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SIAM Journal on Numerical Analysis, Volume 29, Issue 1 (Feb., 1992), 57-77.

SIAM J. NUMER. ANAL.
Vol. 29, No. 1, pp. 57–77, February 1992

ON ERROR ESTIMATES OF PROJECTION METHODS FOR NAVIER–STOKES EQUATIONS: FIRST-ORDER SCHEMES*

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Abstract. In this paper projection methods (or fractional step methods) are studied in the semidiscretized form for the Navier–Stokes equations in a two- or three-dimensional bounded domain. Error estimates for the velocity and the pressure of the classical projection scheme are established via the energy method. A modified projection scheme which leads to improved error estimates is also proposed.

Key words. projection method, Navier–Stokes equations, rate of convergence

AMS(MOS) subject classifications. 35A40, 35Q10, 65J15

1. Introduction.

We consider the time-dependent Navier–Stokes equations in the primitive variable formulation

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \forall (x,t) \quad \text{in } \ \mathrm{Q} = \Omega\times[0,\mathrm{T}], \\ \operatorname{div} u = 0 \quad \forall\,(x,t) \quad \text{in } \ \mathrm{Q}, \\ u(0) = u_0, \end{cases}$$

where Ω is an open bounded domain in R^d ($d = 2$ or 3) with a sufficiently smooth boundary Γ. The unknowns are the vector function u (velocity) and the scalar function p (pressure).

The equations (1.1) should be completed with appropriate boundary conditions for the velocity u. For the sake of simplicity, we will consider the homogeneous boundary condition $u(t)|_\Gamma = 0$ for all $t \in [0, T]$.

In (1.1), the velocity u and the pressure p are coupled together by the incompressibility condition $\operatorname{div} u = 0$ which makes the equations difficult to solve numerically. In the late 60s, Chorin [2] and Temam [15] constructed the so-called projection method (or fractional step method), which decoupled the velocity and the pressure. The semidiscretized version of the projection method can be written as follows.

We start with $u^0 = u_0$ and solve successively $\tilde{u}^{n+1}$ and $\{u^{n+1}, \phi^{n+1}\}$ by

$$(1.2)\qquad \begin{cases} \dfrac{1}{k}(\tilde{u}^{n+1} - u^n) - \nu\Delta\tilde{u}^{n+1} + (u^n\cdot\nabla)\tilde{u}^{n+1} = f(t_{n+1}), \\ \tilde{u}^{n+1}|_\Gamma = 0, \end{cases}$$

* Received by the editors September 17, 1990; accepted for publication (in revised form) February 8, 1991. This work was supported in part by National Science Foundation grant DMS-8802596.

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and

$$(1.3)\qquad \begin{cases} \frac{1}{k}(u^{n+1} - \tilde{u}^{n+1}) + \nabla\phi^{n+1} = 0, \\ \text{div} u^{n+1} = 0, \\ u^{n+1} \cdot \vec{n}|_\Gamma = 0, \end{cases}$$

where k is the timestep, $t_{n+1} = (n+1)k$, and $\vec{n}$ is the normal vector to Γ. Note that we have omitted the dependency to x of the function f to simplify our notations; we will do so for $\{u, p\}$ as well.

The scheme (1.2)–(1.3) is in a slightly different form than that of [15] and it has two advantages: (i) (1.2) is a linear elliptic equation; (ii) the artificial stabilizer introduced in [15] is not required.

If we denote

- $H = \{u \in (L^2(\Omega))^d : \text{div} u = 0, u \cdot \vec{n}|_\Gamma = 0\}$,
- P_H: the orthogonal projector in $(L^2(\Omega))^d$ onto H,

we can readily check that (1.3) is equivalent to

$$u^{n+1} = P_H \tilde{u}^{n+1}.$$

In the first step (1.2), we solve an intermediate velocity $\tilde{u}^{n+1}$, which does not satisfy the incompressibility condition. Then, in the second step we project $\tilde{u}^{n+1}$ onto the divergence free space H to get an adequate velocity approximation u^{n+1}. However, by definition, u^{n+1} is only in H. Consequently, it does not necessarily satisfy the homogeneous boundary condition. This particularity will eventually prevent us from obtaining error estimates in a stronger norm.

The numerical efficiency of the scheme is obvious since the velocity and the pressure are totally decoupled: (1.2) is a second-order linear elliptic problem for $\tilde{u}^{n+1}$ which can be solved by a standard procedure; by applying the divergence operator to (1.3), we find that (1.3) is equivalent to the following Poisson equation for ϕ^{n+1}:

$$(1.4)\qquad \begin{cases} \Delta\phi^{n+1} = \frac{1}{k}\text{div}\tilde{u}^{n+1}, \\ \frac{\partial\phi^{n+1}}{\partial\vec{n}}|_\Gamma = 0, \end{cases}$$

and

$$u^{n+1} = \tilde{u}^{n+1} - k\nabla\phi^{n+1}.$$

We immediately remark that ϕ^{n+1} satisfies the homogeneous Neumann boundary condition, which is not necessarily verified by the exact pressure. Nevertheless, Chorin [3] and Temam [15] were still able to prove the convergence of $\tilde{u}^{n+1}$ and u^{n+1} towards $u(t_{n+1})$ in appropriate norms. However, there has been confusion about whether ϕ^{n+1} is, and how we can get, a proper pressure approximation. We will clarify this point during the course of our presentation (see, in particular, Remark 2).

The scheme (1.2)–(1.3) and its variations have been widely used in practice (see, among others, [1], [2], [4], [7], [10], [11], [19]). The numerical experiences suggest that the scheme provides a first-order (in an appropriate sense) approximation for the velocity u. However, to the author's knowledge, there is still no rigorous theoretical justification that proves the scheme is of first-order accuracy. Chorin in [3] proved that the rate of convergence of the projection method with a finite difference space discretization, applied to Navier–Stokes equations with *periodical boundary conditions*, was of order 1; for *Dirichlet boundary conditions* he only mentioned that he could prove the rate of convergence of the scheme to be $O(k^{\frac{1}{4}})$. In [14], Shen and Temam proposed a more complicated fractional step scheme, providing a better pressure approximation, and proved its rate of convergence to be $O(k^{\frac{1}{2}})$. The author was recently informed that Lu, Neittaanmäki, and Tai [8] proved that the convergence rate of the scheme (1.2)–(1.3) to be $O(k^{\frac{1}{2}})$ with a rather restrictive condition on the solution of (1.1). Orszag, Israeli, and Deville in [9] analyzed the scheme (1.2)–(1.3) applied to a one-dimensional linear model, i.e., *the two-dimensional Stokes equations with Dirichlet boundary condition in one direction and periodical boundary condition in the other.* They used normal mode analyses to show that for this simple model case, the rate of convergence of the scheme was of order 1.

In this paper, we will consider the full nonlinear Navier–Stokes equations with Dirichlet boundary conditions and derive rigorously precise error estimates for both the velocity and pressure. To classify order of the precision of a scheme, the following terminology will be used.

DEFINITION. *Let X be a Banach space equipped with norm $\|\cdot\|_X$ and $f: [0,T] \to X$ is continuous. Let $\{t_n^{(k)}\}_{n=0}^{n=T/k}$ be a family of discretization of $[0,T]$ such that*

$$0 = t_0^{(k)} < \cdots < t_n^{(k)} < t_{n+1}^{(k)} < \cdots < t_{T/k}^{(k)} = T; \quad \text{and}$$

$$\max_{0 \le n \le T/k-1} |t_{n+1}^{(k)} - t_n^{(k)}| \le \delta_k \to 0 \quad (\text{as } k \to 0).$$

Then, we say f_k is a weakly order α approximation of f in X if there exists a constant c independent of k such that

$$k \sum_{n=0}^{T/k} \|f_k(t_n^{(k)}) - f(t_n^{(k)})\|_X^2 \le ck^{2\alpha};$$

and we say f_k is a strongly order α approximation of f in X if there exists a constant c independent of k and n such that

$$\|f_k(t_n^{(k)}) - f(t_n^{(k)})\|_X^2 \le ck^{2\alpha} \quad \forall 0 \le n \le T/k.$$

Our results can then be summarized as follows.

We will prove that both $\tilde{u}^{n+1}$ and u^{n+1} are *weakly* first-order approximations to $u(t_{n+1})$ in $L^2(\Omega)^d$. And despite the incompatible Neumann boundary condition of ϕ^{n+1}, we will prove that ϕ^{n+1} as well as $(I - k\nu\Delta)\phi^{n+1}$ are weakly order $\frac{1}{2}$ approximations to $p(t_{n+1})$ in $L^2(\Omega)/R$. We will also propose a modified scheme that provides *strongly* first order approximations to the velocity and *weakly* first-order approximations to the pressure.

Let us emphasize that higher-order projection schemes can be constructed while keeping the simplicity of (1.2)–(1.3). Orszag, Israeli, and Deville described in [9] how to use extrapolations and improved pressure boundary conditions to achieve higher accuracy. Kim and Moin [7] proposed a second-order projection scheme (based on the Crank and Nicolson–Adams and Bashforth scheme) which removes the large splitting errors at the boundary by imposing an appropriate boundary condition for the intermediate velocity $\tilde{u}^{n+1}$. Satisfactory results were obtained by using this scheme with various space discretizations (see, for instance, [7], [11]). Van Kan [19] proposed an interesting second-order variant of the projection method via pressure correction. A similar idea was also used by Bell, Colella, and Glaz [1] to construct a second-order projection scheme. In fact, when applied to the linear Stokes equations, the scheme in [1] with one iteration for each timestep is essentially identical to the scheme in [19]. In a forthcoming paper [13], we will study several higher order projection schemes.

The purpose of this paper is to derive error estimates for the projection methods. We will study the semidiscretized scheme (1.2)–(1.3) directly instead of considering fully discrete schemes, since the technicalities vary depending on the spatial discretizations and may obscure the essential goal of the paper. To simplify our presentation, we will only consider a finite time interval, i.e., $0 < T < +\infty$. Related uniform (in time) stability and convergence analyses for $T = +\infty$ can be done as in [12]. As for uniform error estimates in time for $T = +\infty$, we need to assume the solutions of (1.1) to be exponentially stable, which is generally not true; we refer to [6] for more details on this subject. Since the stability and convergence of the velocity approximations have been established in [16], our focus will be on error estimates, which certainly requires some sort of regularity of the solution (u, p). Sufficient regularity is provided by assuming the data u_0 and f are sufficiently smooth (but without any compatibility condition of the data). A by-product of the error analysis is that it automatically implies the stability and convergence of the scheme. The problem with rough data will not be addressed here.

The paper is organized as follows: in the next section, we begin with some notation and lay out some assumptions which enable us to derive some regularity results required by error analyses. In §3, we will establish a first error estimate for both $\tilde{u}^{n+1}$ and u^{n+1}. Then in §4, we will improve the error estimate of §3 for the velocity, and provide an error estimate for the pressure as well. Finally, in §5, we will propose a modified projection scheme and show that it provides improved error estimates for both the velocity and pressure.

2. Preliminaries. Let $|\cdot|, \|\cdot\|$ denote, respectively, the norms in $L^2(\Omega)$ and $H_0^1(\Omega)$, i.e.,

$$|u|^2 = \int_\Omega |u(x)|^2 dx \quad \text{and} \quad \|u\|^2 = \int_\Omega |\nabla u(x)|^2 dx.$$

The norm in $H^s(\Omega)$ (for all s) will be denoted simply by $\|\cdot\|_s$. We will use, respectively, $(\cdot,\cdot)$ to denote the inner product in $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ to denote the duality between H^{-s} and $H_0^s(\Omega)$ for all $s > 0$.

In addition to H and P_H, which are already defined in §1, we define

$$V = \{v \in (H_0^1(\Omega))^d : \text{div} v = 0\},$$

and the Stokes operator

$$Au = -P_H \Delta u \quad \forall u \in D(A) = V \cap (H^2(\Omega))^d.$$

The Stokes operator A is an unbounded positive self-adjoint closed operator in H with domain $D(A)$, and its inverse A^{-1} is compact in H.

Let us prove first the following relations which will be used in the sequel:

$$(2.1)\ \exists\, c_1, c_2 > 0, \quad \text{such that } \forall u \in H : \begin{cases} \|A^{-1}u\|_s \le c_1\|u\|_{s-2} & \text{for } s = 1, 2; \\ c_2\|u\|_{-1}^2 \le (A^{-1}u, u) \le c_1^2\|u\|_{-1}^2. \end{cases}$$

Given $u \in H$, by definition of A, $v = A^{-1}u$ is the solution of the following Stokes equations:

$$(2.2) \qquad \begin{cases} -\Delta v + \nabla p = u, \\ \text{div} v = 0, \\ v|_\Gamma = 0. \end{cases}$$

The regularity results for (2.2) immediately give

$$\|A^{-1}u\|_s = \|v\|_s \le c_1\|u\|_{s-2} \quad \text{for } s = 1, 2; \quad \text{and}$$

$$(A^{-1}u, u) = (v, u) = -(\Delta v, v) + (\nabla p, v) = \|v\|^2 \le c_1^2\|u\|_{-1}^2.$$

On the other hand, since $u = Av$,

$$\|u\|_{-1} \le c \sup_{w \in H_0^1(\Omega)} \frac{\langle u, w\rangle}{\|w\|} = c \sup_{w \in H_0^1(\Omega)} \frac{\langle Av, w\rangle}{\|w\|} \le c\|v\|.$$

This completes the proof of (2.1). For the sake of simplicity, we will use $(A^{-1}u, u)^{\frac{1}{2}}$ as an equivalent norm of $H^{-1}(\Omega)^d$ for $u \in H$.

We define the trilinear form $b(\cdot,\cdot,\cdot)$ by

$$b(u, v, w) = \int_\Omega (u \cdot \nabla)v \cdot w dx.$$

It is a easy matter to verify that the trilinear form $b(\cdot,\cdot,\cdot)$ is skew-symmetric with respect to its last two arguments, i.e.,

$$(2.3) \qquad b(u, v, w) = -b(u, w, v) \quad \forall u \in H, \quad v, w \in (H_0^1(\Omega))^d.$$

In particular, we have

$$(2.4) \qquad b(u, v, v) = 0 \quad \forall u \in H, \quad v \in (H_0^1(\Omega))^d.$$

We can also readily check (see, for instance, [18]) that $b(\cdot,\cdot,\cdot)$ is continuous in $H^{m_1}(\Omega) \times H^{m_2+1}(\Omega) \times H^{m_3}(\Omega)$ provided

$$m_1 + m_2 + m_3 \ge \frac{d}{2} \quad \text{if } m_i \ne \frac{d}{2}, \quad i = 1, 2, 3.$$

In particular, for $d \le 4$, we have

$$(2.5) \qquad b(u, v, w) \le \begin{cases} c\|u\|\|v\|\|w\|, \\ c|u|\|v\|_2\|w\|, \\ c\|u\|\|v\|_2|w|, \\ c|u|\|v\|\|w\|_2, \\ c\|u\|_2\|v\||w|. \end{cases}$$

Under the above notation, the system (1.1) is equivalent to the following abstract equation:

$$(2.6)\qquad \begin{cases} \dfrac{du}{dt} + \nu A u + P_H(u\cdot\nabla)u = P_H f, \\ u(0) = u_0. \end{cases}$$

To simplify our presentation, we will assume that the data u_0 and f are sufficiently smooth; and if $d = 3$ we will also assume that a strong solution of (2.6) exists in the whole interval [0,T]. More precisely, we assume

$$(A1)\qquad u_0 \in (H^2(\Omega))^d \cap V, \qquad f \in L^\infty(0,T;(L^2(\Omega))^d) \cap L^2(0,T;(H^1(\Omega))^d);$$

in the three-dimensional case, we assume additionally

$$(A2)\qquad \sup_{t\in[0,T]} \|u(t)\| \le M_1.$$

We recall that (A2) is automatically satisfied with some appropriate constant M_1 when $d = 2$.

Hereafter, we will use c to denote a generic positive constant which depends only on Ω, ν, T, and constants from various Sobolev inequalities. We will use M as a generic positive constant which may additionally depend on u_0, f and the solution u through the constant M_1 in (A2).

Under the assumption (A1) and (A2), we can show that (see, for instance, Heywood and Rannacher [5])

$$(2.7)\qquad \sup_{t\in[0,T]} \{\|u(t)\|_2 + |u_t(t)| + |\nabla p(t)|\} \le M$$

and

$$(2.8)\qquad \int_0^T \|u_t(t)\|^2 dt \le M.$$

For our purpose, we also need the following regularity result.

LEMMA 1. *Assuming* (A1), (A2), *and* $f_t \in L^2(0,T;H^{-1})$, *we have*

$$\int_0^T \|u_{tt}\|_{-1}^2 dt \le M.$$

Proof. Taking the time derivative in (2.6), we find

$$(2.9)\qquad \langle u_{tt}, v\rangle + \nu(\nabla u_t, \nabla v) + b(u_t,u,v) + b(u,u_t,v) = \langle f_t, v\rangle \quad \forall v \in V.$$

Taking $v = A^{-1}u_{tt}$, using (2.1) and the Schwarz inequality, we obtain

$$\begin{aligned}\|u_{tt}\|_{-1}^2 + \frac{\nu}{2}\frac{d}{dt}|u_t|^2 &\le b(u_t,u,A^{-1}u_{tt}) + b(u,u_t,A^{-1}u_{tt}) + \|f_t\|_{-1}\|A^{-1}u_{tt}\| \\ &\le b(u_t,u,A^{-1}u_{tt}) + b(u,u_t,A^{-1}u_{tt}) + \|f_t\|_{-1}^2 + \frac{1}{4}\|u_{tt}\|_{-1}^2.\end{aligned}$$

The nonlinear terms on the right-hand side can be majorized as follows. Using (2.5), (2.1), and the regularity result (2.7), we have

$$|b(u_t, u, A^{-1}u_{tt})| \le c\|u_t\|\|u\|\|A^{-1}u_{tt}\| \le c\|u_t\|\|u\|\|u_{tt}\|_{-1}$$

$$\le M\|u_t\|\|u_{tt}\|_{-1} \le M\|u_t\|^2 + \tfrac{1}{4}\|u_{tt}\|_{-1}^2;$$

similarly, by using (2.3) and (2.5), we derive

$$b(u, u_t, A^{-1}u_{tt}) = -b(u, A^{-1}u_{tt}, u_t) \le \|u\|\|A^{-1}u_{tt}\|\|u_t\|$$

$$\le M\|u_{tt}\|_{-1}\|u_t\| \le M\|u_t\|^2 + \tfrac{1}{4}\|u_{tt}\|_{-1}^2.$$

The summation of the last three inequalities leads to

$$\frac{1}{4}\|u_{tt}\|_{-1}^2 + \frac{\nu}{2}\frac{d}{dt}|u_t|^2 \le \|f_t\|_{-1}^2 + M\|u_t\|^2.$$

Integrating the above inequality over [0,T], using (2.7)–(2.8), we derive

$$\frac{1}{4}\int_0^T \|u_{tt}\|_{-1}^2 dt + \frac{\nu}{2}|u_t(T)|^2 \le M\int_0^T \|u_t\|^2 dt + \int_0^T \|f_t\|_{-1}^2 dt + \frac{\nu}{2}|u_t(0)|^2 \le M. \quad \square$$

3. A first error estimate. Equation (1.2) is a linear elliptic system for $\tilde{u}^{n+1}$ whose existence and uniqueness are ensured by the classical Lax–Milgram theorem thanks to (2.4). Our purpose in this section is to show that $\tilde{u}^{n+1}$ and u^{n+1} are both strongly order $\frac{1}{2}$ approximations to $u(t_{n+1})$ in $L^2(\Omega)^d$. This result will be needed to improve the error estimates to weakly first order in the next section. More precisely, we want to establish the following lemma.

LEMMA 2. *Let us denote*

$$e^{n+1} = u(t_{n+1}) - u^{n+1} \quad \textit{and} \quad \tilde{e}^{n+1} = u(t_{n+1}) - \tilde{u}^{n+1}.$$

Then under the assumptions of Lemma 1, *we have*

$$|e^{N+1}|^2 + |\tilde{e}^{N+1}|^2 + k\nu\sum_{n=0}^{N}\{\|\tilde{e}^{n+1}\|^2 + \|e^{n+1}\|^2\}$$

$$+\sum_{n=0}^{N}\{|e^{n+1} - \tilde{e}^{n+1}|^2 + |\tilde{e}^{n+1} - e^n|^2\} \le Mk$$

$$\forall 0 \le N \le T/k - 1.$$

Proof. Let R^n be the truncation error defined by

$$\frac{1}{k}(u(t_{n+1}) - u(t_n)) - \nu\Delta u(t_{n+1}) + (u(t_{n+1})\cdot\nabla)u(t_{n+1}) + \nabla p(t_{n+1}) = f(n+1) + R^n, \tag{3.1}$$

where R^n is the integral residual of the Taylor series, i.e.,

$$R^n = \frac{1}{k}\int_{t_n}^{t_{n+1}} (t - t_n)u_{tt}(t)dt. \tag{3.2}$$

By subtracting (1.2) from (3.1), we obtain

$$\frac{1}{k}(\tilde{e}^{n+1}-e^n)-\nu\Delta\tilde{e}^{n+1}=(u^n\cdot\nabla)\tilde{u}^{n+1}-(u(t_{n+1})\cdot\nabla)u(t_{n+1}) +R^n-\nabla p(t_{n+1}). \tag{3.3}$$

The nonlinear terms on the right-hand side can be split up into three terms:

$$(u^n\cdot\nabla)\tilde{u}^{n+1}-(u(t_{n+1})\cdot\nabla)u(t_{n+1}) = -(e^n\cdot\nabla)\tilde{u}^{n+1}+(u(t_n)-u(t_{n+1})\cdot\nabla)\tilde{u}^{n+1}-(u(t_{n+1})\cdot\nabla)\tilde{e}^{n+1}. \tag{3.4}$$

We now take the inner product of (3.3) with $2k\tilde{e}^{n+1}$; using the identity

$$(a-b,2a)=|a|^2-|b|^2+|a-b|^2, \tag{3.5}$$

we derive

$$\begin{aligned}|\tilde{e}^{n+1}|^2-|e^n|^2+|\tilde{e}^{n+1}-e^n|^2+2k\nu\|\tilde{e}^{n+1}\|^2 &= 2k\langle R^n,\tilde{e}^{n+1}\rangle+2k(\nabla p(t_{n+1}),\tilde{e}^{n+1})-2kb(e^n,\tilde{u}^{n+1},\tilde{e}^{n+1})\\ &+2kb(u(t_n)-u(t_{n+1}),\tilde{u}^{n+1},\tilde{e}^{n+1})-2kb(u(t_{n+1}),\tilde{e}^{n+1},\tilde{e}^{n+1}).\end{aligned} \tag{3.6}$$

We majorize the right-hand side as follows:

$$\begin{aligned}2k\langle R^n,\tilde{e}^{n+1}\rangle &\le \frac{\nu}{4}k\|\tilde{e}^{n+1}\|^2+ck\|R^n\|_{-1}^2\\ &=\frac{\nu}{4}k\|\tilde{e}^{n+1}\|^2+ck^{-1}\|\int_{t_n}^{t_{n+1}}(t-t_n)u_{tt}dt\|_{-1}^2\\ &\le\frac{\nu}{4}k\|\tilde{e}^{n+1}\|^2+ck^{-1}\int_{t_n}^{t_{n+1}}\|u_{tt}\|_{-1}^2dt\int_{t_n}^{t_{n+1}}(t-t_n)^2dt\\ &\le\frac{\nu}{4}k\|\tilde{e}^{n+1}\|^2+ck^2\int_{t_n}^{t_{n+1}}\|u_{tt}\|_{-1}^2dt.\end{aligned}$$

Since $e^n\in H$, we have

$$\begin{aligned}2k(\nabla p(t_{n+1}),\tilde{e}^{n+1}) &= 2k(\nabla p(t_{n+1}),\tilde{e}^{n+1}-e^n)\\ &\le\tfrac{1}{2}|\tilde{e}^{n+1}-e^n|^2+2k^2|\nabla p(t_{n+1})|^2.\end{aligned}$$

We will see later that the last term on the right-hand side actually prevents us from obtaining a first-order error estimate in this section.

By using (2.3), (2.5), and (2.7),

$$\begin{aligned}2kb(e^n,\tilde{u}^{n+1},\tilde{e}^{n+1}) &= -kb(e^n,\tilde{e}^{n+1},\tilde{u}^{n+1})=-kb(e^n,\tilde{e}^{n+1},u(t_{n+1}))\\ &\le ck|e^n|\|\tilde{e}^{n+1}\|\|u(t_{n+1})\|_2\le Mk|e^n|\|\tilde{e}^{n+1}\|\\ &\le\frac{\nu k}{4}\|\tilde{e}^{n+1}\|^2+Mk|e^n|^2.\end{aligned}$$

Likewise, we have

$$\begin{aligned}2kb(u(t_n)-u(t_{n+1}),\tilde{u}^{n+1},\tilde{e}^{n+1}) &= -2kb(u(t_n)-u(t_{n+1}),\tilde{e}^{n+1},u(t_{n+1}))\\ &\le ck|u(t_n)-u(t_{n+1})|\|\tilde{e}^{n+1}\|\|u(t_{n+1})\|_2\\ &\le Mk\|\tilde{e}^{n+1}\||\int_{t_n}^{t_{n+1}} u_t dt|\\ &\le \frac{\nu}{4}k\|\tilde{e}^{n+1}\|^2 + Mk^2\int_{t_n}^{t_{n+1}}|u_t|^2dt,\end{aligned}$$

and by (2.3),

$$2kb(u(t_{n+1}),\tilde{e}^{n+1},\tilde{e}^{n+1})=0.$$

Combining the above estimates into (3.6), we obtain

$$\begin{aligned}&|\tilde{e}^{n+1}|^2-|e^n|^2+\nu k\|\tilde{e}^{n+1}\|^2+\frac{1}{2}|\tilde{e}^{n+1}-e^n|^2\\ &\le Mk^2\left(\int_{t_n}^{t_{n+1}}\|u_{tt}\|_{-1}^2dt+\int_{t_n}^{t_{n+1}}|u_t|^2dt\right)\\ &\quad +2k^2|\nabla p(t_{n+1})|^2+Mk|e^n|^2.\end{aligned}\tag{3.7}$$

On the other hand, we derive from (1.3) that

$$\frac{1}{k}(e^{n+1}-\tilde{e}^{n+1})-\nabla\phi^{n+1}=0.\tag{3.8}$$

Taking the inner product of the above equality with $2ke^{n+1}$, since $\operatorname{div}e^{n+1}=0$, we obtain

$$|e^{n+1}|^2-|\tilde{e}^{n+1}|^2+|e^{n+1}-\tilde{e}^{n+1}|^2=0.\tag{3.9}$$

Taking the sum of (3.7) and (3.9) for n from zero to N (for all $0\le N\le T/k-1$), using the regularity of u, we arrive at

$$\begin{aligned}&|e^{N+1}|^2+\sum_{n=0}^{N}\left\{|e^{n+1}-\tilde{e}^{n+1}|^2+\frac{1}{2}|\tilde{e}^{n+1}-e^n|^2+k\nu\|\tilde{e}^{n+1}\|^2\right\}\\ &\le Mk\sum_{n=0}^{N}|e^n|^2+Mk\left(k\int_0^T\|u_{tt}\|_{-1}^2dt+k\int_0^T|u_t|^2dt+\sup_{t\in[0,T]}|\nabla p(t)|^2\right)\\ &\le Mk\sum_{n=0}^{N}|e^n|^2+Mk.\end{aligned}$$

By applying the discrete Gronwall lemma to the last inequality, we derive

$$|e^{N+1}|^2+\sum_{n=0}^{N}\{k\nu\|\tilde{e}^{n+1}\|^2+|e^{n+1}-\tilde{e}^{n+1}|^2+|\tilde{e}^{n+1}-e^n|^2\}\le Mk$$
$$\forall 0\le N\le T/k-1.$$

Thanks to (3.9) and the inequality (see [16, Remark 1.6]

$$\|P_H u\|_{H^1(\Omega)} \le c(\Omega)\|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \tag{3.10}$$

we also have

$$|\tilde{e}^{N+1}|^2 + k\nu \sum_{n=0}^{N} \|e^{n+1}\|^2 \le Mk \quad \forall 0 \le N \le T/k - 1.$$

The proof of the lemma is complete. □

Remark 1. *It is unlikely that the right-hand side of the estimate of Lemma* 2 *can be improved to* Mk^2, *because such an estimate and* (3.8) *would lead us to derive*

$$k \sum_{n=0}^{N} |\nabla \phi^{n+1}|^2 = \frac{1}{k} \sum_{n=0}^{N} |e^{n+1} - \tilde{e}^{n+1}|^2 \le Mk \to 0 \quad (\text{as} \quad k \to 0).$$

Yet this is in contradiction with the pressure error estimate in the next section. Therefore, we will try to improve the error estimate in a weaker norm.

4. Improved error estimate. In this section, we will use the results of §3 to improve the error estimates for the velocity and establish an error estimate for the pressure as well. Our main result in this section is the following theorem.

THEOREM 1. *Under the assumptions of Lemma* 1, *both* $\tilde{u}^{n+1}$ *and* u^{n+1} *are weakly first-order approximations to* $u(t_{n+1})$ *in* $L^2(\Omega)^d$, *and* ϕ^{n+1} *as well as* $(I - k\nu\Delta)\phi^{n+1}$ *are weakly order* $\frac{1}{2}$ *approximations to* $p(t_{n+1})$ *in* $L^2(\Omega)/R$. *More precisely, we have*

$$k \sum_{n=0}^{T/k-1} \left\{ |e^{n+1}|^2 + |\tilde{e}^{n+1}|^2 \right\} \le Mk^2; \tag{4.1}$$

$$k \sum_{n=0}^{T/k-1} \left\{ |\phi^{n+1} - p(t_{n+1})|^2_{L^2(\Omega)/R} + |(I - k\nu\Delta)\phi^{n+1} - p(t_{n+1})|^2_{L^2(\Omega)/R} \right\} \le Mk. \tag{4.2}$$

Proof. (i) *Error estimate for the velocity.* Taking the sum of (1.2) and (1.3), we obtain

$$\frac{1}{k}(u^{n+1} - u^n) - \nu\Delta\tilde{u}^{n+1} + (u^n \cdot \nabla)\tilde{u}^{n+1} + \nabla\phi^{n+1} = f(t_{n+1}). \tag{4.3}$$

Let us denote

$$\tilde{q}^{n+1} = p(t_{n+1}) - \phi^{n+1}.$$

Subtracting (4.3) from (3.1), we obtain

$$\begin{cases} \frac{1}{k}(e^{n+1} - e^n) - \nu\Delta\tilde{e}^{n+1} + \nabla\tilde{q}^{n+1} \\ = (u^n \cdot \nabla)\tilde{u}^{n+1} - (u(t_{n+1}) \cdot \nabla)u(t_{n+1}) + R^n, \\ \operatorname{div} e^{n+1} = 0, \\ e^{n+1} \cdot \vec{n}|_\Gamma = 0. \end{cases} \tag{4.4}$$

As mentioned in §1, e^{n+1} is only in H and consequently it is not appropriate to take the inner product of (4.4) with e^{n+1} due to the extra tangential boundary terms appearing when integrating by parts. Hence, we will take the inner product of (4.4) with a smoother function $A^{-1}e^{n+1}$. The main difficulty here is how to treat the term $-\langle\Delta\tilde{e}^{n+1}, A^{-1}e^{n+1}\rangle$.

Let $u = e^{n+1}$ in (2.2) and $\{v, p\}$ be the solution of the corresponding Stokes equations. Then we have $v = A^{-1}e^{n+1}$ and

$$\|v\|_2 + |\nabla p| \leq c|e^{n+1}|;$$

$$-\Delta A^{-1}e^{n+1} = -\Delta v = e^{n+1} - \nabla p.$$

Since $\tilde{e}^{n+1}, A^{-1}e^{n+1} \in H_0^1(\Omega)^d$, by integration by parts,

$$\begin{aligned}-\langle\Delta\tilde{e}^{n+1}, A^{-1}e^{n+1}\rangle &= (\tilde{e}^{n+1}, -\Delta A^{-1}e^{n+1}) = (\tilde{e}^{n+1}, e^{n+1} - \nabla p)\\ &= (\tilde{e}^{n+1}, e^{n+1}) - (\tilde{e}^{n+1}, \nabla p)\\ &= |e^{n+1}|^2 - (\tilde{e}^{n+1} - e^{n+1}, \nabla p).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\tilde{e}^{n+1} - e^{n+1}, \nabla p) &\leq |e^{n+1} - \tilde{e}^{n+1}||\nabla p| \leq c|e^{n+1} - \tilde{e}^{n+1}||e^{n+1}|\\ &\leq \frac{1}{16}|e^{n+1}|^2 + c|e^{n+1} - \tilde{e}^{n+1}|^2.\end{aligned}$$

We derive from the above relations that

$$-\langle\Delta\tilde{e}^{n+1}, A^{-1}e^{n+1}\rangle \geq \frac{15}{16}|e^{n+1}|^2 - c|e^{n+1} - \tilde{e}^{n+1}|^2. \tag{4.5}$$

We now take the inner product of (4.4) with $2kA^{-1}e^{n+1}$, splitting the nonlinear terms into three parts as in §3, using (2.1), and noticing that

$$(A^{-1}u, \nabla p) = 0 \quad \forall u \in H,$$

we obtain

$$\begin{aligned}&\|e^{n+1}\|_{-1}^2 - \|e^n\|_{-1}^2 + \|e^{n+1} - e^n\|_{-1}^2 + \frac{15k\nu}{8}|e^{n+1}|^2\\ &\leq 2k\langle R^n, A^{-1}e^{n+1}\rangle \qquad (4.6)\\ &\quad -2kb(e^n, \tilde{u}^{n+1}, A^{-1}e^{n+1}) - 2kb(u(t_{n+1}), \tilde{e}^{n+1}, A^{-1}e^{n+1})\\ &\quad +2kb(u(t_n) - u(t_{n+1}), \tilde{u}^{n+1}, A^{-1}e^{n+1}) + ck|e^{n+1} - \tilde{e}^{n+1}|^2.\end{aligned}$$

We will majorize the right-hand side as follows: Using (2.1), (3.2), and the fact that

$$\|u\|_{-1} \leq c|u| \quad \forall u \in L^2(\Omega), \tag{4.7}$$

we have

$$2k\langle R^n, A^{-1}e^{n+1}\rangle \le ck\|R^n\|_{-1}\|A^{-1}e^{n+1}\|$$

$$\le ck\|\frac{1}{k}\int_{t_n}^{t_{n+1}}(t-t_n)u_{tt}dt\|_{-1}\|e^{n+1}\|_{-1}$$

$$\le \frac{\nu k}{4}|e^{n+1}|^2 + Mk^2\int_{t_n}^{t_{n+1}}\|u_{tt}\|_{-1}^2 dt.$$

Using (2.3), we derive

$$2kb(e^n, \tilde{u}^{n+1}, A^{-1}e^{n+1}) = -2kb(e^n, A^{-1}e^{n+1}, \tilde{u}^{n+1})$$

$$= 2kb(e^n, A^{-1}e^{n+1}, \tilde{e}^{n+1})$$

$$-2kb(e^n, A^{-1}e^{n+1}, u(t_{n+1}))$$

$$= I_{11} + I_{12},$$

by using (2.5) and (2.1),

$$I_{11} \le ck|e^n|\|A^{-1}e^{n+1}\|_2\|\tilde{e}^{n+1}\| \le ck|e^n||e^{n+1}|\|\tilde{e}^{n+1}\|$$

$$(\text{since } |e^n| \le Mk^{\frac{1}{2}} \text{ from Lemma 2})$$

$$\le Mk^{\frac{3}{2}}|e^{n+1}|\|\tilde{e}^{n+1}\| \le Mk^2\|\tilde{e}^{n+1}\|^2 + \frac{\nu k}{8}|e^{n+1}|^2,$$

and by (2.5),

$$I_{12} = -2kb(e^n, A^{-1}e^{n+1}, u(t_{n+1})) = 2kb(e^n, u(t_{n+1}), A^{-1}e^{n+1})$$

$$\le ck|e^n|\|u(t_{n+1})\|_2\|A^{-1}e^{n+1}\| \le Mk|e^n|\|e^{n+1}\|_{-1}$$

$$\le Mk\{|e^{n+1}| + |e^{n+1} - \tilde{e}^{n+1}| + |\tilde{e}^{n+1} - e^n|\}\|e^{n+1}\|_{-1}$$

$$\le \frac{k\nu}{8}\{|e^{n+1}|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2 + |\tilde{e}^{n+1} - e^n|^2\} + Mk\|e^{n+1}\|_{-1}^2.$$

Also by using (2.3), (2.5), and (3.9), we derive

$$2kb(u(t_{n+1}), \tilde{e}^{n+1}, A^{-1}e^{n+1}) = -kb(u(t_{n+1}), A^{-1}e^{n+1}, \tilde{e}^{n+1})$$

$$\le ck\|u(t_{n+1})\|_2\|A^{-1}e^{n+1}\||\tilde{e}^{n+1}|$$

$$\le Mk\|e^{n+1}\|_{-1}|\tilde{e}^{n+1}|$$

$$\le \frac{\nu k}{8}(|e^{n+1}|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2) + Mk\|e^{n+1}\|_{-1}^2.$$

Similarly,

$$\begin{aligned}2kb(u(t_n)-u(t_{n+1}),\tilde{u}^{n+1},A^{-1}e^{n+1}) &= -2kb(u(t_n)-u(t_{n+1}),A^{-1}e^{n+1},\tilde{u}^{n+1})\\ &= 2kb(u(t_n)-u(t_{n+1}),A^{-1}e^{n+1},\tilde{e}^{n+1})\\ &\quad -2kb(u(t_n)-u(t_{n+1}),A^{-1}e^{n+1},u(t_{n+1}))\\ &= I_{21}+I_{22};\end{aligned}$$

using (2.1) and (2.7), we have

$$\begin{aligned}I_{21} &\le ck|u(t_n)-u(t_{n+1})||e^{n+1}|\|\tilde{e}^{n+1}\|\\ &\le Mk|u(t_n)-u(t_{n+1})|^2\|\tilde{e}^{n+1}\|^2+\frac{\nu k}{8}|e^{n+1}|^2\\ &\le \frac{\nu k}{8}|e^{n+1}|^2+Mk^2\int_{t_n}^{t_{n+1}}|u_t|^2dt\|\tilde{e}^{n+1}\|^2\\ &\le \frac{\nu k}{8}|e^{n+1}|^2+Mk^3\|\tilde{e}^{n+1}\|^2.\end{aligned}$$

Likewise,

$$\begin{aligned}I_{22} &\le ck|u(t_n)-u(t_{n+1})||e^{n+1}|\|u(t_{n+1})\|\\ &\le Mk|u(t_n)-u(t_{n+1})|^2+\frac{\nu k}{8}|e^{n+1}|^2\\ &\le Mk^2\int_{t_n}^{t_{n+1}}|u_t|^2dt+\frac{\nu k}{8}|e^{n+1}|^2.\end{aligned}$$

Combining these inequalities into (4.6), we arrive at

$$\begin{aligned}&\|e^{n+1}\|_{-1}^2-\|e^n\|_{-1}^2+\nu k|e^{n+1}|^2+\|e^{n+1}-e^n\|_{-1}^2\\ &\le Mk\|e^{n+1}\|_{-1}^2+M(k^2+k^3)\|\tilde{e}^{n+1}\|^2+Mk|\tilde{e}^{n+1}-e^n|^2\\ &\quad +Mk|e^{n+1}-\tilde{e}^{n+1}|^2+Mk^2\int_{t_n}^{t_{n+1}}(\|u_{tt}\|_{-1}^2+|u_t|^2)dt.\end{aligned}$$

Taking the sum of the last inequality for n from zero to N (for all $0\le N\le T/k-1$), we derive from Lemma 2 that

$$\|e^{N+1}\|_{-1}^2+\sum_{n=0}^{N}\{\|e^{n+1}-e^n\|_{-1}^2+k\nu|e^{n+1}|^2\}\le Mk^2+Mk\sum_{n=0}^{N}\|e^n\|_{-1}^2.$$

By applying the discrete Gronwall lemma to the last inequality, we obtain

$$\|e^{N+1}\|_{-1}^2+\sum_{n=0}^{N}\{\|e^{n+1}-e^n\|_{-1}^2+k\nu|e^{n+1}|^2\}\le Mk^2$$

$$\forall 0 \le N \le T/k - 1. \tag{4.8}$$

We then derive from (3.9) and Lemma 2 that

$$k\sum_{n=0}^{N} |\tilde{e}^{n+1}|^2 = k\sum_{n=0}^{N}\{|e^{n+1}|^2 + |\tilde{e}^{n+1} - e^{n+1}|^2\} \le Mk^2$$
$$\forall 0 \le N \le T/k - 1.$$

(ii) *Error estimate for the pressure.* We rearrange (4.4) to

$$\nabla q_*^{n+1} = \frac{1}{k}(e^{n+1} - e^n) - \nu\Delta e_*^{n+1}$$
$$+(u(t_{n+1})\cdot\nabla)u(t_{n+1}) - (u^n\cdot\nabla)\tilde{u}^{n+1} - R^n, \tag{4.9}$$

where $\{e_*^{n+1}, q_*^{n+1}\} = \{\tilde{e}^{n+1}, \tilde{q}^{n+1}\}$.

If we denote $q^{n+1} = p(t_{n+1}) - (I - k\nu\Delta)\phi^{n+1}$, we derive that (4.9) is also true for $\{e_*^{n+1}, q_*^{n+1}\} = \{e^{n+1}, q^{n+1}\}$. Hence we can consider simultaneously the two pressure approximations.

We first split up the nonlinear term on the right-hand side of (4.9) as

$$(u(t_{n+1})\cdot\nabla)u(t_{n+1}) - (u^n\cdot\nabla)\tilde{u}^{n+1}$$
$$= ((u(t_{n+1}) - u(t_n))\cdot\nabla)u(t_{n+1}) + (e^n\cdot\nabla)u(t_{n+1}) + (u^n\cdot\nabla)\tilde{e}^{n+1}. \tag{4.10}$$

We derive from Lemma 2 that

$$\|u^n\| \le \|e^n\| + \|u(t_n)\| \le M \quad \forall n. \tag{4.11}$$

Hence, by using (2.5), we can derive that, for all $v \in H_0^1(\Omega)^d$,

$$((u(t_{n+1})\cdot\nabla)u(t_{n+1}) - (u^n\cdot\nabla)\tilde{u}^{n+1}, v)$$
$$\le c|u(t_{n+1}) - u(t_n)|\|u(t_{n+1})\|_2\|v\|$$
$$+ c\|e^n\|\|u(t_{n+1})\|\|v\| + c\|u^n\|\|\tilde{e}^{n+1}\|\|v\| \tag{4.12}$$
$$\le M\{\|\tilde{e}^{n+1}\| + \|e^n\| + |u(t_{n+1}) - u(t_n)|\}\|v\|.$$

Using the Schwarz inequality, we have also, for all $v \in H_0^1(\Omega)^d$,

$$\left(\frac{1}{k}(e^{n+1} - e^n) - \nu\Delta e_*^{n+1} - R^n, v\right)$$
$$\le \left(\frac{1}{k}\|e^{n+1} - e^n\|_{-1} + \|R^n\|_{-1} + \nu\|e_*^{n+1}\|\right)\|v\|. \tag{4.13}$$

We then derive from the inequality

$$|p|_{L^2(\Omega)/R} \le c \sup_{v\in H_0^1(\Omega)^d} \frac{(\nabla p, v)}{\|v\|}, \tag{4.14}$$

and (4.9), (4.12), (4.13) that

$$|q_*^{n+1}|_{L^2(\Omega)/R} \le c \sup_{v\in H_0^1(\Omega)^d} \frac{(\nabla q_*^{n+1}, v)}{\|v\|} \le \frac{M}{k}\|e^{n+1}-e^n\|_{-1}$$

$$+M(\|R^n\|_{-1} + \|\tilde{e}^{n+1}\| + \|e^{n+1}\|$$

$$+\|e^n\| + |u(t_{n+1}) - u(t_n)|).$$

Therefore, by using Lemma 2 and (4.8), we derive

$$k\sum_{n=0}^{T/k-1} |q_*^{n+1}|^2_{L^2(\Omega)/R} \le Mk\sum_{n=0}^{T/k-1} \{\|\tilde{e}^{n+1}\|^2 + \|e^{n+1}\|^2$$

$$+\|R^n\|^2_{-1} + |u(t_{n+1}) - u(t_n)|^2\}$$

$$+\frac{1}{k}\sum_{n=0}^{T/k-1} \|e^{n+1}-e^n\|^2_{-1} \le Mk.$$

The proof of Theorem 1 is then complete. □

Remark 2. We emphasize that $\tilde{u}^{n+1}$ is not merely an auxiliary velocity introduced for computing u^{n+1}. It is in fact an approximation to $u(t_{n+1})$, most likely as accurate asymptotically as u^{n+1}. This is also true for our modified scheme presented in the next section.

Despite the incompatible Neumann boundary condition of ϕ^{n+1}, we showed that ϕ^{n+1} was still a legitimate approximation to $p(t_{n+1})$. Given the velocity error estimate (4.1), the pressure error estimate (4.2) seems to be the best we can expect. Hence, whether $(I - k\nu\Delta)\phi^{n+1}$ is a better approximation to $p(t_{n+1})$ than ϕ^{n+1}, as is believed by many authors, needs more careful analysis and must be tested in practice.

5. A modified scheme. The scheme (1.2)–(1.3) can be slightly modified such that it becomes strongly first order while keeping the simplicity of the classical scheme. We modify the scheme (1.2)–(1.3) as follows.

We start with $u^0 = u_0$ and an arbitrary ϕ^0, then we solve successively $\tilde{u}^{n+1}$ and $\{u^{n+1}, \phi^{n+1}\}$ by

$$\text{(5.1)} \qquad \begin{cases} \frac{1}{k}(\tilde{u}^{n+1} - u^n) - \nu\Delta\tilde{u}^{n+1} + (u^n\cdot\nabla)\tilde{u}^{n+1} + \nabla\phi^n = f(t_{n+1}), \\ \tilde{u}^{n+1}|_\Gamma = 0, \end{cases}$$

and

$$\text{(5.2)} \qquad \begin{cases} \frac{1}{k}(u^{n+1} - \tilde{u}^{n+1}) + \alpha\nabla(\phi^{n+1} - \phi^n) = 0, \\ \mathrm{div} u^{n+1} = 0, \\ u^{n+1}\cdot\vec{n}|_\Gamma = 0, \end{cases}$$

where α can be any constant greater than or equal to one.

Once again, the existence and uniqueness of $\tilde{u}^{n+1}$ defined by (5.1) are ensured by the Lax–Milgram theorem, and $u^{n+1} = P_H\tilde{u}^{n+1}$. It is clear that the scheme (5.1)–(5.2) is numerically equally as effective as scheme (1.2)–(1.3). Furthermore, if we make the additional assumption (see Remark 3 below)

$$\text{(A3)} \qquad \int_0^T |\nabla p_t(t)|^2 dt \le M \quad \text{and} \quad \sup_{t\in(0,T]} \|u_{tt}(t)\|_{-1} \le M,$$

we can prove the following theorem.

THEOREM 2. *Let us denote*

$$\tilde{p}^{n+1} = \phi^n + \alpha(\phi^{n+1} - \phi^n),\ p^{n+1} = \tilde{p}^{n+1} - k\alpha\nu\Delta(\phi^{n+1} - \phi^n).$$

Then, under the assumptions of Lemma 1 *and* (A3), $\tilde{u}^{n+1}$ (*in case* $\alpha > 1$) *and* u^{n+1} *from* (5.1)–(5.2) *are strongly first-order approximations to* $u(t_{n+1})$ *in* $L^2(\Omega)^d$ *and weakly first-order approximations to* $u(t_{n+1})$ *in* $H^1(\Omega)^d$. $\tilde{p}^{n+1}$ *as well as* p^{n+1} *are weakly first-order approximations to* $p(t_{n+1})$ *in* $L^2(\Omega)/R$. *More precisely, we have*

$$|e^{N+1}|^2 + (\alpha - 1)|\tilde{e}^{N+1}|^2 + k\nu\sum_{n=0}^{N}(\|\tilde{e}^{n+1}\|^2 + \|e^{n+1}\|^2) \le Mk^2$$

$$\text{(5.3)} \qquad \forall 0 \le N \le T/k - 1;$$

$$k\sum_{n=0}^{N}\{|p^{n+1} - p(t_{n+1})|^2_{L^2(\Omega)/R} + |\tilde{p}^{n+1} - p(t_{n+1})|^2_{L^2(\Omega)/R}\} \le Mk^2$$

$$\text{(5.4)} \qquad \forall 0 \le N \le T/k - 1.$$

Remark 3. The verification of (A3) involves some compatibility conditions of the data which are not generally satisfied (see, for instance, [17], [6]). We make this assumption merely to simplify the presentation. In fact, assuming only (A1)–(A2), it can be proven that (see, for instance, [6])

$$\int_0^T \min(t,1)|\nabla p_t(t)|^2 dt \le M \quad \text{and} \quad \sup_{t\in(0,T]} \min(t,1)\|u_{tt}(t)\|_{-1} \le M.$$

Consequently, without assuming (A3), we can prove the following alternative to Theorem 2:

$$t_{N+1}(|e^{N+1}|^2 + |\tilde{e}^{N+1}|^2) + k\nu\sum_{n=0}^{N} t_{n+1}\{\|\tilde{e}^{n+1}\|^2 + \|e^{n+1}\|^2\} \le Mk^2$$
$$\forall 0 \le N \le T/k - 1;$$

$$k\sum_{n=0}^{N} t_{n+1}\{|p^{n+1} - p(t_{n+1})|^2_{L^2(\Omega)/R} + |\tilde{p}^{n+1} - p(t_{n+1})|^2_{L^2(\Omega)/R}\} \le Mk^2$$
$$\forall 0 \le N \le T/k - 1.$$

Proof of Theorem 2. (i) *Error estimate for the velocity.* The procedure is similar as in the proof of Lemma 2 except for the treatment of the pressure terms. As in §3, we can formulate the following error equations:

$$\frac{1}{k}(\tilde{e}^{n+1}-e^n)-\nu\Delta\tilde{e}^{n+1} = (u^n\cdot\nabla)\tilde{u}^{n+1}-(u(t_{n+1})\cdot\nabla)u(t_{n+1})+R^n+\nabla(\phi^n-p(t_{n+1})), \tag{5.5}$$

$$\frac{1}{k}(e^{n+1}-\tilde{e}^{n+1})=\alpha\nabla(\phi^{n+1}-\phi^n). \tag{5.6}$$

Taking the inner product of (5.5) with $2k\tilde{e}^{n+1}$, repeating the computation in §3 except for the pressure terms, we can derive the following inequality similar to (3.7):

$$\begin{aligned}|\tilde{e}^{n+1}|^2-|e^n|^2+\nu\|\tilde{e}^{n+1}\|^2+\frac{1}{2}|\tilde{e}^{n+1}-e^n|^2 &\le Mk^2\left(\int_{t_n}^{t_{n+1}}\|u_{tt}\|_{-1}^2dt+\int_{t_n}^{t_{n+1}}|u_t|^2dt\right)\\ &+2k(\nabla(\phi^n-p(t_{n+1})),\tilde{e}^{n+1})+Mk|e^n|^2.\end{aligned} \tag{5.7}$$

Taking the inner product of (5.6) with $\frac{2(\alpha-1)k}{\alpha^2}e^{n+1}$, we obtain

$$\frac{\alpha-1}{\alpha}(|e^{n+1}|^2-|\tilde{e}^{n+1}|^2+|e^{n+1}-\tilde{e}^{n+1}|^2)=0. \tag{5.8}$$

Taking the inner product of (5.6) with $\frac{k}{\alpha}(e^{n+1}+\tilde{e}^{n+1})$, because of (2.4), we derive

$$\frac{1}{\alpha}(|e^{n+1}|^2-|\tilde{e}^{n+1}|^2)=k(\nabla(\phi^{n+1}-\phi^n),\tilde{e}^{n+1}). \tag{5.9}$$

Adding (5.7), (5.8), and (5.9), we arrive at

$$\begin{aligned}|e^{n+1}|^2-|e^n|^2+k\nu\|\tilde{e}^{n+1}\|^2+\frac{1}{2}|\tilde{e}^{n+1}-e^n|^2+\frac{\alpha-1}{\alpha}|e^{n+1}-\tilde{e}^{n+1}|^2 &\le Mk^2\left(\int_{t_n}^{t_{n+1}}\|u_{tt}\|_{-1}^2dt+\int_{t_n}^{t_{n+1}}|u_t|^2dt\right)\\ &+Mk|e^n|^2+k(\nabla(\phi^{n+1}+\phi^n-2p(t_{n+1})),\tilde{e}^{n+1}).\end{aligned} \tag{5.10}$$

Now we have to deal with the last term on the right-hand side.

Let us denote now $q^n=p(t_n)-\phi^n$, so we can write

$$\begin{aligned}&k(\nabla(\phi^{n+1}+\phi^n-2p(t_{n+1})),\tilde{e}^{n+1})\\ &\quad=-k(\nabla(q^{n+1}+q^n+p(t_{n+1})-p(n)),\tilde{e}^{n+1}).\end{aligned} \tag{5.11}$$

On the other hand, we derive from (5.6) that

$$\begin{aligned}\tilde{e}^{n+1}&=e^{n+1}-2k\nabla(\phi^{n+1}-\phi^n)\\ &=e^{n+1}+2k\left\{\nabla(q^{n+1}-q^n)-\nabla(p(n+1)-p(n))\right\}.\end{aligned}$$

Putting this into (5.11), using (2.4) and the estimate

$$|\nabla(p(n+1)-p(n))|^2 = \left|\int_{t_n}^{t_{n+1}} \nabla p_t(t)dt\right|^2 \le k\int_{t_n}^{t_{n+1}} |\nabla p_t(t)|^2 dt,$$

we obtain

$$\begin{aligned} &k(\nabla(\phi^{n+1}+\phi^n-2p(n+1)),\tilde{e}^{n+1}) \\ &\quad = 2k^2(|\nabla q^n|^2-|\nabla q^{n+1}|^2)+2k^2|\nabla(p(n+1)-p(n))|^2 \\ &\qquad +4k^2(\nabla q^n,\nabla(p(n+1)-p(n))) \\ &\quad \le 2k^2(|\nabla q^n|^2-|\nabla q^{n+1}|^2) \\ &\qquad +2k^3|\nabla q^n|^2+2(k^2+k^3)\int_{t_n}^{t_{n+1}}|\nabla p_t(t)|^2dt. \end{aligned} \tag{5.12}$$

Now, taking the sum of (5.10) and (5.12) for n from zero to N, because of (A3), we arrive at

$$\begin{aligned} &|e^{N+1}|^2+k^2|\nabla q^{N+1}|^2+\sum_{n=0}^{N}\left\{\frac{\alpha-1}{\alpha}|\tilde{e}^{n+1}-e^{n+1}|^2+\frac{1}{2}|\tilde{e}^{n+1}-e^n|^2+k\nu\|\tilde{e}^{n+1}\|^2\right\} \\ &\qquad \le Mk\sum_{n=0}^{N}\{|e^n|^2+k^2|\nabla q^n|^2\}+k^2|\nabla q^0|^2 \\ &\qquad\quad +Mk^2\int_0^T(\|u_{tt}\|_{-1}^2+|u_t|^2+|\nabla p_t(t)|^2)dt \\ &\qquad \le Mk\sum_{n=0}^{N}\{|e^n|^2+k^2|\nabla q^n|^2\}+Mk^2. \end{aligned}$$

By applying the discrete Gronwall lemma to the last inequality, we derive

$$\begin{aligned} &|e^{N+1}|^2+k\nu\sum_{n=0}^{N}\|\tilde{e}^{n+1}\|^2+\sum_{n=0}^{N}\left\{\frac{\alpha-1}{\alpha}|e^{n+1}-\tilde{e}^{n+1}|^2+\frac{1}{2}|\tilde{e}^{n+1}-e^n|^2\right\} \\ &\qquad \le Mk^2 \quad \forall 0\le N\le T/k-1. \end{aligned} \tag{5.13}$$

In case $\alpha > 1$, thanks to the last inequality and (5.8), we also have

$$|\tilde{e}^{n+1}|^2 = |e^{n+1}|^2+|\tilde{e}^{n+1}-e^{n+1}|^2 \le Mk^2 \quad \forall 0\le N\le T/k-1.$$

The proof of (5.3) is complete thanks to (3.10).

(ii) *Error estimate for the pressure.* Taking the sum of (5.1) and (5.2), we obtain

$$\frac{1}{k}(u^{n+1}-u^n)-\nu\Delta\tilde{u}^{n+1}+(u^n\cdot\nabla)\tilde{u}^{n+1}+\nabla\tilde{p}^{n+1}=f(t_{n+1}). \tag{5.14}$$

Subtracting (5.14) from (3.1), we obtain

(5.15)

$$\begin{cases} \dfrac{1}{k}(e^{n+1}-e^n)-\nu\Delta\tilde{e}^{n+1}+\nabla\tilde{q}^{n+1}=(u^n\cdot\nabla)\tilde{u}^{n+1}-(u(t_{n+1})\cdot\nabla)u(t_{n+1})+R^n, \\ \operatorname{div} e^{n+1}=0, \\ e^{n+1}\cdot\vec{n}|_\Gamma=0, \end{cases}$$

where $\tilde{q}^{n+1} = p(t_{n+1}) - \tilde{p}^{n+1}$.

Let us first prove the following lemma.

LEMMA 3. *Under the assumption of Theorem* 2, *we have*

$$\sum_{n=0}^{N} \|e^{n+1} - e^n\|_{-1}^2 \leq Mk^3 \quad \forall 0 \leq N \leq T/k - 1.$$

Proof. Taking the inner product of (5.15) with $kA^{-1}(e^{n+1} - e^n)$, using (2.1), we have

$$(\nabla \tilde{q}^{n+1}, A^{-1}(e^{n+1} - e^n)) = 0,$$

$$(e^{n+1} - e^n, A^{-1}(e^{n+1} - e^n)) = \|e^{n+1} - e^n\|_{-1}^2, \quad \text{and}$$

$$k(R^n, A^{-1}(e^{n+1} - e^n)) \leq k\|R^n\|_{-1}\|e^{n+1} - e^n\|_{-1} \leq \tfrac{1}{4}\|e^{n+1} - e^n\|_{-1}^2 + k^2\|R^n\|_{-1}^2.$$

As for the nonlinear terms, we again use the splitting of (4.10); using (2.1), (2.5), and (4.11), we can derive

$$\begin{aligned}
&k((u^n \cdot \nabla)\tilde{u}^{n+1} - (u(t_{n+1}) \cdot \nabla)u(t_{n+1}), A^{-1}(e^{n+1} - e^n)) \\
&= kb(u(t_{n+1}) - u(t_n), u(t_{n+1}), A^{-1}(e^{n+1} - e^n)) \\
&\quad + kb(e^n, u(t_{n+1}), A^{-1}(e^{n+1} - e^n)) + kb(u^n, \tilde{e}^{n+1}, A^{-1}(e^{n+1} - e^n)) \\
&\leq k|u(t_{n+1}) - u(t_n)|\|u(t_{n+1})\|\|e^{n+1} - e^n\|_{-1} \\
&\quad + k|e^n|\|u(t_{n+1})\|_2\|e^{n+1} - e^n\|_{-1} + k\|u^n\|\|\tilde{e}^{n+1}\|\|e^{n+1} - e^n\|_{-1} \\
&\leq \tfrac{1}{4}\|e^{n+1} - e^n\|_{-1} + Mk^2\{|u(t_{n+1}) - u(t_n)|^2 + |e^n|^2 + \|\tilde{e}^{n+1}\|^2\}.
\end{aligned}$$

It remains to estimate $-\langle \Delta \tilde{e}^{n+1}, kA^{-1}(e^{n+1} - e^n)\rangle$. As in the last section, we take $u = e^{n+1} - e^n$ in (2.2) and let $\{v, p\}$ be the solution of the corresponding Stokes equations; we have

$$\begin{aligned}
-\langle \Delta \tilde{e}^{n+1}, kA^{-1}(e^{n+1} - e^n)\rangle &= \langle \tilde{e}^{n+1}, -k\Delta A^{-1}(e^{n+1} - e^n)\rangle \\
&= k(\tilde{e}^{n+1}, e^{n+1} - e^n - \nabla p) \\
&= k(e^{n+1}, e^{n+1} - e^n) - k(\tilde{e}^{n+1}, \nabla p) \\
&= \frac{k}{2}\{|e^{n+1}|^2 - |e^n|^2 + |e^{n+1} - e^n|^2\} - k(\tilde{e}^{n+1}, \nabla p).
\end{aligned}$$

Since $|\nabla p| \leq c|e^{n+1} - e^n|$, the last term on the right-hand side can be majorized by

$$k(\tilde{e}^{n+1}, \nabla p) = k(\tilde{e}^{n+1} - e^n, \nabla p) \leq k|\tilde{e}^{n+1} - e^n||\nabla p|$$

$$\le \frac{k}{2}(|\tilde{e}^{n+1} - e^n|^2 + |\nabla p|^2)$$

$$\le Mk(|\tilde{e}^{n+1} - e^n|^2 + |e^{n+1} - e^n|^2).$$

Finally, thanks to (5.13), we can derive from (5.15) and the above inequalities that

$$\sum_{n=0}^{N} \|e^{n+1} - e^n\|_{-1}^2 \le Mk^2 \sum_{n=0}^{N} \{\|R^n\|_{-1}^2 + |u(t_{n+1}) - u(t_n)|^2 + \|\tilde{e}^{n+1}\|^2\}$$
$$+ Mk^2 \sum_{n=0}^{N} \{|\tilde{e}^{n+1} - e^n|^2 + |e^{n+1} - e^n|^2\} \le Mk^3$$
$$\forall 0 \le n \le T/k - 1. \qquad \square$$

We now return to the proof of Theorem 2.

Let us denote $q^{n+1} = p(t_{n+1}) - p^{n+1}$. As in the last section, we have

$$\nabla q_*^{n+1} = \frac{1}{k}(e^{n+1} - e^n) - \nu \Delta e_*^{n+1} + (u(t_{n+1}) \cdot \nabla) u(t_{n+1}) - (u^n \cdot \nabla)\tilde{u}^{n+1} - R^n, \tag{5.16}$$

where $\{e_*^{n+1}, q_*^{n+1}\} = \{e^{n+1}, q^{n+1}\}$ or $\{\tilde{e}^{n+1}, \tilde{q}^{n+1}\}$. Exactly as in the last section, we can derive

$$|q_*^{n+1}|_{L^2(\Omega)/R} \le c \sup_{v \in H_0^1(\Omega)^d} \frac{(\nabla q_*^{n+1}, v)}{\|v\|}$$
$$\le M \left\{ \frac{1}{k} \|e^{n+1} - e^n\|_{-1} + \|R^n\|_{-1} + \|\tilde{e}^{n+1}\| + \|e^{n+1}\| + \|e^n\| \right\}.$$

Hence, by using (5.13), Lemma 3, and (A3), we finally obtain

$$k \sum_{n=0}^{N} |q_*^{n+1}|_{L^2(\Omega)/R}^2 \le Mk \sum_{n=0}^{N} \left\{ \frac{1}{k^2} \|e^{n+1} - e^n\|_{-1}^2 + \|R^n\|_{-1}^2 + \|\tilde{e}^{n+1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 \right\}$$
$$\le Mk^2 \quad \forall 0 \le n \le T/k - 1.$$

The proof of Theorem 2 is complete. $\square$

Remark 4. It is interesting to observe that the choice of α does not affect the precision of the scheme as long as $\alpha \ge 1$. We note that the choice of ϕ^0 only introduces an extra error term of order $O(k|\nabla(p(0) - \phi^0)|)$. Hence, it does not affect the precision of first order schemes.

We immediately observe from (5.2) that once again we have

$$\frac{\partial \phi^{n+1}}{\partial \vec{n}} = \frac{\partial \phi^n}{\partial \vec{n}} = \cdots = \frac{\partial \phi^0}{\partial \vec{n}}.$$

Therefore,

$$\frac{\partial \tilde{p}^{n+1}}{\partial \vec{n}} = \frac{\partial(\phi^{n+1} + \alpha(\phi^{n+1} - \phi^n))}{\partial \vec{n}} = \frac{\partial \phi^0}{\partial \vec{n}},$$

which is certainly not satisfied by the exact pressure. However, as for the classical projection scheme (1.2)–(1.3), we were still able to prove that $\tilde{p}^{n+1}$ was a legitimate approximation to $p(t_{n+1})$.

If we consider a coupled first-order backward Euler scheme for (1.1), we would end up with exactly the same error estimates as stated in Theorem 2. Consequently, we have showed the decoupled projection scheme (5.1)–(5.2) has the same order of accuracy as the coupled conventional scheme.

Acknowledgments. The author thanks R. Temam for his valuable suggestions and encouragement.

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