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Remarks on the pressure error estimates for the projection methods*

Jie Shen

Department of Mathematics, Penn State University, University Park, Penn 16802, USA

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The purpose of this note is to correct some errors in the proofs of pressure error estimates in [1] and [2]. The origin of the errors is the following incorrect inequality[1]

$$c_2 \|u\|_{-1}^2 \le (A^{-1}u, u) \,, \tag{1}$$

(see (2.1) in [1] and (2.7) in [2]), which the author incorrectly derived by identifying $\|u\|_{V'}$ with $\|u\|_{-1}$ for $u \in H$.[2] The correct inequality is:

$$c_2 \|u\|_{V'}^2 \le (A^{-1}u, u) \,. \tag{2}$$

However, the incorrect proofs induced by the error (1) can all be fixed as indicated below. More precisely, the proofs for the velocity error estimates in [1, 2] are all valid provided some minor changes of notations; the pressure error estimates still hold, but their proofs necessitate some additional estimates. In summary, *all the results presented in [1, 2] remain valid* provided some additional regularity assumptions on the exact solution are made. In the following we provide the details for the aforementioned modifications and corrections.

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[1] The incorrectness of (1) was pointed out to me by R. Temam to whom I am grateful. Later on J.L. Guermond has also expressed to me his doubts about (1)

[2] The confusion can be partly explained by a technical sublety. We know that $V \subset H_0^1(\Omega)^d$ and hence $H_0^1(\Omega)^d = V \oplus V^\perp$. By the Riesz theorem and elementary properties of Hilbert spaces, V' is isomorphic to and could be identified with a subspace of $H^{-1}(\Omega)^d$. However this identification must not be made because, as it is usual with evolution equations, we already identified H with a subspace of $V'(V \subset H \subset V')$. Hence this double identification is not allowed

Velocity error estimates

We need to make the following modifications for the proofs of the velocity error estimates in [1] and [2]:

(i) The norm $\|\cdot\|_{-1}$ should be replaced by $\|\cdot\|_{V'}$ in the following places: Lemma 1 in [1] and its proof, the proof of (4.1) in [1], Lemma 3 and its proof in both [1] and [2], Lemma 6 in [2] and its proof.

(ii) In the proof of Lemma 2 in [1], do not use $\int_0^T \|u_{tt}(t)\|_{-1}^2 dt$, which is not necessarily bounded with the assumptions of Lemma 2, but use instead the following crude estimate which is sufficient:

$$k \sum_{k=0}^{T/k-1} \|R^n\|_X^2 \le C \int_0^T t\|u_{tt}(t)\|_X^2 dt \le Mk \ ,$$

where X can be either $L^2(\boldsymbol{\Omega})^d$ or $H^{-1}(\boldsymbol{\Omega})^d$.

Proof of (3). By definition and by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|R^n\|_X^2 &= \frac{1}{k}\int_{t_n}^{t_{n+1}} (t-t_n)u_{tt}(t)dt\|_X^2 \\
&\le \frac{1}{k^2}\int_{t_n}^{t_{n+1}} (t-t_n)\|u_{tt}(t)\|_X^2 dt \int_{t_n}^{t_{n+1}} (t-t_n)dt \\
&\le \frac{1}{2}\int_{t_n}^{t_{n+1}} t\|u_{tt}(t)\|_X^2 dt \ .
\end{aligned}$$

Since $\int_0^T t\|u_{tt}(t)\|_X^2 dt \le M$ with the assumptions of Lemma 2 (see for instance the reference [5] in [1]), we conclude by summing up the last inequalities. □

Pressure error estimates

Let us first deal with the original projection scheme (1.2)–(1.3) in [1]. It is clear that the proof of (4.2) in [1] will become valid if we can prove

$$\sum_{k=0}^{T/k-1} \|e^{n+1}-e^n\|_{-1}^2 \le Mk^2 \ . \tag{4}$$

However due to the aforementioned modifications, we only proved in [1] that

$$\sum_{k=0}^{T/k-1} \|e^{n+1}-e^n\|_{V'}^2 \le Mk^2 \ , \tag{5}$$

and this is not sufficient for obtaining the pressure error estimate (4.2). We now establish a stronger result.

Lemma A1. *In addition to the assumptions of Theorem 1, we assume that*

$$\int_0^T \|p_t(t)\|^2 dt \le M \ . \tag{6}$$

Then we have

$$\sum_{k=0}^{T/k-1} |\tilde{e}^{n+1} - \tilde{e}^n|^2 \leq Mk^2 \ . \tag{7}$$

Remark 1. The above Lemma implies in particular (4) since

$$\|e^{n+1} - e^n\|_{-1} \leq C|e^{n+1} - e^n| \leq C|\tilde{e}^{n+1} - \tilde{e}^n| \ .$$

Hence the proof of (4.2) in [1] becomes complete, provided with the additional assumption (6).

Proof. It is clear that the essential difficulty comes from the linear Stokes operator rather than from the nonlinear term. Thus to simplify the presentation, we shall focus on the linearized (around $u = 0$) Navier-Stokes equations. With the same notations as in [1], we have the following error equations:

$$\frac{\tilde{e}^{n+1} - e^n}{k} - \nu\Delta\tilde{e}^{n+1} + \nabla p(t_{n+1}) = R^n \ , \quad \tilde{e}^{n+1}|_{\partial\Omega} = 0 \ , \tag{8}$$

$$\frac{e^{n+1} - \tilde{e}^{n+1}}{k} - \nabla\phi^{n+1} = 0 \ , \quad \text{div}\, e^{n+1} = 0 \ , \quad e^{n+1} \cdot \mathbf{n}|_{\partial\Omega} = 0 \ . \tag{9}$$

Replacing e^n in (8) by $\tilde{e}^n + k\nabla\phi^n$ [obtained from (9)], we obtain

$$\frac{\tilde{e}^{n+1} - \tilde{e}^n}{k} - \nu\Delta\tilde{e}^{n+1} + \nabla(p(t_{n+1}) - \phi^n) = R^n \ , \quad \tilde{e}^{n+1}|_{\partial\Omega} = 0 \ . \tag{10}$$

We also derive from (9) that

$$\text{div}(\tilde{e}^{n+1} - \tilde{e}^n) + k\Delta(\phi^{n+1} - \phi^n) = 0 \ , \quad \frac{\partial\phi^{n+1}}{\partial\mathbf{n}}\Big|_{\partial\Omega} = 0 \ . \tag{11}$$

Taking the scalar product of (10) with $k(\tilde{e}^{n+1} - \tilde{e}^n)$, we obtain

$$\begin{aligned}
&|\tilde{e}^{n+1} - \tilde{e}^n|^2 + \frac{k\nu}{2}\left\{\|\tilde{e}^{n+1}\|^2 - \|\tilde{e}^n\|^2 + \|\tilde{e}^{n+1} - \tilde{e}^n\|^2\right\} \\
&\quad = k(R^n, \tilde{e}^{n+1} - \tilde{e}^n) + k(p(t_{n+1}) - \phi^n, \text{div}(\tilde{e}^{n+1} - \tilde{e}^n)) \\
&\quad = k(R^n, \tilde{e}^{n+1} - \tilde{e}^n) + k(q^n, \text{div}(\tilde{e}^{n+1} - \tilde{e}^n)) + k(p(t_{n+1}) - p(t_n), \text{div}(\tilde{e}^{n+1} - \tilde{e}^n)) \\
&\quad = I_1 + I_2 + I_3 \ .
\end{aligned} \tag{12}$$

In what follows, we shall use frequently integration by parts and the Cauchy-Schwarz inequality to estimate the three terms above.

$$I_1 \leq \tfrac{1}{8}|\tilde{e}^{n+1} - \tilde{e}^n|^2 + 2k^2|R^n|^2 \ . \tag{13}$$

Using (11), we find

$$
\begin{aligned}
I_2 &= -k^2(q^n, \Delta(\phi^{n+1} - \phi^n)) = k^2(\nabla q^n, \nabla(\phi^{n+1} - \phi^n)) \\
&= -k^2(\nabla q^n, \nabla(q^{n+1} - q^n)) + k^2(\nabla q^n, \nabla(p(t_{n+1}) - p(t_n))) \\
&= -\frac{k^2}{2}\left\{\|q^{n+1}\|^2 - \|q^n\|^2 - \|q^{n+1} - q^n\|^2\right\} + k^2\left(\nabla q^n, \nabla \int_{t_n}^{t_{n+1}} p_t(t)dt\right) \\
(14) \quad &\le -\frac{k^2}{2}\left\{\|q^{n+1}\|^2 - \|q^n\|^2 - \|q^{n+1} - q^n\|^2\right\} + k^3\|\nabla q^n\|^2 + k^2 \int_{t_n}^{t_{n+1}} \|p_t(t)\|^2 t
\end{aligned}
$$

The "bad" term $\frac{k^2}{2}\|q^{n+1} - q^n\|^2$ in the above relation can be controlled as follows: We infer from (11) that

$$
k\Delta(q^{n+1} - q^n) = \operatorname{div}(\tilde{e}^{n+1} - \tilde{e}^n) + k\Delta(p(t_{n+1}) - p(t_n)) \; . \tag{15}
$$

Taking the scalar product of (15) with $-(q^{n+1} - q^n)$, we obtain

$$
\begin{aligned}
k\|q^{n+1} - q^n\|^2 &= (\tilde{e}^{n+1} - \tilde{e}^n, \nabla(q^{n+1} - q^n)) + k(\nabla(p(t_{n+1}) - p(t_n)), \nabla(q^{n+1} - q^n)) \\
&\le \frac{k}{2}\|q^{n+1} - q^n\|^2 + 2k\|p(t_{n+1}) - p(t_n)\|^2 + \frac{2}{3k}|\tilde{e}^{n+1} - \tilde{e}^n|^2 \\
&\le \frac{k}{2}\|q^{n+1} - q^n\|^2 + 2k^2 \int_{t_n}^{t_{n+1}} \|p_t(t)\|^2 dt + \frac{2}{3k}|\tilde{e}^{n+1} - \tilde{e}^n|^2 \; .
\end{aligned}
$$

Therefore

$$
\frac{k^2}{2}\|q^{n+1} - q^n\|^2 \le 2k^3 \int_{t_n}^{t_{n+1}} \|p_t(t)\|^2 dt + \frac{2}{3}|\tilde{e}^{n+1} - \tilde{e}^n|^2 \; . \tag{16}
$$

For the last term I_3 we have

$$
\begin{aligned}
I_3 &= -k(\nabla(p(t_{n+1}) - p(t_n)), \tilde{e}^{n+1} - \tilde{e}^n) = -k\left(\int_{t_n}^{t_{n+1}} \nabla p_t(t)dt, \tilde{e}^{n+1} - \tilde{e}^n\right) \\
(17) \quad &\le \frac{1}{8}|\tilde{e}^{n+1} - \tilde{e}^n|^2 + 2k^3 \int_{t_n}^{t_{n+1}} \|p_t(t)\|^2 dt \; .
\end{aligned}
$$

Combining (12–14, 16 and 17) together, dropping some unnecessary terms, we arrive to:

$$
\begin{aligned}
&|\tilde{e}^{n+1} - \tilde{e}^n|^2 + k\nu\left\{\|\tilde{e}^{n+1}\|^2 - \|\tilde{e}^n\|^2\right\} + k^2\left\{\|q^{n+1}\|^2 - \|q^n\|^2\right\} \\
&\le C\left\{k^3\|q^n\|^2 + k^2 \int_{t_n}^{t_{n+1}} \|p_t(t)\|^2 dt + k^2|R^n|^2\right\} \; .
\end{aligned}
$$

Summing up the last inequalities for n from 0 to $T/k - 1$, using the Gronwall inequality, (3) and the assumption (6), we obtain in particular

$$
\sum_{n=0}^{T/k-1} |\tilde{e}^{n+1} - \tilde{e}^n|^2 \le C\left\{k^2 \int_0^T \|p_t(t)\|^2 dt + k^2 \sum_{n=0}^{T/k-1} |R^n|^2\right\} \le Mk^2 \; . \qquad \square
$$

We shall now deal with the modified scheme (5.1)–(5.2) in [1] and the higher order schemes in [2]. Similar as above, the proofs of (5.4) in [1] and of (3.4) in [2] will become valid if we can prove that

(18) $$\sum_{k=0}^{T/k-1} \|e^{n+1} - e^n\|_{-1}^2 \le Mk^3 \ .$$

However considering the modifications previously indicated, what we proved in Lemma 3 of [1] and [2] was not (18) but instead

(19) $$\sum_{k=0}^{T/k-1} \|e^{n+1} - e^n\|_{V'}^2 \le Mk^3 \ .$$

We shall prove below a stronger result which implies in particular (18), and hence completes the proofs of the pressure estimates for the scheme (5.1)–(5.2) in [1] and for the higher order schemes in [2].

Lemma A2. *In addition to the assumptions for Theorem 2 in [1] and for Theorem 1 in [2], we assume that*

(20) $$u_{tt} \in C([0,T],H); p_t \in C([0,T];H^1(\Omega)), \|\phi^0 - p(0)\| \le Ck \ ,$$

and

(21) $$u_{ttt} \in L^2(0,T;H), p_{tt} \in L^2(0,T;H^1(\Omega)) \ .$$

Then for the schemes (5.1)–(5.2) in [1], (3.1)–(3.2) and (4.1)–(4.2) in [2], we have

(22) $$\left|e^{n+1} - e^n\right| \le Mk^2 \ , \quad \forall\ 0 \le n \le T/k - 1 \ .$$

Proof. It is transparent that the three schemes can be treated by similar arguments. Thus we shall only provide the proof for the scheme (5.1)–(5.2) in [1]. In the absence of the nonlinear term, the error equations read

(23) $$\frac{\tilde{e}^{n+1} - e^n}{k} - \nu\Delta\tilde{e}^{n+1} = R^n + \nabla(\phi^n - p(t_{n+1})), \tilde{e}^{n+1}|_{\partial\Omega} = 0 \ ,$$

(24) $$\frac{e^{n+1} - \tilde{e}^{n+1}}{k} = \alpha\nabla(\phi^{n+1} - \phi^n), \operatorname{div} e^{n+1} = 0 \ , \quad e^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0 \ .$$

Summing up the two relations, we obtain

(25) $$\frac{e^{n+1} - e^n}{k} - \nu\Delta\tilde{e}^{n+1} = R^n + \nabla(\phi^n - p(t_{n+1})) + \alpha\nabla(\phi^{n+1} - \phi^n) \ .$$

On the other hand, taking the divergence of (24), we find

(26) $$\operatorname{div}\tilde{e}^{n+1} = -\alpha k\Delta(\phi^{n+1} - \phi^n) \ , \quad \frac{\partial(\phi^{n+1} - \phi^n)}{\partial\mathbf{n}}\Big|_{\partial\Omega} = 0 \ .$$

Now setting

$$\varepsilon^n = e^n - e^{n-1},\ \tilde{\varepsilon}^n = \tilde{e}^n - \tilde{e}^{n-1},\ r^n = q^n - q^{n-1} \ ,$$

taking the difference of (25) with two consecutive indices, we find

$$\frac{\varepsilon^{n+1}-\varepsilon^n}{k} - \nu\Delta\tilde{\varepsilon}^{n+1} = R^n - R^{n-1} - \nabla r^n - \alpha\nabla(r^{n+1}-r^n) + (\alpha-1)\nabla(p(t_{n+1}) - 2p(t_n) + p(t_{n-1})) \,. \tag{27}$$

Similarly, we derive from (26) that

$$\begin{aligned} \operatorname{div}(\tilde{e}^{n+1} - \tilde{e}^n) &= -\alpha k\Delta(\phi^{n+1} - 2\phi^n + \phi^{n-1}) \\ &= \alpha k\Delta(q^{n+1} - 2q^n + q^{n-1}) - \alpha k\Delta(p(t_{n+1}) - 2p(t_n) + p(t_{n-1})) \,. \end{aligned} \tag{28}$$

Taking the scalar product of (27) with $k\tilde{\varepsilon}^{n+1}$, since $\varepsilon^n = P_H\tilde{\varepsilon}^n$, we obtain

$$\begin{aligned} &\tfrac{1}{2}\left\{|\varepsilon^{n+1}|^2 - |\varepsilon^n|^2 + |\varepsilon^{n+1}-\varepsilon^n|^2\right\} + k\nu\|\tilde{\varepsilon}^{n+1}\|^2 = -k(\nabla r^n, \tilde{\varepsilon}^{n+1}) \\ &\quad - \alpha k(\nabla(r^{n+1}-r^n), \tilde{\varepsilon}^{n+1}) + k(R^n - R^{n-1}, \tilde{\varepsilon}^{n+1}) \\ &\quad + (\alpha-1)k(\nabla(p(t_{n+1}) - 2p(t_n) + p(t_{n-1})), \tilde{\varepsilon}^{n+1}) = J_1 + J_2 + J_3 + J_4 \,. \end{aligned} \tag{29}$$

In what follows, we shall use frequently integration by parts and the Cauchy-Schwarz inequality to estimate the four terms above.

From the Taylor expansion formula with the integral residue, we find

$$p(t_{n+1}) = p(t_n) + kp_t(t_n) + \int_{t_n}^{t_{n+1}} (t_{n+1}-t)p_{tt}(t)dt \,,$$

$$p(t_{n-1}) = p(t_n) - kp_t(t_n) + \int_{t_{n-1}}^{t_n} (t-t_{n-1})p_{tt}(t)dt \,.$$

Therefore

$$\begin{aligned} \|p(t_{n+1}) - 2p(t_n) + p(t_{n-1})\|^2 &= \left\|\int_{t_n}^{t_{n+1}} (t_{n+1}-t)p_{tt}(t)dt - \int_{t_{n-1}}^{t_n} (t-t_{n-1})p_{tt}(t)dt\right\|^2 \\ &\le 2\int_{t_n}^{t_{n+1}} (t_{n+1}-2)^2 dt \int_{t_n}^{t_{n+1}} \|p_{tt}\|^2 dt + 2\int_{t_{n-1}}^{t_n} (t-t_{n-1})^2 dt \int_{t_{n-1}}^{t_n} \|p_{tt}(t)\|^2 dt \\ &= \frac{2}{3}k^3 \int_{t_{n-1}}^{t_{n+1}} \|p_{tt}(t)\|^2 dt \,. \end{aligned} \tag{30}$$

Similarly, expanding the terms in $R^n - R^{n-1}$ at t_n by Taylor formula with the integral residue, we can derive that

$$\begin{aligned} |R^n - R^{n-1}|^2 &= \left|\frac{1}{k}(u(t_{n+1}) - 2u(t_n) + u(t_{n-1})) - (u_t(t_{n+1}) - u_t(t_n))\right|^2 \\ &\le 2k^3 \int_{t_{n-1}}^{t_{n+1}} |u_{ttt}(t)|^2 dt \,. \end{aligned} \tag{31}$$

By using (28) and (30), we find

$$\begin{aligned} J_1 = k(r^n, \operatorname{div}\tilde{\varepsilon}^{n+1}) &= -\alpha k^2(\nabla r^n, \nabla(r^{n+1}-r^n)) \\ &\quad + \alpha k^2(\nabla r^n, \nabla(p(t_{n+1}) - 2p(t_n) + p(t_{n-1}))) \\ &\le -\frac{\alpha k^2}{2}\left\{\|r^{n+1}\|^2 - \|r^n\|^2 - \|r^{n+1} - r^2\|^2\right\} + k^3\|r^n\|^2 \\ &\quad + Ck^4 \int_{t_{n-1}}^{t_{n+1}} \|p_{tt}(t)\|^2 dt \,. \end{aligned} \tag{32}$$

Similarly,

$$\begin{aligned} J_2 &= \alpha k(r^{n+1} - r^n\,,\ \operatorname{div}\tilde{\varepsilon}^{n+1}) = -\alpha^2 k^2 \|r^{n+1} - r^n\|^2 \\ &\quad + \alpha^2 k^2 (\nabla(r^{n+1} - r^n), \nabla(p(t_{n+1}) - 2p(t_n) + p(t_{n-1}))) \\ &\leq -\frac{\alpha^2 k^2}{2}\|r^{n+1} - r^n\|^2 + Ck^5 \int_{t_{n-1}}^{t_{n+1}} \|p_{tt}(t)\|^2 dt\,. \end{aligned} \tag{33}$$

We infer from (31) that

$$\begin{aligned} J_3 &= k(R^n - R^{n-1}, \tilde{\varepsilon}^{n+1}) \leq \frac{k\nu}{4}\|\tilde{\varepsilon}^{n+1}\|^2 + Ck|R^n - R^{n-1}|^2 \\ &\leq \frac{k\nu}{4}\|\tilde{\varepsilon}^{n+1}\|^2 + Ck^4 \int_{t_{n-1}}^{t_{n+1}} |u_{ttt}(t)|^2 dt\,. \end{aligned} \tag{34}$$

We derive from (30) that

$$\begin{aligned} J_4 &= (\alpha - 1)k(\nabla(p(t_{n+1}) - 2p(t_n) + p(t_{n-1})), \tilde{\varepsilon}^{n+1}) \\ &\leq \frac{k\nu}{4}\|\tilde{\varepsilon}^{n+1}\|^2 + Ck^4 \int_{t_{n-1}}^{t_{n+1}} \|p_{tt}(t)\|^2 dt\,. \end{aligned} \tag{35}$$

Combining the inequalities (32–35) into (29) and dropping some unnecessary terms, since $\alpha > 1$, we arrive at

$$\begin{aligned} &|\varepsilon^{n+1}|^2 - |\varepsilon^n|^2 + |\varepsilon^{n+1} - \varepsilon^n|^2 + k\nu\|\tilde{\varepsilon}^{n+1}\|^2 \\ &\quad + k^2\left\{\|r^{n+1}\|^2 - \|r^n\|^2 + \|r^{n+1} - r^n\|^2\right\} \\ &\quad \leq Ck^3\|r^n\|^2 + Ck^4 \int_{t_{n-1}}^{t_{n+1}} \left\{\|p_{tt}(t)\|^2 + |u_{ttt}|^2\right\} dt\,. \end{aligned}$$

Summing up the last inequalities for n from 1 to N, using the discrete Gronwall inequality and the assumption (21), we find

$$\begin{aligned} &|\varepsilon^{N+1}|^2 + k^2\|r^{N+1}\|^2 + \sum_{n=1}^{N}\left\{|\varepsilon^{n+1} - \varepsilon^n|^2 + k\nu\|\tilde{\varepsilon}^{n+1}\|^2 + k^2\|r^{n+1} - r^n\|^2\right\} \\ &\leq |\varepsilon^1|^2 + k^2\|r^1\|^2 + Ck^4\,, \quad \forall\, 1 \leq N \leq T/k - 1\,. \end{aligned} \tag{36}$$

It remains to estimate $|\varepsilon^1|^2$ and $\|r^1\|^2$.

Since $e^0 = 0$, we infer from (23) that

$$\tilde{e}^1 - k\nu\Delta\tilde{e}^1 = kR^1 - k\nabla(q^0 + p(k) - p(0))\,. \tag{37}$$

Taking the scalar product of (37) with $\tilde{e}^1$, thanks to the assumption (20), we derive

$$\begin{aligned} |\tilde{e}^1|^2 + k\nu\|\tilde{e}^1\|^2 &= k(\tilde{e}^1, R^1) - k(\nabla q^0, \tilde{e}^1) - k(\nabla(p(k) - p(0)), \tilde{e}^1) \\ &\leq \tfrac{1}{2}|\tilde{e}^1|^2 + Ck^2\left\{|R^1|^2 + \|q^0\|^2 + \|p(k) - p(0)\|^2\right\} \\ &\leq \tfrac{1}{2}|\tilde{e}^1|^2 + Ck\,. \end{aligned}$$

We derive from the last relation that $|\varepsilon^1|^2 = |e^1|^2 \leq \tilde{e}^1|^2 \leq Ck^4$. On the other hand, we derive from this last result and (24) that

$$\alpha\|r^1\| \le \frac{1}{k}|e^1 - \tilde{e}^1| \le \frac{2}{k}|\tilde{e}^1| \le Ck\ .$$

We have thus proved that $|\varepsilon^1|^2 + k^2\|r^1\|^2 \le Ck^4$. The proof is then complete thanks to the last inequality and (36). □

References

1. Shen J. (1992): On error estimates of the projection method for the Navier-Stokes equations: first order schemes. Siam J. Numer. Anal. **29**, 57–77
2. Shen, J. (1992): On error estimates of some higher oder projection and penalty-projection schemes for the Navier-Stokes equations. Numer. Math. **62**, 49–73