$$\nabla \cdot \boldsymbol{e} + \varepsilon q_t = p_t, \tag{4.5}$$

with $\boldsymbol{e}(t_0) = 0$ and $q(t_0) = 0$. Taking the inner product of (4.4) with $\boldsymbol{e}$ and of (4.5) with q, thanks to (2.3), (3.4) and (2.7), we derive that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}\|^2 + \nu\|\nabla\boldsymbol{e}\|^2 + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 &= \varepsilon(p_t, q) - \tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{e}) \\ &\leqslant \varepsilon(p_t, q) + C\|\boldsymbol{u}\|_2\|\boldsymbol{e}\|\,\|\nabla\boldsymbol{e}\| \\ &\leqslant \varepsilon(p_t, q) + \frac{\nu}{4}\|\nabla\boldsymbol{e}\|^2 + C\|\boldsymbol{e}\|^2.\end{aligned} \tag{4.6}$$

Since $p_t, p_{tt} \in C([t_0, T_1]; L^2(\Omega))$ (cf. (3.4)), by the stability of the divergence operator (cf. [9,30]), we can find, for each t, $\boldsymbol{\varphi}(t) \in \boldsymbol{H}_0^1(\Omega)$ such that

$$\nabla \cdot \boldsymbol{\varphi}(t) = p_t(t), \qquad \|\boldsymbol{\varphi}(t)\|_1 \leqslant C\|p_t(t)\|, \quad \forall t \in [t_0, T_1], \tag{4.7}$$

and

$$\nabla \cdot \boldsymbol{\varphi}_t(t) = p_{tt}(t), \qquad \|\boldsymbol{\varphi}_t(t)\|_1 \leqslant C\|p_{tt}(t)\|, \quad \forall t \in [t_0, T_1]. \tag{4.8}$$

Therefore, we infer from (4.4) that

$$\begin{aligned}\varepsilon(p_t, q) &= \varepsilon(\nabla \cdot \boldsymbol{\varphi}, q) = -\varepsilon(\boldsymbol{\varphi}, \nabla q) \\ &= \varepsilon(\boldsymbol{\varphi}, \boldsymbol{e}_t) + \varepsilon(\nabla\boldsymbol{\varphi}, \nabla\boldsymbol{e}) + \varepsilon\tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{e}, \boldsymbol{\varphi}) + \varepsilon\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{\varphi}) \\ &= \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\varphi}, \boldsymbol{e}) - \varepsilon(\boldsymbol{\varphi}_t, \boldsymbol{e}) + \varepsilon(\nabla\boldsymbol{\varphi}, \nabla\boldsymbol{e}) + \varepsilon\tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{e}, \boldsymbol{\varphi}) + \varepsilon\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{\varphi}) \\ &\leqslant \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\varphi}, \boldsymbol{e}) + \frac{\nu}{4}\|\nabla\boldsymbol{e}\|^2 + C\varepsilon^2\big(\|\boldsymbol{\varphi}_t\|^2 + \|\boldsymbol{\varphi}\|_1^2\big) + \varepsilon\tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{e}, \boldsymbol{\varphi}) + \varepsilon\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{\varphi}).\end{aligned}$$

In view of (2.7), (4.7) and (3.4), we have

$$\begin{aligned}\varepsilon\tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{e}, \boldsymbol{\varphi}) &= \varepsilon\tilde{b}(\boldsymbol{u} - \boldsymbol{e}, \boldsymbol{e}, \boldsymbol{\varphi}) \leqslant C\varepsilon\big(\|\boldsymbol{u}\|_1 + \|\boldsymbol{e}\|_1\big)\|\boldsymbol{e}\|_1\|\boldsymbol{\varphi}\|_1 \leqslant \Big(\frac{\nu}{8} + C\varepsilon\Big)\|\nabla\boldsymbol{e}\|^2 + C\varepsilon^2; \\ \varepsilon\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{\varphi}) &\leqslant C\varepsilon\|\boldsymbol{u}\|_1\|\boldsymbol{e}\|_1\|\boldsymbol{\varphi}\|_1 \leqslant \frac{\nu}{8}\|\nabla\boldsymbol{e}\|^2 + C\varepsilon^2.\end{aligned}$$

Collecting the above results in (4.6) and integrating in time, we derive that for ε sufficient small,

$$\|e(t)\|^2 + \nu \int_{t_0}^{t} \|\nabla e(s)\|^2\,\mathrm{d}s + \varepsilon\|q(t)\|^2 \leqslant C\varepsilon\big(\varphi(t), e(t)\big) + C\varepsilon^2$$

$$\leqslant \frac{1}{2}\|e(t)\|^2 + C\varepsilon^2\|\varphi(t)\|^2 + C\varepsilon^2, \quad \forall t \in [t_0, T_1].$$

We conclude from (4.7) and (3.4) that

$$\|e(t)\|^2 + \nu \int_{t_0}^{t} \|\nabla e(s)\|^2\,\mathrm{d}s + \varepsilon\|q(t)\|^2 \leqslant C\varepsilon^2, \quad \forall t \in [t_0, T_1]. \qquad \square$$

Remark 4.1. As an approximation to (1.1)–(1.2), (4.1)–(4.2) appears to be more accurate than (1.3)–(1.4) is. However, for the reason mentioned above, appropriate time discretizations of (1.3)–(1.4) such as (1.5) and (1.6)–(1.7) are more efficient and more flexible than time discretizations of (4.1)–(4.2) such as (4.3).

References

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems* (Elsevier, New York, 1965).

[2] F. Brezzi and J. Douglas, Stabilized mixed methods for the Stokes problem, *Numer. Math.* 53 (1988) 225–235.

[3] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods* (Springer, New York, 1991).

[4] F. Brezzi and J. Pitkaranta, On the stabilization of finite element approximation of the Stokes problem, in: W. Hackbusch, ed., *Efficient Solutions of Elliptic Systems*, Notes on Numerical Fluid Mechanics 10 (Vieweg, Braunschweig, 1984) 11–19.

[5] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods on Fluid Dynamics* (Springer, Berlin, 1987).

[6] A.J. Chorin, Numerical solution of the Navier–Stokes equations, *Math. Comp.* 22 (1968) 745–762.

[7] A.J. Chorin, On the convergence of discrete approximations to the Navier–Stokes equations, *Math. Comp.* 23 (1969) 341–353.

[8] W. E and J.G. Liu, Projection method I: Convergence and numerical boundary layers, *SIAM J. Numer. Anal.* 32 (1995) 1017–1057.

[9] V. Girault and P.A. Raviart, *Finite Element Methods for Navier–Stokes Equations* (Springer, Berlin, 1986).

[10] J.G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization, *SIAM J. Numer. Anal.* 19 (1982) 275–311.

[11] T.J.R. Hughes, L.P. Franca and M. Balestra, A new finite element formulation for computational fluid mechanics: V. Circumventing the Babuška–Brezzi condition: a stable Petrov–Galerkin formulation of the Stokes problem accommodating equal order interpolation, *Comput. Methods Appl. Mech. Engrg.* 59 (1986) 85–99.

[12] T.J.R. Hughes, W.T. Liu and A.J. Brooks, Finite element analysis of incompressible viscous flows by the penalty function formulation, *J. Comput. Phys.* 30 (1979) 1–60.

[13] J. Kim and P. Moin, Application of a fractional-step method to incompressible Navier–Stokes equations, *J. Comput. Phys.* 59 (1985) 308–323.

[14] S. Klainerman and A. Majda, Compressible and incompressible fluids, *Comm. Pure Appl. Math.* 35 (1982) 629–651.

[15] H.O. Kreiss, J. Lorenz and J.M. Naughton, Convergence of the solutions of the compressible to the solution of the incompressible Navier–Stokes equations, *Adv. in Appl. Math.* (1991).
[16] J.L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéaires* (Dunod, Paris, 1969).
[17] J. Morgan and R. Scott, A nodal basis for C^1 piecewise polynomials of degree $n \geqslant 5$, *Math. Comp.* 29 (1975) 736–740.
[18] R. Rannacher, On Chorin's projection method for the incompressible Navier–Stokes equations, in: *The Navier–Stokes Equations* II. *Theory and Numerical Methods*, Lecture Notes in Mathematics 1530 (1991) 167–183.
[19] R. Scott and M. Vogelius, Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, *Math. Model. Numer. Anal.* 19 (1985) 111–143.
[20] J. Shen, On error estimates of the projection methods for the Navier–Stokes equations: first-order schemes, *SIAM J. Numer. Anal.* 29 (1992) 57–77.
[21] J. Shen, On pressure stabilization method and projection method for unsteady Navier–Stokes equations, in: R. Vichnevetsky, D. Knight and G. Richter, eds., *Advances in Computer Methods for Partial Differential Equations* (IMACS, 1992) 658–662.
[22] J. Shen, On error estimates of some higher order projection and penalty-projection methods for Navier–Stokes equations, *Numer. Math.* 62 (1992) 49–73.
[23] J. Shen, A remark on the projection-3 method, *Internat. J. Numer. Methods Fluids* 16 (1993) 249–253.
[24] J. Shen, On error estimates of the penalty method for the unsteady Navier–Stokes equations, *SIAM J. Numer. Anal.* 32 (2) (1995).
[25] J. Shen, On error estimates of the projection method for the Navier–Stokes equations: second order schemes, *Math. Comp.* 65 (1996).
[26] R. Temam, Une méthode d'approximation des solutions des équations de Navier–Stokes, *Bull. Soc. Math. France* 98 (1968) 115–152.
[27] R. Temam, Sur l'approximation de la solution des équations de Navier–Stokes par la méthode des pas fractionnaires II, *Arch. Rational Mech. Anal.* 33 (1969) 377–385.
[28] R. Temam, Sur l'approximation de la solution des équations de Navier–Stokes par la méthode des pas fractionnaires I, *Arch. Rational Mech. Anal.* 32 (1969) 135–153.
[29] R. Temam, Navier–Stokes equations and nonlinear functional analysis, in: CBMS-NSF Regional Conference Series in Applied Mathematics 41 (SIAM, Philadelphia, 1983).
[30] R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis* (North-Holland, Amsterdam, 1984).
[31] J. van Kan, A second-order accurate pressure-correction scheme for viscous incompressible flow, *SIAM J. Sci. Statist. Comput.* 7 (1986) 870–891.