On Pressure Stabilization Method and Projection Method for Unsteady Navier-Stokes Equations[1]

Jie Shen
Department of Mathematics, Penn State University
University Park, PA 16802

Abstract. We present some recent results on the pressure stabilization method and the projection method. The relation between the two methods is exploited to derive an improved error estimate for the projection method.

1. Introduction. In this paper, we consider the numerical approximation of the unsteady Navier-Stokes equations:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f},\ \text{in } \Omega\times[0,T], \tag{1.1}$$

$$\text{div}\boldsymbol{u} = 0, \qquad \text{in } \Omega\times[0,T], \tag{1.2}$$

$$\boldsymbol{u}|_{\partial\Omega} = 0, \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \tag{1.3}$$

where Ω is a open bounded set in $\mathbb{R}^d$ with $d = 2$ or 3.

One of the main difficulties in numerical approximation of the unsteady Navier-Stokes equations is how to treat the incompressibility constraint "div$\boldsymbol{u} = 0$", which not only couples the velocity and the pressure, but also requires that the solution spaces, to which the velocity and the pressure belong, verify the so called inf-sup condition. A popular strategy to overcome this difficulty is to relax the incompressibility constraint in an appropriate way, resulting in a class of pseudo-compressibility methods, among which are *the penalty method, the artificial compressibility method, pressure stabilization method and the projection method.*

A semi-discretized version of the projection method can be written as follows: let $\boldsymbol{u}^0 = \boldsymbol{u}_0$, we solve successively $\tilde{\boldsymbol{u}}^{n+1}$ and $\{\boldsymbol{u}^{n+1}, p^{n+1}\}$ by

$$\begin{aligned} &\frac{(\tilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n)}{\Delta t} - \nu\Delta\tilde{\boldsymbol{u}}^{n+1} + (\tilde{\boldsymbol{u}}^{n+1}\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1} = \boldsymbol{f}^{n+1},\ \text{in } \Omega,\\ &\tilde{\boldsymbol{u}}^{n+1}|_{\partial\Omega} = 0, \end{aligned} \tag{1.4}$$

$$\begin{aligned} &\frac{(\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1})}{\Delta t} + \nabla p^{n+1} = 0,\ \ \text{div}\boldsymbol{u}^{n+1} = 0,\ \text{in } \Omega,\\ &\boldsymbol{u}^{n+1}\cdot\boldsymbol{n}|_{\partial\Omega} = 0. \end{aligned} \tag{1.5}$$

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The numerical efficiency of the scheme (1.4)-(1.5) is obvious since the velocity approximation and the pressure approximation in (1.4)-(1.5) are totally decoupled. Furthermore, the solution spaces for the velocity and the pressure need not to satisfy the Babuska-Brezzi inf-sup condition.

It is interesting to notice that $\{\boldsymbol{u}^n\}$ in (1.4)-(1.5) can be eliminated to form a system only involving $\{\tilde{\boldsymbol{u}}^n\}$. taking the sum of (1.4) at step n and (1.5) at step $n-1$, taking the divergence of (1.5), we obtain

$$\frac{(\tilde{\boldsymbol{u}}^{n+1}-\tilde{\boldsymbol{u}}^n)}{\Delta t} - \nu\Delta\tilde{\boldsymbol{u}}^{n+1} + (\tilde{\boldsymbol{u}}^{n+1}\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1} + \nabla p^n = \boldsymbol{f}^{n+1}, \tag{1.6}$$

$$\text{div}\tilde{\boldsymbol{u}}^{n+1} - \Delta t\Delta p^{n+1} = 0,\ \frac{\partial p^{n+1}}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0. \tag{1.7}$$

(1.6)-(1.7) reminds us the so called pressure stabilization method, initially introduced by Brezzi and Pitkäranta [1] for the approximation of the steady Stokes equation. When applied to the unsteady Navier-Stokes equations, the pressure stabilization method takes the form:

$$\boldsymbol{u}^\varepsilon_t - \Delta\boldsymbol{u}^\varepsilon + (\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon) + \nabla p^\varepsilon = \boldsymbol{f}, \tag{1.8}$$

$$\text{div}\boldsymbol{u}^\varepsilon - \varepsilon\Delta p^\varepsilon = 0,\ \frac{\partial p^\varepsilon}{\partial\boldsymbol{n}}|_{\partial\Omega} = 0. \tag{1.9}$$

Therefore the projection method, written in the form of (1.6)-(1.7), is simply a decoupled time discretization of the pressure stabilization method (1.8)-(1.9) with $\varepsilon = \Delta t$. We will present below some recent results on the pressure stabilization method and the projection method, part of which improves previous results presented in [4].

We will use the standard natations $L^2(\Omega)$, $H^k(\Omega)$ and $H^k_0(\Omega)$ to denote the Sobolev spaces over Ω. The norm corresponding to $H^k(\Omega)$ will be denoted by $\|\cdot\|_k$, in particular, $\|\cdot\|$ will be denoting the norm in $L^2(\Omega)$. The vector functions and vector spaces will be indicated by bold face type. We will use c as a generic constant depending only on the data.

In the three dimensional case, we assume a strong solution exists for $0\le t\le T$, i.e. $\|u\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))}\le c$. It is then easy to show that for sufficiently smooth data, the solution $(\boldsymbol{u},p)$ of (1.1)-(1.2) satisfies (see for instance [6], [2])

$$\boldsymbol{u}\in L^2(0,T;\boldsymbol{H}^2(\Omega)),\ p\in L^2(0,T;H^1(\Omega))\ \text{ and }\ t^2p_t\in L^2(0,T;H^1(\Omega)). \tag{1.10}$$

2. Error estimates of the pressure stabilization method and projection method.

THEOREM 1. *Let* $(\boldsymbol{u},p)$ *be the solution of* (1.1)-(1.2), $(\boldsymbol{u}^\varepsilon,p^\varepsilon)$ *be the solution of* (1.8)-(1.9), *then*

$$\sqrt{t}\|\boldsymbol{u}(t)-\boldsymbol{u}^\varepsilon(t)\|_1 + t\|p(t)-p^\varepsilon(t)\| \le c\sqrt{\varepsilon}\ ,\ \forall\, 0\le t\le T,$$

$$\sqrt{t}\|\boldsymbol{u}(t)-\boldsymbol{u}^\varepsilon(t)\| \le c\varepsilon\ ,\ \forall\, 0\le t\le T.$$

Sketch of the proof. We will only provide the proof in the linear case, which is indeed responsible for the error. A complete proof will be given in [3].

Let us denote $\boldsymbol{e}=\boldsymbol{u}-\boldsymbol{u}^\varepsilon$, $q=p-p^\varepsilon$. Dropping out the nonlinear term and setting $\nu=1$ for simplicity, we obtain the following error equations.

$$\boldsymbol{e}_t-\Delta\boldsymbol{e}+\nabla q=0, \tag{2.1}$$

$$\mathrm{div}\boldsymbol{e}-\varepsilon\Delta q=-\varepsilon\Delta p,\ \frac{\partial q}{\partial \boldsymbol{n}}|_{\partial\Omega}=\frac{\partial p}{\partial \boldsymbol{n}}|_{\partial\Omega}. \tag{2.2}$$

We will derive the desired results through a series of optimal estimates.

Taking the scalar product of (2.1) with $\boldsymbol{e}$ and (2.2) with q, summing up the two relations, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}\|^2+\|\nabla\boldsymbol{e}\|^2+\varepsilon\|\nabla q\|^2=\varepsilon(\nabla p,\nabla q)\le\frac{\varepsilon}{2}\|\nabla q\|^2+\frac{\varepsilon}{2}\|\nabla p\|^2.$$

Integrating the above relation from 0 to t, since $\boldsymbol{e}(0)=0$, we obtain

$$\|\boldsymbol{e}(t)\|^2+\int_0^t(\|\nabla\boldsymbol{e}\|^2+\varepsilon\|\nabla q\|^2)ds\le c\varepsilon\ ,\ \forall\, 0\le t\le T. \tag{2.3}$$

Next, we differentiate (2.2) with respect to t to obtain

$$\mathrm{div}\boldsymbol{e}_t-\varepsilon\Delta q_t=-\varepsilon\Delta p_t,\ \frac{\partial q_t}{\partial \boldsymbol{n}}|_{\partial\Omega}=\frac{\partial p_t}{\partial \boldsymbol{n}}|_{\partial\Omega}. \tag{2.4}$$

Taking the scalar product of (2.1) with $2t\boldsymbol{e}_t$ and (2.4) with $2tq$, we derive

$$\begin{aligned}2t\|\boldsymbol{e}_t\|^2+\frac{d}{dt}(t\|\nabla\boldsymbol{e}\|^2)+\varepsilon\frac{d}{dt}(t\|\nabla q\|^2)&=\|\nabla\boldsymbol{e}\|^2+\varepsilon\|\nabla q\|^2+2\varepsilon t(\nabla p_t,\nabla q)\\&\le\|\nabla\boldsymbol{e}\|^2+2\varepsilon\|\nabla q\|^2+\varepsilon t^2\|\nabla p_t\|^2.\end{aligned}$$

Integrating the above relation from 0 to t, using (1.10) and the estimate (2.3), we derive

$$t\|\nabla\boldsymbol{e}\|^2+\varepsilon t\|\nabla q\|^2+\int_0^t s\|\boldsymbol{e}(s)\|^2ds\le c\varepsilon+\varepsilon\int_0^t s^2\|\nabla p_t\|^2ds\le c\varepsilon. \tag{2.5}$$

We now use the standard parabolic duality argument. For $s\le T$ fixed, let $(\boldsymbol{w},r)$ be the solution of the following system.

$$\boldsymbol{w}_t+\Delta\boldsymbol{w}-\nabla r=\boldsymbol{e}(t)\ ,\ \forall\, 0\le t\le s, \tag{2.6}$$

$$\mathrm{div}\boldsymbol{w} = 0, \quad \boldsymbol{w}(s) = 0. \tag{2.7}$$

It is standard to show that

$$\int_0^s (\|\boldsymbol{w}\|_2^2 + \|\nabla r\|^2)dt \le c\int_0^s \|\boldsymbol{e}\|^2 dt. \tag{2.8}$$

Taking the scalar product of (2.6) with $\boldsymbol{e}(t)$, using the error equations (2.1)-(2.2) and the fact $\mathrm{div}\boldsymbol{w} = 0$, we derive

$$\begin{aligned}\|\boldsymbol{e}(t)\|^2 &= (\boldsymbol{w}_t, \boldsymbol{e}) + (\boldsymbol{w}, \Delta\boldsymbol{e}) + (r, \mathrm{div}\boldsymbol{e}) = \frac{d}{dt}(\boldsymbol{e}, \boldsymbol{w}) + (\Delta\boldsymbol{e}, \boldsymbol{w}) - (\boldsymbol{e}_t, \boldsymbol{w}) + \varepsilon(\nabla r, \nabla p^\varepsilon)\\ &= \frac{d}{dt}(\boldsymbol{e}, \boldsymbol{w}) + (\nabla q, \boldsymbol{w}) + \varepsilon(\nabla r, \nabla p^\varepsilon) = \frac{d}{dt}(\boldsymbol{e}, \boldsymbol{w}) + \varepsilon(\nabla r, \nabla p^\varepsilon).\end{aligned}$$

Integrating from 0 to s, since $\boldsymbol{w}(s) = \boldsymbol{e}(0) = 0$, we derive that for $\delta > 0$, we have

$$\int_0^s \|\boldsymbol{e}(t)\|^2 dt \le \delta \int_0^s \|\nabla r\|^2 dt + \frac{\varepsilon^2}{\delta}\int_0^s \|\nabla p^\varepsilon\| dt.$$

Choose δ sufficiently small, using (2.3) and (2.8), we obtain

$$\int_0^s \|\boldsymbol{e}(t)\|^2 dt \le c\varepsilon^2. \tag{2.9}$$

To derive the improved estimates for $\boldsymbol{e}$ and q, we need the following estimate on $\boldsymbol{e}_t$ and q_t.

$$t^2\|\boldsymbol{e}_t\|^2 + \varepsilon\int_0^t s^2\|\nabla q_t\|^2 ds \le c\varepsilon\ , \ \forall\, 0 \le t \le T. \tag{2.10}$$

To prove (2.10), we differentiate (2.1) with respect to t to obtain

$$\boldsymbol{e}_{tt} - \Delta\boldsymbol{e}_t + \nabla q_t = 0. \tag{2.11}$$

Taking the scalar product of (2.11) with $t^2\boldsymbol{e}_t$ and (2.4) with $t^2 q_t$, we derive

$$\frac{1}{2}\frac{d}{dt}(t^2\|\boldsymbol{e}\|^2) + t^2\|\nabla\boldsymbol{e}_t\|^2 + \varepsilon t^2\|\nabla q_t\|^2 = t\|\boldsymbol{e}\|^2 + \varepsilon t^2(\nabla p_t, \nabla q_t).$$

Taking into account (1.10) and (2.3), (2.10) follows immediately after integrating the above relation.

We now take the scalar product of (2.6) with $t\boldsymbol{e}_t$, using again the error equations (2.1)-(2.2), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(t\|\boldsymbol{e}\|^2) =& \frac{1}{2}\|\boldsymbol{e}\|^2 + (\boldsymbol{e}_t, t\boldsymbol{w}_t) + t(\Delta\boldsymbol{e}_t, \boldsymbol{w}) + t(r, \mathrm{div}\boldsymbol{e}_t)\\ =& \frac{1}{2}\|\boldsymbol{e}\|^2 + (\boldsymbol{e}_t, t\boldsymbol{w}_t) + \frac{d}{dt}t(\Delta\boldsymbol{e}, \boldsymbol{w}) - (\Delta\boldsymbol{e}, t\boldsymbol{w}_t) - (\boldsymbol{e}, \Delta\boldsymbol{w}) + t(r, \mathrm{div}\boldsymbol{e}_t)\\ =& \frac{1}{2}\|e\|^2 + \frac{d}{dt}t(\Delta\boldsymbol{e}, \boldsymbol{w}) - (\nabla q, t\boldsymbol{w}_t) - (\boldsymbol{e}, \Delta\boldsymbol{w}) + \varepsilon t(\nabla r, \nabla p_t^\varepsilon).\end{aligned}$$

Integratin from 0 to s, since div$\boldsymbol{w}_t = 0$, using (2.8), (2.9) and (2.10), we derive

$$s\|\boldsymbol{e}\|^2 \le c\int_0^s \|\boldsymbol{e}\|dt + c\varepsilon^2 \int_0^s t^2\|\nabla p_t^\varepsilon\|^2 dt \le c\varepsilon^2.$$

Finally for the pressure estimate, we simply use the error equation (2.1) to derive

$$\|q\| \le c\|\nabla q\|_{-1} \le c(\|\boldsymbol{e}_t\|_{-1} + \|\Delta \boldsymbol{e}\|_{-1}) \le c(\|\boldsymbol{e}_t\| + \|\nabla \boldsymbol{e}\|),$$

Hence using (2.5) and (2.10), we arrive to

$$t^2\|q\|^2 \le ct^2(\|e_t\|^2 + \|\nabla e\|^2) \le c\varepsilon. \qquad \square$$

Considering the projection method (1.6)-(1.7) as a time discretization for (1.8)-(1.9), we have the following results, which will be proved elsewhere.

THEOREM 2. *Let* $(\tilde{\boldsymbol{u}}^n, p^n)$ *be the solution of* (1.6)-(1.7) *and let* $(\boldsymbol{u}, p)$ *be the solution of* (1.1)-(1.2), *then*

$$\sqrt{t_n}\|\boldsymbol{u}(t_n) - \tilde{\boldsymbol{u}}^n\|_1 + \sqrt{t^n}\|p(t_n) - p^n\| \le c\sqrt{\Delta t}\ ,\ \forall\, 0 \le n \le T/\Delta t,$$

$$\sqrt{t_n}\|\boldsymbol{u}(t_n) - \tilde{\boldsymbol{u}}^n\| \le c\Delta t\ ,\ \forall\, 0 \le n \le T/\Delta t.$$

REMARK 1. *The results in Theorem 2 improve the results presented in [4] by taking into account the smoothing property at* $t = 0$ *of the Navier-Stokes equations.*

3. Finite element approximation for the pressure stabilization method. We now consider the semi-discretization of (1.8)-(1.9) by using a finite element method. For simplicity, we assume Ω to be a convex polygonal domain. Given a quasi uniform triangulation $\mathcal{T}_h$ for Ω, let W_h consist of C^0 piecewise polynomial functions over the triangulation $\mathcal{T}_h$ such that $W_h \subset H_0^1(\Omega)$ and for some $m \ge 2$,

$$\inf_{v_h \in W_h} \{\|v - v_h\| + h\|\nabla(v - v_h)\|\} \le ch^m\|v\|_m\ ,\ \forall\, v \in H_0^1(\Omega) \cap H^m(\Omega),$$

let Q_h consist of C^0 piecewise polynomial functions over the triangulation $\mathcal{T}_h$ such that $Q_h \in H^1(\Omega) \cap L_0^2(\Omega)$ and for some $k \ge 2$,

$$\inf_{q_h \in Q_h} \{\|q - q_h\| + h\|\nabla(q - q_h)\|\} \le ch^k\|q\|_k\ ,\ \forall\, q \in L_0^2(\Omega) \cap H^k(\Omega).$$

We denote $\boldsymbol{W}_h = (W_h)^d$. Then the finite element approximation of (1.8)-(1.9) is to seek $(\boldsymbol{u}^{\varepsilon,h}, p^{\varepsilon,h}) \in (\boldsymbol{W}^h, Q_h)$ such that

$$(\boldsymbol{u}_t^{\varepsilon,h}, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{u}^{\varepsilon,h}, \nabla \boldsymbol{v}_h) + ((\boldsymbol{u}^{\varepsilon,h}\cdot\nabla)\boldsymbol{u}^{\varepsilon,h}, \boldsymbol{v}_h) - (p^{\varepsilon,h}, \mathrm{div}\boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h \in \boldsymbol{W}_h, \tag{3.1}$$

$$(\mathrm{div}\boldsymbol{u}^{\varepsilon,h}, q_h) + \varepsilon(\nabla p^{\varepsilon,h}, \nabla q_h) = 0,\ \forall q_h \in Q_h. \tag{3.2}$$

We have the following results, which will be proved in [3].

Theorem 3. *Let $k \geq \min\{2, m-1\}$, then*

$$\sqrt{t^{m-1}}\|\nabla(\boldsymbol{u} - \boldsymbol{u}^{\varepsilon,h})(t)\| \leq c\sqrt{\varepsilon} + c_m(\boldsymbol{u}, p)(h^{m-1} + \frac{h^m}{\sqrt{\varepsilon}}) \ , \ \forall\, 0 \leq t \leq T.$$

Remark 2.

(i) *$C_m(\boldsymbol{u}, p)$ is a constant depending on the solution $(\boldsymbol{u}, p)$. In case Ω is of class C^m, $C_m(\boldsymbol{u}, p)$ can be bounded by a constant only depending on the data $\boldsymbol{u}_0$, $\boldsymbol{f}$ and Ω.*

(ii) *Due to the $O(\sqrt{\varepsilon})$ error committed by the continuous pressure stabilization method, $\boldsymbol{u}^{\varepsilon,h}(t)$ is not an optimal approximation to $\boldsymbol{u}(t)$ in $\boldsymbol{W}_h$. The optimal choice for ε is $\varepsilon = \sqrt{h^m}$, which results in the error estimate*

$$\sqrt{t^{m-1}}\|\nabla(u - \boldsymbol{u}^{\varepsilon,h})(t)\| \leq c\sqrt{h^m}.$$

(iii) *With $\varepsilon = h^m$, we can also prove*

$$\|p(t) - p^{\varepsilon,h}(t)\| \leq ch \ , \ \forall\, 0 \leq t \leq T.$$

Due to the incompatible Neumann boundary condition imposed by the pressure stabilization method, the error estimate for $\|p(t) - p^{\varepsilon,h}(t)\|$ does not improve when higher order methods are used.

The space discretization of the projection scheme (1.4)-(1.5) and its higher order extensions (see [5]) will be studied in a future paper.

References

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