PSEUDO-COMPRESSIBILITY METHODS FOR THE UNSTEADY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We present in this paper a up-to-date review on the error analysis of a class of pseudo-compressibility methods and their time discretizations for the unsteady incompressible Navier-Stokes equations.

1. Introduction. Numerical simulation of the Navier-Stokes equations, being an essential part of the computational fluid dynamics, plays an important role in numerous scientific and industrial applications of current interest, including aeronautical sciences, meteorology, thermo-hydraulics, petroleum reservoir modeling and climatology. Although important advances have been made in recent years on both the hardware (high-performance computing) and software (new algorithms), the numerical simulation of the Navier-Stokes equations remains a challenging problem, especially when the flow governed by the Navier-Stokes equations exhibits complex transient or turbulent behaviors. It is clear that improvements in computational methods will play a major role in advancing our understanding of complex flow phenomena. The purpose of this paper is to provide a up-to-date review on a class of pseudo-compressibility method and their time discretizations for the unsteady incompressible Navier-Stokes equations.

Let $\Omega \in \mathbb{R}^d$ (with $d = 2$ or 3) be an open bounded set with a sufficiently smooth boundary. The Navier-Stokes equations governing the motion of an incompressible viscous fluid inside Ω read:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \ \text{ in } \ Q = \Omega \times (0, T], \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \ \text{ in } \ Q. \tag{1.2}$$

The unknowns are the velocity $\boldsymbol{u}$ and the pressure p. The system (1.1)-(1.2) should be completed with an initial condition for the velocity: $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ and an appropriate boundary condition. For the sake of simplicity, we shall consider only the homogeneous Dirichlet boundary condition for the velocity, i.e. $\boldsymbol{u}|_{\partial\Omega} = 0$, $\forall\, t \in [0, T]$.

One of the main difficulties in a numerical procedure for approximating the solution of the Navier-Stokes equations is introduced by the incompressibility constraint

"$\nabla \cdot \boldsymbol{u} = 0$", which not only couples the velocity $\boldsymbol{u}$ and the pressure p, but also requires that the approximation spaces for the velocity and the pressure satisfy the so called Babuska-Brezzi inf-sup condition. There exists a vast literature on numerical approximations of the Stokes equations and the incompressible Navier-Stokes equations. Excluding those associated with non primitive-variable formulations, the numerical methods for incompressible flows can be classified in three categories according to how the incompressibility constraint is treated:
(i) Using a divergence-free subspace for the velocity approximation (see for instance [**18**]): the pressure is then eliminated from the system, resulting in a well-behaved positive definite discrete system with a significant smaller number of unknowns compared to a coupled formulation. However, the divergence-free subspaces are usually not easy to construct and they involve in general tedious programming.
(ii) Using a pair of compatible pressure and velocity spaces that satisfy the Babuska-Brezzi inf-sup condition (cf. [**12**] and [**5**] for finite element methods and cf. [**7**] and [**3**] for spectral methods): the resulting discrete system is coupled and indefinite. How to efficiently solve the resulting discrete system is still a challenging task.
(iii) Relaxing the incompressibility constraint in an appropriate way: a number of possible approaches are listed below:

$$\nabla \cdot \boldsymbol{u}^\varepsilon + \varepsilon p^\varepsilon = 0 \text{ in } Q; \tag{1.3}$$

$$\nabla \cdot \boldsymbol{u}^\varepsilon + \varepsilon p_t^\varepsilon = 0 \text{ in } Q; \tag{1.4}$$

$$\nabla \cdot \boldsymbol{u}^\varepsilon - \varepsilon \Delta p^\varepsilon = 0 \text{ in } Q, \ \frac{\partial p^\varepsilon}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0; \tag{1.5}$$

$$\nabla \cdot \boldsymbol{u}^\varepsilon - \varepsilon \Delta p_t^\varepsilon = 0 \text{ in } Q, \ \frac{\partial p_t^\varepsilon}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0. \tag{1.6}$$

The first three versions are well known while the last one was recently introduced in [**22**]. We will refer them as a class of pseudo-compressibility methods for the unsteady incompressible Navier-Stokes equations. The aim of this paper is to present some recent analyses on this class of pseudo-compressibility methods as the perturbation parameter $\varepsilon \to 0$. An important aspect of this work is the error analysis of time discretization schemes (they can be employed, in principle, with any consistent space discretization) associated with the pseudo-compressibility methods.

We describe below some of the notations which will be frequently used in this paper. We will use bold face letters to denote vector functions and vector spaces, and standard notations $L^2(\Omega)$, $H^k(\Omega)$ and $H_0^k(\Omega)$ to denote the usual Sobolev spaces. The norm corresponding to $H^k(\Omega)$ will be denoted by $\|\cdot\|_k$. In particular, we will use $\|\cdot\|$ to denote the norm in $L^2(\Omega)$ and $(\cdot,\cdot)$ to denote the scalar product in $L^2(\Omega)$. To simplify the notation, we shall omit the space variables from the notation, i.e. $\boldsymbol{v}(t)$ should be considered as a function of t with value in a Sobolev space. We will use C to denote a generic constant which may depend on the data Ω, ν, $\boldsymbol{f}, \cdots$, but will be *independent* of the perturbation parameter ε.

We now introduce some operators associated with the Navier-Stokes equations and its approximations. We denote $B(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}$. It is an easy matter to verify that

$$(B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{v}) = 0, \quad \forall \boldsymbol{u} \in \boldsymbol{H},\ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{1.7}$$

where

$$\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) :\ \nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\} \tag{1.8}$$

with $\boldsymbol{n}$ to the unit normal vector at $\partial\Omega$. Following Temam [**24**], we introduce the modified bilinear form

$$\tilde{B}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \frac{1}{2}(\nabla\cdot\boldsymbol{u})\boldsymbol{v}$$

to deal with a non divergent-free approximation of the Navier-Stokes equations. Obviously $\tilde{B}(\boldsymbol{u},\boldsymbol{v}) = B(\boldsymbol{u},\boldsymbol{v})$ if $\boldsymbol{v} \in H$ and more importantly, we have

$$(\tilde{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{v}) = 0, \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{1.9}$$

We now recall some classical results for the Navier-Stokes equations.

Proposition 1.1. *(see for instance [**28**] and [**13**]) Given*

$$\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{V}, \quad \boldsymbol{f} \in C([0,T];\boldsymbol{L}^2(\Omega)), \tag{1.10}$$

there exists $T_0 \le T$ ($T_0 = T$ if $d = 2$) such that the variational formulation of (1.1)-(1.2) *admits a unique solution satisfying*

$$\|\boldsymbol{u}(t)\|_2 + \|\boldsymbol{u}_t(t)\| + \|p(t)\|_1 \le C,\ \forall\, t \in [0,T_0]. \tag{1.11}$$

Higher regularity for the solution at $t = 0$ requires that the data Ω, $\boldsymbol{u}_0$ and $\boldsymbol{f}(0)$ satisfy certain non local compatibility conditions (see [**27**] and [**13**]). However, thanks to the smoothing property of the Navier-Stokes equations, the solution becomes as smooth as the data allows for $t > 0$. In particular, we have the following results which are sufficient for our error analysis.

Proposition 1.2. *(see [**27**] and [**13**]) In addition to* (1.10)*, we assume that*

$$\boldsymbol{f}_t \in C([0,T];\boldsymbol{L}^2(\Omega)), \tag{1.12}$$

then

$$t\|\boldsymbol{u}_t(t)\|_1^2 + \int_0^t s^2(\|\boldsymbol{u}_{tt}(s)\|^2 + \|\boldsymbol{u}_t(s)\|_2^2 + \|p_t(s)\|_1^2)ds \le C,\ \forall\, t \in (0,T_0]. \tag{1.13}$$

Furthermore, if we have additionally

$$\boldsymbol{f}_{tt} \in C([0,T];\boldsymbol{L}^2(\Omega)), \tag{1.14}$$

then

$$t^2\|\boldsymbol{u}_t(t)\|_2^2 + \int_0^t s^3(\|\boldsymbol{u}_{ttt}(s)\|^2 + \|\boldsymbol{u}_{tt}(s)\|_2^2 + \|p_{tt}(s)\|_1^2)ds \le C,\ \forall\, t \in (0,T_0]. \tag{1.15}$$

2. Penalty Method. The penalty method, in which the incompressibility constraint is relaxed by using (1.3), was first used for approximating the Navier-Stokes equations in [**24**]. More precisely, we look for $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ such that

$$\boldsymbol{u}_t^\varepsilon - \nu\,\Delta\,\boldsymbol{u}^\varepsilon + \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) + \nabla p^\varepsilon = \boldsymbol{f} \ \text{ in } \ Q, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u}^\varepsilon + \varepsilon p^\varepsilon = 0 \ \text{ in } \ Q, \ \ \boldsymbol{u}^\varepsilon(0) = \boldsymbol{u}_0. \tag{2.2}$$

A distinguished feature of the penalty method is that the pressure approximation p^ε can be eliminated from the above system. In particular, when applied to the (linear, steady) Stokes equations, it results in a positive definite system for the velocity, although severely ill-conditioned when $\varepsilon \ll 1$. The penalty method has been used extensively for the approximation of the Stokes equations (see [**2**], [**16**]). As for the unsteady Navier-Stokes equations, a convergence result was proved in [**24**] and the following error estimate was recently established in [**23**].

Theorem 2.1. *Assuming* (1.10) *and* (1.12), *there exists* $T_1 \le T_0$ *(*$T_1 = T$ *if* $d = 2$*) such that*

$$\sqrt{t}\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\| + t\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\|_1 + \left(\int_0^t s^2\|p(t) - p^\varepsilon(t)\|^2 ds\right)^{\frac{1}{2}} \le C\varepsilon, \ \ \forall 0 < t \le T_1.$$

We note that the smoothing property of (2.1)-(2.2) played an important role in proving the above result.

Let us consider now the time discretization of the penalized system (2.1)-(2.2) by a backward Euler scheme:

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{k} - \nu\Delta\boldsymbol{u}^{n+1} + \nabla p^{n+1} + \tilde{B}(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}) = \boldsymbol{f}(t_{n+1}), \tag{2.3}$$

$$\nabla \cdot \boldsymbol{u}^{n+1} + \varepsilon p^{n+1} = 0, \ \ \text{with } \boldsymbol{u}^0 = \boldsymbol{u}_0, \tag{2.4}$$

where k is the time step size and $t_n = nk$. Since (2.3)-(2.4) is simply a first-order time discretization of the penalized system (2.1)-(2.2), using the triangular inequality

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\| \le \|\boldsymbol{u}(t_n) - \boldsymbol{u}^\varepsilon(t_n)\| + \|\boldsymbol{u}^\varepsilon(t_n) - \boldsymbol{u}^n\|$$

and the estimate in Theorem 2.1, we can prove (see [**23**])

Theorem 2.2. *Under the assumption of Theorem 2.1, we have*

$$\sqrt{t_m}\|\boldsymbol{u}(t_m) - \boldsymbol{u}^m\| + t_m\|\boldsymbol{u}(t_m) - \boldsymbol{u}^m\|_1 + \left(k\sum_{n=1}^{m} t_n^2\|p(t_n) - p^n\|^2\right)^{\frac{1}{2}} \le C(k + \varepsilon),$$

for all $0 < m \le T_1/k$.

In general, when a pth-order ($p > 1$) time discretization scheme is used instead of (2.3)-(2.4), it can be shown (with necessary smoothness assumption) that the right hand side of the estimate in Theorem 2.2 becomes $O(k^p + \varepsilon)$. Hence, we should choose $\varepsilon = O(k^p)$ which results in a severely ill-conditioned linear system.

3. Artificial Compressibility Method. The artificial compressibility method, in which the incompressibility constraint is relaxed by using (1.4), was introduced by Chorin [**8**] and Temam [**26**]. We look for $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ such that

$$\boldsymbol{u}_t^\varepsilon - \nu \Delta \boldsymbol{u}^\varepsilon + \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) + \nabla p^\varepsilon = \boldsymbol{f} \text{ in } Q, \tag{3.1}$$

$$\nabla \cdot \boldsymbol{u}^\varepsilon + \varepsilon p_t^\varepsilon = 0 \text{ in } Q, \quad \boldsymbol{u}^\varepsilon|_{\partial\Omega} = 0. \tag{3.2}$$

A fundamental difficulty associated with this system (compare to the penalized system (2.1)-(2.2)) is the lack of dissipative mechanism for p^ε and consequently the lack of smoothing property at $t = 0$ (see also [**11**]). It was proved in [**26**] (see also Section 3.8 in [**28**]) that $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ in $L^2(0, T; \boldsymbol{H}^1(\Omega))$ and $\nabla p^\varepsilon \to \nabla p$ in $H^{-1}(Q)$. But an error estimate requires further regularity or smoothing property at $t = 0$ which is not available for (3.1)-(3.2). To avoid making non local compatibility assumptions on the data (see [**13**]), we consider the system (3.1)-(3.2) for $t \ge t_0$ (with some $t_0 > 0$) and with the initial conditions $\boldsymbol{u}^\varepsilon(t_0) = \boldsymbol{u}_0^\varepsilon$ and $p^\varepsilon(t_0) = p_0^\varepsilon$. The following result is proved in [**22**].

Theorem 3.1. *We assume* (1.10), (1.12), (1.14) *and*

$$\|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}(t_0)\| \le C\varepsilon, \quad \|p_0^\varepsilon - p(t_0)\| \le C\sqrt{\varepsilon}. \tag{3.3}$$

Then there exists $T_2 \le T_0$ ($T_2 = T$ if $d = 2$) such that

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\| + \left(\int_{t_0}^t \|\boldsymbol{u}(s) - \boldsymbol{u}^\varepsilon(s)\|_1^2 ds \right)^{\frac{1}{2}} + \sqrt{\varepsilon}\|p(t) - p^\varepsilon(t)\| \le C\varepsilon, \ \forall\, t \in [t_0, T_2].$$

We observe from Theorems 2.1 and 3.1 that as an approximation to the Navier-Stokes equations in term of the perturbation parameter ε, the system (3.1)-(3.2) offers no improvement over the penalized system (2.1)-(2.2). However, the results below indicate that time discretizations of (3.1)-(3.2) lead to substantially better conditioned linear systems than those of (2.1)-(2.2). In fact, let us consider for instance a first-order scheme for (3.1)-(3.2):

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{k} - \nu \Delta \boldsymbol{u}^{n+1} + \tilde{B}(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}) + \nabla p^{n+1} = \boldsymbol{f}(t_{n+1}), \tag{3.4}$$

$$\nabla \cdot \boldsymbol{u}^{n+1} + \varepsilon \frac{p^{n+1} - p^n}{k} = 0, \tag{3.5}$$

with the initial conditions: $\boldsymbol{u}^0 = \boldsymbol{u}_0^\varepsilon$, $p^0 = p_0^\varepsilon$. Similar to (2.3)-(2.4), at each time step p^{n+1} can be eliminated from the above system to obtain a linear system for $\boldsymbol{u}^{n+1}$ only. However, the condition number of (3.4)-(3.5) is related to $\frac{k}{\varepsilon}$ and is considerably smaller than the condition number for (2.3)-(2.4) which is related to $\frac{1}{\varepsilon}$. The following error estimates (see [**23**]) hold.

Theorem 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\|\boldsymbol{u}(t_m)-\boldsymbol{u}^m\|+\left(k\sum_{n=0}^{m}\|\boldsymbol{u}(t_n)-\boldsymbol{u}^n\|_1^2\right)^{\frac{1}{2}}+\sqrt{\varepsilon}\|p(t_m)-p^m\|\le C(k+\varepsilon),$$

for all $0<m\le (T_2-t_0)/k$.

It can also be shown that for a pth-order time discretization of (3.1)-(3.2), the right hand side of the estimate in Theorem 3.2 become $C(k^p+\varepsilon)$. Thus, with the same accuracy requirement, time discretizations of the artificial compressibility method always lead to better conditioned linear systems than those of the penalty method.

In addition to the ill-conditioning, another difficulty associated with (2.3)-(2.4) or (3.4)-(3.5) is that different components of the velocity $\boldsymbol{u}^{n+1}$ are coupled together and there is no direct fast method available for solving this coupled system for $\boldsymbol{u}^{n+1}$.

4. Pressure Stabilization Method and First-order Projection Method. The projection method was introduced by Chorin [**9**] and Temam [**25**] to decouple the computation of the pressure and the velocity. A semi-discretized version of the classical projection method can be written as follows:

$$\frac{\tilde{\boldsymbol{u}}^{n+1}-\boldsymbol{u}^n}{k}-\nu\Delta\tilde{\boldsymbol{u}}^{n+1}+B(\boldsymbol{u}^n,\tilde{\boldsymbol{u}}^{n+1})=\boldsymbol{f}(t_{n+1}), \tag{4.1}$$

$$\frac{\boldsymbol{u}^{n+1}-\tilde{\boldsymbol{u}}^{n+1}}{k}+\nabla p^{n+1}=0, \tag{4.2}$$
$$\nabla\cdot\boldsymbol{u}^{n+1}=0,\quad \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}|_{\partial\Omega}=0.$$

The numerical efficiency of the scheme (4.1)-(4.2) is evident, since different components of the velocity $\boldsymbol{u}^{n+1}$ and the pressure p^{n+1} are totally decoupled. In fact, applying the divergence operator to the first equation in (4.2), we find that (4.2) is equivalent to a Poisson equation for p^{n+1} with homogeneous Neumann boundary condition. Furthermore, if we treat the nonlinear term in (4.1) explicitly, then (4.1) reduces to a vector Helmholtz equation for $\boldsymbol{u}^{n+1}$. The convergence of the scheme (4.1)-(4.2) was established in [**25**], while a first rigorous error estimate was provided in [**19**].

The pressure stabilization method, in which the incompressibility constraint is relaxed by using (1.5), was first introduced in [**6**] for the approximation of the Stokes equations. It results in a positive definite (though usually non-symmetric) linear coupled system for almost any type of discretization. In particular, it allows the use of convenient equal-order approximations for the velocity and the pressure (see for instance [**4**] and [**14**]).

It is interesting to observe that $\{\boldsymbol{u}^n\}$ in (4.1)-(4.2) can be eliminated to obtain a system for $(\tilde{\boldsymbol{u}}^n,p^n)$ only. In fact, taking the sum of (4.1) at step n and (4.2) at step $n-1$, and taking the divergence of (4.2), we obtain

$$\frac{\tilde{\boldsymbol{u}}^{n+1}-\tilde{\boldsymbol{u}}^{n}}{k} - \nu\Delta\tilde{\boldsymbol{u}}^{n+1} + B(\boldsymbol{u}^n, \tilde{\boldsymbol{u}}^{n+1}) + \nabla p^n = \boldsymbol{f}^{n+1}, \tag{4.3}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}^{n+1} - k\Delta p^{n+1} = 0, \ \frac{\partial p^{n+1}}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0, \tag{4.4}$$

where P is the orthogonal projector in $\boldsymbol{L}^2(\Omega)$ onto H (see the definition (1.8)). Hence, we can interpret (4.3)-(4.4) as a *decoupled* first-order time discretization for the following pressure stabilization method with $\varepsilon = k$ (see [**17**] and [**20**]):

$$\boldsymbol{u}_t^\varepsilon - \nu\Delta\boldsymbol{u}^\varepsilon + \tilde{B}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) + \nabla p^\varepsilon = \boldsymbol{f} \text{ in } Q, \ \boldsymbol{u}^\varepsilon|_{t=0} = \boldsymbol{u}_0, \tag{4.5}$$

$$\nabla\cdot\boldsymbol{u}^\varepsilon - \varepsilon\,\Delta\, p^\varepsilon = 0 \text{ in } Q, \ \frac{\partial p^\varepsilon}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0. \tag{4.6}$$

Hence, the error behavior of the projection method (4.1)-(4.2) is dictated by that of (4.5)-(4.6). The following results were proved in [**17**] (see also [**20**]).

Theorem 4.1. *Assuming* (1.10) *and* (1.12), *there exists* $T_3 \le T_0$ *(*$T_3 = T$ *if* $d = 2$*) such that*

$$\sqrt{t}\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\|_1 + t\|p(t) - p^\varepsilon(t)\| \le C\sqrt{\varepsilon}, \ \forall\, 0 \le t \le T_3,$$

$$\sqrt{t}\|\boldsymbol{u}(t) - \boldsymbol{u}^\varepsilon(t)\| \le C\varepsilon, \ \forall\, 0 \le t \le T_3.$$

Theorem 4.2. *Let* $(\boldsymbol{u}^n, \tilde{\boldsymbol{u}}^n, p^n)$ *be the solution of* (4.1)-(4.2) *and let* $(\boldsymbol{u}, p)$ *be the solution of* (1.1)-(1.2). *Under the assumption of Theorem 4.1, we have*

$$\sqrt{t_n}(\|\boldsymbol{u}(t_n) - \tilde{\boldsymbol{u}}^n\|_1 + \|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|_1) + \sqrt{t^n}\|p(t_n) - p^n\| \le C\sqrt{k},$$
$$\sqrt{t_n}(\|\boldsymbol{u}(t_n) - \tilde{\boldsymbol{u}}^n\| + \|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|) \le Ck,$$

for all $0 \le n \le T_3/k$.

Unfortunately, any *decoupled* time discretization scheme of (4.5)-(4.6) will be unstable if $\varepsilon = O(k^p)$ with any $p > 1$. Thus, a *decoupled* higher-order scheme for (4.5)-(4.6) will not yield a higher-order accuracy for the approximation of the Navier-Stokes equations due to the perturbation error of order $\varepsilon = O(k)$ in Theorem 4.1.

5. A New Pseudo-compressibility Method and Second-order Projection Method. Let us consider the following second-order projection scheme (see [**15**] and [**1**] for similar schemes):

$$\begin{cases} \dfrac{\tilde{\boldsymbol{u}}^{n+1}-\boldsymbol{u}^{n}}{k} - \dfrac{\nu}{2}\Delta(\tilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n) + \tilde{B}(\dfrac{\tilde{\boldsymbol{u}}^{n+1}+\boldsymbol{u}^{n}}{2}, \dfrac{\tilde{\boldsymbol{u}}^{n+1}+\boldsymbol{u}^{n}}{2}) \\ \qquad\qquad\qquad\qquad + \nabla p^n = \boldsymbol{f}(t_{n+\frac{1}{2}}), \\ (\tilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n)|_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

$$
(5.2)\qquad \begin{cases} \dfrac{\boldsymbol{u}^{n+1}-\tilde{\boldsymbol{u}}^{n+1}}{k}+\dfrac{1}{2}\nabla(p^{n+1}-p^n)=0,\\ \nabla\cdot\boldsymbol{u}^{n+1}=0,\quad \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}|_{\partial\Omega}=0. \end{cases}
$$

As we did for the first-order scheme (4.1)-(4.2), we can eliminate $\{\boldsymbol{u}^n\}$ from (5.1)-(5.2) to form the following system for $(\tilde{\boldsymbol{u}}^n,p^n)$:

$$
(5.3)\qquad \begin{cases} \dfrac{\tilde{\boldsymbol{u}}^{n+1}-\tilde{\boldsymbol{u}}^n}{k}-\dfrac{\nu}{2}\Delta(\tilde{\boldsymbol{u}}^{n+1}+P\tilde{\boldsymbol{u}}^n)+\tilde{B}(\dfrac{\tilde{\boldsymbol{u}}^{n+1}+P\tilde{\boldsymbol{u}}^n}{2},\dfrac{\tilde{\boldsymbol{u}}^{n+1}+P\tilde{\boldsymbol{u}}^n}{2})\\ \qquad\qquad\qquad\qquad\qquad\qquad\dfrac{1}{2}\nabla(3p^n-p^{n-1})=\boldsymbol{f}(t_{n+\frac{1}{2}}),\\ (\tilde{\boldsymbol{u}}^{n+1}+P\tilde{\boldsymbol{u}}^n)|_{\partial\Omega}=0. \end{cases}
$$

$$
(5.4)\qquad \nabla\cdot\tilde{\boldsymbol{u}}^{n+1}-\frac{1}{2}k\Delta(p^{n+1}-p^n)=0,\ \frac{\partial p^{n+1}}{\partial\boldsymbol{n}}|_{\partial\Omega}=\frac{\partial p^n}{\partial\boldsymbol{n}}|_{\partial\Omega}.
$$

The above system can be interpreted in particular as a second-order time discretization of the following new pseudo-compressibility method:

$$
(5.5)\qquad \boldsymbol{u}^\varepsilon_t-\nu\Delta\boldsymbol{u}^\varepsilon+\tilde{B}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}^\varepsilon)+\nabla p^\varepsilon=\boldsymbol{f},
$$

$$
(5.6)\qquad \mathrm{div}\boldsymbol{u}^\varepsilon-\varepsilon\,\Delta\,p^\varepsilon_t=0,\ \frac{\partial p^\varepsilon_t}{\partial\boldsymbol{n}}|_{\partial\Omega}=0,
$$

with $\varepsilon=\frac{1}{2}k^2$ (compare to $\varepsilon=k$ in the previous section).

Similar to the artificial compressibility method (3.1)-(3.2), there is neither dissipative mechanism for p^ε nor smoothing effect at $t=0$. An error estimate which is valid as $\varepsilon\to 0$ is only possible provided with further regularities which involve non local compatibility assumptions on the data. Therefore, we consider (5.5)-(5.6) only for $t\ge t_0$ for some $t_0>0$ and with the initial conditions $\boldsymbol{u}^\varepsilon|_{t=0}=\boldsymbol{u}^\varepsilon_0,\ p^\varepsilon|_{t=0}=p^\varepsilon_0$.

The following result was established in [**22**].

Theorem 5.1. *We assume* (1.10), (1.12), (1.14) *and that*

$$
(5.7)\qquad \|\boldsymbol{u}(t_0)-\boldsymbol{u}^\varepsilon_0\|\le C\varepsilon,\ \ \|p(t_0)-p^\varepsilon_0\|_1\le C\sqrt{\varepsilon}.
$$

Then there exist $T_4\le T_0$ ($T_4=T_0$ *if* $d=2$) *such that*

$$
\|\boldsymbol{u}(t)-\boldsymbol{u}^\varepsilon(t)\|_1+\|p(t)-p^\varepsilon(t)\|\le C\sqrt{\varepsilon},\ \forall\, t\in(t_0,T_4],
$$
$$
\Big(\int_{t_0}^t\|\boldsymbol{u}(s)-\boldsymbol{u}^\varepsilon(s)\|^2ds\Big)^{\frac{1}{2}}\le C\varepsilon,\ \forall\, t\in(t_0,T_4].
$$

We also have the following results for the projection scheme (5.1)-(5.2) (see [**21**], see also [**10**] for a similar result):

Theorem 5.2. *Let* $(\boldsymbol{u}^n, \tilde{\boldsymbol{u}}^n, p^n)$ *be the solution of* (5.1)-(5.2) *and let* $(\boldsymbol{u}, p)$ *be the solution of* (1.1)-(1.2). *Under the assumption of Theorem 5.1, we have*

$$\|\boldsymbol{u}(t_m) - \boldsymbol{u}^m\|_1 + \|\boldsymbol{u}(t_m) - \tilde{\boldsymbol{u}}^m\|_1 + \|p(t_m) - p^m\| \le Ck, \ \forall\, 1 \le m \le \frac{T_4 - t_0}{k},$$
$$\left\{ k \sum_{n=1}^{m} (\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|^2 + \|\boldsymbol{u}(t_n) - \tilde{\boldsymbol{u}}^n\|^2) \right\}^{\frac{1}{2}} \le Ck^2, \ \forall\, 1 \le m \le \frac{T_4 - t_0}{k}.$$

We observe from Theorems 4.1 and 5.1 that as an approximation to the Navier-Stokes equation, (5.5)-(5.6) is no better than the pressure stabilization method (4.5)-(4.6). However, as Theorem 5.2 indicates, the *decoupled* time discretization (5.3)-(5.4) for (5.5)-(5.6) is of second-order accurate for the velocity, while any *decoupled* time discretization of (4.5)-(4.6) is at best of first-order. It is worthwhile to note that a *decoupled* time discretization of (5.5)-(5.6) can be at best of second-order due to the perturbation error $\varepsilon = O(k^2)$, since it will be unstable if $\varepsilon = O(k^p)$ with any $p > 2$.

6. Concluding remarks. Based on the error analysis presented above, we conclude with the following remarks:
(i) The penalty method (2.1)-(2.2) (resp. the pressure stabilization method (4.5)-(4.6)) is a slightly better approximation, in term of the perturbation parameter ε, to the Navier-Stokes equations than the artificial compressibility method (3.1)-(3.2) (resp. the new pseudo-compressibility method (5.5)-(5.6)). However, time discretizations of (3.1)-(3.2) (resp. (5.5)-(5.6)) results in substantially better numerical schemes than those of (2.1)-(2.2) (resp. (4.5)-(4.6)).
(ii) All numerical schemes based on pseudo-compressibility methods decouple the computation of the pressure and the velocity, and they result in positive definite systems at each time step. In addition, the Babuska-Brezzi inf-sup condition between the pressure space and the velocity space is not required. Hence, they are more efficient and more flexible than solving the coupled indefinite systems.
(iii) The projection schemes, which can be interpreted as *decoupled* time discretizations of (4.5)-(4.6) or (5.5)-(5.6), decouple a generalized Stokes system to a vector Helmholtz equation for the velocity and a Poisson equation for the pressure. They are more efficient (but slightly less accurate) than the schemes based on (2.1)-(2.2) or (3.1)-(3.2), especially when fast Poisson solvers are available.

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