

## EFFICIENT SPECTRAL-GALERKIN METHODS III: POLAR AND CYLINDRICAL GEOMETRIES\*

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**Abstract.** We present in this paper several extremely efficient and accurate spectral-Galerkin methods for second- and fourth-order equations in polar and cylindrical geometries. These methods are based on appropriate variational formulations which incorporate naturally the pole condition(s). In particular, the computational complexities of the Chebyshev–Galerkin method in a disk and the Chebyshev–Legendre–Galerkin method in a disk or a cylinder are quasi-optimal (optimal up to a logarithmic term). As an indication of efficiency, the CPU time for the Poisson solver on a disk by our Chebyshev–Galerkin method is only about 70% of the corresponding finite-difference code in FISHPACK.

**Key words.** spectral-Galerkin, polar coordinates, Legendre polynomial, Chebyshev polynomial

**AMS subject classifications.** 65N35, 65N22, 65F05, 35J05

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**1. Introduction.** This paper is the third part in a series for developing efficient spectral-Galerkin methods for elliptic problems. In the first part [14] and the second part [15], we presented efficient algorithms for solving elliptic equations in rectangular domains. In this part, we shall deal with polar and cylindrical geometries, which have important applications in many fields of science and engineering, especially in computational fluid dynamics.

The most effective way to deal with polar or cylindrical geometries is to map them to rectangular domains by using polar or cylindrical coordinates. Thus, spectral methods are extremely well suited in this case, since the actual computational domains are of rectangular type, and the axial direction can be treated efficiently by fast Fourier transform (FFT). There exist already a number of good algorithms based on the spectral-collocation or spectral-tau methods (see, for instance, [9], [5], [6], [10], and [7]), but they may not be entirely satisfactory in terms of either efficiency or accuracy. To the best of my knowledge, only the algorithm in [6], which is limited to second-order equations and polar geometries, has a quasi-optimal computational complexity. However, the algorithm may not be easy to generalize and to implement because it is based on an odd–even parity argument and the tau method. The aim of this paper is to develop spectral-Galerkin algorithms whose computational complexities are quasi-optimal for second- and fourth-order equations in polar and cylindrical geometries.

There are apparently two difficulties in developing efficient spectral methods for polar and cylindrical geometries. Namely, (i) the polar transformation introduces singularity at the pole(s) so that appropriate pole condition(s) should be imposed to ensure the desired smoothness in the Cartesian coordinates, and (ii) even simple equations with constant coefficients in Cartesian coordinates will have variable coefficients of the form  $r^{\pm k}$  in polar coordinates. However, these two difficulties can be naturally resolved by using a variational formulation which incorporates the pole condition(s) and by choosing a set of appropriate basis functions for the spectral-Galerkin methods

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which would lead to sparse matrices even for problems with variable coefficients of the form  $r^{\pm k}$ .

The rest of the paper is organized as follows. In the next section, we provide a detailed presentation for the Legendre, Chebyshev, and Chebyshev–Legendre–Galerkin methods for a model Helmholtz equation in a disk. In section 3, we deal with the same model equation but in a cylinder. In section 4, we treat fourth-order problems. In section 5, we present various extensions including, in particular, problems with variable coefficients. Numerical results demonstrating the efficiency and accuracy of our methods are presented in section 6 followed by some concluding remarks.

**2. Polar geometries.** As a model problem, we consider the Helmholtz equation on a unit disk

$$(2.1) \quad \begin{aligned} -\Delta U + \alpha U &= F \text{ in } \Omega = \{(x, y) : x^2 + y^2 < 1\}, \\ U &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Its variational formulation is to find  $U \in H_0^1(\Omega)$  such that

$$(2.2) \quad A(U, V) := \int_{\Omega} \nabla U \nabla V \, dx \, dy + \alpha \int_{\Omega} U V \, dx \, dy = \int_{\Omega} F V \, dx \, dy \quad \forall V \in H_0^1(\Omega).$$

We will develop approximation schemes which are consistent with (2.2) in a sense to be specified below.

Applying the polar transformation  $x = r \cos \theta$ ,  $y = r \sin \theta$  to (2.1), and setting  $u(r, \theta) = U(r \cos \theta, r \sin \theta)$ ,  $f(r, \theta) = F(r \cos \theta, r \sin \theta)$ , we obtain

$$(2.3) \quad \begin{aligned} -u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta} + \alpha u &= f, & (r, \theta) \in Q = (0, 1) \times [0, 2\pi), \\ u(1, \theta) &= 0, & \theta \in [0, 2\pi), \quad u \text{ periodic in } \theta. \end{aligned}$$

Similarly, the variational formulation (2.2) becomes

$$(2.4) \quad \begin{aligned} a(u, v) &:= \int_Q u_r v_r \, r \, dr \, d\theta + \int_Q \frac{1}{r} u_{\theta} v_{\theta} \, dr \, d\theta \\ &+ \alpha \int_Q u v \, r \, dr \, d\theta = \int_0^1 f v \, r \, dr \, d\theta. \end{aligned}$$

Since the polar transformation is singular at the pole  $r = 0$ , additional pole conditions should be imposed for the solution of (2.3) to have desired regularity in the Cartesian coordinates. In fact, if the function

$$(2.5) \quad u(r, \theta) = \sum_{|m|=0}^{\infty} (u_{1m}(r) \cos m\theta + u_{2m}(r) \sin m\theta)$$

were to be infinitely differentiable in the Cartesian coordinates, the following pole conditions would need to be satisfied (cf. [13]):

$$(2.6) \quad u_{1m}(r) = O(r^{|m|}), \quad u_{2m}(r) = O(r^{|m|}) \quad \text{as } r \rightarrow 0, \quad |m| = 1, 2, \dots$$

Obviously, it is computationally impractical to impose all the pole conditions in (2.6). Since our approximations will be based on the variational formulation (2.4) which is well defined if  $u_{\theta}(0, \theta) = 0$  for  $\theta \in [0, 2\pi)$ , or equivalently

$$(2.7) \quad u_{1m}(0) = u_{2m}(0) = 0 \quad \text{for } m \neq 0,$$

we will term (2.7) as the *essential* pole condition for (2.3), while all other conditions in (2.6) will be termed as *natural* or *nonessential*. Although it is possible to impose any given number of pole conditions in (2.6) in a numerical scheme, it is generally inefficient and may lead to ill-posed linear systems if *more than necessary* pole conditions are imposed so that the total number of boundary conditions in the radial direction *exceeds* the order of the underlying differential equation. On the other hand, ignoring the *essential* pole condition(s) will lead to inaccurate results.

We emphasize that the accuracy of a spectral approximation to (2.4) is only affected by the smoothness of the solution  $u$  in polar coordinates. In particular, the singularity of the solution  $u$  at the pole in Cartesian coordinates will not degrade the accuracy of the spectral Galerkin schemes presented below (see also the numerical results in Table 6.2).

We now describe our spectral approximations to (2.3). Given a cutoff number  $M > 0$ , let  $(f^{1m}(r), f^{2m}(r))$  be defined by the discrete (in  $\theta$ ) Fourier expansion

$$(2.8) \quad f(r, \theta_j) = \sum_{m=0}^M (f^{1m}(r) \cos(m\theta_j) + f^{2m}(r) \sin(m\theta_j)),$$

$$\theta_j = \frac{j\pi}{M}, \quad j = 0, 1, \dots, 2M - 1.$$

We define a Fourier-spectral approximation to the solution  $u$  of (2.3) by

$$(2.9) \quad u_M(r, \theta) = \sum_{m=0}^M (u^{1m}(r) \cos(m\theta) + u^{2m}(r) \sin(m\theta)),$$

where  $(u^{1m}(r), u^{2m}(r))$  ( $m = 0, 1, \dots, M$ ) are solutions of the following system:

$$(2.10) \quad -u_{rr}^{1m} - \frac{1}{r}u_r^{1m} + \left(\frac{m^2}{r^2} + \alpha\right)u^{1m} = f^{1m}(r), \quad 0 < r < 1,$$

$$-u_{rr}^{2m} - \frac{1}{r}u_r^{2m} + \left(\frac{m^2}{r^2} + \alpha\right)u^{2m} = f^{2m}(r), \quad 0 < r < 1,$$

$$u^{1m}(0) = u^{2m}(0) = 0 \text{ if } m \neq 0, \quad u^{1m}(1) = u^{2m}(1) = 0.$$

*Remark 2.1.* The extra pole condition  $u_r^{1,0}(0) = u_r^{2,0}(0) = 0$  used by many authors (see, for instance, [9], [5], [6], [10], and [7]) is derived from the parity argument on the expansion (2.9). It is, however, not part of the essential pole condition for (2.3). Although in most cases there is no harm to impose the extra pole condition, we choose not to do so since its implementation is more complicated and it may fail to give accurate results in some extreme (but still legitimate) cases, e.g., when the exact solution is a function of  $r - 1$ .  $\square$

Now, we only have to consider the approximation of the following prototypical one-dimensional equation:

$$(2.11) \quad -u'' - \frac{1}{r}u' + \left(\frac{m^2}{r^2} + \alpha\right)u = f, \quad 0 < r < 1; \quad u(0) = 0 \text{ if } m \neq 0, \quad u(1) = 0,$$

where  $u$  and  $f$  now represent generic functions. Most of the existing spectral algorithms treat (2.11) directly by using either a collocation (cf. [10] and [7]) or a tau method exploring the inherent parity conditions in (2.11) (cf. [13], [6]). However, it is important to note that a direct treatment of (2.11) is not quite appropriate since

the measure  $rdrd\theta$  related to the polar coordinate transformation is not taken into account. Instead, it is more appropriate to use the following variational formulation originated from (2.4): find  $u \in Y(m)$  such that

$$(2.12) \quad \int_0^1 u_r v_r r dr + \int_0^1 \left( \frac{m^2}{r^2} + \alpha \right) u v r dr = \int_0^1 f v r dr \quad \forall v \in Y(m),$$

where  $Y(m) = H_0^1(0, 1)$  if  $m \neq 0$  and  $Y(0) = \{v \in H^1(0, 1) : u(1) = 0\}$ .

The formulation (2.12) has the advantage of being symmetric, while direct treatment of (2.11) leads to nonsymmetric systems which may introduce additional difficulties in practice.

Now, let us derive a weighted variational formulation which is suitable for both the Legendre and Chebyshev methods. Although it is possible to extend (2.11) to the interval  $(-1, 0)$  by parity argument (cf. [6] and [7]), it is more efficient, at least from our point of view, to make a coordinate transformation  $r = (t + 1)/2$  in (2.11). Setting  $v(t) = u((t + 1)/2)$ , we get

$$(2.13) \quad -v'' - \frac{1}{t+1}v' + \left( \frac{m^2}{(t+1)^2} + \frac{\alpha}{4} \right) v = \frac{1}{4}f \left( \frac{t+1}{2} \right), \quad t \in I = (-1, 1),$$

$$v(-1) = 0 \text{ if } m \neq 0, \quad v(1) = 0.$$

Thus, a weighted variational formulation for (2.13) (compare with (2.12)) is to find  $v \in X(m)$  such that

$$(2.14) \quad ((t+1)v', (w\omega)') + \left( \frac{m^2}{t+1}v, w \right)_\omega + \beta((t+1)v, w)_\omega = (g, w)_\omega \quad \forall w \in X(m),$$

where  $X(m) = H_{0,\omega}^1(I)$  if  $m \neq 0$ , and  $X(0) = \{v \in H_\omega^1(I) : u(1) = 0\}$ ,  $\beta = \frac{\alpha}{4}$ ,  $g(t) = \frac{1}{4}(t+1)f((t+1)/2)$ .  $\omega$  is a certain weight function and  $(u, v)_\omega = \int_I u v \omega dt$ .

Let  $P_N$  be the space of polynomials of degree less than or equal to  $N$ , and let

$$X_N(m) = \{v \in P_N : v(\pm 1) = 0\} \text{ for } m \neq 0, \quad X_N(0) = \{v \in P_N : v(1) = 0\}.$$

The spectral-Galerkin methods will seek approximations in  $X_N(m)$ . The key for efficient implementation of the spectral-Galerkin methods is to choose appropriate basis functions for  $X_N(m)$ . In most cases, we use basis functions of the form

$$(2.15) \quad \phi_i(t) = p_i(t) + e_{i1}p_{i+1}(t) + \dots + e_{ik}p_{i+k}(t), \quad i = 0, 1, \dots, N - k,$$

where  $p_i(t)$  is either the Legendre or Chebyshev, or, more generally, Jacobi polynomial of degree  $i$ ,  $k$  is the number of underlying boundary conditions, and  $\{e_{ij}, i = 1, \dots, k\}$  should be chosen so that  $\phi_i(t)$  satisfies the corresponding homogeneous boundary conditions.

*Remark 2.2.* The basis functions in (2.15) usually lead to matrices with the simplest structure. However, in some special cases, other basis functions may be more suitable; see Remark 2.3 and Lemma 4.2 for two examples.  $\square$

We now consider several alternative spectral approximations for (2.14).

**2.1. Legendre–Galerkin approximation.** Setting  $\omega \equiv 1$  in (2.14), the Legendre–Galerkin approximation to (2.14) is to find  $v_N \in X_N(m)$  such that  $\forall w \in X_N(m)$

$$(2.16) \quad ((t+1)v'_N, w') + m^2 \left( \frac{1}{t+1}v_N, w \right) + \beta((t+1)v_N, w) = (I_N g, w),$$

where  $I_N$  is the operator of interpolation based on the Legendre–Gauss–Lobatto points, i.e.,  $(I_N g)(t_i) = g(t_i)$ ,  $i = 0, 1, \dots, N$ , where  $\{t_i\}$  are the roots of  $(1 - t^2)L'_N(t)$ .

We now treat the cases  $m \neq 0$  and  $m = 0$  separately.

**2.1.1. Case  $m \neq 0$ .** Let  $L_k(t)$  be the  $k$ th degree Legendre polynomial; it is easy to verify that

$$X_N(m) = \text{span}\{\phi_i(t) = L_i(t) - L_{i+2}(t), i = 0, 1, \dots, N - 2\}.$$

Setting

$$\begin{aligned} a_{ij} &= \int_I (t + 1) \phi'_j \phi'_i dt, & A &= (a_{ij})_{i,j=0,1,\dots,N-2}, \\ b_{ij} &= \int_I \frac{1}{t + 1} \phi_j \phi_i dt, & B &= (B_{ij})_{i,j=0,1,\dots,N-2}, \\ (2.17) \quad c_{ij} &= \int_I (t + 1) \phi_j \phi_i dt, & C &= (C_{ij})_{i,j=0,1,\dots,N-2}, \\ f_i &= \int_I I_N g \phi_i dt, & \mathbf{f} &= (f_i)_{i=0,1,\dots,N-2}, \\ v_N &= \sum_{i=0}^{N-2} x_i \phi_i(t), & \mathbf{x} &= (x_i)_{i=0,1,\dots,N-2}, \end{aligned}$$

then (2.16) becomes the matrix equation

$$(2.18) \quad (A + m^2 B + \beta C) \mathbf{x} = \mathbf{f}.$$

It turns out that the variable coefficients of the form  $(t \pm 1)^{\pm 1}$  do not lead to dense matrices. In fact, we have the following.

LEMMA 2.1. *The matrices  $A$  and  $B$  are symmetric tri-diagonal with*

$$(2.19) \quad a_{ij} = \begin{cases} 2i + 4, & j = i + 1, \\ 4i + 6, & j = i, \end{cases} \quad b_{ij} = \begin{cases} -\frac{2}{i + 2}, & j = i + 1, \\ \frac{2(2i + 3)}{(i + 1)(i + 2)}, & j = i. \end{cases}$$

*The matrix  $C$  is symmetric seven-diagonal with*

$$(2.20) \quad c_{ij} = \begin{cases} -\frac{2(i + 3)}{(2i + 5)(2i + 7)}, & j = i + 3, \\ -\frac{2}{2i + 5}, & j = i + 2, \\ \frac{2}{(2i + 1)(2i + 5)} + \frac{2(i + 3)}{(2i + 5)(2i + 7)}, & j = i + 1, \\ \frac{2}{2i + 1} + \frac{2}{2i + 5}, & j = i. \end{cases}$$

*Proof.* It is obvious from the definition that the matrices  $A$ ,  $B$ , and  $C$  are symmetric positive definite. The formula for  $a_{ij}$  can be easily established by using the following properties of Legendre polynomials:

$$(2.21) \quad \phi'_i(t) = -(2i + 3)L_{i+1}(t),$$

$$(2.22) \quad (i + 1)L_{i+1}(t) = (2i + 1)t L_i(t) - i L_{i-1}(t).$$

We can also easily verify that

$$(2.23) \quad \phi_i(t) = \frac{2i + 3}{(i + 1)(i + 2)}(1 - t^2)L'_{i+1}(t).$$

Therefore, temporarily setting  $\alpha_j = \frac{2j+3}{(j+1)(j+2)}$ , and using successively (2.23), integration by parts, and (2.21), we have

$$\begin{aligned} b_{ij} &= \int_I \frac{1}{t+1} \phi_j \phi_i dt = \alpha_j \int_I (1-t) L'_{j+1} \phi_i dt \\ &= -\alpha_j \int_I L_{j+1} \frac{d}{dt}((1-t)\phi_i) dt \\ &= \alpha_j \int_I L_{j+1} ((2i+3)(1-t)L_{i+1} + \phi_i) dt. \end{aligned}$$

The formula for  $b_{ij}$  can now be easily established by using (2.22). The formula for  $c_{ij}$  can be derived similarly.  $\square$

**2.1.2. Case  $m = 0$ .** In this case, (2.16) becomes

$$(2.24) \quad ((t + 1)v'_N, w') + \beta((t + 1)v_N, w) = (I_N g, w) \quad \forall w \in X_N(0).$$

It is easy to verify that

$$X_N(0) = \text{span}\{\phi_i(t) = L_i(t) - L_{i+1}(t) : i = 0, 1, \dots, N - 1\}.$$

Using again the definitions in (2.18) with the index  $i$  and  $j$  extending to  $N - 1$ , we find that (2.24) is equivalent to the matrix equation

$$(A + \beta C)\mathbf{x} = \mathbf{f}.$$

LEMMA 2.2. *The matrix  $A$  is diagonal with  $a_{ii} = 2i + 2$ . The matrix  $C$  is symmetric penta-diagonal with*

$$c_{ij} = \begin{cases} -\frac{2(i+2)}{(2i+3)(2i+5)}, & j = i + 2, \\ \frac{4}{(2i+1)(2i+3)(2i+5)}, & j = i + 1, \\ \frac{4(i+1)}{(2i+1)(2i+3)}, & j = i. \end{cases}$$

*Proof.* It is easy to see that  $a_{ij} = 0$  for  $i \neq j$ . On the other hand, we have

$$a_{ii} = ((t + 1)\phi'_i, \phi'_i) = -(\phi'_i, \phi_i) - ((t + 1)\phi''_i, \phi_i).$$

Direct computations using (2.22) will lead to  $a_{ii} = 2i + 2$  and the formula for  $c_{ij}$ .  $\square$

**2.2. Chebyshev–Galerkin approximation.** The Legendre–Galerkin method suffers from the lack of fast transform between the physical and frequency spaces. Therefore, it is natural to use the Chebyshev polynomials for which a fast transform

is available. Setting  $\omega = (1 - t^2)^{-\frac{1}{2}}$  in (2.14), the Chebyshev–Galerkin approximation to (2.14) is to find  $v_N \in X_N(m)$  such that

$$(2.25) \quad \begin{aligned} & ((t + 1)v'_N, (w\omega)') + m^2 \left( \frac{1}{t + 1} v_N, w \right)_\omega \\ & + \beta((t + 1)v_N, w)_\omega = (I_N^c g, w)_\omega \quad \forall w \in X_N(m), \end{aligned}$$

where  $I_N^c$  is the operator of interpolation based on the Chebyshev–Gauss–Lobatto points, i.e.,  $I_N^c f \in P_N$  with  $(I_N^c f)(t_i) = f(t_i)$ ,  $t_i = \cos(\frac{i\pi}{N})$ ,  $i = 0, 1, \dots, N$ .

**2.2.1. Case  $m \neq 0$ .** Let  $T_k(t)$  be the  $k$ th degree Chebyshev polynomial; we have

$$X_N(m) = \text{span}\{\phi_i(t) = T_i(t) - T_{i+2}(t) : i = 0, 1, \dots, N - 2\}.$$

Setting

$$(2.26) \quad \begin{aligned} a_{ij} &= \int_I (t + 1) \phi'_j (\phi_i \omega)' dt, & A &= (a_{ij})_{i,j=0,1,\dots,N-2}, \\ b_{ij} &= \int_I \frac{1}{t + 1} \phi_j \phi_i \omega dt, & B &= (B_{ij})_{i,j=0,1,\dots,N-2}, \\ c_{ij} &= \int_I (t + 1) \phi_j \phi_i \omega dt, & C &= (C_{ij})_{i,j=0,1,\dots,N-2}, \\ f_i &= \int_I I_N^c g \phi_i \omega dt, & \mathbf{f} &= (f_i)_{i=0,1,\dots,N-2}, \\ v_N &= \sum_{i=0}^{N-2} x_i \phi_i(t), & \mathbf{x} &= (x_i)_{i=0,1,\dots,N-2}, \end{aligned}$$

then (2.25) becomes the matrix equation

$$(2.27) \quad (A + m^2 B + \beta C) \mathbf{x} = \mathbf{f}.$$

The direct computation of elements of  $A$ ,  $B$ , and  $C$  is very involved, but it can be substantially simplified by using the following results (cf. (2.6), (2.5), and (4.6) in [15]):

$$(2.28) \quad \tilde{a}_{ij} = - \int_I \phi''_j \phi_i \omega dt = \begin{cases} 2\pi(i + 1)(i + 2), & j = i, \\ 4\pi(i + 1), & j = i + 2, i + 4, i + 6, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

$$(2.29) \quad \tilde{b}_{ij} = \int_I \phi_j \phi_i \omega dt = \begin{cases} \frac{d_i + 1}{2} \pi, & j = i, \\ -\frac{\pi}{2}, & j = i - 2 \text{ or } i + 2, \\ 0, & \text{otherwise,} \end{cases}$$

$$(2.30) \quad \tilde{c}_{ij} = \int_I \phi'_j \phi_i \omega dt = \begin{cases} \pi(i + 1), & j = i + 1, \\ -\pi(i + 1), & j = i - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $d_0 = 2$  and  $d_i = 1$  for  $i \geq 1$ .

LEMMA 2.3. *A is an upper Heisenberg matrix with*

$$(2.31) \quad a_{ij} = \begin{cases} (i+1)^2\pi, & j = i-1, \\ 2(i+1)(i+2)\pi, & j = i, \\ (i+1)(i+5)\pi, & j = i+1, \\ 4(i+1)\pi, & j \geq i+2. \end{cases}$$

*B is a symmetric tri-diagonal matrix with*

$$(2.32) \quad b_{ij} = \begin{cases} 2\pi, & j = i, \\ -\pi, & j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

*C is a symmetric seven-diagonal matrix with*

$$(2.33) \quad c_{ij} = \begin{cases} \frac{d_i+1}{2}\pi, & j = i, \\ \frac{d_{i-1}}{4}\pi, & j = i-1, \\ -\frac{\pi}{2}, & j = i-2, \\ -\frac{\pi}{4}, & j = i-3. \end{cases}$$

*Proof.*

$$(2.34) \quad \begin{aligned} a_{ij} &= -\int_I ((t+1)\phi'_j)' \phi_i \omega dt = -\int_I \phi''_j \phi_i \omega dt - \int_I (t\phi'_j)' \phi_i \omega dt \\ &= \tilde{a}_{ij} - \tilde{c}_{ij} - \int_I \phi''_j t \phi_i \omega dt. \end{aligned}$$

It is clear from the definition that  $a_{ij} = 0$  if  $j < i-1$ . On the other hand, from the relations above and

$$(2.35) \quad t\phi_i = t(T_i - T_{i+2}) = \frac{1}{2}(T_{i-1} - T_{i+3}) = \frac{1}{2}(\phi_{i-1} + \phi_{i+1}), \quad i \geq 1,$$

we derive immediately that

$$a_{ij} = \tilde{a}_{ij} - \tilde{c}_{ij} + \frac{1}{2}(\tilde{a}_{i-1,j} + \tilde{a}_{i+1,j}).$$

The formula (2.31) is then a direct consequence of the above relation, (2.28), and (2.30).

It is easy to see from the definition that  $b_{ij} = 0$  if  $|i-j| > 1$ . By using the formula (A.22) in [9] we can show that

$$(2.36) \quad \phi_j(t) = (1-t^2) \sum_{\substack{0 \leq k \leq j \\ k+j \text{ even}}} \frac{4}{d_k} T_k(t).$$



Therefore,

$$\begin{aligned}
 (2.37) \quad b_{ij} &= \int_I \frac{1}{t+1} \phi_j \phi_i \omega dt = \sum_{\substack{0 \leq k \leq j \\ k+j \text{ even}}} \frac{4}{d_k} \int_I (1-t) \phi_i T_k \omega dt \\
 &= \sum_{\substack{0 \leq k \leq j \\ k+j \text{ even}}} \frac{4}{d_k} \int_I \left( \phi_i - \frac{1}{2}(\phi_{i-1} + \phi_{i+1}) \right) T_k \omega dt.
 \end{aligned}$$

$b_{ii}$  and  $b_{i,i+1}$  can then be easily computed from the above relation.

Finally, by using (2.35), we find

$$c_{ij} = \tilde{b}_{ij} + \frac{1}{2}(\tilde{b}_{i-1,j} + \tilde{b}_{i+1,j}).$$

Hence, (2.33) is a direct consequence of the above relation and (2.29).  $\square$

*Remark 2.3.* Although the matrix  $A$  is not sparse, (2.27) can still be solved in  $O(N)$  operations by taking advantage of the special structure of  $A$ , namely,  $a_{ij} = 4(i+1)\pi$  for  $j \geq i+2$ .

An alternative is to use a new set of basis functions  $\phi_i(t) = (1-t^2)T_i(t)$ . It is easy to see that in this case  $B$  and  $C$  are symmetric sparse matrices with  $b_{ij} = 0$  for  $|i-j| > 3$  and  $c_{ij} = 0$  for  $|i-j| > 5$ . One can also show by using integration by parts that  $A$  is a nonsymmetric sparse matrix with  $a_{ij} = 0$  for  $|i-j| > 3$ . Thus, (2.27) can also be solved in  $O(N)$  operations.  $\square$

**2.2.2. Case  $m = 0$ .** In this case, (2.25) becomes

$$(2.38) \quad ((t+1)v'_N, (w\omega)') + \beta((t+1)v_N, w)_\omega = (I_N g, w)_\omega \quad \forall w \in X_N(0),$$

where

$$X_N(0) = \text{span}\{\phi_i(t) = T_i(t) - T_{i+1}(t) : i = 0, 1, \dots, N-1\}.$$

Using again the definitions in (2.26) with the index  $i$  and  $j$  extending to  $N-1$ , (2.38) is equivalent to the matrix equation

$$(A + \beta C)\mathbf{x} = \mathbf{f}.$$

LEMMA 2.4.  $A$  is an upper-triangular matrix with

$$(2.39) \quad a_{ij} = \begin{cases} (i+1)\pi^2, & j = i, \\ (i-j)\pi, & j = i+1, i+3, i+5, \dots, \\ (i+j+1)\pi, & j = i+2, i+4, i+6, \dots \end{cases}$$

$C$  is a symmetric penta-diagonal matrix with nonzero elements

$$\begin{aligned}
 (2.40) \quad c_{ii} &= \frac{\pi}{2}, \quad i = 0, 1, \dots, N-1, \\
 c_{i,i+2} &= c_{i+2,i} = -\frac{\pi}{4}, \quad i = 0, 1, \dots, N-3, \\
 c_{01} &= c_{10} = \frac{\pi}{4}.
 \end{aligned}$$

*Proof.* The computation of  $c_{ij}$  is straightforward by using the orthogonality of the Chebyshev polynomials and the relation

$$t\phi_i(t) = t(T_i(t) - T_{i+1}(t)) = \frac{1}{2}(\phi_{i-1}(t) + \phi_{i+1}(t)), \quad i \geq 1.$$

The computation of  $a_{ij}$  is quite involved. The idea is to use the relations (A.9)–(A.10) in [9] to expand  $((t+1)\phi_j'(t))'$  in Chebyshev series. The details are left to the interested readers.  $\square$

*Remark 2.4.* Once again, the matrix  $A$  is not sparse. But (2.38) can still be solved in  $O(N)$  operations by exploring the special structure of  $A$ . We refer to pages 80–81 in [15] for more details on this procedure for a similar problem.  $\square$

**2.3. Chebyshev–Legendre–Galerkin approximation.** The Chebyshev–Legendre–Galerkin method was introduced in [16] to take advantage of both the Legendre and Chebyshev polynomials. The idea is to use the Legendre–Galerkin formulation which preserves the symmetry of the underlying problem and leads to a simple sparse linear system, while the physical values are evaluated at the Chebyshev–Gauss-type points. Thus, we can replace the expensive Legendre transform by a fast Chebyshev–Legendre transform (cf. [1] and [16]) between the coefficients of Legendre expansions and the values at the Chebyshev–Gauss-type points. More precisely, the Chebyshev–Legendre–Galerkin approximation for (2.14) is to find  $v_N \in X_N(m)$  such that  $\forall w \in X_N(m)$

$$(2.41) \quad ((t+1)v_N', w') + m^2 \left( \frac{1}{t+1} v_N, w \right) + \beta((t+1)v_N, w) = (I_N^c g, w),$$

where  $I_N^c$  is the interpolation operator based on the Chebyshev–Gauss–Lobatto points. The only difference with the Legendre–Galerkin approximation (2.16) is that the interpolation operator  $I_N$  is replaced by  $I_N^c$  which allows fast discrete transform.

**2.4. Computational complexity.** The computational complexity for each of the methods presented above is  $O(NM) + 2T(NM)$ , where  $N$  and  $M$  are, respectively, the cutoff number of the spectral expansion in radial and axial directions, and  $T(NM)$  is the cost of one forward or inverse discrete transform of the form

$$(2.42) \quad g(t_i, \theta_j) = \sum_{n=0}^N \left( \sum_{m=0}^M (g_n^{1m} \cos m\theta_j + g_n^{2m} \sin m\theta_j) \right) p_n(t_j), \\ i = 0, 1, \dots, N, \quad j = 0, 1, \dots, 2M - 1,$$

where  $p_n(t)$  is the  $n$ th degree Chebyshev or Legendre polynomial. Therefore,

$$T(NM) = N^2 M + O(NM \log_2 M)$$

for the Legendre–Galerkin method, while

$$T(NM) = O(NM \log_2 M) + O(NM \log_2 N)$$

for the Chebyshev–Galerkin or Chebyshev–Legendre–Galerkin method. Thus, the computational complexity of both the Chebyshev–Galerkin and Chebyshev–Legendre–Galerkin methods is quasi-optimal.

**3. Cylindrical geometries.** Consider the Helmholtz equation in a cylinder

$$(3.1) \quad \begin{aligned} -\Delta U + \alpha U &= F \text{ in } \Omega = \{(x, y, z) : x^2 + y^2 < 1, z \in I\}, \\ U &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Applying the cylindrical transformation  $x = r \cos \theta, y = r \sin \theta, z = z$ , and setting  $u(r, \theta, z) = U(r \cos \theta, r \sin \theta, z), f(r, \theta, z) = F(r \cos \theta, r \sin \theta, z)$ , we obtain

$$(3.2) \quad \begin{aligned} -u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta} - u_{zz} + \alpha u &= f, \quad (r, \theta, z) \in (0, 1) \times [0, 2\pi) \times I, \\ u &= 0 \text{ at } r = 1 \text{ or } z = \pm 1, \quad u \text{ periodic in } \theta, \end{aligned}$$

with the essential pole condition

$$(3.3) \quad \frac{\partial}{\partial \theta} u(0, \theta, z) = 0, \quad (\theta, z) \in [0, 2\pi) \times I.$$

As before, let  $(f^{1m}(r, z), f^{2m}(r, z))$  be defined by

$$(3.4) \quad \begin{aligned} f(r, z, \theta_j) &= \sum_{m=0}^M (f^{1m}(r, z) \cos(m\theta_j) + f^{2m}(r, z) \sin(m\theta_j)), \\ \theta_j &= \frac{j\pi}{M}, \quad j = 0, 1, \dots, 2M - 1. \end{aligned}$$

Then, a Fourier-spectral approximation to  $u$  is given by

$$u_M(r, \theta, z) = \sum_{m=0}^M (u^{1m}(r, z) \cos(m\theta) + u^{2m}(r, z) \sin(m\theta))$$

with  $u^{im}(i = 1, 2)$  satisfying the following two-dimensional equation

$$(3.5) \quad \begin{aligned} -u_{zz}^{im} - u_{rr}^{im} - \frac{1}{r}u_r^{im} + \left(\frac{m^2}{r^2} + \alpha\right) u^{im} &= f^{im}(r, z) \text{ in } \Omega = (0, 1) \times I, \\ u^{im} &= 0 \text{ at } r = 0 \text{ if } m \neq 0, \quad u^{im} = 0 \text{ at } r = 1 \text{ or } z = \pm 1. \end{aligned}$$

We then make a coordinate transformation  $r = (t + 1)/2$ , denoting  $v(t, z) = u^{im}(r, z), g(t, z) = \frac{1}{4}(t + 1)f^{im}(r, z)$ , and  $\beta = \frac{\alpha}{4}$ , we obtain the prototypical two-dimensional equation

$$(3.6) \quad \begin{aligned} -\frac{t+1}{4}v_{zz} - ((t+1)v_t)_t + \left(m^2 \frac{1}{t+1} + \beta(t+1)\right) v &= g, \quad (t, z) \in I \times I, \\ v &= 0 \text{ at } t = -1 \text{ if } m \neq 0, \quad v = 0 \text{ at } t = 1 \text{ or } z = \pm 1. \end{aligned}$$

Let us denote  $\psi_i(z) = p_i(z) - p_{i+2}(z)$  and  $\phi_i(t) = p_i(t) - p_{i+s(m)}(t)$ , where  $s(m) = 2$  if  $m \neq 0$  and  $s(0) = 1$ , and  $p_j$  is either the  $j$ th degree Legendre or Chebyshev polynomial. Let

$$X_N(m) = \text{span}\{\phi_i(t)\psi_j(z) : 0 \leq i \leq N - s(m), 0 \leq j \leq N - 2\}.$$

Then a spectral-Galerkin approximation to (3.6) is to find  $v_N \in X_N(m)$  such that

$$(3.7) \quad \begin{aligned} \frac{1}{4}((t+1)\partial_z v_N, \partial_z(w\omega)) + ((t+1)\partial_t v_N, \partial_t(w\omega)) + m^2 \left(\frac{1}{t+1}v_N, w\right)_\omega \\ + \beta((t+1)v_N, w)_\omega = (I_N g, w)_\omega \quad \forall w \in X_N(m), \end{aligned}$$

where  $\omega \equiv 1$  in the Legendre case and  $w = ((1-t^2)(1-z^2))^{-\frac{1}{2}}$  in the Chebyshev case,  $(\cdot, \cdot)_\omega$  is the weighted  $L^2$ -inner product in  $I \times I$ , and  $I_N$  is the interpolation operator based on the Legendre- or Chebyshev–Gauss-type points. Setting

$$\begin{aligned}
 a_{ij} &= \int_I (t+1) \phi'_j (\phi_i \omega(t))' dt, & A &= (a_{ij})_{i,j=0,1,\dots,N-s(m)}, \\
 b_{ij} &= \int_I \frac{1}{t+1} \phi_j \phi_i \omega(t) dt, & B &= (B_{ij})_{i,j=0,1,\dots,N-s(m)}, \\
 c_{ij} &= \frac{1}{4} \int_I (t+1) \phi_j \phi_i \omega(t) dt, & C &= (C_{ij})_{i,j=0,1,\dots,N-s(m)}, \\
 d_{ij} &= \int_I \psi_j \psi_i \omega(z) dz, & D &= (D_{ij})_{i,j=0,1,\dots,N-2}, \\
 e_{ij} &= \int_I \psi'_j (\psi_i \omega(z))' dz, & E &= (E_{ij})_{i,j=0,1,\dots,N-2},
 \end{aligned}
 \tag{3.8}$$

and

$$\begin{aligned}
 f_{ij} &= \int_I \int_I I_N g \phi_i(t) \psi_j(z) \omega(t, z) dt dz, & F &= (f_{ij}), \\
 v_N &= \sum_{i=0}^{N-s(m)} \sum_{j=0}^{N-2} x_{ij} \phi_i(t) \psi_j(z), & X &= (x_{ij}), \\
 i &= 0, 1, \dots, N-s(m), & j &= 0, 1, \dots, N-2.
 \end{aligned}
 \tag{3.9}$$

Then (3.7) becomes the matrix equation

$$CXE + (A + m^2B + \beta C)XD = F.
 \tag{3.10}$$

The entries of  $A$ ,  $B$ , and  $C$  in the Legendre or Chebyshev case are explicitly given in the previous section, while those of  $D$  and  $E$  are given in [14] for the Legendre case and [15] for the Chebyshev case. This matrix equation can be efficiently solved, in particular, by using the tensor product (or matrix decomposition) method (see, for instance, [12], [14]). More precisely, we consider the following generalized eigenvalue problem  $E^t \mathbf{g} = \lambda D \mathbf{g}$  ( $E^t$  being the transpose of  $E$ ), and let  $\Lambda$  be the diagonal matrix formed by the eigenvalues and let  $G$  be the matrix formed by the corresponding eigenvectors. Then

$$E^t G = D G \Lambda \quad \text{or} \quad G^t E = \Lambda G^t D.
 \tag{3.11}$$

It is well known that the eigenvalues are all real positive (the Legendre case is trivial while the Chebyshev case can be proved as in [8]). Making a change of variable  $X = YG^t$  in (3.10), we find

$$CYG^t E + (A + m^2B + \beta C)YG^t D = F.$$

We then derive from (3.11) that

$$CY\Lambda + (A + m^2B + \beta C)Y = FD^{-1}G^{-t}.
 \tag{3.12}$$

The above matrix equation is nothing but a sequence of  $N - 1$  one-dimensional equations considered in the previous section. In summary, after the preprocessing for the computation of the eigenpair  $(\Lambda, G)$  and  $G^{-1}$  (in the Legendre case,  $G$  is a orthonormal matrix, i.e.,  $G^{-1} = G^t$ ), the solution of (3.7) for each  $m$  consists of three steps:

1. Compute  $FD^{-1}G^{-t}$ :  $N^3 + O(N^2)$  flops.
2. Solving  $Y$  from (3.12):  $O(N^2)$  flops.
3. Set  $X = YG^t$ :  $N^3$  flops.

The above matrix decomposition method is very easy to implement and quite efficient for small to moderate  $N$ . But its computational complexity is not optimal. However, for the Legendre or Chebyshev–Legendre method, the matrix equation (3.10) can be solved by the generalized cyclic reduction method (cf. [18]) whose computational complexity is  $O(N^2 \log_2 N)$ . Hence, the computational complexity of the Chebyshev–Legendre–Galerkin method for the three-dimensional equation (3.1) will be  $O(N^2 M \log_2(NM))$  which is again quasi-optimal. We refer to [16] for a performance comparison of Poisson solvers in rectangular domains by Legendre, Chebyshev, and Chebyshev–Legendre–Galerkin methods.

The above method can be applied in particular to the axisymmetric problem in a cylinder; see [11] for an example of solving the axisymmetric Navier–Stokes equations.

**4. Fourth-order equations.** Consider the biharmonic equation on a unit disk

$$(4.1) \quad \Delta^2 U = F \text{ in } \Omega = \{(x, y) : x^2 + y^2 < 1\}, \quad U = \frac{\partial U}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega,$$

whose variational formulation is to find  $U \in H_0^2(\Omega)$  such that

$$(4.2) \quad A(U, V) := \int_{\Omega} \Delta U \Delta V \, dx \, dy = \int_{\Omega} F V \, dx \, dy \quad \forall V \in H_0^2(\Omega).$$

Let us denote

$$\tilde{\Delta} v := \frac{1}{r}(rv_r)_r + \frac{1}{r^2}v_{\theta\theta}.$$

Applying the polar transformation  $x = r \cos \theta$ ,  $y = r \sin \theta$  to (4.1), we obtain

$$(4.3) \quad \begin{aligned} \tilde{\Delta}^2 u &= f, & (r, \theta) \in Q &= (0, 1) \times [0, 2\pi), \\ u(1, \theta) &= u_r(1, \theta) = 0, & \theta \in [0, 2\pi), & \quad u \text{ periodic in } \theta, \end{aligned}$$

where  $u(r, \theta) = U(r \cos \theta, r \sin \theta)$  and  $f(r, \theta) = F(r \cos \theta, r \sin \theta)$ . The corresponding variational formulation is

$$(4.4) \quad \begin{aligned} a(u, v) &:= \int_Q \tilde{\Delta} u \tilde{\Delta} v \, r \, dr \, d\theta \\ &= \int_Q \left( \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} \right) \left( \frac{1}{r}(rv_r)_r + \frac{1}{r^2}v_{\theta\theta} \right) r \, dr \, d\theta = \int_Q f \, v \, r \, dr \, d\theta. \end{aligned}$$

Thus, the *essential* pole conditions, which make the bilinear form  $a(\cdot, \cdot)$  meaningful, are

$$(4.5) \quad (ru_r)_r|_{(0,\theta)} = u_{\theta\theta}|_{(0,\theta)} = (u_{\theta\theta})_r|_{(0,\theta)} = 0, \quad \theta \in [0, 2\pi).$$

Since  $(ru_r)_r = ru_{rr} + u_r$ , the above conditions reduce to

$$(4.6) \quad u_r(0, \theta) = u_{\theta\theta}(0, \theta) = u_{\theta\theta r}(0, \theta) = 0, \quad \theta \in [0, 2\pi).$$

As before, let

$$u_M(r, \theta) = \sum_{m=0}^M (u^{1m}(r) \cos(m\theta) + u^{2m}(r) \sin(m\theta))$$

be a Fourier-spectral approximation to  $u$ . Then the pole conditions in (4.6) imply that

$$u^{1m}(0) = u^{2m}(0) = 0 \quad \text{if } m \neq 0, \quad u_r^{1m}(0) = u_r^{2m}(0) = 0.$$

Let  $\tilde{\delta}_m v = \frac{1}{r}(rv_r)_r - \frac{m^2}{r^2}v$ . Then,  $u^{im}$  ( $i = 1, 2$ ) will satisfy the equation

$$(4.7) \quad \begin{aligned} \tilde{\delta}_m^2 u^{im} &= f^{im}(r), & r \in (0, 1), \\ u^{im}(0) &= 0 \quad \text{if } m \neq 0, & u^{im}(1) = u_r^{im}(0) = u_r^{im}(1) = 0, \end{aligned}$$

where  $(f^{1m}(r), f^{2m}(r))$  are defined in (2.8). Hence, we only have to consider the following prototypical one-dimensional fourth-order equation:

$$(4.8) \quad \begin{aligned} \tilde{\delta}_m^2 v &= g, & r \in (0, 1), \\ v(0) &= 0 \quad \text{if } m \neq 0, & v(1) = v'(0) = v'(1) = 0. \end{aligned}$$

The variational formulation corresponding to (4.2) is to find  $v \in Y(m)$  such that

$$(4.9) \quad \tilde{a}_m(v, w) := \int_0^1 \tilde{\delta}_m v \tilde{\delta}_m w r \, dr = \int_0^1 g w r \, dr \quad \forall w \in Y(m),$$

where  $Y(m) = H_0^2(0, 1)$  if  $m \neq 0$  and  $Y(0) = \{w \in H^2(0, 1) : w(1) = w'(0) = w'(1) = 0\}$ .

LEMMA 4.1.

$$(4.10) \quad \begin{aligned} \tilde{a}_m(u, v) &= \int_0^1 r u'' v'' \, dr + (2m^2 + 1) \int_0^1 \frac{1}{r} u' v' \, dr \\ &\quad + (m^4 + 4m^2) \int_0^1 \frac{1}{r^3} u v \, dr \quad \forall u, v \in Y(m). \end{aligned}$$

*Proof.* By definition

$$(4.11) \quad \begin{aligned} a_m(u, v) &= \int_0^1 \frac{1}{r} \left[ (ru_r)_r - \frac{m^2}{r}u \right] \left[ (rv_r)_r - \frac{m^2}{r}v \right] \, dr \\ &= \int_0^1 \frac{1}{r} (ru_r)_r (rv_r)_r \, dr + m^4 \int_0^1 \frac{1}{r^3} u v \, dr \\ &\quad - \int_0^1 \frac{m^2}{r^2} [(ru_r)_r v + (rv_r)_r u] \, dr. \end{aligned}$$

Thanks to the homogeneous boundary conditions in  $Y(m)$ , we can integrate by parts to get

$$\begin{aligned} -m^2 \int_0^1 \frac{1}{r^2} [(ru_r)_r v + (rv_r)_r u] \, dr &= m^2 \int_0^1 \left[ ru_r \left( \frac{v}{r^2} \right)_r + rv_r \left( \frac{u}{r^2} \right)_r \right] \, dr \\ &= 2m^2 \int_0^1 \left[ \frac{1}{r} u_r v_r - \frac{1}{r^2} (uv)_r \right] \, dr = 2m^2 \int_0^1 \left[ \frac{1}{r} u_r v_r + 2 \frac{1}{r^3} uv \right] \, dr, \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \frac{1}{r} (ru_r)_r (rv_r)_r dr &= \int_0^1 \left[ ru_{rr}v_{rr} + (u_rv_r)_r + \frac{1}{r}u_rv_r \right] dr \\ &= \int_0^1 r u_{rr} v_{rr} dr + \int_0^1 \frac{1}{r} u_r v_r dr. \end{aligned}$$

Then (4.10) is a direct consequence of the above relations.  $\square$

Setting  $X_N(m) = P_N \cap H_0^2(I)$  if  $m \neq 0$  and  $X_N(0) = P_N \cap \{w \in H^2(I) : w(1) = w'(\pm 1) = 0\}$ , and making a coordinate transformation  $r = (t + 1)/2$  in (4.9), the Legendre- or Chebyshev-Legendre-Galerkin method for (4.9) is to find  $v_N \in X_N(m)$  such that

$$\begin{aligned} (4.12) \quad & \int_I (t+1) v_N'' w'' dt + (2m^2 + 1) \int_I \frac{1}{t+1} v_N' w' dt \\ & + (m^4 + 4m^2) \int_I \frac{1}{(t+1)^3} v_N w dt = \int_I I_N h w dt \quad \forall w \in X_N(m), \end{aligned}$$

where  $h(t) = \frac{1}{16}(t+1)g((t+1)/2)$ ,  $I_N$  is the interpolation operator based on the Legendre- or Chebyshev-Gauss-type points.

Thus, by setting

$$\begin{aligned} (4.13) \quad & a_{ij} = \int_I (t+1) \phi_j'' \phi_i'' dt, \quad A = (a_{ij}), \\ & b_{ij} = \int_I \frac{1}{t+1} \phi_j' \phi_i' dt, \quad B = (b_{ij}), \\ & c_{ij} = \int_I \frac{1}{(t+1)^3} \phi_j \phi_i dt, \quad C = (c_{ij}), \\ & h_i = \int_I I_N h \phi_i dt, \quad \mathbf{h} = (h_i), \\ & v_N = \sum x_i \phi_i(t), \quad \mathbf{x} = (x_i), \end{aligned}$$

(4.12) becomes the matrix equation

$$(A + (2m^2 + 1)B + (m^4 + 4m^2)C)\mathbf{x} = \mathbf{h}.$$

It is clear that the matrices  $A$ ,  $B$ , and  $C$  are symmetric and positive definite. However, the exact structures of these matrices depend on the choice of basis functions.

Case  $m \neq 0$ . If we look for basis functions in the form (2.15), we obtain

$$\phi_j(t) = L_j(t) - \frac{2(2j+5)}{2j+7}L_{j+2}(t) + \frac{2j+3}{2j+7}L_{j+4}(t),$$

and

$$(4.14) \quad X_N(m) = \text{span}\{\phi_0(t), \phi_1(t), \dots, \phi_{N-4}(t)\}.$$

But, unfortunately,  $C$  will be a full matrix in this case. Hence, we shall use the basis functions defined by

$$(4.15) \quad \phi_j(t) = (1 - t^2)^2 L'_{j+1}(t)$$

which will lead to a sparse matrix (see Lemma 4.2 below).

Case  $m = 0$ . In this case,  $C$  is not needed so we can use the basis functions defined by

$$(4.16) \quad \begin{aligned} \phi_j(t) = & L_j(t) - \frac{(2j+3)(j+4)}{2j+5} L_{j+1}(t) \\ & - \frac{j(j+1)}{(j+2)(j+3)} L_{j+2}(t) + \frac{(j+1)(j+2)(2j+3)}{(j+3)(2j+5)} L_{j+3}(t), \end{aligned}$$

and

$$(4.17) \quad X_N(0) = \text{span}\{\phi_0(t), \phi_1(t), \dots, \phi_{N-3}(t)\}.$$

LEMMA 4.2. Case  $m \neq 0$ . For the basis functions (4.15), the matrices  $A$ ,  $B$ , and  $C$  are symmetric banded matrices such that

$$a_{ij} = b_{ij} = 0 \text{ for } |i - j| > 5; \quad c_{ij} = 0 \text{ for } |i - j| > 3.$$

Case  $m = 0$ . For the basis functions (4.16), the matrix  $A$  is diagonal and  $b_{ij} = 0$  for  $|i - j| > 3$ .

*Proof.* Thanks to the symmetry, we only have to consider the case  $i \geq j$ .

Case  $m \neq 0$ . By definition and thanks to (2.23)

$$c_{ij} = \int_I (1+t)(1-t)^4 L'_{i+1} L'_{j+1} dt = \frac{(i+1)(i+2)}{2i+3} \int_I (1-t)^3 (L_i - L_{i+2}) L'_{j+1} dt.$$

Thus,  $c_{ij} = 0$  for  $j < i - 3$  since  $(1-t)^3 L'_{j+2}$  is a polynomial of degree  $\leq j + 3$ .

Likewise, we can show by integration by parts that  $a_{ij} = b_{ij} = 0$  for  $|i - j| > 5$ .

Case  $m = 0$ . Thanks to the boundary conditions satisfied by  $\phi_j(t)$  in (4.16), we can integrate by parts twice for  $a_{ij}$  and once for  $b_{ij}$  without introducing new boundary terms. In virtue of the orthogonality relations of the Legendre polynomials, we can then conclude that  $a_{ij} = 0$  for  $i \neq j$ ;  $b_{ij} = 0$  for  $|i - j| > 3$ .

The evaluation of the entries of  $A$ ,  $B$ , and  $C$  is tedious by hand, but these entries can be precomputed numerically by using Gaussian quadrature formulas. The details are left to the interested readers.  $\square$

Thus, the above matrix equation can be solved in  $O(N)$  operations. In particular, the computational complexity of the Chebyshev–Legendre–Galerkin method for (4.1) is quasi-optimal.

*Remark 4.1.* The biharmonic equation in a cylinder can be decomposed, as in the previous section, into a set of two-dimensional fourth-order equations which can also be solved efficiently; see [14, 3] for more details.

The above method can be used to solve the two-dimensional Stokes problem in a disk by either transforming it to a biharmonic equation for the stream function or using the Fourier expansion in  $\theta$  to reduce it to a sequence of one-dimensional fourth-order equations (see [9, pp. 146–148] and [15, pp. 83–84]).

We can also develop a Chebyshev–Galerkin method for (4.1). More precisely, let  $w(r) = (1 - (2r - 1))^{-\frac{1}{2}}$  be the Chebyshev weight function in  $(0, 1)$ , then the bilinear form associated to the Chebyshev–Galerkin approximation for (4.8) is

$$\tilde{a}_m^c(u, v) = \int_0^1 \tilde{\delta}_m u \tilde{\delta}_m (v w(r)) r dr. \quad \square$$

**5. Extensions.** The spectral-Galerkin methods presented above can be used to treat more general problems. Below are some immediate extensions.



**5.1. Other boundary conditions.** For problems with homogeneous Neumann- or Robin-type boundary conditions, the procedure is exactly the same as with the homogeneous Dirichlet boundary condition except that the basis functions should be chosen to satisfy the underlying homogeneous Neumann- or Robin-type boundary conditions. It is shown in [16] that such basis functions in the form of (2.15) can be uniquely determined so long as the underlying boundary conditions leads to a well-posed problem.

For problems with nonhomogeneous boundary conditions, we need to construct a simple *discrete* function satisfying the *discrete* nonhomogeneous boundary conditions and reduce the original problem to a problem with homogeneous boundary conditions. To illustrate this procedure, we consider the case where the homogeneous boundary condition in (2.3) is replaced by  $u(1, \theta) = h(\theta)$ . Let  $(h^{1m}, h^{2m})$  be defined by the discrete Fourier expansion

$$(5.1) \quad h(\theta_j) = \sum_{m=0}^M (h^{1m} \cos(m\theta_j) + h^{2m} \sin(m\theta_j)),$$

$$\theta_j = \frac{j\pi}{M}, \quad j = 0, 1, \dots, 2M - 1.$$

Then the boundary condition for  $u^{im}(r)$  at  $r = 1$  in (2.10) should be replaced by  $u^{im}(1) = h^{im}$ ,  $i = 1, 2$ . Therefore,  $\tilde{u}^{im}(r) := u_{im}(r) - h^{im}r$  will satisfy the following equation with homogeneous boundary condition:

$$-\tilde{u}_{rr}^{im} - \frac{1}{r}\tilde{u}_r^{im} + \left(\frac{m^2}{r^2} + \alpha\right)\tilde{u}_{im} = f^{im}(r) - \frac{h^{im}}{r}(m^2 + \alpha r^2 - 1), \quad r \in (0, 1),$$

$$\tilde{u}_{im}(0) = 0 \text{ if } m \neq 0, \quad \tilde{u}_{im}(1) = 0.$$

**5.2. Other domains in polar and cylindrical coordinates.** The spectral-Galerkin methods can also be used for problems in other domains that can be conveniently represented by polar and cylindrical coordinates.

Consider for instance the Helmholtz equation in an annular domain  $\Omega = \{a < x^2 + y^2 < b\}$  with homogeneous Dirichlet boundary conditions

$$(5.2) \quad -\Delta U + \alpha U = F \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega.$$

Then we need to solve (2.11) with  $a < r < b$ . Applying the coordinate transformation  $r = \frac{b-a}{2}(t+c)$  with  $c = \frac{b+a}{b-a}$  and setting  $v(t) = u(r)$ ,  $g(t) = \frac{4}{(b-a)^2}f(r)$ , then (2.11) becomes

$$(5.3) \quad -\delta_m v := -v'' - \frac{1}{t+c}v' + \left(\frac{m^2}{(t+c)^2} + \frac{4}{(b-a)^2}\alpha\right)v = g(t), \quad v(\pm 1) = 0.$$

Unfortunately, the standard bilinear form  $a_m(\phi_j(t), \phi_i(t)) = -\int_I \delta_m \phi_j(t) \phi_i(t) (t+c) dt$  cannot be evaluated exactly and would result in a linear system with full matrices, since  $(t+c)$  is generally not a factor of the basis functions when  $c \neq pm1$ . One can, of course, consider the problem as having a variable coefficient  $(t+c)^{-1}$  and solve it by using an iterative method, but a more efficient way is to use the nonstandard bilinear form

$$b_m(\phi_j(t), \phi_i(t)) := -\int_I \delta_m \phi_j(t) \phi_i(t) (t+c)^2 dt$$

which would lead to a *sparse* but *nonsymmetric* system.

With the above in mind, problems in a domain between two cylinders

$$\Omega = \{(x, y, z) : 0 < a < x^2 + y^2 < b, \quad c < z < d\}$$

can be easily treated, for instance, by using the matrix decomposition method in section 3.

Efficient spectral-Galerkin algorithms for problems in unbounded polar domains such as  $\Omega = \{(x, y) : x^2 + y^2 \geq 1\}$  can be similarly constructed by using, for the radial direction, Laguerre polynomials if the solution converges to zero exponentially at infinity, or orthogonal rational functions introduced in [4].

For a pie-shaped domain  $\Omega = \{(x, y) = (r \cos \theta, r \sin \theta) : 0 \leq a < r < b, \quad 0 < c < \theta < d < 2\pi\}$ , one can no longer apply the Fourier method in the  $\theta$  direction; instead, double Legendre or Chebyshev expansion, similar to those used in [14] and [15], should be directly used.

**5.3. Problems with variable coefficients.** For problems with variable coefficients in the Cartesian coordinates, the unmodified spectral-Galerkin method is usually impractical since the stiffness and mass matrices are dense and expensive to evaluate. However, such problems are usually spectrally equivalent to a problem with constant coefficients (in the Cartesian coordinates) which can be efficiently solved by a spectral-Galerkin method. Therefore, one can use the constant-coefficient problem as a preconditioner for the variable-coefficient problem and apply a conjugate-gradient-type iterative method. Thanks to the spectral equivalence, the iterative procedure will converge in a finite number of steps, independent of the discretization parameters. Let us illustrate the procedure by considering the following elliptic problem with variable coefficients in a unit disk:

$$(5.4) \quad -\nabla \cdot (D(\mathbf{x})\nabla U) + \beta(\mathbf{x})U = F \text{ in } \Omega = \{(x, y) : x^2 + y^2 < 1\}, \\ u = 0 \text{ on } \partial\Omega,$$

where  $\beta(\mathbf{x}) \geq 0$  in  $\Omega$ , and  $D(\mathbf{x})$  is a  $2 \times 2$  matrix satisfying the usual elliptic condition.

Let  $B, u, \alpha$ , and  $f$  be, respectively,  $D, U, \beta$ , and  $F$  under the transformation

$$(5.5) \quad x = \frac{t+1}{2} \cos \theta, \quad y = \frac{t+1}{2} \sin \theta.$$

Let  $I_{NM}$  be the interpolation operator based on the Fourier–Chebyshev Gauss–Lobatto points, i.e.,

$$(I_{NM} f)(t_i, \theta_j) = f(t_i, \theta_j), \quad i = 0, 1, \dots, N, \quad j = 0, 1, \dots, 2M - 1,$$

with  $t_i = \cos \frac{i\pi}{N}$  and  $\theta_j = \frac{j\pi}{M}$ , and

$$X_{NM} = \text{span}\{\phi_n^m(t) \cos m\theta, \phi_n^m(t) \sin m\theta : n = 0, 1, \dots, N - s(m), m = 0, 1, \dots, M\}$$

with  $\phi_n^m(t) = T_n(t) - T_{n+2}(t)$  if  $m \neq 0$  and  $\phi_n^0(t) = T_n(t) - T_{n+1}(t)$ ,  $s(m) = 2$  if  $m \neq 0$  and  $s(0) = 1$ . Then a *pseudospectral* Fourier–Chebyshev approximation to (5.4) is to find  $u_N \in X_{NM}$  such that

$$(A_{ps} u_N, v) := - \int_0^{2\pi} \int_{-1}^1 (t+1) \tilde{\nabla} \cdot (I_{NM}(B\tilde{\nabla} u_N)) v \omega \, dr \, d\theta \\ + \int_0^{2\pi} \int_{-1}^1 (t+1) I_{NM}(\alpha u_N) v \omega \, dr \, d\theta = \int_0^{2\pi} \int_{-1}^1 (t+1) I_{NM} f v \omega \, dr \, d\theta \quad \forall v \in X_{NM},$$

where  $\omega(t) = (1 - t^2)^{-\frac{1}{2}}$ ,  $\tilde{\nabla}$  and  $\tilde{\nabla} \cdot$  are, respectively,  $\nabla$  and  $\nabla \cdot$  under the transformation (5.5).

It is easy to see that the operator  $A_{ps}$  is spectrally equivalent to  $H_{sp}$  defined by

$$(H_{sp}v, w) := - \int_0^{2\pi} \int_{-1}^1 (t+1) \tilde{\nabla} \cdot (\tilde{\nabla}v) w \omega dr d\theta.$$

Hence, we can use  $H_{sp}$  as a preconditioner for  $A_{ps}$ . Since both  $A_{ps}v$  and  $H_{sp}^{-1}v$  can be evaluated in  $O(NM \log_2(NM))$  operations, and the number of iterations required is independent of  $N$  and  $M$ , the computational complexity of a conjugate-gradient-type method (e.g., CGS [17]) for problems with variable coefficients is still quasi-optimal. We refer to [16] for a more detailed presentation on a similar problem.

**5.4. Collocation approach.** Other types of approximations, including spectral-collocation, finite-difference or finite-element, can also be easily developed by using the variational formulations. For instance, a spectral-collocation method based on a variational formulation differs from a spectral-Galerkin method in only two aspects:

1. It uses Lagrange polynomials based on Gaussian-type collocation points as basis functions. For instance, a Chebyshev-collocation method for (2.13) should use Chebyshev-Gauss-Lobatto points when  $m \neq 0$ , while Chebyshev-Gauss-Radau points should be used when  $m = 0$  to avoid explicit computation at the singular points  $t = -1$ .
2. The integrals are approximated by using Gaussian-type quadrature rules.

The main advantage of the collocation method is that variable coefficients can be easily handled without extra effort, while its main disadvantage is that the corresponding linear system has a full matrix, even for problems with constant coefficients; hence, it is usually more expensive than the spectral-Galerkin methods presented above, even for problems with variable coefficients.

**6. Numerical results.** In this section, we present some numerical experiments and compare them with existing algorithms. All computations are performed in double precision on a SUN Sparc 10 workstation Model 30 with standard optimization option “-O”.

The first example is the computation of the eigenvalues of Bessel’s equation

$$(6.1) \quad -u_{rr} - \frac{1}{r}u_r + \frac{m^2}{r^2}u = \lambda u$$

subject to the conditions that  $u(1) = 0$  and that  $u(r)$  be finite in  $[0, 1]$ . This problem has been the standard test problem for algorithms dealing with the pole singularity (see, for instance, [9], [6], [10], and [7]). We approximate (6.1) (when  $m \neq 0$ ) by the following spectral approximations: find  $u_N \in X_N = P_N \cap H_0^1(0, 1)$  such that

$$(6.2) \quad \int_0^1 r u_N' (v\omega)' dr + m^2 \int_0^1 \frac{1}{r} u_N v \omega dr = \lambda \int_0^1 r u_N v \omega dr \quad \forall u_N \in X_N,$$

where  $\omega = 1$  in the Legendre case and  $\omega = (1 - (2r - 1)^2)^{-\frac{1}{2}}$  in the Chebyshev case. The matrix form of (6.2) is a generalized eigenvalue problem of the form  $A\mathbf{u} = \lambda B\mathbf{u}$  which will be solved by using LAPACK (cf. [2]) routines dsygv.f and dgeev.f for the Legendre and Chebyshev methods, respectively. In Table 6.1, we list the relative errors for the approximation of the first eigenvalue when  $m = 7$  by using, respectively, the Legendre-Galerkin, Legendre-collocation, Chebyshev-Galerkin and Chebyshev-collocation methods. The collocation approximations are obtained by simply replacing

TABLE 6.1  
Approximation of the eigenvalue of the Bessel's equation.

$N$	4	8	12	16	20	128	256
Legen-G	1.8E-2	6.5E-5	3.9E-9	1.1E-14	1.2E-16	2.7E-14	3.4E-14
Legen-c	6.6E-2	8.7E-5	2.8E-9	9.9E-15	5.4E-15	8.8E-13	1.1E-11
Cheby-G	6.1E-2	4.9E-5	4.4E-7	4.4E-10	1.6E-13	3.0E-14	1.4E-13
Cheby-c	2.0E-2	8.5E-5	6.6E-7	7.2E-10	2.8E-13	1.6E-13	1.4E-12
[10]	$\sim 10^{-2}$	$\sim 10^{-3}$	$\sim 10^{-5}$	$\sim 10^{-8}$	$\sim 10^{-11}$		

TABLE 6.2  
Approximations of the Poisson's equation.

$N$	8	16	32	64	128	256
	$u(r, \theta) = e^{r \cos \theta + r \sin \theta}$					
Cheby-G	2.6E-8	1.8E-15	1.8E-15	2.7E-15	2.7E-15	4.4E-15
FISHPACK	1.6E-2	3.8E-3	9.2E-4	2.3E-4	5.6E-5	1.4E-5
[6]	2.9E-8	2.7E-15	2.7E-15			
$N$	8	16	32	8	16	32
	$u(r, \theta) = r^{2.5}$			$u(r, \theta) = r^3$		
Cheby-G	1.3E-4	5.9E-6	2.3E-7	3.8E-16	3.3E-16	1.3E-15
[6]	7.7E-2	1.6E-2	3.4E-3	2.9E-2	4.2E-3	6.0E-4

TABLE 6.3  
CPU time of the Poisson solvers.

$N$	64	128	256	512
Cheby-G	0.10	0.46	2.02	9.23
FISHPACK	0.14	0.62	2.90	14.53

the integrals in (6.2) by Gaussian quadrature rules. For the sake of comparison, we also list the results in [10] obtained by a direct Chebyshev-collocation approximation to (6.1).

A few remarks are in order: all four methods produce significantly more accurate results than those reported in [10]. The fact that our Chebyshev-collocation method is more accurate than the Chebyshev-collocation method in [10] indicates another advantage of using algorithms based on variational formulations. The Galerkin methods lead to better results than the collocation methods, especially for large  $N$ , since the integrals in the Galerkin methods are computed exactly while those in the collocation methods are approximated by Gaussian quadrature rules whose accuracy may decrease as  $N$  increases due to the singularity at the pole.

The second example is the Poisson equation on a unit disk. For the sake of comparison, we consider several exact solutions which were used in [6]. In Table 6.2, we list the maximum errors by the Chebyshev-Galerkin method, which is the most efficient in this case, and by FISHPACK (cf. [20]) routine `hwsplr.f` which implements the second-order finite-difference algorithm presented in [19].  $N$  is the number of points used in the radial direction, while  $2N - 1$  points are used in the axial direction. The FFTPACK routines are used for the discrete Fourier and Chebyshev transforms. The CPU times for both Poisson solvers are listed in Table 6.3.

For  $u = e^{r \cos \theta + r \sin \theta}$  which is smooth in both the Cartesian and polar coordinates, all spectral methods converge exponentially fast. For  $u = r^3$  which is smooth in the polar coordinates but has a singularity at the pole in Cartesian coordinates, the rate of convergence of our Chebyshev-Galerkin method is still exponential while that of

the method in [6] is only algebraic. For  $u = r^{2.5}$  which has a singularity in the polar (and Cartesian) coordinates, the rate of convergences of the two methods are algebraic as expected, but the Chebyshev–Galerkin method still gives much better results than those in [6]. The poor performance of the method in [6] for problems with singularity at the pole is caused by the *parity argument* which assumes implicitly the smoothness of the solution at the pole.

We also note that the effect of roundoff errors is very limited in the Chebyshev–Galerkin method since the approximate solutions have almost full machine accuracy for  $N$  as large as 256. A more striking fact is that, in addition to its superior accuracy, the Chebyshev–Galerkin method only consumes about 70% of the CPU time of the finite difference code `hwsplr.f` in FISHPACK.

**7. Concluding remarks.** We have presented in this paper several extremely efficient and accurate spectral-Galerkin methods for some second- and fourth-order problems in polar and cylindrical geometries. These methods are based on variational formulations which incorporate the essential pole condition(s) for the underlying equation. Their rate of convergence only depends on the smoothness of the solution in the polar coordinates, while the singularity at the pole in the Cartesian coordinates does not affect their accuracy. The computational complexity of the Chebyshev–Legendre–Galerkin method is quasi-optimal for two-dimensional polar and three-dimensional cylindrical domains, while that of the Chebyshev–Galerkin method is quasi-optimal only for two-dimensional polar domains. The Chebyshev–Galerkin method, being less expensive than the finite difference code `hwsplr.f` in FISHPACK, is the most efficient for two-dimensional polar domains while the Chebyshev–Legendre–Galerkin method will become the most efficient for three-dimensional cylindrical domains as the number of unknowns increases. Furthermore, the algorithms presented in this paper are easy to implement, very stable to the propagation of roundoff errors, and more efficient and/or more accurate than other spectral algorithms.

A potential disadvantage of these methods is that the Gaussian-type collocation points are clustered near the pole(s), which would result in a severe CFL condition if an explicit scheme is used to solve a time-dependent problem. However, this difficulty can be alleviated by using a semi-implicit scheme, i.e., treating the principle elliptic operator implicitly and other lower-order operators explicitly, whose costs at each time step is about the same as an explicit scheme, thanks to the fast direct solvers presented in this paper.

In a forthcoming paper, we shall develop efficient spectral methods for spherical geometries.

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