NUMERICAL SIMULATION OF INCOMPRESSIBLE FLOWS IN CYLINDRICAL GEOMETRIES USING A SPECTRAL PROJECTION METHOD

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Abstract. An efficient and accurate spectral projection scheme for numerical simulations of incompressible flows in cylindrical geometries is presented and implemented for studying a number of canonical rotating flows.

1. Introduction

We present in this paper an efficient and accurate numerical scheme for numerical simulations of incompressible flows in cylindrical geometries: the time variable is discretized by using a second-order semi-implicit projection method with which only a sequence of Helmholtz/Poisson equations needs to be solved at each time step; the space variables are discretized by using a new spectral-Galerkin method with which Helmholtz/Poisson equations can be solved very efficiently and accurately using a significantly less number of grid points than that required by finite difference or finite element methods. The scheme is implemented to simulate a number of canonical rotating flows which will help us to isolate generic mechanisms which lead to transitions to turbulence in rotating flows and to design corresponding dynamic control mechanisms to either delay transition or limit the intensity of the resultant instabilities. The very efficient spectral-Galerkin method takes advantage of the particular geometry (i.e. a enclosed cylinder or two concentric cylinders), so that the available computational resources can be focused to push the system to large driving forces and to three-dimensions.

Turbulence dominated by strong rotation is still poorly understood, yet rotation plays a large role in many flows that affect our everyday lives, whether it is the threat of a hurricane or tornado bearing down upon us, or the flow in a turbine engine powering the aircraft we're flying in, or the the flow between the hard-disks in the computer on our desk. An understanding of transition to turbulence in rotating systems requires a fundamental knowledge of the vortical waves present on the axis, in the endwall and sidewall boundary layers, and in free shear layers in the interior inclined at arbitrary angles to the rotation axis, and the interactions and couplings between them. The considerations in this study will provide a basic understanding of how in rotating flows, the secondary induced motions interact with the primary rotating flow to produce flows which are non-intuitive from a non-rotating flow point-of-view, but are clearly involved in the transitions leading to turbulence. Three-dimensional flows which are often considered to be inhibited by rotation (e.g. due to Taylor–Proudman theorem) are produced by the interaction between the primary and secondary flow and the details of the geometry bounding and driving the flows and play a crucial role in the transition process.

Traditionally, rotation dominated flows have been treated theoretically by reducing the Navier–Stokes equations or by considering flow idealizations where the PDEs reduce to ODEs through similarity considerations, or by reducing the order of the PDEs by boundary layer or quasi-geostrophic approximations, as well as imposing certain symmetries. In many practical situations, the physically realized flow undergoes transitions whereby the resulting flow no longer obeys the simplifications made to the PDEs. To explore transition in these flows, one must consider the PDEs, and due to their nonlinearity, one is often faced with the only available tools being numerical computation together with laboratory experiments as the means to investigate the flows in a controlled systematic fashion, together with guidance from asymptotics.

A complicating property of rotating flows is that distant boundaries may be surprisingly important (when compared to situations in non-rotating flows). One cannot automatically suppose that, so long as the size of the system is large compared to the length scale of the shearing region, one is dealing with an effectively infinite expanse of fluid. In some circumstances, such an assumption has proven very useful, e.g. the von Kármán solution [25] for the flow due to a rotating disk of infinite radius is relevant in many practical situation (i.e. finite enclosed disk flows). However, when the rotation is sufficiently strong, the self-similar flow loses stability and the details of the finiteness of the disk and its housing become important to the transition process.

The philosophy behind the present investigation of instabilities and transitions in rotation dominated flows is to undertake computations of canonical flows for which laboratory experiments have control over external forcings and the boundary and initial conditions are well defined, and then together with theoretical knowledge of the base flow from analytical, similarity, or asymptotic considerations, establish a physical conceptual framework where hypothesis based on the computational and theoretical results are tested against corresponding experiments.

The rest of the paper is organized as follows. In the next section, we present the governing equations in axisymmetric, velocity-pressure formulation and introduce a second-order semi-implicit projection method for the time variable. Then in Section 3 we describe the spectral-Galerkin method for solving Helmholtz/Poisson equations in cylindrical geometries. Numerical results and discussions for two rotating flows are presented in Section 4. Finally, we end the paper with some concluding remarks.

2. Navier-Stokes equations and a projection scheme

We consider the motion of an incompressible fluid confined in a cylinder or between two concentric cylinders. The equations governing the flow are the Navier-Stokes equations, together with initial and boundary conditions. For such cylindrical geometries, it is convenient to use a cylindrical coordinate system (r, θ, z). We shall first consider the axisymmetric case, and indicate later how our schemes can be easily generalized to the non-axisymmetric case. Due to the azimuthal symmetry, the flow depends spatially on only two cylindrical coordinates (r, z), and the equations governing the axisymmetric flows in the velocity-pressure formulation are

$$u_t - \frac{1}{Re}(\tilde{\nabla}^2 u - \frac{1}{r^2}u) + P_r + uu_r + wu_z - \frac{1}{r}v^2 = 0, \tag{2.1}$$

$$v_t - \frac{1}{Re}(\tilde{\nabla}^2 v - \frac{1}{r^2}v) + uv_r + wv_z + \frac{1}{r}uv = 0, \tag{2.2}$$

$$w_t - \frac{1}{Re}\tilde{\nabla}^2 w + P_z + uw_r + ww_z = 0, \tag{2.3}$$

$$\frac{1}{r}(ru)_r + w_z = 0, \tag{2.4}$$

where

$$\tilde{\nabla}^2 = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 \tag{2.5}$$

is the Laplace operator in axisymmetric cylindrical coordinates. The axisymmetric Navier–Stokes equations (NSE) (2.1–2.4) have been non-dimensionalized with the length scale R, the radius of the cylinder or the difference of the radius of the two cylinders, and the time scale $1/\Omega$, where Ω rad s^{-1} is a characteristic rotation rate of the system. The Reynolds number is $Re = \Omega R^2/\nu$, where ν is the kinematic viscosity. The flow is governed by another non-dimensional parameter, the aspect ratio of the cylinder $\Lambda = H/R$, where H is the height of the cylinder. Therefore, the domain for the space variables (r, z) is the rectangle $(r_i, r_o) \times (0, \Lambda)$. The equations are to be completed with admissible initial and boundary conditions.

Although a finite difference or finite element approximation can be used for the space variables, it appears that a spectral approximation (e.g. [7], [3]) is more appealing in this case because of its ability to resolve thin boundary layers of viscous flows with relatively few collocation points, and because of the simplicity of the computational domain. Hence, we shall use a spectral approximation for the space variables.

In order to solve the time dependent problem (2.1–2.4) efficiently, it is a general practice, especially for spectral approximations, to treat the nonlinear terms explicitly. With this in mind, we still face the difficulty associated with the incompressibility constraint (2.4) which couples the two velocity components u, w and the pressure p. This difficulty can be overcome by using the so-called influence matrix method [23]. However, this approach may become prohibitively expensive for long time and three-dimensional computations. A more efficient way to deal with this coupling is to use a projection (fractional step) method which was originally proposed by Chorin [4] and Temam [22]. In the next section, we will introduce a second-order semi-implicit projection scheme for the axisymmetric NSE. In addition to its remarkable efficiency and accuracy, the scheme has the distinct advantage that it can be easily extended to non-axisymmetric three-dimensional cases. Note that the apparent coordinate singularity (at $r = 0$) is not of an essential nature and can be handled naturally by using an appropriate variational formulation [19]. In short, we shall develop a spectral-projection scheme which consists of a time discretization by a second-order projection scheme and a space discretization by a spectral-Galerkin method.

To simplify the presentation, we introduce the following notations

$$\tilde{\Delta} = \begin{pmatrix} \tilde{\nabla}^2 - 1/r^2, & 0, & 0 \\ 0, & \tilde{\nabla}^2 - 1/r^2, & 0 \\ 0, & 0, & \tilde{\nabla}^2 \end{pmatrix}, \quad \tilde{\nabla} = \begin{pmatrix} \partial_r \\ 0 \\ \partial_z \end{pmatrix}, \tag{2.6}$$

$$\mathcal{D} = \{(r, z) : r \in (r_i, r_o) \text{ and } z \in (0, \Lambda)\}, \tag{2.7}$$

and rewrite the equations (2.1–2.4) in vector form

$$\begin{aligned} &\boldsymbol{u}_t - \frac{1}{Re}\tilde{\Delta}\boldsymbol{u} + \tilde{\nabla}p + \boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{0}, \\ &\tilde{\nabla} \cdot \boldsymbol{u} := \frac{1}{r}(ru)_r + w_z = 0, \end{aligned} \tag{2.8}$$

where $\boldsymbol{u} = (u, v, w)^T$ and $\boldsymbol{N}(\boldsymbol{u})$ is the vector containing the nonlinear terms in (2.1–2.3).

To overcome the difficulties associated with the nonlinearity and the coupling of velocity components and the pressure, we propose to use a projection scheme. Assuming the velocity $\boldsymbol{u}$ is subjected to the boundary condition

$$B(t)\boldsymbol{u}|_{\partial\mathcal{D}} = 0,$$

(where $B(t)$ is a given operator), the semi-implicit second-order projection scheme for the system of equations (2.8) can be rewritten as follows

$$\begin{aligned} &\frac{1}{2\delta t}(3\tilde{\boldsymbol{u}}^{k+1} - 4\boldsymbol{u}^k + \boldsymbol{u}^{k-1}) - \frac{1}{Re}\tilde{\Delta}\tilde{\boldsymbol{u}}^{k+1} = -\tilde{\nabla}p^k - (2\boldsymbol{N}(\boldsymbol{u}^k) - \boldsymbol{N}(\boldsymbol{u}^{k-1})),\\ &B(t^{k+1})\tilde{\boldsymbol{u}}^{k+1}|_{\partial\mathcal{D}} = 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\frac{1}{2\delta t}(\boldsymbol{u}^{k+1} - \tilde{\boldsymbol{u}}^{k+1}) + \tilde{\nabla}(p^{k+1} - p^k) = \boldsymbol{0},\\ &\tilde{\nabla}\cdot\boldsymbol{u}^{k+1} = 0,\\ &(\boldsymbol{u}^{k+1} - \tilde{\boldsymbol{u}}^{k+1})\cdot\boldsymbol{n}|_{\partial\mathcal{D}} = 0, \end{aligned} \tag{2.10}$$

where δt is the time step, $\boldsymbol{n}$ is the outward normal at the boundary, and $\tilde{\boldsymbol{u}}^{k+1} = (\tilde{u}^{k+1}, \tilde{v}^{k+1}, \tilde{w}^{k+1})^T$ and $\boldsymbol{u}^{k+1} = (u^{k+1}, v^{k+1}, w^{k+1})^T$ are respectively the intermediate and final approximations of $\boldsymbol{u}$ at time $t^{k+1} = (k+1)\delta t$.

The scheme is in the same class as the second-order pressure-correction projection scheme of [24] (see also [1]). The linear parabolic operator here is approximated by a second-order backward scheme which appears to be more stable than the Crank-Nicholson scheme, while the nonlinear terms are approximated by a second-order extrapolation to avoid solving a nonlinear system at each time step. It is easy to see that $\tilde{\boldsymbol{u}}^{k+1}$ can be determined from (2.9) by solving three Helmholtz-type equations. Instead of solving for $(\boldsymbol{u}^{k+1}, p^{k+1})$ from the coupled first-order differential equations (2.10), we apply the operator "$\tilde{\nabla}\cdot$" (see the definition in (2.8)) to the first equation in (2.10) to obtain an equivalent system

$$\begin{aligned} &-\tilde{\nabla}^2(p^{k+1} - p^k) = -\frac{1}{2\delta t}\tilde{\nabla}\cdot\tilde{\boldsymbol{u}}^{k+1},\\ &\frac{\partial}{\partial\boldsymbol{n}}(p^{k+1} - p^k)|_{\partial\mathcal{D}} = 0, \end{aligned} \tag{2.11}$$

and

$$\boldsymbol{u}^{k+1} = \tilde{\boldsymbol{u}}^{k+1} - 2\delta t\tilde{\nabla}(p^{k+1} - p^k). \tag{2.12}$$

Thus, $(\boldsymbol{u}^{k+1}, p^{k+1})$ can be obtained by solving an additional Poisson equation (2.11). Note that the equivalence between (2.11–2.12) and (2.10) will be no longer valid once the space variables are discretized. However, numerous numerical experiments and the theoretical justification in [20] indicate that this approach does not affect the second-order accuracy in time for the velocity.

The main advantage of the projection methods is that at each time step one only needs to solve a sequence of Helmholtz/Poisson equations for which fast solution techniques, in particular the spectral-Galerkin method presented below, can be used. Furthermore, since the projection method is based on the velocity-pressure formulation, it is quite obvious that it can be extended with relative ease to three-dimensional cases. The implementation of the projection method for three-dimensional non-axisymmetric flows in cylindrical geometries is current underway and will be reported elsewhere.

3. Weighted spectral-Galerkin method

We first transform the domain $\mathcal{D}$ to the unit square $\mathcal{D}^* = (-1,1)\times(-1,1)$ by using the transformations

$$r = \frac{r_o - r_i}{2}\Big(y + \frac{r_o + r_i}{r_o - r_i}\Big), \quad z = \frac{\Lambda}{2}(x+1). \tag{3.1}$$

Setting

$$R = r_o - r_i, \; c = \frac{r_o + r_i}{r_o - r_i}, \; \alpha = \frac{3R^2 Re}{8\delta t}, \; \beta = \frac{R^2}{\Lambda^2}. \tag{3.2}$$

Then, at each time step, the systems (2.9) and (2.11) lead to the following four Poisson-type equations:

$$\begin{aligned} &\alpha u - \beta u_{xx} - \frac{1}{y+c}((y+c)u_y)_y + \frac{1}{(y+c)^2}u = f \;\text{ in } \mathcal{D}^*,\\ &B_1 u|_{\partial\mathcal{D}*} = 0. \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\alpha v - \beta v_{xx} - \frac{1}{y+c}((y+c)v_y)_y + \frac{1}{(y+c)^2}v = g \;\text{ in } \mathcal{D}^*,\\ &B_2 v|_{\partial\mathcal{D}*} = 0. \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\alpha w - \beta w_{xx} - \frac{1}{y+c}((y+c)w_y)_y = h, \;\text{ in } \mathcal{D}^*,\\ &B_3 w|_{\partial\mathcal{D}*} = 0. \end{aligned} \tag{3.5}$$

$$\begin{aligned} &-\beta p_{xx} - \frac{1}{y+c}((y+c)p_y)_y = q \;\text{ in } \mathcal{D}^*,\\ &\frac{\partial p}{\partial \boldsymbol{n}}|_{\partial\mathcal{D}*} = 0. \end{aligned} \tag{3.6}$$

In the above, B_1, B_2 and B_3 are operators describing the boundary conditions for u, v and w respectively, and f, g, h, q are functions depending on the solutions at the two previous time steps.

In [19], an efficient and accurate spectral-Galerkin method was proposed for solving elliptic equations in polar and cylindrical geometries. It was found that the spectral-Galerkin method in [19] is as good, if not more efficient and accurate, as other spectral methods (see, for instance, [6] and [13]) which take into accounts the parity factor (about $r = 0$) satisfied by the solutions. It should also be noted that the clustering of the collocation points near $r = 0$ in this case will not introduce unreasonable time step restriction as long as the principle linear operator is treated implicitly (cf. [14]).

The spectral-Galerkin method is based on a variational formulation which naturally incorporates the pole conditions and takes care of the coordinate singularity at

$r = 0$. For axisymmetric problems, there are no pole conditions but the coordinate singularity at $r = 0$ is still present. The spectral-Galerkin method of [19] can be directly applied to (3.3–3.6). We shall discuss the method for solving (3.3) in some detail. The three other equations can be treated similarly.

Let P_K be the space of all polynomials of degree less than or equal to K. Assuming that $B_1 u|_{\partial\mathcal{D}*} = 0$ is expressed in the following general form:

$$\begin{aligned} &a_- u(-1,y) + b_- u_x(-1,y) = s_-,\ a_+ u(1,y) + b_+ u_x(1,y) = s_+,\ \ y \in (-1,1), \\ &c_- u(x,-1) + d_- u_y(x,-1) = t_-,\ c_+ u(x,1) + d_+ u_y(x,1) = t_+,\ \ x \in (-1,1), \end{aligned} \tag{3.7}$$

where $a_{\pm 1},\ b_{\pm 1},\ c_{\pm 1},\ d_{\pm 1},\ s_{\pm 1},\ t_{\pm 1}$ are given constants. Since the non-homogeneous case can be easily handled by constructing a simple particular solution satisfying the non-homogeneous boundary conditions (see [17]), we only need to consider the homogeneous case, i.e. $s_{\pm 1} = t_{\pm 1} = 0$.

Let us denote

$$\begin{aligned} X_N &= \{w \in P_N :\ a_- w(-1) + b_- w_x(-1) = 0,\ a_+ w(1) + b_+ w_x(1) = 0\}, \\ Y_M &= \{w \in P_M :\ c_- w(-1) + d_- w_y(-1) = 0,\ c_+ w(1) + d_+ w_y(1) = 0\}, \\ Z_{NM} &= X_N \times Y_M. \end{aligned} \tag{3.8}$$

Then, we look for $u_{NM} \in Z_{NM}$ such that $\forall\, v \in Z_{NM}$,

$$\begin{aligned} &\alpha\big((y+c)^s u_{NM}, v\big)_{\tilde\omega} - \beta\big((y+c)^s \partial_x^2 u_{NM}, v\big)_{\tilde\omega} - \Big((y+c)^{s-1}\big((y+c)\partial_y u_{NM}\big)_y, v\Big)_{\tilde\omega} \\ &\qquad + \big((y+c)^{s-2} u_{NM}, v\big)_{\tilde\omega} = \big((y+c)^s I_{NM} f, v\big)_{\tilde\omega} \end{aligned} \tag{3.9}$$

where $(u,v)_{\tilde\omega} = \int_{\mathcal{D}^*} u\, v\, \omega(x)\, \omega(y)\, dxdy$ with $\omega(z)$ to be respectively 1 or $(1-z^2)^{-\frac{1}{2}}$ depending on whether Legendre or Chebyshev polynomials are used, I_{NM} is a polynomial interpolation operator, based on the Legendre or Chebyshev Gauss-Lobatto points, from $C(D^*)$ to $P_N \times R_M$, and s is chosen such that (3.9) leads to a sparse or simple linear system, namely, we set $s = 1$ if $r_i = 0$ and $s = 2$ if $r_i > 0$.

The above formulation is obtained by multiplying (3.3) by $v(y+c)^s \omega(x)\omega(y)$ and then integrating over $\mathcal{D}^*$. Hence, it can be interpreted as a weighted spectral-Galerkin method for (3.3).

The efficiency of the method depends on the choice of basis function for X_{NM}. The general strategy for choosing basis functions was discussed in [18] and [19]. It is shown in [18] that there exist $\{a_k,\ b_k,\ c_k,\ d_k\}$ such that

$$\begin{aligned} \phi_k(x) &:= p_k(x) + a_k p_{k+1}(x) + b_k p_{k+2}(x) \in X_N \ \text{ for } k = 0, 1, \ldots, N-2, \\ \rho_k(y) &:= p_k(y) + c_k p_{k+1}(y) + d_k p_{k+2}(y) \in Y_M \ \text{ for } k = 0, 1, \ldots, M-2, \end{aligned} \tag{3.10}$$

where $p_k(\cdot)$ is either the k-th degree Legendre or Chebyshev polynomials, Hence, by dimension argument, we have

$$X_N = \text{span}\{\phi_k(x): \ k = 0, 1, \ldots, N-2\},$$
$$Y_M = \text{span}\{\rho_k(y): \ k = 0, 1, \ldots, M-2\}. \tag{3.11}$$

Then, setting

$$\begin{aligned}
a_{ij} &= \int_{-1}^{1} \phi_j(x)\, \phi_i(x)\, \omega(x)\, dx, \\
b_{ij} &= -\int_{-1}^{1} \phi_j''(x)\, \phi_i(x)\, \omega(x)\, dx, \\
c_{ij} &= \int_{-1}^{1} (y+c)^s\, \rho_j(y)\, \rho_i(y)\, \omega(y)\, dy, \\
d_{ij} &= -\int_{-1}^{1} (y+c)^{s-1}((y+c)\, \rho_j'(y))'\, \rho_i(y)\, \omega(y)\, dy, \\
e_{ij} &= \int_{-1}^{1} (y+1)^{s-2}\, \rho_j(y)\, \rho_i(y)\, \omega(y)\, dy, \\
f_{ij} &= \int_{\mathcal{D}} (y+c)^s f\, \phi_i(x) \rho_j(y)\, \omega(x)\, \omega(y)\, dxdy,
\end{aligned} \tag{3.12}$$

and letting A, B, C, D, E, F and U be the corresponding matrices with entries given above, we find that (3.9) is equivalent to the following matrix system:

$$\alpha AUC + \beta BUC + AUD^T + AUE = F. \tag{3.13}$$

Note that in all cases A, C, and E are symmetric, and it is easy to see that they are sparsely banded due to the orthogonality of the Legendre or Chebyshev polynomials. However, the structure of the matrices B and D will depend on a number of factors:

1. For Legendre (i.e. $\omega(z) \equiv 1$) case and $r_i = 0$ (i.e. $s = 1$): B and D are symmetric and banded;
2. For Legendre (i.e. $\omega(z) \equiv 1$) case and $r_i \neq 0$ (i.e. $s = 2$): B is symmetric and banded while D is non-symmetric but banded;
3. For the Chebyshev (i.e. $\omega(z) = (1-z^2)^{-\frac{1}{2}}$) case: B and D are non-symmetric but possess special structures such that the equation $(c_1 B + c_2 D)\boldsymbol{x} = \boldsymbol{f}$ can be efficiently solved (see [19] for more details).

Hence, (3.13) can be efficiently solved by using the matrix diagonalization method (e.g. [11], [19]) at a cost of $4NM \min(N, M) + O(NM)$ operations. Note however that in the Legendre case this operation count can be reduced to $O(NM \log(N+M))$ (see [18] for further details).

Notice that the above spectral-Galerkin method can be easily extended to non-axisymmetric three-dimensional cases (see [19] for further details).

4. Numerical results and discussions

In this section, we apply the above scheme to two typical rotating flows which are of current research interest. The Legendre formulation ($\omega(z) \equiv 1$ in (3.9)) is used for all simulations.

We begin with a flow that tests the spatial and temporal fidelity of the code without the complications associated with discontinuous boundary conditions. The flow is the impulsive spin-down of fluid in a rotating circular cylinder. In this case, $r_i = 0$ so $s = 1$ is used in (3.9). Then we consider a flow where there are discontinuities in the fixed and time-dependent boundary conditions. This flow is a parametrically excited Taylor–Couette flow, where the inner cylinder not only rotates at a constant rate, but also undergoes a harmonic oscillation in the axial direction. In this case, $r_i \neq 0$ so $s = 2$ is used in (3.9).

4.1. **Case 1. Endwall boundary layer flow.** We consider the impulsive spin-down of a fluid in solid body rotation within a circular cylinder, which is governed by the axisymmetric Navier-Stokes equations (2.1–2.4) with homogeneous Dirichlet boundary conditions for all components of the velocity and the initial condition $v|_{t=0} = r$. This flow leads to the rapid development of thin boundary layers whose nature and stability are strongly influenced by the rotation. Similarity solutions for these types of flows in idealized configurations have been known for a long time (see [2]), and their stability has been studied experimentally (see [15]). It is only very recently that further insight into the dynamics of these flows has been available from computations (e.g. [8], [10]). Continued interest in these flows is reflected in a number of recent studies (e.g. [16]). These flows test not only the code's ability to resolve the boundary layer flows, but also the bulk flow as the secondary motions produced following the impulsive spindown are not confined solely to the boundary layer region. As well, they test the temporal accuracy of the code; the group and phase speeds of the various instability waves need to be captured accurately. A fuller account of the flow physics involved is given in [10]; here we compare the new spectral-projection code with the finite difference code used in [10] for this class of flows.

Figure 1*a* shows details of the boundary layer flow (contours of η, the azimuthal component of vorticity) near the top endwall; the left-hand-sides of the plots are the axis and the right-hand sides are the sidewall. The figure shows snap-shots at 2, 4, and 6 radians of time (time is scaled by $1/\Omega$, Ω is the rotation rate in radians per second of the system for $t < 0$), for $Re = \Omega R^2/\mu = 9632$ and aspect ration $\Lambda = 2$. These parameters correspond to an experiment of Savaş [15], which was simulated using finite differences by [10]. Comparing this figure with figure 3 of [10] (reproduced here as figure 1*b*) shows very close agreement between the two totally different methods (and with the experiment, as discussed in [10]). The finite difference code used a stretched grid with 301 nodes in the radial and axial directions (note that Z_2-symmetry was imposed, so that only a half-cylinder computation was performed). The spectral code used 121 Legendre polynomials in the radial direction and 181 in the axial direction, without imposing Z_2-symmetry. Thus, the spectral code requires significantly fewer degrees of freedom to capture the dynamics of the flow. Furthermore, unlike the stream function-vorticity formulation used in [10], the projection method applied to the velocity-pressure formulation can be extended to treat the three-dimensional cases which are necessary for investigating larger Re flows for which experiments (e.g. [15]) show that the boundary layer waves become spirals, breaking symmetry as the flow transitions to turbulence.

4.2. **Case 2. Parametrically forced Taylor–Couette flow.** The second test case consists of a Taylor–Couette flow with real endwall effects and harmonic oscillations of the inner cylinder in the axial direction. There have been numerous studies of the Taylor–Couette flow, including several nonlinear numerical studies. Most do not include real endwall effects, instead they impose periodic boundary conditions and certain symmetries which allow the use of Fourier modes in the axial direction and a significant reduction of the number of modes used (e.g. [5]). Studies that have included real endwall effects (e.g. [21]) have been presented for small aspect ratio systems in order to reduce the degrees of freedom needed in the finite element computations; their techniques become prohibitively expensive for large aspect ratio systems.

The system is governed by a large number of dynamic and geometric parameters. These depend on the radius of the inner and outer cylinders r_i and r_o, the length of the cylinders H, the angular velocity of the inner cylinder Ω (the out cylinder is stationary in the present study), the amplitude and frequency of the harmonic axial motion of the inner cylinder given by $U \sin \tilde{\omega} t$, and the kinematic viscosity of the fluid μ. These are combined to give the governing parameters

$$\Lambda = H/(r_o - r_i), \quad \text{the aspect ratio,}$$

$$e = r_i/r_o, \quad \text{the radius ratio,}$$

$$Re_i = (r_o - r_i) r_i \Omega/\mu, \quad \text{the Couette} - \text{flow Reynolds number,}$$

$$Re_a = (r_o - r_i) U/\mu, \quad \text{the axial Reynolds number,}$$

$$\omega = (r_o - r_i)^2 \tilde{\omega}/\mu, \quad \text{the non} - \text{dimensional forcing frequency.}$$

The flow starts from rest with the boundary conditions:

$$\begin{aligned}
&u(r_i) = 0,\ v(r_i) = Re_i,\ w(r_i) = Re_a \sin\omega t \qquad \text{at } r = r_i; \\
&u(r_o) = 0,\ v(r_o) = Re_o,\ w(r_o) = 0 \qquad \text{at } r = r_o; \\
&u = w = 0,\ v(r) = \frac{Re_o}{r_o} r \qquad \text{at } z = 0,\ \Lambda.
\end{aligned} \tag{4.1}$$

This flow has discontinuous boundary conditions for v and w, the azimuthal and axial components of velocity, at the corners where the inner cylinder meets the stationary endwalls. Since spectral methods are very sensitive to the smoothness of the solutions, it is crucial to design a sensible treatment for the singular boundary conditions. We emphasize that the singular boundary conditions are usually a mathematical idealization of the physical situation. The singularity can never be realized in experiments nor in numerical computations. Therefore, it is appropriate to use a regularized boundary boundary layer function to approximate the actual physical situation. For example, the singular boundary condition

$$v(z) = 1 \text{ for } z \in [-1, 1), \qquad v(z) = 0 \text{ for } z = 1$$

can be approximated by

$$v_\varepsilon(z) = 1 - \exp(-\frac{1-z}{\varepsilon})$$

to within any prescribed accuracy by choosing an appropriate ε. Such an approach has been proven successful in [9], where the vortex breakdown flow has been computed to capture the details of the axisymmetric waves/recirculation zones on the axis with high fidelity. Here we apply the same Regularization technique.

Experiments in a very large aspect ratio system are detailed in [26], and [12] give a detailed linear stability analysis of the system. These studies have shown that parametric excitation through the harmonic axial motion of the inner cylinder, the onset of centrifugal instabilities (Taylor cells) can be controlled. The stabilization is due to waves of azimuthal vorticity (shear waves) propagating radially outwards from the boundary layer on the inner cylinder, and when endwall effects are included, there are also waves propagating inwards from the outer cylinder. These waves act to nullify the azimuthal vorticity associated with the centrifugal instability of the circular Couette flow. There are still several open questions regarding this dramatic stabilization of the flow well beyond its critical state. Nonlinear aspects beyond criticality, the effects of finite aspect ratios and no-slip endwalls (which break the often imposed axial periodicity and translation and reflection symmetries), and nonlinear modal interactions between distinct modes with different axial and azimuthal wave numbers, are some of the issues that have not been addressed. Some aspects of these can be addressed with the present axisymmetric code that includes real endwall effects, but most require the full three-dimensional code. These questions are currently under investigation and will be reported elsewhere. Here, we give the first nonlinear results for the parametrically excited Taylor–Couette flow, with the restriction to axisymmetric flow.

Figure 2 shows contours of the stream-function ψ, azimuthal vorticity η, and angular momentum $\Gamma = rv$ of the steady Taylor flow in the absence of harmonic axial motion for $\Lambda = 10$, $e = 0.905$, $Re_i = 200$, and $Re_a = 0$. The system is fully resolved with $n = 64$, $m = 32$, and $\delta t = 0.001$; note that the large aspect ratio does not require a comparatively large number of modes. The objective is to select the amplitude and frequency of the parametric excitation in order to obtain a solution that is z-independent over a large portion of the flow domain. The sequences in figure 3 show snap-shots of ψ, η, and Γ over approximately one period $(2\pi/\omega)$ of the nonlinear solution when the system shown in figure 2 is subjected to various combinations of Re_a and ω. Over the range of parameters considered, the resultant flow is synchronous with the forcing frequency.

The linear Floquet analysis in [12] shows that for $Re_a \approx 70$ and $\omega \approx 30$, the critical value of Re_i for the onset of centrifugal instability is approximately 200, i.e. for $Re_i = 200$ and Re_a less than about 70 and/or ω greater than about 30, the flow will have axial variations associated with the centrifugal instability (Taylor vortex-like flow). It also shows that the wavelength of these variations is roughly 20% larger than at onset when $Re_a = 0$. The results shown in figure 3 are all consistent with these linear theory results. For the case with $Re_a = 50$, $\omega = 20$, a wave of azimuthal vorticity is seen to propagate from the inner to outer cylinder; also for this case, the Taylor vortex cells only exist during a part of the period, also as suggested by the linear analysis. Even when the centrifugal instability has been quenched over most of the flow domain, there remain cells located near the endwalls. The linear analysis does not capture these effects and they warrant further investigation.

5. Concluding remarks

We present in this paper an efficient and accurate spectral projection scheme for the numerical simulations of incompressible flows in cylindrical geometries. The scheme is second-order accurate in time and spectrally accurate (i.e. convergence rate increases as the smoothness of the solution increases) in space. Furthermore, the time step size of this semi-implicit scheme is only limited by the physical parameters but not restricted by the spatial resolution, i.e. the scheme is unconditionally stable in the sense of numerical analysis.

Preliminary numerical results reported here for two canonical rotating axisymmetric flows demonstrate that the scheme is well suited for the studying of flows with complex temporal and spatial structures, even with singular and time-dependent boundary conditions. In the Taylor–Couette problem with axial motion of the inner cylinder, we have reproduced numerically the precise control of the centrifugal instability in a highly nonlinear system that has been previously demonstrated experimentally. With the level of accuracy and efficiency in the code for dealing with systems of spatio-temporal complexity, we are now in the position to numerically design and test nonlinear control mechanisms and to study the intricate nonlinear dynamics involved. On the other hand, the spatio-temporal complexity in less restricted regimes of parameter space will eventually generate three-dimensional flows. This crucial step for the study of the transition to turbulence and its control is currently being pursued by extending the present spectral-projection scheme from axisymmetric to three dimensions.

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