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ON ERROR ESTIMATES OF THE PENALTY METHOD FOR UNSTEADY NAVIER–STOKES EQUATIONS*

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Abstract. The penalty method has been widely used for numerical computations of the unsteady Navier–Stokes equations. However, the best error estimates available to the author's knowledge were not optimal and could have led to an improper choice of the penalty coefficient ε for time discretizations of the penalized system. Optimal error estimates for the penalized system and its time discretizations for the unsteady Navier–Stokes equations are derived.

Key words. artificial compressibility method, error estimates, Navier–Stokes equations, penalty method

AMS subject classifications. 35A40, 35Q30, 65J15, 65M15

1. Introduction.

In this article, we consider the approximations by the penalty method for the unsteady Navier–Stokes equations:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega\times[0,T], \tag{1.1}$$

$$\text{div}\boldsymbol{u} = 0 \text{ in } \Omega\times[0,T], \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \tag{1.2}$$

where Ω is an open bounded set in $\mathbb{R}^d$ ($d = 2$ or 3) with a sufficiently smooth boundary. For the sake of simplicity, we shall only consider the homogeneous Dirichlet boundary condition for the velocity, i.e., $\boldsymbol{u}|_{\partial\Omega} = 0$.

We note that the velocity $\boldsymbol{u}$ and the pressure p in the above equations are coupled together by the incompressibility constraint "div$\boldsymbol{u} = 0$," which makes the system difficult to solve numerically. A popular strategy to overcome this difficulty is to relax the incompressibility constraint in an appropriate way, resulting in a class of pseudocompressibility methods, among which are *the penalty method, the artificial compressibility method, the pressure stabilization method, and the projection method* (see for instance [5], [6], [17], [19], [18], [4], and [16]).

The penalty method was introduced by Courant [7] in the context of the calculus of variations. Its application to the Navier–Stokes equations was initiated in Temam [17]. When applied to the unsteady Navier–Stokes equations, the penalty method is to approximate the solution $(\boldsymbol{u},p)$ of (1.1)–(1.2) by $(\boldsymbol{u}^\varepsilon,p^\varepsilon)$ satisfying the following penalized system:

$$\boldsymbol{u}_t^\varepsilon - \nu\Delta\boldsymbol{u}^\varepsilon + \tilde{B}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}^\varepsilon) + \nabla p^\varepsilon = \boldsymbol{f}, \tag{1.3}$$

$$\text{div}\boldsymbol{u}^\varepsilon + \varepsilon p^\varepsilon = 0, \quad \boldsymbol{u}^\varepsilon|_{t=0} = \boldsymbol{u}_0, \tag{1.4}$$

where $\tilde{B}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \frac{1}{2}(\text{div}\boldsymbol{u})\boldsymbol{v}$ is the modified bilinear term, introduced by Temam [17] to ensure the dissipativity of the system (1.3)–(1.4). We note also that p^ε in (1.3)–(1.4) can be eliminated to obtain a system of $\boldsymbol{u}^\varepsilon$ only, which is much easier to

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solve than the original system (1.1)–(1.2). Hence the penalty method has been widely used in many areas of computational fluid dynamics (see for instance [2], [12]).

It is well known (cf. [17]) that $\lim_{\varepsilon\to 0}(\boldsymbol{u}^\varepsilon, p^\varepsilon) = (\boldsymbol{u}, p)$, the solution of (1.1)–(1.2). It has also been shown (cf. [3]) that the attractors generated by the penalized system converge to the attractor of the original Navier–Stokes equations. As for the error analysis, it has been thoroughly studied in the context of steady Stokes and Navier–Stokes equations (see for instance [1], [13], and [14]). For instance, the following optimal error estimate holds for the penalized system applied to the steady Stokes equations:

$$\|\boldsymbol{u} - \boldsymbol{u}^\varepsilon\|_1 + \|p - p^\varepsilon\| \le C\varepsilon, \tag{1.5}$$

where $\|\cdot\|_1$ and $\|\cdot\|$ denote the norm in $H^1(\Omega)^d$ and $L^2(\Omega)$, respectively. However, in the unsteady case, to the author's knowledge, the best error estimate available is (see [10] for the linearized Navier–Stokes equations and [15] for fully nonlinear Navier–Stokes equations)

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\| \le C\sqrt{\varepsilon} \quad \forall t \in [0, T], \tag{1.6}$$

which is certainly not satisfactory in view of the error estimate (1.5) for the steady case. Furthermore, the estimate (1.6) is misleading when it is used to choose the value of ε for a time-discretized penalized system. For instance, when the backward Euler scheme (see (5.5) below) is applied to the penalized system (1.3)–(1.4), the estimate (1.6) would lead to

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\| \le C(\Delta t + \sqrt{\varepsilon}) \quad \forall n \le T/\Delta t. \tag{1.7}$$

Hence it suggests the choice $\varepsilon = \Delta t^2$, which would results in a very ill conditioned system when (5.5) is further discretized in space variables. The situation may become even worse when higher-order time-discretization schemes are used (see Thm. 4 in [15]). In general, the optimal choice of ε will vary according to different time discretization schemes (as we can see from the results in §5) as well as to different space-discretization schemes. But unfortunately in previous implementations (see for instance [2], [12]) the relation between the penalty coefficient ε and the time step Δt of a time-discretization scheme was not addressed because of lack of proper error estimates. Our aim in this article is to derive optimal error estimates for the penalty system and its time discretizations. The main results are stated in Theorem 4.1 for the error estimate of the penalized system and in Theorem 5.1 and Proposition 5.1 for error estimates of two time-discretized schemes of the penalized system. We note that these results are optimal and substantially improve the previous results (1.6) and (1.7). In particular, our error estimates lead to proper choices of ε for time discretizations of the penalized system.

The rest of the article is organized as follows. In the next section, we introduce notations and recall some preliminary results. In §3, we analyze the error behavior of the penalty method for the Navier–Stokes equations linearized at $\boldsymbol{u} = 0$. Then in §4, we consider the penalty method for the fully nonlinear Navier–Stokes equations. Finally in §5, we analyze two time-discretized schemes for the penalized system.

2. Preliminaries. We describe below some of the notations and results which will be frequently used in this paper.

We will use the standard notations $L^2(\Omega)$, $H^k(\Omega)$, and $H^k_0(\Omega)$ to denote the usual Sobolev spaces over Ω. The norm corresponding to $H^k(\Omega)$ will be denoted by $\|\cdot\|_k$. In particular, we will use $\|\cdot\|$ to denote the norm in $L^2(\Omega)$ and $(\cdot,\cdot)$ to denote the scalar product in $L^2(\Omega)$. The vector functions and vector spaces will be indicated by boldface type.

We denote
$\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \operatorname{div}\boldsymbol{v} = 0, \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0\}$,
$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \operatorname{div}\boldsymbol{v} = 0\}$,
$H^{-2}(\Omega)$: the dual space of $H^2(\Omega) \cap H_0^1(\Omega)$,
$P_{\boldsymbol{H}}$: the orthogonal projector in $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{H}$,
$A = -P_H\Delta$: the Stokes operator, which is an unbounded positive self-adjoint closed operator in $\boldsymbol{H}$ with domain $D(A) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}$.

We now introduce some operators usually associated with the Navier–Stokes equations and their approximations.

$$B(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\ \ \tilde{B}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \frac{1}{2}(\operatorname{div}\boldsymbol{u})\boldsymbol{v},$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = (B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}),\ \ \tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = (\tilde{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}).$$

We note that

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0 \quad \forall \boldsymbol{u} \in \boldsymbol{H}, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{2.1}$$

One can also easily check by integration by parts that

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \frac{1}{2}\{b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) - b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\}\ \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega).$$

Therefore, we have

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0 \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{2.2}$$

The following two inequalities will be used repeatedly in the upcoming sections. They can be proved by using a combination of integration by parts, Holder's inequality, and Sobolev inequalities (see for instance Lemma 2.1 in [20]).

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le C\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1^{\frac{1}{2}}\|\boldsymbol{v}\|_2^{\frac{1}{2}}\|\boldsymbol{w}\|\ \forall \boldsymbol{v} \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega),\ \boldsymbol{u},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \tag{2.3}$$

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le \begin{cases} \|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1\ \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \\ \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|\|\boldsymbol{w}\|_1\ \forall \boldsymbol{u} \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega),\ \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \\ \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|\ \forall \boldsymbol{u} \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega),\ \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \\ \|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|\ \forall \boldsymbol{v} \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega),\ \boldsymbol{u},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \end{cases} \tag{2.4}$$

We note that (2.3) is valid for $d \le 3$ and sharp for $d = 3$; (2.4) is valid for $d \le 4$. In most cases, (2.4) is sufficient for our purposes.

We define $A_\varepsilon\boldsymbol{u} = -\nu\Delta\boldsymbol{u} - \frac{1}{\varepsilon}\nabla\operatorname{div}\boldsymbol{u}$, which is the operator associated with the penalty method. It is clear that A_ε is a positive self-adjoint operator from $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ onto $\boldsymbol{L}^2(\Omega)$ and that the powers A^α of A ($\alpha \in \boldsymbol{R}$) are well defined. Furthermore, we have the following lemma.

LEMMA 2.1. *There exists a constant $C > 0$ such that for ε sufficiently small, we have*

(a) $\|\Delta\boldsymbol{u}\| \le C\|A_\varepsilon\boldsymbol{u}\| \quad \forall \boldsymbol{u} \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1(\Omega)$;

(b) $\|\nabla\boldsymbol{u}\| \le C\|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}\| \quad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$;

(c) $\|A_\varepsilon^{-1}\boldsymbol{u}\| \le C\|\boldsymbol{u}\|_{-2} \quad \forall \boldsymbol{u} \in \boldsymbol{H}^{-2}(\Omega).$

Proof. Let us consider the equations

$$A_\varepsilon \boldsymbol{u} = \boldsymbol{f}, \qquad \boldsymbol{u} = 0|_{\partial\Omega}. \tag{2.5}$$

It can be reformulated as

$$\begin{aligned} &-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ &\operatorname{div}\boldsymbol{u} = -\varepsilon p, \qquad \boldsymbol{u} = 0|_{\partial\Omega}. \end{aligned} \tag{2.6}$$

From the regularity result of the Stokes equations (see for instance [21]), we have

$$\|\boldsymbol{u}\|_2 + \|p\|_1 \le C(\|\boldsymbol{f}\| + \varepsilon\|p\|_1).$$

Thus, for ε sufficient small such that $C\varepsilon \le \frac{1}{2}$, we have

$$\|\boldsymbol{u}\|_2 + \|p\|_1 \le 2C\|\boldsymbol{f}\| = 2C\|A_\varepsilon \boldsymbol{u}\|.$$

For proving inequality (b), we take the inner product of (2.5) with $\boldsymbol{u}$ to obtain

$$(\boldsymbol{u}, \boldsymbol{f}) = (\boldsymbol{u}, A_\varepsilon \boldsymbol{u}) = \|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}\|^2 = \nu\|\nabla\boldsymbol{u}\|^2 + \frac{1}{\varepsilon}\|\operatorname{div}\boldsymbol{u}\|^2.$$

Finally, let $\boldsymbol{v}$ be the solution of

$$A_\varepsilon \boldsymbol{v} = A_\varepsilon^{-1}\boldsymbol{u}, \qquad \boldsymbol{v} = 0|_{\partial\Omega}. \tag{2.7}$$

Using (a), we derive

$$\begin{aligned} \|A_\varepsilon^{-1}\boldsymbol{u}\|^2 = (A_\varepsilon\boldsymbol{v}, A_\varepsilon^{-1}\boldsymbol{u}) = (\boldsymbol{v}, \boldsymbol{u}) &\le \|\boldsymbol{u}\|_{-2}\|\boldsymbol{v}\|_2 \\ &\le C\|\boldsymbol{u}\|_{-2}\|A_\varepsilon\boldsymbol{v}\| = C\|\boldsymbol{u}\|_{-2}\|A_\varepsilon^{-1}\boldsymbol{u}\|. \end{aligned}$$ □

We recall that (see for instance [21]) for

$$\boldsymbol{u}_0 \in \boldsymbol{V}, \qquad \boldsymbol{f} \in L^\infty(0, T; \boldsymbol{L}^2(\Omega)), \tag{A1}$$

there exists a $T_1 < T$ such that

$$\boldsymbol{u} \in C([0, T_1]; \boldsymbol{V}) \cap L^2(0, T_1; \boldsymbol{H}^2(\Omega)), \qquad p \in L^2(0, T_1; H^1(\Omega)/\mathbb{R}). \tag{2.8}$$

In some cases, we will assume additionally

$$t\boldsymbol{f}_t \in L^2(0, T; \boldsymbol{L}^2(\Omega)), \tag{A2}$$

which enables us to prove, by using the smoothing property of the Navier–Stokes equations at $t = 0$, that (see for instance [11])

$$tp_t \in L^2(0, T_1; H^1(\Omega)). \tag{2.9}$$

Using the operator A_ε, we can rewrite the penalized system (1.3)–(1.4) as

$$\boldsymbol{u}_t^\varepsilon + A_\varepsilon\boldsymbol{u}^\varepsilon + \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) = \boldsymbol{f}, \qquad \boldsymbol{u}^\varepsilon|_{\partial\Omega} = 0. \tag{2.10}$$

Similar to the Navier–Stokes equations, one can show (see for instance [3]) that assuming (A1), there exists a $T_2 < T$ and a constant C independent of ε such that

$$\|\boldsymbol{u}^\varepsilon(t)\|_1^2 + \int_0^t \|\boldsymbol{u}^\varepsilon(s)\|_2^2 ds \le C, \qquad t \in [0, T_2].$$

In the sequel, we restrict ourselves to the interval $[0, T_0]$ with $T_0 = \min\{T_1, T_2\}$. For simplicity, we will use $\|v\|_{L^p(X)}$ to denote the norm $(\int_0^{T_0} \|v\|_X^p dt)^{1/p}$ in $L^p(0, T_0; X)$.

We will use C as a generic constant depending only on the data: $\Omega, \boldsymbol{u}_0, \boldsymbol{f}$, T_0, and ν. C may grow exponentially with T_0, but it will only grow algebraically with ν^{-1}.

For the readers' convenience, we recall two lemmas of Gronwall type which will be frequently used.

LEMMA 2.2 (Gronwall lemma). *Let $y(t), h(t), g(t), f(t)$ be nonnegative functions satisfying*

$$y(t) + \int_0^t h(s)ds \leq y(0) + \int_0^t (g(s)y(s) + f(s))ds \; \forall 0 \leq t \leq T, \text{ with } \int_0^T g(t)dt \leq M.$$

Then

$$y(t) + \int_0^t h(s)ds \leq \exp(M)(y(0) + \int_0^t f(s)ds) \; \forall 0 < t \leq T.$$

LEMMA 2.3 (discrete Gronwall lemma). *Let y^n, h^n, g^n, f^n be nonnegative series satisfying*

$$y^m + k\sum_{n=0}^{m} h^n \leq k\sum_{n=0}^{m}(g^n y^n + f^n), \text{ with } k\sum_{n=0}^{T/k} g^n \leq M \; \forall 0 \leq m \leq T/k.$$

Assume $kg^n < 1 \forall n$ and let $\sigma = \max_{0\leq n \leq T/R}(1 - kg^n)^{-1}$, then

$$y^m + k\sum_{n=1}^{m} h^n \leq \exp(\sigma M)(B + k\sum_{n=0}^{m} f^n) \;\; \forall m \leq T/k.$$

3. Linearized problem. In this section, we will consider the following Navier–Stokes equations linearized at $\boldsymbol{u} = 0$:

$$\begin{aligned} &\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ &\text{div}\boldsymbol{u} = 0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{aligned} \tag{3.1}$$

The results in this section will then be used in the next section as an intermediate step for analyzing the fully nonlinear Navier–Stokes equations.

The penalty method applied to (3.1) is

$$\begin{aligned} &\boldsymbol{u}_t^\varepsilon - \nu\Delta\boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{f}, \\ &\text{div}\boldsymbol{u}^\varepsilon + \varepsilon p^\varepsilon = 0, \qquad \boldsymbol{u}^\varepsilon(0) = \boldsymbol{u}_0. \end{aligned} \tag{3.2}$$

We shall derive a sequence of estimates for the penalty errors $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}^\varepsilon$ and $q = p - p^\varepsilon$.

Subtracting (3.2) from (3.1), we obtain

$$\boldsymbol{e}_t - \nu\Delta\boldsymbol{e} + \nabla q = 0, \tag{3.3}$$

$$\text{div}\boldsymbol{e} + \varepsilon q = \varepsilon p, \qquad \boldsymbol{e}(0) = 0. \tag{3.4}$$

LEMMA 3.1. *Assuming* (A1), *we have*

$$\|\boldsymbol{e}\|_{L^\infty(\boldsymbol{L}^2)} + \sqrt{\nu}\|\boldsymbol{e}\|_{L^2(\boldsymbol{H}^1)} + \sqrt{\varepsilon}\|q\|_{L^2(L^2)} \leq C\sqrt{\varepsilon}, \tag{3.5}$$

$$\|\boldsymbol{e}\|_{L^2(\boldsymbol{L}^2)} \leq C\varepsilon. \tag{3.6}$$

Proof. Taking the inner product of (3.3) with $\boldsymbol{e}$ and (3.4) with q, summing up the two relations, we derive

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}\|^2 + \nu\|\nabla\boldsymbol{e}\|^2 + \varepsilon\|q\|^2 = \varepsilon(p, q) \leq \frac{\varepsilon}{2}\|q\|^2 + \frac{\varepsilon}{2}\|p\|^2.$$

Integrating the above inequality from 0 to $t \le T_0$, thanks to $\boldsymbol{e}(0) = 0$ and (2.8), we obtain

$$\|\boldsymbol{e}\|^2_{L^\infty(\boldsymbol{L}^2)} + \nu\|\boldsymbol{e}\|^2_{L^2(\boldsymbol{H}^1)} + \varepsilon\|q\|^2_{L^2(L^2)} \le C\varepsilon,$$

which is equivalent to (3.5).

We now use the standard parabolic duality argument. For any $0 < t \le T_0$, we define $(\boldsymbol{w}, \phi)$ by

$$\text{(3.7)} \qquad \begin{aligned} &\boldsymbol{w}_s + \nu\Delta\boldsymbol{w} + \nabla\phi = \boldsymbol{e}(s) \quad \forall 0 < s \le t, \\ &\text{div}\boldsymbol{w} = 0, \qquad \boldsymbol{w}(t) = 0. \end{aligned}$$

Let us first establish the following inequality:

$$\text{(3.8)} \qquad \nu\|\boldsymbol{w}\|_{L^2(\boldsymbol{H}^2)} + \|\nabla\phi\|_{L^2(\boldsymbol{L}^2)} \le C\|\boldsymbol{e}\|_{L^2(\boldsymbol{L}^2)}.$$

Taking the inner product of (3.7) with $A\boldsymbol{w}$, integrating from 0 to t, we derive immediately

$$\nu\|\boldsymbol{w}\|_{L^2(\boldsymbol{H}^2)} + \|\nabla\boldsymbol{w}(0)\|_{L^2(\boldsymbol{L}^2)} \le C\|\boldsymbol{e}\|_{L^2(\boldsymbol{L}^2)}.$$

Applying the projection operator $P_{\boldsymbol{H}}$ on (3.7), we derive

$$\|\boldsymbol{w}_s\|_{L^2(\boldsymbol{L}^2)} \le C\|\boldsymbol{e}\|_{L^2(\boldsymbol{L}^2)}.$$

We then use the equation (3.7) again to obtain

$$\|\nabla\phi\|_{L^2(\boldsymbol{L}^2)} \le C\|\boldsymbol{e}\|_{L^2(\boldsymbol{L}^2)},$$

which completes the proof of (3.8).

We now take the inner product of (3.7) with $\boldsymbol{e}(s)$, in light of (3.3) and $\text{div}\boldsymbol{w} = 0$, and derive

$$\begin{aligned} \|\boldsymbol{e}\|^2 &= (\boldsymbol{w}_s, \boldsymbol{e}) + \nu(\Delta\boldsymbol{w}, \boldsymbol{e}) + (\nabla\phi, \boldsymbol{e}) \\ &= \frac{d}{ds}(\boldsymbol{w}, \boldsymbol{e}) - (\boldsymbol{e}_s, \boldsymbol{w}) - \nu(\nabla\boldsymbol{e}, \nabla\boldsymbol{w}) + (\nabla\phi, \boldsymbol{e}) \\ &= \frac{d}{ds}(\boldsymbol{w}, \boldsymbol{e}) + (\nabla q, \boldsymbol{w}) - (\phi, \text{div}\boldsymbol{e}) = \frac{d}{ds}(\boldsymbol{w}, \boldsymbol{e}) - \varepsilon(\phi, p^\varepsilon). \end{aligned}$$

Integrating from 0 to t, since $\boldsymbol{w}(t) = \boldsymbol{e}(0) = 0$, we obtain

$$\int_0^t \|\boldsymbol{e}\|^2 ds = -\int_0^t \varepsilon(\phi, p^\varepsilon) ds \le \delta\|\phi\|^2_{L^2(\boldsymbol{L}^2)} + C_\delta\varepsilon^2\|p^\varepsilon\|^2_{L^2(L^2)},$$

where C_δ is a constant depending on δ only. Thanks to (3.8) and Lemma 3.1, we can choose δ sufficiently small such that

$$\int_0^t \|\boldsymbol{e}\|^2 ds \le C\varepsilon^2\|p^\varepsilon\|^2_{L^2(L^2)} \le C\varepsilon^2 \quad \forall t \in [0, T_0]. \qquad \square$$

The following results related to (2.9) will be needed to derive improved error estimates.

LEMMA 3.2. *Assuming* (A1) *and* (A2), *we have*

$$\int_0^t s^2\|p_t^\varepsilon\|^2 ds \le C \quad \forall\, t \in [0, T_0].$$

Proof. Taking the partial derivative with respect to t on (3.3) and (3.4), we obtain

$$e_{tt} - \nu\Delta e_t + \nabla q_t = 0, \tag{3.9}$$

$$\text{div} e_t + \varepsilon q_t = \varepsilon p_t, \qquad e(0) = 0. \tag{3.10}$$

Taking the inner products of (3.3) with $te_t(t)$, (3.10) with tq, and summing up the two relations we get

$$t\|e_t\|^2 + \frac{\nu}{2}\frac{d}{dt}t\|\nabla e\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}t\|q\|^2 = \frac{\nu}{2}\|\nabla e\|^2 + \frac{\varepsilon}{2}\|q\|^2 + \varepsilon t(p_t, q)$$
$$\leq \frac{\nu}{2}\|\nabla e\|^2 + \varepsilon\|q\|^2 + \frac{\varepsilon}{2}t^2\|p_t\|^2.$$

Integrating over $[0,t]$, using Lemma 3.1, (2.9), and the Gronwall lemma, we derive

$$\int_0^t s\|e_t\|^2 ds + t\|e\|_1^2 + \varepsilon t\|q\|^2 \leq C\varepsilon. \tag{3.11}$$

We then take the inner products of (3.9) with $t^2 e_t$, (3.10) with $t^2 q_t$, and adding them up, we obtain

$$\frac{1}{2}\frac{d}{dt}t^2\|e_t\|^2 + \nu t^2\|\nabla e_t\|^2 + \varepsilon t^2\|q_t\|^2 = t\|e_t\|^2 + \varepsilon t^2(p_t, q_t)$$
$$\leq t\|e_t\|^2 + \frac{\varepsilon t^2}{2}\|q_t\|^2 + \frac{\varepsilon t^2}{2}\|p_t\|^2.$$

Integrating the above relation from 0 to t, using (3.11) and (2.9), we get

$$\varepsilon\int_0^t s^2\|q_t\|^2 ds \leq C\int_0^t s\|e_t\|^2 ds + C\varepsilon\int_0^t s^2\|p_t\|^2 ds \leq C\varepsilon,$$

The desired result follows from the above inequality and (2.9). ∎

LEMMA 3.3. *Assuming* (A1) *and* (A2), *we have*

$$t\|e(t)\|^2 + \nu\int_0^t s\|\nabla e(s)\|^2 ds + \varepsilon\int_0^t s\|q(s)\|^2 ds \leq C\varepsilon^2 \quad \forall t \in [0, T_0], \tag{3.12}$$

$$t^2\|\nabla e(t)\|^2 + \int_0^t s^2\|q(s)\|^2 ds \leq C\varepsilon^2 \quad \forall t \in [0, T_0]. \tag{3.13}$$

Proof. Let us consider the decomposition (cf. [9])

$$\boldsymbol{H}_0^1(\Omega) = V \oplus V^{\perp}, \quad \text{where } V^{\perp} = \{(-\Delta)^{-1}\nabla q : q \in L^2(\Omega)\}$$

and $v = (-\Delta)^{-1}\nabla q$ iff $-\Delta v = \nabla q$ and $v|_{\partial\Omega} = 0$. It is well known (cf. [9]) that for $p(t) \in L^2(\Omega)/\mathbb{R}$, there exists a unique $\varphi(t) \in V^{\perp}$ such that $\text{div}\varphi(t) = p(t)$ with

$$\|\varphi(t)\|_1 \leq C\|p(t)\| \quad \forall t \in [0, T_0]. \tag{3.14}$$

Furthermore, if $p_t(t) \in L^2(\Omega)/\mathbb{R}$, we then have $\text{div}\varphi_t(t) = p_t(t)$ with

$$\|\varphi_t(t)\|_1 \leq C\|p_t(t)\| \quad \forall t \in [0, T_0]. \tag{3.15}$$

Taking the inner product of (3.3) with $t\boldsymbol{e}$ and of (3.4) with tq, summing up the two relations, and using (3.3), we derive

$$(3.16)\quad \begin{aligned}\frac{1}{2}\frac{d}{dt}(t\|\boldsymbol{e}\|^2)+t\nu\|\nabla \boldsymbol{e}\|^2+\varepsilon t\|q\|^2 &= \frac{1}{2}\|\boldsymbol{e}\|^2+\varepsilon t(p,q)=\frac{1}{2}\|\boldsymbol{e}\|^2+\varepsilon t(\mathrm{div}\boldsymbol{\varphi},q)\\ &= \frac{1}{2}\|\boldsymbol{e}\|^2-\varepsilon t(\nabla q,\boldsymbol{\varphi})=\frac{1}{2}\|\boldsymbol{e}\|^2+\varepsilon t(\boldsymbol{e}_t,\boldsymbol{\varphi})+\varepsilon\nu t(\nabla \boldsymbol{e},\nabla\boldsymbol{\varphi})\\ &= \frac{1}{2}\|\boldsymbol{e}\|^2+\varepsilon\frac{d}{dt}t(\boldsymbol{e},\boldsymbol{\varphi})-\varepsilon(\boldsymbol{e},\boldsymbol{\varphi})-\varepsilon t(\boldsymbol{e},\boldsymbol{\varphi}_t)+\varepsilon\nu t(\nabla \boldsymbol{e},\nabla\boldsymbol{\varphi}).\end{aligned}$$

With the assumption (A1), we can show (cf. [11]) that $\sqrt{t}p \in L^\infty(0,T_0;H^1(\Omega)/\mathbb{R})$, therefore using (3.14), we get

$$\varepsilon t(\boldsymbol{e}(t),\boldsymbol{\varphi}(t)) \le \frac{t}{4}\|\boldsymbol{e}(t)\|^2+\varepsilon^2 t\|\boldsymbol{\varphi}\|^2 \le \frac{t}{4}\|\boldsymbol{e}(t)\|^2+C\varepsilon^2.$$

Hence, integrating (3.16) from 0 to t, using the above relation, the Cauchy–Schwarz inequality, Lemma 3.1, (2.9), (3.14), and (3.15), we derive

$$\begin{aligned}t\|\boldsymbol{e}(t)\|^2&+\int_0^t(\nu s\|\nabla \boldsymbol{e}(s)\|^2+\varepsilon s\|q(s)\|^2)ds\\ &\le C\varepsilon^2+C\int_0^t\|\boldsymbol{e}(s)\|^2ds+C\varepsilon^2\int_0^t\|\boldsymbol{\varphi}(s)\|^2ds+C\varepsilon^2\int_0^t s^2\|\boldsymbol{\varphi}_t(s)\|^2\\ &\le C\varepsilon^2+C\varepsilon^2\int_0^t(\|p\|^2+s^2\|p_t\|^2)ds\le C\varepsilon^2.\end{aligned}$$

Taking the inner products of (3.3) with $t^2\boldsymbol{e}_t$, (3.10) with t^2q, and summing up the two relations, we obtain

$$(3.17)\quad t^2\|\boldsymbol{e}_t\|^2+\frac{1}{2}\frac{d}{dt}t^2\|\nabla \boldsymbol{e}\|^2+\frac{\varepsilon}{2}\frac{d}{dt}t^2\|q\|^2=t\|\boldsymbol{e}\|_1^2+\varepsilon t\|q\|^2+\varepsilon t^2(p_t,q).$$

Using equations (3.3) and (3.14), we get

$$\begin{aligned}\varepsilon t^2(p_t,q)=\varepsilon t^2(\mathrm{div}\boldsymbol{\varphi}_t,q) &= -\varepsilon t^2(\boldsymbol{\varphi}_t,\nabla q)=\varepsilon t^2(\boldsymbol{e}_t,\boldsymbol{\varphi}_t)+\varepsilon t^2(\nabla \boldsymbol{e},\nabla\boldsymbol{\varphi}_t)\\ &\le \frac{t^2}{2}\|e_t\|^2+C\varepsilon^2t^2\|\boldsymbol{\varphi}_t\|^2+t^2\|\nabla \boldsymbol{e}\|^2+\varepsilon^2t^2\|\nabla\boldsymbol{\varphi}_t\|^2\\ &\le \frac{t^2}{2}\|\boldsymbol{e}_t\|^2+t^2\|\nabla \boldsymbol{e}\|^2+C\varepsilon^2t^2\|p_t\|^2.\end{aligned}$$

Integrating (3.17) and using the Gronwall lemma, we obtain

$$\int_0^t s^2\|\boldsymbol{e}_t(s)\|^2ds+t^2\|\nabla \boldsymbol{e}(t)\|^2+\varepsilon t^2\|q(t)\|^2\le C\varepsilon^2\quad \forall t\in[0,T_0].$$

Finally we derive from equation (3.3) that

$$\|q\|^2\le C\|\nabla q\|_{-1}^2\le C(\|\Delta \boldsymbol{e}\|_{-1}^2+\|\boldsymbol{e}_t\|_{-1}^2)\le C(\|\boldsymbol{e}\|_1^2+\|\boldsymbol{e}_t\|^2).$$

Therefore

$$\int_0^{T_0}s^2\|q\|^2ds\le C\int_0^{T_0}s^2(\|\boldsymbol{e}\|_1^2+\|\boldsymbol{e}_t\|^2)ds\le C\varepsilon^2. \qquad \square$$

To summarize, we have proved the following theorem.

THEOREM 3.1. *Assuming* (A1), *we have*

$$\|\boldsymbol{e}(t)\| + \left(\int_0^t \|\boldsymbol{e}(s)\|_1^2 ds\right)^{\frac{1}{2}} \le C\sqrt{\varepsilon} \ \ \forall t \in [0, T_0].$$

Assuming (A1) *and* (A2), *we have*

$$\sqrt{t}\|\boldsymbol{e}(t)\| + t\|e(t)\|_1 + \left(\int_0^t s^2\|q\|^2 ds\right)^{\frac{1}{2}} \le C\varepsilon \ \ \forall t \in [0, T_0].$$

4. Nonlinear Navier–Stokes equations. We consider first the following intermediate linear equations:

$$\boldsymbol{v}_t - \nu\Delta\boldsymbol{v} + \nabla\gamma = \boldsymbol{f} - B(\boldsymbol{u}, \boldsymbol{u}), \tag{4.1}$$

$$\text{div}\boldsymbol{v} + \varepsilon\gamma = 0, \qquad \boldsymbol{v}(0) = u_0, \tag{4.2}$$

where $\boldsymbol{u}$ is the solution of Navier–Stokes equations (1.1)–(1.2).

Letting $\boldsymbol{\xi} = \boldsymbol{v} - \boldsymbol{u}, \phi = \gamma - p$, and subtracting (4.1)–(4.2) from (1.1)-(1.2), we obtain

$$\begin{aligned} &\boldsymbol{\xi}_t - \nu\Delta\boldsymbol{\xi} + \nabla\phi = 0, \\ &\text{div}\boldsymbol{\xi} + \varepsilon\phi = -\varepsilon p, \qquad \boldsymbol{\xi}(0) = 0. \end{aligned} \tag{4.3}$$

LEMMA 4.1. *Assuming* (A1) *and* (A2), *we have*

$$\left(\int_0^t \|\boldsymbol{\xi}(s)\|^2 ds\right)^{\frac{1}{2}} + \sqrt{t}\|\boldsymbol{\xi}(t)\| + t\|\boldsymbol{\xi}(t)\|_1 + \left(\int_0^t s^2\|\phi(s)\|^2 ds\right)^{\frac{1}{2}} \le C\varepsilon \ \ \forall t \in [0, T_0].$$

Proof. Thanks to (2.8), we have $\boldsymbol{f} - B(\boldsymbol{u}, \boldsymbol{u}) \in L^2(0, T_0; \boldsymbol{L}^2(\Omega))$. We note that the assumption (A1) for a linear problem can be replaced by the weaker condition $\boldsymbol{f} \in L^2(0, T_0; \boldsymbol{L}^2(\Omega))$. On the other hand, it can be easily shown (see for instance [11] that $t\boldsymbol{u}_t \in L^2(0, T_0; \boldsymbol{H}_0^1(\Omega))$. Hence

$$t\frac{\partial}{\partial t}(\boldsymbol{f} - B(\boldsymbol{u}, \boldsymbol{u})) = t(\boldsymbol{f}_t - B(\boldsymbol{u}_t, \boldsymbol{u}) - B(\boldsymbol{u}, \boldsymbol{u}_t)) \in L^2(0, T_0; \boldsymbol{L}^2(\Omega)).$$

The lemma is then a direct consequence of Lemma 3.1 and Theorem 3.1 applied to (4.3). □

Now, letting $\boldsymbol{\eta} = \boldsymbol{u}^\varepsilon - \boldsymbol{v}$, $q = p^\varepsilon - \gamma$, and subtracting (4.1)–(4.2) from (1.3)–(1.4), we get

$$\boldsymbol{\eta}_t - \nu\Delta\boldsymbol{\eta} + \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) - \tilde{B}(\boldsymbol{u}, \boldsymbol{u}) + \nabla q = 0, \tag{4.4}$$

$$\text{div}\boldsymbol{\eta} + \varepsilon q = 0, \qquad \boldsymbol{\eta}(0) = 0. \tag{4.5}$$

Since

$$\begin{aligned} \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) - \tilde{B}(\boldsymbol{u}, \boldsymbol{u}) &= \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon - \boldsymbol{u}) + \tilde{B}(\boldsymbol{u}^\varepsilon - \boldsymbol{u}, \boldsymbol{u}) \\ &= \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{\xi} + \boldsymbol{\eta}) + \tilde{B}(\boldsymbol{\xi} + \boldsymbol{\eta}, \boldsymbol{u}), \end{aligned} \tag{4.6}$$

we can rewrite (4.4) as

$$\boldsymbol{\eta}_t + A_\varepsilon\boldsymbol{\eta} + \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{\xi} + \boldsymbol{\eta}) + \tilde{B}(\boldsymbol{\xi} + \boldsymbol{\eta}, \boldsymbol{u}) = 0. \tag{4.7}$$

THEOREM 4.1. *Assuming* (A1) *and* (A2), *we have the following estimates:*

$$\sqrt{t}\|\boldsymbol{u}(t)-\boldsymbol{u}^\varepsilon(t)\|+\sqrt{\nu}t\|\boldsymbol{u}(t)-\boldsymbol{u}^\varepsilon(t)\|_1+\left(\int_0^t s^2\|p(t)-p^\varepsilon(t)\|^2 ds\right)^{\frac{1}{2}} \le C\varepsilon \ \ \forall 0<t\le T_0.$$

Proof. Taking the inner product of (4.7) with $A_\varepsilon^{-1}\boldsymbol{\eta}$, we get

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{-\frac{1}{2}}\boldsymbol{\eta}\|^2+\nu\|\boldsymbol{\eta}\|^2=-\tilde{b}(\boldsymbol{u}^\varepsilon,\boldsymbol{\xi}+\boldsymbol{\eta},A_\varepsilon^{-1}\boldsymbol{\eta})-\tilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta},\boldsymbol{u},A_\varepsilon^{-1}\boldsymbol{\eta})\equiv I_1+I_2.$$

Thanks to (2.4) and Lemma 2.1, using Schwarz's inequality we derive that

$$\begin{aligned}I_1 &\le C\|\boldsymbol{u}^\varepsilon\|_2\|\boldsymbol{\xi}+\boldsymbol{\eta}\|\|A_\varepsilon^{-1}\boldsymbol{\eta}\|_1\le C\|\boldsymbol{u}^\varepsilon\|_2\|\boldsymbol{\xi}+\boldsymbol{\eta}\|\|\nabla A_\varepsilon^{-1}\boldsymbol{\eta}\|\\ &\le C\|\boldsymbol{u}^\varepsilon\|_2(\|\boldsymbol{\xi}\|+\|\boldsymbol{\eta}\|)\|A_\varepsilon^{-\frac{1}{2}}\boldsymbol{\eta}\|\le\frac{\nu}{4}\|\boldsymbol{\eta}\|^2+C\|\boldsymbol{\xi}\|^2+C\|\boldsymbol{u}^\varepsilon\|_2^2\|A_\varepsilon^{-\frac{1}{2}}\boldsymbol{\eta}\|^2.\end{aligned}$$

Similarly, we have

$$I_2\le\frac{\nu}{4}\|\boldsymbol{\eta}\|^2+C\|\boldsymbol{\xi}\|^2+C\|\boldsymbol{u}\|_2^2\|A_\varepsilon^{-\frac{1}{2}}\boldsymbol{\eta}\|^2.$$

Combining the above inequalities, we arrive at

$$\frac{d}{dt}\|A_\varepsilon^{-\frac{1}{2}}\boldsymbol{\eta}\|^2+\nu\|\boldsymbol{\eta}\|^2\le C\|\boldsymbol{\xi}\|^2+C(\|\boldsymbol{u}\|_2^2+\|\boldsymbol{u}^\varepsilon\|_2^2)\|A_\varepsilon^{-\frac{1}{2}}\boldsymbol{\eta}\|^2. \tag{4.8}$$

Since $\int_0^{T_0}(\|\boldsymbol{u}\|_2^2+\|\boldsymbol{u}^\varepsilon\|_2^2)dt\le C$, we can apply the Gronwall lemma to (4.8), using Lemma 4.1, and obtain

$$\|A_\varepsilon^{-\frac{1}{2}}\boldsymbol{\eta}(t)\|^2+\nu\int_0^t\|\boldsymbol{\eta}(s)\|^2 ds\le C\int_0^t\|\boldsymbol{\xi}(s)\|^2 ds\le C\varepsilon^2 \ \ \forall t\in[0,T_0]. \tag{4.9}$$

Now, taking the inner products of (4.4) with $t\boldsymbol{\eta}$, (4.5) with tq, and adding them up using (2.4) and Schwarz's inequality, we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}t\|\boldsymbol{\eta}\|^2+\nu t\|\nabla\boldsymbol{\eta}\|^2+\varepsilon t\|q\|^2&=\frac{1}{2}\|\boldsymbol{\eta}\|^2-t\tilde{b}(\boldsymbol{u}^\varepsilon,\boldsymbol{\xi}+\boldsymbol{\eta},\boldsymbol{\eta})-t\tilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta},\boldsymbol{u},\boldsymbol{\eta})\\ &\le\frac{1}{2}\|\boldsymbol{\eta}\|^2+Ct\|\boldsymbol{u}^\varepsilon\|_2\|\boldsymbol{\xi}+\boldsymbol{\eta}\|_1\|\boldsymbol{\eta}\|+Ct\|\boldsymbol{u}\|_2\|\boldsymbol{\xi}+\boldsymbol{\eta}\|_1\|\boldsymbol{\eta}\|\\ &\le\frac{1}{2}\|\boldsymbol{\eta}\|^2+\frac{\nu t}{2}\|\nabla\boldsymbol{\eta}\|^2+Ct\|\boldsymbol{\xi}\|_1^2+Ct(\|\boldsymbol{u}^\varepsilon\|_2^2+\|\boldsymbol{u}\|_2^2)\|\boldsymbol{\eta}\|^2.\end{aligned}$$

Integrating over [0, t], using (4.9), Lemma 4.1, and the Gronwall lemma, we obtain

$$t\|\boldsymbol{\eta}(t)\|^2+\nu\int_0^t s\|\nabla\boldsymbol{\eta}(s)\|^2 ds+\varepsilon\int_0^t s\|q(s)\|^2 ds\le C\varepsilon^2. \tag{4.10}$$

Next we take the partial derivative with respect to t of (4.5) to obtain

$$\mathrm{div}\boldsymbol{\eta}_t+\varepsilon q_t=0. \tag{4.11}$$

Taking the inner products of of (4.4) with $t^2\boldsymbol{\eta}_t$, (4.11) with t^2q, adding them up, we get

$$\begin{aligned}t^2\|\boldsymbol{\eta}_t\|^2+\frac{\nu}{2}\frac{d}{dt}t^2\|\nabla\boldsymbol{\eta}\|^2+\frac{\varepsilon}{2}\frac{d}{dt}t^2\|q\|^2&=\nu t\|\nabla\boldsymbol{\eta}\|^2+\varepsilon t\|q\|^2\\ &\quad-t^2\tilde{b}(\boldsymbol{u}^\varepsilon,\boldsymbol{\xi}+\boldsymbol{\eta},\boldsymbol{\eta}_t)-t^2\tilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta},\boldsymbol{u},\boldsymbol{\eta}_t).\end{aligned} \tag{4.12}$$

We treat the nonlinear terms on the right-hand side as follows.

Using (2.4) and Lemma 4.1, we derive

$$\begin{aligned} t^2\tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{\eta}_t) &\le t^2\|\boldsymbol{u}^\varepsilon\|_2\|\boldsymbol{\xi}+\boldsymbol{\eta}\|_1\|\boldsymbol{\eta}_t\| \\ &\le \frac{t^2}{4}\|\boldsymbol{\eta}_t\|^2 + Ct^2\|\boldsymbol{u}^\varepsilon\|_2^2(\|\boldsymbol{\xi}\|_1^2+\|\boldsymbol{\eta}\|_1^2) \\ &\le \frac{t^2}{4}\|\boldsymbol{\eta}_t\|^2 + C\varepsilon^2\|\boldsymbol{u}^\varepsilon\|_2^2 + Ct^2\|\boldsymbol{u}^\varepsilon\|_2^2\|\nabla\boldsymbol{\eta}\|^2. \end{aligned}$$

Similarly, we can derive

$$t^2\tilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u}, \boldsymbol{\eta}_t) \le \frac{t^2}{4}\|\boldsymbol{\eta}_t\|^2 + C\varepsilon^2\|\boldsymbol{u}\|_2^2 + Ct^2\|\boldsymbol{u}\|_2^2\|\nabla\boldsymbol{\eta}\|^2.$$

Combining the above inequalities into (4.12), we obtain

$$\begin{aligned} t^2\|\boldsymbol{\eta}_t\|^2 + \nu\frac{d}{dt}t^2\|\nabla\boldsymbol{\eta}\|^2 + \varepsilon\frac{d}{dt}t^2\|q\|^2 \le& \nu t\|\nabla\boldsymbol{\eta}\|^2 + \varepsilon t\|q\|^2 \\ &+ C(\varepsilon^2 + t^2\|\nabla\boldsymbol{\eta}\|^2)(\|\boldsymbol{u}^\varepsilon\|_2^2 + \|\boldsymbol{u}\|_2^2). \end{aligned}$$

Integrating over $[0,t]$, using (4.10) and the Gronwall lemma, we get

$$\int_0^t s^2\|\boldsymbol{\eta}_t(s)\|^2 ds + \nu t^2\|\nabla\boldsymbol{\eta}(t)\|^2 + \varepsilon t^2\|q(t)\|^2 \le C\varepsilon^2. \tag{4.13}$$

Finally by (2.4), we have

$$\|\tilde{B}(\boldsymbol{e}^\varepsilon, \boldsymbol{\xi}+\boldsymbol{\eta})\|_{-1} \le C(\|\boldsymbol{u}^\varepsilon\|_1\|\boldsymbol{\xi}+\boldsymbol{\eta}\|_1) \le C(\|\boldsymbol{u}^\varepsilon\|_1\|\boldsymbol{\xi}\|_1 + \|\boldsymbol{\eta}\|_1),$$

$$\|\tilde{B}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u})\|_{-1} \le C(\|\boldsymbol{u}\|_1\|\boldsymbol{\xi}+\boldsymbol{\eta}\|_1) \le C(\|\boldsymbol{u}\|_1\|\boldsymbol{\xi}\|_1 + \|\boldsymbol{\eta}\|_1).$$

From (4.4) and (4.6), we get

$$\nabla q = -\boldsymbol{\eta}_t + \nu\Delta\boldsymbol{\eta} - \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{\xi}+\boldsymbol{\eta}) - \tilde{B}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u}).$$

Therefore by using previous estimates on the above equation, we derive

$$\int_0^{T_0} s^2\|q\|^2 ds \le \int_0^{T_0} s^2\|\nabla q\|_{-1}^2 ds \le C\varepsilon^2,$$

which completes the proof of Theorem 4.1. □

5. Time discretizations of the penalized system. The main purpose for introducing the penalized system is to alleviate some difficulties related to the numerical solution of the Navier–Stokes equations. To simplify our presentation, we shall focus on semidiscretization (only the time variable will be discretized) of the penalized system instead of considering full discretizations, since the technicalities vary on different spatial discretizations and may obscure the essential goal of the paper. Note in particular that since the pressure approximation p^ε is given as the divergence of the velocity approximation $\boldsymbol{u}^\varepsilon$, the discrete spaces for the velocity and the pressure must satisfy the Babuska–Brezzi inf-sup condition to have a well-posed system and an optimal convergence rate in space variables for the velocity as well as for the pressure. (We refer to [9] for more details on this aspect.)

We shall analyze two time-discretization schemes for the penalized system. The first is the backward Euler scheme; the second is a modified scheme which leads to improved error estimates. To simplify the analysis, we need more regularities on the solutions of the Navier–Stokes equations and its penalized system.

LEMMA 5.1. *In addition to* (A1) *and* (A2), *we assume* $\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega)$. *Then the solution* $\boldsymbol{u}^\varepsilon$ *of* (2.10) *satisfies*

$$\boldsymbol{u}^\varepsilon \in C([0,T_0];\boldsymbol{H}^2(\Omega)), \tag{5.1}$$

(5.2)
$$\boldsymbol{u}_t^\varepsilon \in L^2(0,T_0;\boldsymbol{H}_0^1(\Omega)),\ \ A_\varepsilon^{-\frac{1}{2}}\boldsymbol{u}_{tt}^\varepsilon \in L^2(0,T_0;\boldsymbol{L}^2(\Omega)),\ \ \sqrt{t}\boldsymbol{u}_{tt}^\varepsilon \in L^2(0,T_0;\boldsymbol{L}^2(\Omega)).$$

Sketch of the proof. The proof of (5.1) is standard. We will only sketch the proof of (5.2).

Taking the partial derivative with respect to t on (2.10), we get

$$\boldsymbol{u}_{tt}^\varepsilon + A_\varepsilon \boldsymbol{u}_t^\varepsilon + \tilde{B}(\boldsymbol{u}_t^\varepsilon,\boldsymbol{u}^\varepsilon) + \tilde{B}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}_t^\varepsilon) = \boldsymbol{f}_t. \tag{5.3}$$

Taking the inner product of (5.3) with $\boldsymbol{u}_t^\varepsilon$, using (2.4) and Lemma 2.1, we obtain

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{u}_t^\varepsilon\|^2 + \nu\|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}_t^\varepsilon\|^2 &\le C\|\boldsymbol{f}_t\|_{-1}^2 + \tilde{b}(\boldsymbol{u}_t^\varepsilon,\boldsymbol{u}^\varepsilon,\boldsymbol{u}_t^\varepsilon)\\ &\le C\|\boldsymbol{f}_t\|_{-1}^2 + C\|\boldsymbol{u}_t^\varepsilon\|\|\boldsymbol{u}^\varepsilon\|_2\|\boldsymbol{u}_t^\varepsilon\|_1\\ &\le C\|\boldsymbol{f}_t\|_{-1}^2 + C\|\boldsymbol{u}^\varepsilon\|_2^2\|\boldsymbol{u}_t^\varepsilon\|^2 + \frac{\nu}{2}\|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}_t^\varepsilon\|^2.\end{aligned}$$

Since $\boldsymbol{u}^\varepsilon \in C([0,T];\boldsymbol{H}^2(\Omega))$, we can show (see for instance [11]) that $\boldsymbol{u}_t^\varepsilon(0)$ is well defined. Hence, integrating over $[0,t]$, using the Gronwall lemma and Lemma 2.1, we derive

$$\|\boldsymbol{u}_t^\varepsilon\|_{L^\infty(\boldsymbol{L}^2)} + \|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}_t^\varepsilon\|_{L^2(\boldsymbol{L}^2)} \le C. \tag{5.4}$$

By Lemma 2.1, we get $\|\boldsymbol{u}_t^\varepsilon\|_{L^2(\boldsymbol{H}^1)} \le C$. Then by using (2.4), we get

$$\|A_\varepsilon^{-\frac{1}{2}}\tilde{B}(\boldsymbol{u}_t^\varepsilon,\boldsymbol{u}^\varepsilon)\| \le \|\tilde{B}(\boldsymbol{u}_t^\varepsilon,\boldsymbol{u}^\varepsilon)\|_{-1} \le \sup_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)} \frac{\tilde{b}(\boldsymbol{u}_t^\varepsilon,\boldsymbol{u}^\varepsilon,\boldsymbol{v})}{\|\boldsymbol{v}\|_1} \le C\|\boldsymbol{u}_t^\varepsilon\|\|\boldsymbol{u}^\varepsilon\|_2.$$

The same is true for $\tilde{B}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}_t^\varepsilon)$. Thus

$$A_\varepsilon^{-\frac{1}{2}}\boldsymbol{u}_{tt} = A_\varepsilon^{-\frac{1}{2}}\{\boldsymbol{f}_t - A_\varepsilon\boldsymbol{u}_t - \tilde{B}(\boldsymbol{u}_t^\varepsilon,\boldsymbol{u}^\varepsilon) - \tilde{B}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}_t^\varepsilon)\} \in L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

Taking the inner product of (5.3) with $t\boldsymbol{u}_{tt}^\varepsilon$, using (2.2) and Schwarz's inequality, we get

$$\begin{aligned}t\|\boldsymbol{u}_{tt}^\varepsilon\|^2 + \frac{1}{2}\frac{d}{dt}t\|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}_t\|^2 &= \frac{1}{2}\|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}_t\|^2 + t(\boldsymbol{f}_t,\boldsymbol{e}_{tt}^\varepsilon) - t\tilde{b}(\boldsymbol{u}_t^\varepsilon,\boldsymbol{u}^\varepsilon,\boldsymbol{u}_{tt}^\varepsilon) - t\tilde{b}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}_t^\varepsilon,\boldsymbol{u}_{tt}^\varepsilon)\\ &\le \frac{1}{2}\|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}_t\|^2 + t\|\boldsymbol{f}_t\|\|\boldsymbol{u}_{tt}^\varepsilon\| + Ct\|\boldsymbol{u}^\varepsilon\|_2\|\boldsymbol{u}_t^\varepsilon\|_1\|\boldsymbol{u}_{tt}^\varepsilon\|\\ &\le \frac{1}{2}\|A_\varepsilon^{\frac{1}{2}}\boldsymbol{u}_t\|^2 + Ct\|\boldsymbol{f}_t\|^2 + \frac{t}{2}\|\boldsymbol{u}_{tt}^\varepsilon\|^2 + Ct\|\boldsymbol{u}^\varepsilon\|^2\|\boldsymbol{u}_t^\varepsilon\|^2.\end{aligned}$$

Integrating over $[0,T_0]$, using the Gronwall lemma, (A1), and (5.4), we derive

$$\int_0^{T_0} t\|\boldsymbol{u}_{tt}^\varepsilon\|^2 dt \le C. \qquad \square$$

5.1. Backward Euler scheme. Let us consider the time discretization of the penalized system (2.10) by the backward Euler scheme

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{k} + A_\varepsilon\boldsymbol{u}^{n+1} + \tilde{B}(\boldsymbol{u}^{n+1},\boldsymbol{u}^{n+1}) = \boldsymbol{f}(t_{n+1}) \text{ with } \boldsymbol{u}^0 = \boldsymbol{u}_0, \tag{5.5}$$

where k is the time-step size and $t_n = nk$.

LEMMA 5.2. *Under the assumption of Lemma* 5.1, *we have*

$$t_m\|\boldsymbol{u}^\varepsilon(t_n)-\boldsymbol{u}^n\|_1^2 + k\sum_{n=1}^{m} t_n\|\boldsymbol{u}^\varepsilon(t_n)-\boldsymbol{u}^n\|_2^2 \le Ck^2 \quad \forall m\le T_0/k.$$

Proof. Letting $\boldsymbol{e}^n = \boldsymbol{u}^\varepsilon(t_n)-\boldsymbol{u}^n$ and subtracting (5.5) from (2.10) at $t=t_{n+1}$, we get

$$\frac{\boldsymbol{e}^{n+1}-\boldsymbol{e}^n}{k} + A_\varepsilon\boldsymbol{e}^{n+1} + \tilde{B}(\boldsymbol{u}^{n+1},\boldsymbol{e}^{n+1}) + \tilde{B}(\boldsymbol{e}^{n+1},\boldsymbol{u}^\varepsilon(t_{n+1})) = \boldsymbol{R}_\varepsilon^n, \tag{5.6}$$

where

$$\boldsymbol{R}_\varepsilon^n = \boldsymbol{u}_t^\varepsilon(t_{n+1}) - \frac{1}{k}(\boldsymbol{u}^\varepsilon(t_{n+1})-\boldsymbol{u}^\varepsilon(t_n)) = \frac{1}{k}\int_{t_n}^{t_{n+1}}(t-t_n)\boldsymbol{u}_{tt}^\varepsilon dt. \tag{5.7}$$

Taking the inner product of (5.6) with $2k\boldsymbol{e}^{n+1}$, thanks to (2.2), we derive

$$\begin{aligned}\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n+1}-\boldsymbol{e}^n\|^2 + 2k\|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\|^2 \\ = 2k(\boldsymbol{R}_\varepsilon^n,\boldsymbol{e}^{n+1}) - 2k\tilde{b}(\boldsymbol{e}^{n+1},\boldsymbol{u}^\varepsilon(t_{n+1}),\boldsymbol{e}^{n+1}).\end{aligned}$$

Using the Schwarz inequality and (5.7), we get

$$k(\boldsymbol{R}_\varepsilon^n,\boldsymbol{e}^{n+1}) = k(A_\varepsilon^{-\frac12}\boldsymbol{R}_\varepsilon^n, A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}) \le Ck^2\int_{t_n}^{t_{n+1}}\|A_\varepsilon^{-\frac12}\boldsymbol{u}_{tt}^\varepsilon\|^2dt + \frac{k}{4}\|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\|^2.$$

Then using (2.4) and (5.1), we have

$$\begin{aligned}2k\tilde{b}(\boldsymbol{e}^{n+1},\boldsymbol{u}^\varepsilon(t_{n+1}),\boldsymbol{e}^{n+1}) &\le Ck\|\boldsymbol{e}^{n+1}\|\|\boldsymbol{e}^{n+1}\|_1\|\boldsymbol{u}^\varepsilon(t_{n+1})\|_2 \\ &\le \frac{k}{4}\|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\|^2 + Ck\|\boldsymbol{e}^{n+1}\|^2.\end{aligned}$$

Taking the summation of (5.6) for n from 0 to m, using the discrete Gronwall lemma and Lemma 5.1, we derive

$$\|\boldsymbol{e}^{m+1}\|^2 + k\sum_{n=0}^{m}\|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\|^2 \le Ck^2 \quad \forall m\le T_0/k. \tag{5.8}$$

Thus

$$\|\boldsymbol{u}^{n+1}\|_1 \le \|\boldsymbol{e}^{n+1}\|_1 + \|\boldsymbol{u}^\varepsilon(t_{n+1})\|_1 \le C\|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\| + C \le C. \tag{5.9}$$

Taking the inner product of (5.6) with $2kt_{n+1}A_\varepsilon\boldsymbol{e}^{n+1}$, we obtain

$$\begin{aligned}t_{n+1}\{\|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\|^2 - \|A_\varepsilon^{\frac12}\boldsymbol{e}^n\|^2 + \|A_\varepsilon^{\frac12}(\boldsymbol{e}^{n+1}-\boldsymbol{e}^n)\|^2\} + 2kt_{n+1}\|A_\varepsilon\boldsymbol{e}^{n+1}\|^2 \\ = 2k(t_{n+1}\boldsymbol{R}_\varepsilon^n, A_\varepsilon\boldsymbol{e}^{n+1}) - 2kt_{n+1}\tilde{b}(\boldsymbol{u}^{n+1},\boldsymbol{e}^{n+1},A_\varepsilon\boldsymbol{e}^{n+1}) \\ - 2kt_{n+1}\tilde{b}(\boldsymbol{e}^{n+1},\boldsymbol{u}^\varepsilon(t_{n+1}),A_\varepsilon\boldsymbol{e}^{n+1}).\end{aligned} \tag{5.10}$$

Using (2.4), (5.9), and Young's inequality, we can derive

$$\begin{aligned}2kt_{n+1}\tilde{b}(\boldsymbol{u}^{n+1},\boldsymbol{e}^{n+1},A_\varepsilon\boldsymbol{e}^{n+1}) &\le Ckt_{n+1}\|A_\varepsilon\boldsymbol{e}^{n+1}\|\|\boldsymbol{u}^{n+1}\|_1\|\boldsymbol{e}^{n+1}\|_1^{\frac12}\|\boldsymbol{e}^{n+1}\|_2^{\frac12} \\ &\le Ckt_{n+1}\|A_\varepsilon\boldsymbol{e}^{n+1}\|^{\frac32}\|\boldsymbol{e}^{n+1}\|_1^{\frac12} \\ &\le Ckt_{n+1}\|\boldsymbol{e}^{n+1}\|_1^2 + \frac{kt_{n+1}}{4}\|A_\varepsilon\boldsymbol{e}^{n+1}\|^2.\end{aligned}$$

$$2kt_{n+1}\tilde{b}(\boldsymbol{e}^{n+1}, \boldsymbol{u}^\varepsilon(t_{n+1}), A_\varepsilon \boldsymbol{e}^{n+1}) \le Ckt_{n+1}\|\boldsymbol{u}^\varepsilon(t_{n+1})\|_2\|\boldsymbol{e}^{n+1}\|_1\|A_\varepsilon \boldsymbol{e}^{n+1}\|$$
$$\le \frac{k}{4}t_{n+1}\|A_\varepsilon \boldsymbol{e}^{n+1}\|^2 + Ckt_{n+1}\|\boldsymbol{e}^{n+1}\|_1^2.$$

Using (5.7) and the Schwarz inequality, we have

$$\begin{aligned}
k(t_{n+1}\boldsymbol{R}_\varepsilon^n, A_\varepsilon \boldsymbol{e}^{n+1}) &\le \frac{kt_{n+1}}{4}\|A_\varepsilon \boldsymbol{e}^{n+1}\|^2 + \frac{Ct_{n+1}}{k}\left(\int_{t_n}^{t_{n+1}} (t-t_n)\|\boldsymbol{u}_{tt}^\varepsilon\|dt\right)^2 \\
&\le \frac{kt_{n+1}}{4}\|A_\varepsilon \boldsymbol{e}^{n+1}\|^2 + \frac{Ct_{n+1}}{k}\int_{t_n}^{t_{n+1}} \frac{(t-t_n)^2}{t}dt \cdot \int_{t_n}^{t_{n+1}} t\|\boldsymbol{u}_{tt}^\varepsilon\|^2 dt \\
&\le \frac{kt_{n+1}}{4}\|A_\varepsilon \boldsymbol{e}^{n+1}\|^2 + \frac{Ck^2t_{n+1}}{t_n}\int_{t_n}^{t_{n+1}} t\|\boldsymbol{u}_{tt}^\varepsilon\|^2 dt \\
&\le \frac{kt_{n+1}}{4}\|A_\varepsilon \boldsymbol{e}^{n+1}\|^2 + Ck^2\int_{t_n}^{t_{n+1}} t\|\boldsymbol{u}_{tt}^\varepsilon\|^2 dt.
\end{aligned}$$

Taking the summation of (5.10) for n from 0 to m, using the above inequalities and the relation

$$t_{n+1}\{\|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\|^2 - \|A_\varepsilon^{\frac12}\boldsymbol{e}^n\|^2\} = t_{n+1}\|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\|^2 - t_n\|A_\varepsilon^{\frac12}\boldsymbol{e}^n\|^2 - k\|A_\varepsilon^{\frac12}\boldsymbol{e}^n\|^2,$$

we obtain

$$\begin{aligned}
t_{m+1}\|A_\varepsilon^{\frac12}\boldsymbol{e}^{m+1}\|^2 + k\sum_{n=0}^m t_{n+1}\|A_\varepsilon \boldsymbol{e}^{n+1}\|^2 &\le Ck\sum_{n=0}^m t_{n+1}\|\boldsymbol{e}^{n+1}\|_1^2 \\
&+ Ck\sum_{n=0}^m \|A_\varepsilon^{\frac12}\boldsymbol{e}^{n+1}\|^2 + Ck^2\int_0^{t_{m+1}} t\|\boldsymbol{u}_{tt}^\varepsilon\|^2 dt.
\end{aligned}$$

Applying the discrete Gronwall lemma to the above inequality, thanks to Lemma 2.1 and Lemma 5.1, we obtain the desired results. □

Finally, combining Theorem 4.1 and Lemma 5.2, we have proved the following theorem.

THEOREM 5.1. *Under the assumption of Lemma* 5.1, *we have*

$$\sqrt{t_n}\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\| + t_n\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|_1 \le C(k+\varepsilon) \;\; \forall n \le T_0/k.$$

Remark 5.1. The pressure approximation in (5.5) can be defined by $p^n = -\frac{1}{\varepsilon}\mathrm{div}\boldsymbol{u}^n$. We can also derive an error estimate for $p(t_n) - p^n$. In fact, taking the inner product of (5.6) with $t_{n+1}^2(\boldsymbol{e}^{n+1} - \boldsymbol{e}^n)$, after some technical endeavors, we can prove

$$k\sum_{n=0}^{T_0/k-1} t_{n+1}^2\left\|\frac{\boldsymbol{e}^{n+1} - \boldsymbol{e}^n}{k}\right\|^2 \le Ck^2.$$

Then using the equation (5.5) and the available estimates for $\boldsymbol{e}^n$, we can prove

$$k\sum_{n=1}^{T_0/k} t_n^2\|p(t_n) - p^n\|^2 \le C(k^2 + \varepsilon^2),$$

which is similar to the estimate in the continuous case (see Theorem 4.1).

5.2. An improved scheme. Theorem 5.1 indicates that the proper choice of ε in (5.5) is $\varepsilon \approx k$. However, one notices that when $\varepsilon \approx k$ is small, the system (5.5) may become severely ill conditioned. We present below a modified scheme which would totally relax the constraint on ε while keeping the same accuracy.

To simplify our presentation, we will restrict ourselves to the linearized equations (3.1), and we will also assume more regularities on the solution $(\boldsymbol{u}, p)$ of (3.1) (see (A3) below). We note that the verification of (A3) involves some nonlocal compatibility conditions on the data (see [11]).

The scheme we propose for (3.1) is

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{k} - \nu\Delta\boldsymbol{u}^{n+1} + \nabla p^{n+1} = \boldsymbol{f}(t_{n+1}), \tag{5.11}$$

$$\text{div}\boldsymbol{u}^{n+1} + \varepsilon(p^{n+1}-p^n) = 0, \quad p^0 \text{ arbitrary.} \tag{5.12}$$

The scheme (5.11)–(5.12) can be considered as the time discretization of the artificial compressibility method (see Remark 5.2 below). Also when we erase the term $\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{k}$ in (5.11), the scheme is in fact the augmented Lagrangian method (or if one prefers, the iterative penalty scheme) [8] applied to the steady Stokes equations.

Let $\boldsymbol{e}^n = \boldsymbol{u}(t_n) - \boldsymbol{u}^n, q^n = p(t_n) - p^n$, Subtracting (5.11)–(5.12) from (3.1), we get

$$\frac{\boldsymbol{e}^{n+1}-\boldsymbol{u}^n}{k} - \nu\Delta\boldsymbol{e}^{n+1} + \nabla q^{n+1} = \boldsymbol{R}^n, \tag{5.13}$$

$$\text{div}\boldsymbol{e}^{n+1} + \varepsilon(q^{n+1}-q^n) = \varepsilon\int_{t_n}^{t_{n+1}} p_t dt, \tag{5.14}$$

where

$$\boldsymbol{R}^n = \boldsymbol{u}_t(t_{n+1}) - \frac{1}{k}(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)) = \frac{1}{k}\int_{t_n}^{t_{n+1}} (t-t_n)\boldsymbol{u}_{tt} dt. \tag{5.15}$$

We will prove the following results for the scheme (5.11)–(5.12).

PROPOSITION 5.1. *In addition to* (A1) *and* (A2), *we assume*

$$p_t \in C([0,T]; L^2(\Omega)), \quad p_{tt} \in C([0,T]; L^2(\Omega)), \quad \boldsymbol{u}_{tt} \in L^2(0,T;\boldsymbol{H}^{-1}). \tag{A3}$$

Then we have the following estimates for the scheme (5.11)–(5.12)*:*

$$\|\boldsymbol{e}^m\|^2 + k\nu\sum_{n=0}^{m} \|\nabla\boldsymbol{e}^n\|^2 \le C(k^2 + \varepsilon^2 k^2) \;\; \forall m \le T_0/k.$$

Proof. Taking the scalar products of (5.13) with $2k\boldsymbol{e}^{n+1}$, (5.14) with $2kq^{n+1}$, and adding up the two relations, we obtain

$$\begin{aligned} &\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n+1}-\boldsymbol{e}^n\|^2 + 2k\nu\|\nabla\boldsymbol{e}^{n+1}\|^2 \\ &\quad + k\varepsilon\{\|q^{n+1}\|^2 - \|q^n\|^2 + \|q^{n+1}-q^n\|^2\} \\ &\quad = (\boldsymbol{R}^n, 2k\boldsymbol{e}^{n+1}) + \varepsilon\left(\int_{t_n}^{t_{n+1}} p_t, 2kq^{n+1}\right). \end{aligned} \tag{5.16}$$

Using the Schwarz inequality and (5.15), we get

$$\begin{aligned} (\boldsymbol{R}_\varepsilon^n, 2k\boldsymbol{e}^{n+1}) &\le Ck\|\boldsymbol{R}^n\|_{-1}^2 + \frac{k\nu}{2}\|\nabla\boldsymbol{e}^{n+1}\|^2 \\ &\le Ck^2\int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_{tt}\|_{-1}^2 dt + \frac{k\nu}{2}\|\nabla\boldsymbol{e}^{n+1}\|^2. \end{aligned}$$

To estimate the second term on the right-hand side of (5.16), we use the same argument as in the proof of Lemma 3.3. Namely for $p_t(t), p_{tt}(t) \in L^2(\Omega)/\mathbb{R}$, there exists a unique $\boldsymbol{\varphi}(t) \in \boldsymbol{H}_0^1(\Omega)$ such that $\mathrm{div}\boldsymbol{\varphi}(t) = p_t(t)$ and $\mathrm{div}\boldsymbol{\varphi}_t(t) = p_{tt}(t)$ with

$$\|\boldsymbol{\varphi}(t)\|_1 \leq C\|p_t(t)\|, \quad \|\boldsymbol{\varphi}_t(t)\|_1 \leq C\|p_{tt}(t)\| \quad \forall\, t \in [0, T_0]. \tag{5.17}$$

Then using equation (5.13), we derive
(5.18)

$$\begin{aligned}
\varepsilon k\left(\int_{t_n}^{t_{n+1}} p_t dt, q^{n+1}\right) &= -\varepsilon k\left(\int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt, \nabla q^{n+1}\right) \\
&= \varepsilon\left(\boldsymbol{e}^{n+1} - \boldsymbol{e}^n, \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt\right) + \varepsilon k\nu\left(\nabla \boldsymbol{e}^{n+1}, \int_{t_n}^{t_{n+1}} \nabla\boldsymbol{\varphi} dt\right) \\
&+ k\varepsilon\left(\boldsymbol{R}^n, \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt\right) \leq \varepsilon\left(\boldsymbol{e}^{n+1} - \boldsymbol{e}^n, \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt\right) \\
&+ \frac{k\nu}{2}\|\nabla \boldsymbol{e}^{n+1}\|^2 + C\varepsilon^2 k^2 \int_{t_n}^{t_{n+1}} \|\boldsymbol{\varphi}\|_1^2 dt + Ck\|\boldsymbol{R}^n\|_{-1}^2.
\end{aligned}$$

The assumption on p_t and p_{tt} implies that $\boldsymbol{\varphi}$, and $\boldsymbol{\varphi}_t \in C([0,T]; \boldsymbol{L}^2(\Omega))$, which then implies that there exist $\eta_n \in [t_n, t_{n+1}], \eta_{n-1} \in [t_{n-1}, t_n]$, and $\xi_n \in [t_{n-1}, t_{n+1}]$ such that

$$\left(\boldsymbol{e}^n, \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt\right) - \left(\boldsymbol{e}^n, \int_{t_{n-1}}^{t_n} \boldsymbol{\varphi} dt\right) = k(\boldsymbol{e}^n, (\boldsymbol{\varphi}(\eta_n) - \boldsymbol{\varphi}(\eta_{n-1})) = k(\eta_n - \eta_{n-1})(\boldsymbol{e}^n, \boldsymbol{\varphi}_t(\xi_n)).$$

Therefore using the above relation, we derive
(5.19)

$$\begin{aligned}
\varepsilon\left(\boldsymbol{e}^{n+1} - \boldsymbol{e}^n, \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt\right) &= \varepsilon\left(\boldsymbol{e}^{n+1}, \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt\right) \\
&- \varepsilon\left(\boldsymbol{e}^n, \int_{t_{n-1}}^{t_n} \boldsymbol{\varphi} dt\right) + \varepsilon\left(\boldsymbol{e}^n, \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt - \int_{t_{n-1}}^{t_n} \boldsymbol{\varphi} dt\right) \\
&\leq \varepsilon\left(\boldsymbol{e}^{n+1}, \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi} dt\right) - \varepsilon\left(\boldsymbol{e}^n, \int_{t_{n-1}}^{t_n} \boldsymbol{\varphi} dt\right) + 2k^2\varepsilon|(\boldsymbol{e}^n, \boldsymbol{\varphi}_t(\xi_n))|.
\end{aligned}$$

Taking the summation of (5.16) for n, using (5.18) and (5.19), we get

$$\begin{aligned}
\|\boldsymbol{e}^{m+1}\|^2 + k\varepsilon\|q^{m+1}\|^2 + k\sum_{n=0}^{m} \|\nabla \boldsymbol{e}^{n+1}\|^2 &\leq \varepsilon\left(\boldsymbol{e}^{m+1}, \int_{t_m}^{t_{m+1}} \boldsymbol{\varphi} dt\right) \\
+ Ck^2 \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_{tt}\|_{-1}^2 dt &+ C\varepsilon^2 k^2 \int_0^{t_{m+1}} \|\boldsymbol{\varphi}_t\|_1^2 dt + 2k^2\varepsilon \sum_{n=0}^{m} |(\boldsymbol{e}^n, \boldsymbol{\varphi}_t(\xi_n))|.
\end{aligned}$$

Using the Schwarz inequality and (5.17), we derive

$$\begin{aligned}
\varepsilon\left(\boldsymbol{e}^{m+1}, \int_{t_m}^{t_{m+1}} \boldsymbol{\varphi} dt\right) &\leq \frac{1}{2}\|\boldsymbol{e}^{m+1}\|^2 + C\varepsilon^2 k^2 \|\boldsymbol{\varphi}(\boldsymbol{\eta}_n)\|^2 \\
&\leq \frac{1}{2}\|\boldsymbol{e}^{m+1}\|^2 + C\varepsilon^2 k^2 \|p_t(\boldsymbol{\eta}_n)\|^2,
\end{aligned}$$

$$2k^2\varepsilon\sum_{n=0}^{m}|(\boldsymbol{e}^n,\varphi_t(\xi_n))|\le k\sum_{n=0}^{m}\|\boldsymbol{e}^n\|^2+Ck^3\varepsilon^2\sum_{n=0}^{m}\|\varphi_t(\xi_n))\|^2$$
$$\le k\sum_{n=0}^{m}\|\boldsymbol{e}^n\|^2+Ck^3\varepsilon^2\sum_{n=0}^{m}\|p_{tt}(\xi_n))\|^2.$$

Therefore using (A3) and Lemma 5.1, we derive

$$\|\boldsymbol{e}^{m+1}\|^2+k\varepsilon\|q^{m+1}\|^2+k\sum_{n=0}^{m}\|\nabla\boldsymbol{e}^{n+1}\|^2\le C(k^2+\varepsilon^2k^2)+k\sum_{n=0}^{m}\|\boldsymbol{e}^n\|^2.$$

We then conclude by consistency the discrete Gronwall lemma. □

Remark 5.2.

(a) Proposition 5.1 indicates that for the scheme (5.11)–(5.12), we can choose ε to be a constant independent of k while keeping the first-order accuracy. The result is not surprising since the continuous form of (5.11)–(5.13) is

$$\boldsymbol{u}_t^\varepsilon-\nu\Delta\boldsymbol{u}^\varepsilon+\nabla p^\varepsilon=f,$$
$$\operatorname{div}\boldsymbol{u}^\varepsilon+\varepsilon kp_t=0,$$

which is in fact a time discretization of the artificial compressibility method (see for instance [19]) applied to the linearized equation (3.1). It has been shown (see [19]) that when $\varepsilon k\to 0$, we have $\boldsymbol{u}^\varepsilon\to\boldsymbol{u}$. In fact, using the techniques in §3, we can prove, under the assumption of Proposition 5.1, that

$$\|\boldsymbol{u}(t)-\boldsymbol{u}^\varepsilon(t)\|\le C\varepsilon k\ \ \forall t\le T.$$

(b) Higher-order time-discretization schemes can also be applied to the penalized system. For instance, for the second-order scheme

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{k}+A_\varepsilon\frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^n}{2}+\tilde{B}\left(\frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^n}{2},\frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^n}{2}\right)=\boldsymbol{f}(t_{n+\frac{1}{2}}),$$

one can show, by using similar procedures and under appropriate assumptions, that the following error estimate holds:

$$\|u(t_n)-u^n\|\le C(k^2+\varepsilon)\ \forall n\le T/k.$$

Accordingly, one can also show that for the second-order version of (5.11)–(5.12),

$$\begin{cases}\dfrac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{k}-\dfrac{\nu}{2}\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)+\tilde{B}(\dfrac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^n}{2},\dfrac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^n}{2})+\nabla p^{n+1}=\boldsymbol{f}(t_{n+\frac{1}{2}}).\\ \operatorname{div}\boldsymbol{u}^{n+1}+\varepsilon(p^{n+1}-p^n)=0,\end{cases}$$

we have the improved error estimate

$$\|u(t_n)-u^n\|\le C(k^2+\varepsilon k)\ \forall n\le T/k.$$

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