SIAM J. NUMER. ANAL.
Vol. 47, No. 1, pp. 271–292


# A LAGUERRE–LEGENDRE SPECTRAL METHOD FOR THE STOKES PROBLEM IN A SEMI-INFINITE CHANNEL*

MEJDI AZAIEZ†, JIE SHEN‡, CHUANJU XU§, AND QINGQU ZHUANG§

**Abstract.** A mixed spectral method is proposed and analyzed for the Stokes problem in a semi-infinite channel. The method is based on a generalized Galerkin approximation with Laguerre functions in the $x$ direction and Legendre polynomials in the $y$ direction. The well-posedness of this method is established by deriving a lower bound on the inf-sup constant. Numerical results indicate that the derived lower bound is sharp. Rigorous error analysis is also carried out.

**Key words.** Laguerre functions, Legendre polynomials, Stokes equations, inf-sup condition, error analysis, spectral method

**AMS subject classifications.** Primary, 65N35; Secondary, 74S25, 76D07

**DOI.** 10.1137/070698269

## 1. Introduction.

The Stokes problem plays an important role in fluid dynamics and in solid mechanics, and its numerical approximation has attracted much attention during the last three decades (see, for instance, [10, 5, 4] and the references therein). Most of these investigations have been concentrated on problems in bounded domains.

There is, however, a need to consider numerical approximations to the Stokes problem in unbounded domains. In particular, the flow in a channel and flow pass a cylinder/sphere have important theoretical and practical applications. In most previous investigations for these problems, an artificial boundary is introduced, and an approximate boundary condition at the artificial boundary has to be used. The accuracy of these methods usually depends on how far downstream the artificial boundary is (cf. [17]). Therefore, even when high-order or spectral methods are applied for these problems, one can not achieve high-order or spectral accuracy for the original problem due to the approximation made to the "unknown" outflow boundary conditions.

We shall take a different approach in this paper. More precisely, we shall consider the problem directly in the unbounded domain without introducing an artificial boundary. In fact, many other problems in science and engineering are also set in unbounded domains, and there have been some investigations in using Laguerre polynomials/functions to approximate PDEs on semi-infinite intervals (see, among others, [8, 12, 18, 11, 15]). However, all these works are concerned with Poisson-type elliptic equations. To the best of our knowledge, no result is available for spectral methods to the Stokes problem in semi-infinite channels. Thus, results in this paper are the first of its kind and will play an important role for the numerical approximation of Stokes and Navier–Stokes equations in unbounded domains.

*Received by the editors July 25, 2007; accepted for publication (in revised form) May 27, 2008; published electronically November 21, 2008.
http://www.siam.org/journals/sinum/47-1/69826.html

†Laboratoire TREFLE (UMR CNRS 8508), ENSCPB, 33607 Pessac, France (azaiez@enscpb.fr).
‡Corresponding author. Department of Mathematics, Purdue University, West Lafayette, IN, 47907 (shen@math.purdue.edu). The work of this author was partially supported by NFS grant DMS-0610646.
§School of Mathematical Sciences, Xiamen University, 361005 Xiamen, China (cjxu@xmu.edu.cn, xmuyfd129@163.com). The research of the third author was partially supported by the NSF of China under grant 10531080, the Excellent Young Teachers Program by the Ministry of Education of China, and the 973 High Performance Scientific Computation Research Program. Part of this work was done when the third author was at the Université de Bordeaux-I as an invited professor.

More precisely, we consider the Stokes equations in a semi-infinite channel and introduce a mixed formulation based on Laguerre functions in the $x$ direction and Legendre polynomials in the $y$ direction. It is worthwhile to emphasize that we use Laguerre functions instead of Laguerre polynomials because the latter behaves wildly at infinity and is not suitable for approximation to flows which are well-behaved at infinity (cf. [18]). The well-posedness of this mixed formulation relies on the verification of the so-called inf-sup condition (cf. [1, 6]). The main contribution of this paper is the derivation of a lower bound on the inf-sup constant. We shall also present numerical results which indicate that the derived lower bound is sharp.

The rest of the paper is organized as follows. In the next section, we introduce some notations, derive some useful inverse inequalities for Laguerre functions and Legendre polynomials, and present the mixed Laguerre–Legendre formulation for the Stokes problem. Section 3 is devoted to deriving a lower bound for the inf-sup constant. In section 4, we carry out a complete error analysis for the mixed Laguerre–Legendre approximation. Finally, we present some implementation details and numerical results in section 5.

## 2. Mixed Laguerre–Legendre approximation.

We start by introducing some notations. Let $R^+ = (0, +\infty)$, $\Lambda = (-1, 1)$, $\Omega = R^+ \times \Lambda$, and $\Gamma = \partial\Omega$. Let $\omega > 0$ be a weight function on $\Omega$; we denote by $(u, v)_{\Omega,\omega} := \int_\Omega uv\omega d\Omega$ the inner product of $L^2_\omega(\Omega)$, whose norm is denoted by $\|\cdot\|_{\omega,\Omega}$. We use $H^m_\omega(\Omega)$ and $H^m_{0,\omega}(\Omega)$ to denote the usual weighted Sobolev spaces, with norm $\|\cdot\|_{m,\omega,\Omega}$. In cases where no confusion would arise, $\omega$ (if $\omega \equiv 1$) and $\Omega$ may be dropped from the notations. Let $M$ and $N$ be the discretization parameters in $x$ and in $y$. We denote by $c$ a generic positive constant independent of the discretization parameters, and we use the expression $A \lesssim B$ to mean that $A \le cB$. Throughout this paper we will use boldface letters to denote vectors and vector functions for ease of reading.

Let $\mathcal{L}_k(x)$ be the Laguerre polynomial of degree $k$; we denote the Laguerre function by

$$\hat{\mathcal{L}}_i(x) = \mathcal{L}_i(x)e^{-x/2}$$

and set

$$\mathbb{P}_M = \text{span}\{\mathcal{L}_i(x), i = 0, 1, \dots, M\}$$

and

$$\hat{\mathbb{P}}_M = \text{span}\{\hat{\mathcal{L}}_i(x), i = 0, 1, \dots, M\}.$$

We now recall some definitions and related results which will be used in what follows.

Let $\omega(x) = e^{-x}$, and let

$$L^2_\omega(R^+) := \left\{v; ve^{-x/2} \in L^2\left(R^+\right)\right\} = \left\{v; \int_0^\infty v^2\omega dx < \infty\right\}.$$

The space $L^2_\omega(R^+)$ is endowed with the norm $\|\cdot\|_{0,\omega,R^+}$ (also denoted by $\|\cdot\|_{0,\omega}$ or $\|\cdot\|_\omega$ when there is no confusion), defined by

$$\|v\|_{0,\omega,R^+} = \left(\int_0^\infty v^2\omega dx\right)^{1/2}.$$

Let $\pi_M^x$ be the $L^2_\omega$-orthogonal projector from $L^2_\omega(R^+)$ into $\mathbb{P}_M(R^+)$ defined by

$$\int_0^\infty (v-\pi_M^x v)\phi_M \omega dx = 0,\ \forall v \in L^2_\omega\left(R^+\right),\ \phi_M \in \mathbb{P}_M\left(R^+\right).$$

The projector $\pi_M^x$ can be characterized by the following expression:

$$\pi_M^x v(x) = \sum_{m=0}^{M} \alpha_m \mathcal{L}_m(x)\ \forall v(x) = \sum_{m=0}^{\infty} \alpha_m \mathcal{L}_m(x). \tag{2.1}$$

We define the operator $\hat{\pi}_M^x$ from $L^2(R^+)$ into $\hat{\mathbb{P}}_M(R^+)$ by (cf. [18])

$$\hat{\pi}_M^x v(x) = e^{-x/2}\pi_M^x(v(x)e^{x/2})\ \forall v \in L^2\left(R^+\right).$$

It can be easily verified that

$$\begin{aligned}\int_0^\infty (\hat{\pi}_M^x v - v)\phi_M dx &= \int_0^\infty \left(\pi_M^x\left(v(x)e^{x/2}\right) - v(x)e^{x/2}\right)e^{-x/2}\phi_M dx \\ &= 0\ \forall \phi_M \in \hat{\mathbb{P}}_M\left(R^+\right).\end{aligned} \tag{2.2}$$

Consequently, $\hat{\pi}_M^x$ is the orthogonal projector from $L^2(R^+)$ into $\hat{\mathbb{P}}_M(R^+)$.

We now present several useful results. We start with an inverse inequality for Laguerre functions.

Lemma 2.1. *For all $\phi_M \in \hat{\mathbb{P}}_M(R^+)$, we have*

$$\|\partial_x \phi_M\|_{0,R^+} \lesssim M\|\phi_M\|_{0,R^+}.$$

*Proof.* Let $\phi_M(x) = \sum_{k=0}^M \tilde{\phi}_k \hat{\mathcal{L}}_k(x)$. Then, $\|\phi_M\|^2_{0,R^+} = \sum_{k=0}^M \tilde{\phi}_k^2$ and

$$\partial_x \phi_M(x) = \sum_{k=0}^M \tilde{\phi}_k \hat{\mathcal{L}}_k'(x) = \sum_{k=0}^M \tilde{\phi}_k \left(\mathcal{L}_k'(x) - \frac{1}{2}\mathcal{L}_k(x)\right)e^{-\frac{x}{2}}.$$

Hence, the desired result is a direct consequence of the above and the inverse inequality for Laguerre polynomials (cf. [4]). ∎

We now denote by $\mathbb{P}_N(\Lambda)$ the space of polynomials of degree less than or equal to $N$ in $\Lambda$, and let $\pi_N^y$ be the standard $L^2$-orthogonal projector from $L^2(\Lambda)$ into $\mathbb{P}_N(\Lambda)$.

Lemma 2.2. *For all $\phi_N \in \mathbb{P}_N(\Lambda) \cap H_0^1(\Lambda)$, we have*

$$\|\phi_N\|_{0,\Lambda} \le N^{1/2}\|\pi_{N-2}^y \phi_N\|_{0,\Lambda}.$$

*Remark* 2.1. A proof of the above result, with a constant in front of $N^{1/2}$, can be found in [3]. In fact, a more precise computation as in [3] shows that the constant can be bounded by one.

A similar result with respect to $\hat{\mathbb{P}}_M(R^+) \cap H_0^1(R^+)$ is as follows.

Lemma 2.3. *For all $\phi_M \in \hat{\mathbb{P}}_M(R^+) \cap H_0^1(R^+)$, we have*

$$\|\phi_M\|_{0,R^+} \le (M+1)^{1/2}\|\hat{\pi}_{M-1}^x \phi_M\|_{0,R^+}.$$

*Proof.* Writing $\phi_M$ in the form

$$\phi_M(x) = \sum_{m=0}^M \alpha_m \mathcal{L}_m(x) e^{-x/2},$$

we derive from (2.1) that

$$\hat{\pi}^x_{M-1}\phi_M(x) = e^{-x/2}\pi^x_{M-1}\left(\sum_{m=0}^{M}\alpha_m\mathcal{L}_m(x)\right) = e^{-x/2}\sum_{m=0}^{M-1}\alpha_m\mathcal{L}_m(x).$$

Hence,

$$\phi_M(x) = \hat{\pi}^x_{M-1}\phi_M(x) + \alpha_M\mathcal{L}_M(x)e^{-x/2},$$

and by using the orthogonality relation,

$$\|\phi_M\|^2_{0,R^+} = \|\hat{\pi}^x_{M-1}\phi_M(x)\|^2_{0,R^+} + \alpha_M^2\int_0^\infty \left(\mathcal{L}_M(x)e^{-x/2}\right)^2 dx. \tag{2.3}$$

Note that $\phi_M(0) = 0$ implies that

$$\alpha_0 + \alpha_1 + \cdots + \alpha_M = 0,$$

from which

$$|\alpha_M| = |\alpha_0 + \cdots + \alpha_{M-1}| \le \left[\sum_{m=0}^{M-1}\alpha_m^2\right]^{1/2} M^{1/2}. \tag{2.4}$$

Combining (2.3) and (2.4) gives

$$\|\phi_M\|^2_{0,R^+} \le \|\hat{\pi}^x_{M-1}\phi_M(x)\|^2_{0,R^+} + M\sum_{m=0}^{M-1}\alpha_m^2 = (M+1)\|\hat{\pi}^x_{M-1}\phi_M(x)\|^2_{0,R^+}. \qquad \square$$

Now we consider the mixed Laguerre–Legendre approximation. Let $\mathbb{P}_{M,N}(\Omega)$ be the space of all polynomials in $\Omega$ of degree $\le M$ in the $x$ direction and $\le N$ in the $y$ direction, i.e.,

$$\mathbb{P}_{M,N}(\Omega) := \text{span}\{\mathcal{L}_i(x)L_j(y), i = 0, 1, \ldots, M; j = 0, 1, \ldots, N\},$$

where $\mathcal{L}_i(x)$ and $L_j(y)$ are, respectively, Laguerre and Legendre polynomials of degree $i$ and $j$, satisfying

$$\int_{-1}^{1}\int_0^\infty \mathcal{L}_i(x)L_j(y)\mathcal{L}_m(x)L_n(y)e^{-x}dxdy = \frac{2}{2n+1}\delta_{im}\delta_{jn}.$$

We also define

$$\hat{\mathbb{P}}_{M,N}(\Omega) := \text{span}\left\{\hat{\mathcal{L}}_i(x)L_j(y), i = 0, 1, \ldots, M; j = 0, 1, \ldots, N\right\}.$$

Let us denote by $\mathcal{N}$ the pair of parameters $(M, N)$ and set

$$X_{\mathcal{N}} = H_0^1(\Omega)^2 \cap \hat{\mathbb{P}}_{M,N}(\Omega)^2, \quad M_{\mathcal{N}} = L^2(\Omega) \cap \hat{\mathbb{P}}_{M-1,N-2}(\Omega).$$

LEMMA 2.4. *For all $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\|\psi\|^2_{2,\Omega} \lesssim \left\|\frac{\partial^2\psi}{\partial y^2}\right\|^2_{0,\Omega} + \left\|\frac{\partial^2\psi}{\partial x^2}\right\|^2_{0,\Omega}. \tag{2.5}$$

*Proof.* For all $\psi \in H^2(\Omega)$, with $\psi(x, \pm 1) = 0 \ \forall x \in (0, +\infty)$, we have

$$|\psi(x,y)|^2 = \left| \int_{-1}^{y} \frac{\partial \psi(x,s)}{\partial s} ds \right|^2 \le \int_{-1}^{y} \left( \frac{\partial \psi(x,s)}{\partial s} \right)^2 ds \int_{-1}^{y} ds$$
$$\le (y+1) \int_{-1}^{1} \left( \frac{\partial \psi(x,s)}{\partial s} \right)^2 ds.$$

Therefore,

$$\int_{-1}^{1} \psi^2(x,y) dy \le \int_{-1}^{1} (y+1) dy \int_{-1}^{1} \left( \frac{\partial \psi(x,s)}{\partial s} \right)^2 ds = 2 \int_{-1}^{1} \left( \frac{\partial \psi(x,y)}{\partial y} \right)^2 dy. \tag{2.6}$$

As a consequence,

$$\|\psi\|_{0,\Omega}^2 = \int_0^\infty \int_{-1}^{1} \psi^2(x,y) dy dx \le 2 \int_0^\infty \int_{-1}^{1} \left( \frac{\partial \psi(x,y)}{\partial y} \right)^2 dy dx = 2 \left\| \frac{\partial \psi}{\partial y} \right\|_{0,\Omega}^2. \tag{2.7}$$

Using (2.7), we find

$$\left\| \frac{\partial \psi}{\partial y} \right\|_{0,\Omega}^2 = \int_0^\infty \int_{-1}^{1} \left( \frac{\partial \psi}{\partial y} \right)^2 dy dx = - \int_0^\infty \int_{-1}^{1} \frac{\partial^2 \psi}{\partial y^2} \psi dy dx$$
$$\le \left\| \frac{\partial^2 \psi}{\partial y^2} \right\|_{0,\Omega} \|\psi\|_{0,\Omega} \le \sqrt{2} \left\| \frac{\partial^2 \psi}{\partial y^2} \right\|_{0,\Omega} \left\| \frac{\partial \psi}{\partial y} \right\|_{0,\Omega},$$

from which we derive

$$\left\| \frac{\partial \psi}{\partial y} \right\|_{0,\Omega}^2 \le 2 \left\| \frac{\partial^2 \psi}{\partial y^2} \right\|_{0,\Omega}^2.$$

We then derive from (2.7) and the above that

$$\|\psi\|_{0,\Omega}^2 \le 4 \left\| \frac{\partial^2 \psi}{\partial y^2} \right\|_{0,\Omega}^2. \tag{2.8}$$

On the other hand, applying (2.7) to $\frac{\partial \psi}{\partial x}$ with $\psi \in H_0^1(\Omega)$, we obtain

$$\left\| \frac{\partial \psi}{\partial x} \right\|_{0,\Omega}^2 \le 2 \left\| \frac{\partial^2 \psi}{\partial y \partial x} \right\|_{0,\Omega}^2. \tag{2.9}$$

Furthermore, we have, for all $\psi \in H_0^1(\Omega)$,

$$2 \left\| \frac{\partial^2 \psi}{\partial y \partial x} \right\|_{0,\Omega}^2 = 2 \int_{-1}^{1} \int_0^\infty \left( \frac{\partial^2 \psi}{\partial x^2} \right) \left( \frac{\partial^2 \psi}{\partial y^2} \right) dx dy \le \left\| \frac{\partial^2 \psi}{\partial x^2} \right\|_{0,\Omega}^2 + \left\| \frac{\partial^2 \psi}{\partial y^2} \right\|_{0,\Omega}^2. \tag{2.10}$$

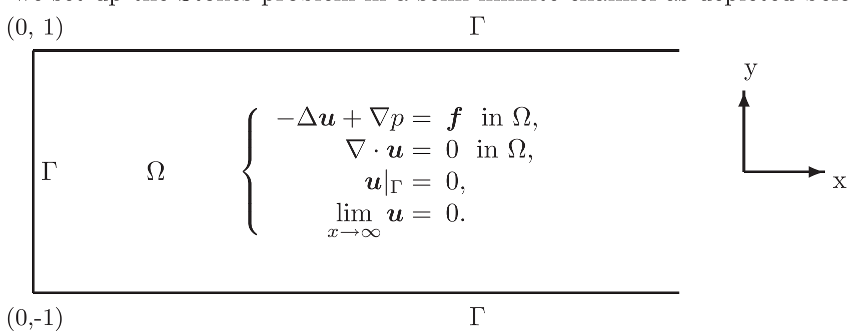
Finally, combining the above inequalities leads to

$$\begin{aligned}
\|\nabla\psi\|_{1,\Omega}^2 &= \left\|\frac{\partial\psi}{\partial x}\right\|_{1,\Omega}^2 + \left\|\frac{\partial\psi}{\partial y}\right\|_{1,\Omega}^2 \\
&= \left\|\frac{\partial\psi}{\partial x}\right\|_{0,\Omega}^2 + \left|\frac{\partial\psi}{\partial x}\right|_{1,\Omega}^2 + \left\|\frac{\partial\psi}{\partial y}\right\|_{0,\Omega}^2 + \left|\frac{\partial\psi}{\partial y}\right|_{1,\Omega}^2 \\
&= \int_{-1}^{1}\int_{0}^{\infty}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial^2\psi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\psi}{\partial y^2}\right)^2\right]dxdy \\
&\le \int_{-1}^{1}\int_{0}^{\infty}\left[4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\psi}{\partial x^2}\right)^2 + 3\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\right]dxdy \\
&\le \int_{-1}^{1}\int_{0}^{\infty}\left[3\left(\frac{\partial^2\psi}{\partial x^2}\right)^2 + 5\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\right]dxdy \\
&= 3\left\|\frac{\partial^2\psi}{\partial x^2}\right\|_{0,\Omega}^2 + 5\left\|\frac{\partial^2\psi}{\partial y^2}\right\|_{0,\Omega}^2 .
\end{aligned}$$

This estimate, together with (2.7), yields (2.5). ∎

Now we set up the Stokes problem in a semi-infinite channel as depicted below:

(2.11)




$$\begin{cases} -\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \ \text{ in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0 \ \text{ in } \Omega, \\ \boldsymbol{u}|_\Gamma = 0, \\ \lim_{x\to\infty} \boldsymbol{u} = 0. \end{cases}$$

Its weak formulation is as follows: Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^2 \times L^2(\Omega)$ such that

$$\begin{cases} (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \ \ \forall \boldsymbol{v} \in H_0^1(\Omega)^2, \\ (q, \nabla\cdot\boldsymbol{u}) = 0 \ \ \forall q \in L^2(\Omega). \end{cases} \tag{2.12}$$

Then, the mixed Laguerre–Legendre spectral approximation to (2.12) is as follows:

Find $\boldsymbol{u}_\mathcal{N} \in X_\mathcal{N}, p_\mathcal{N} \in M_\mathcal{N}$ such that

$$\begin{cases} (\nabla\boldsymbol{u}_\mathcal{N}, \nabla\boldsymbol{v}_\mathcal{N})_\mathcal{N} - (p_\mathcal{N}, \nabla\cdot\boldsymbol{v}_\mathcal{N})_\mathcal{N} = (\boldsymbol{f}, \boldsymbol{v}_\mathcal{N})_\mathcal{N} \ \ \forall \boldsymbol{v}_\mathcal{N} \in X_\mathcal{N}, \\ (q_\mathcal{N}, \nabla\cdot\boldsymbol{u}_\mathcal{N})_\mathcal{N} = 0 \ \ \forall q_\mathcal{N} \in M_\mathcal{N}, \end{cases} \tag{2.13}$$

where the discrete inner product $(\cdot,\cdot)_\mathcal{N}$ is defined by

$$(\phi, \psi)_\mathcal{N} = \sum_{p=0}^{M}\sum_{q=0}^{N} \phi\big(\hat{\xi}_p, \xi_q\big)\psi\big(\hat{\xi}_p, \xi_q\big)\hat{\omega}_p\omega_q,$$

where $\{\hat{\xi}_p, \hat{\omega}_p\}_{p=0,1,\ldots,N}$ are the Laguerre–Gauss–Radau points and the associated weights, such that the following quadrature rule holds:

$$\int_0^\infty \varphi(x)dx = \sum_{p=0}^{M} \varphi\big(\hat{\xi}_p\big)\hat{\omega}_p \quad \forall\varphi(x) \in \hat{\mathbb{P}}_{2M}\left(R^+\right);$$

$\{\xi_q, \omega_q\}_{q=0,1,\ldots,N}$ are the Legendre–Gauss–Lobatto points and the associated weights, such that the following quadrature rule holds:

$$\int_{-1}^{1} \varphi(y)dy = \sum_{q=0}^{N} \varphi(\xi_q)\omega_q, \quad \forall \varphi(y) \in \mathbb{P}_{2N-1}(\Lambda).$$

It is well known that, since the coercivity and continuity of the bilinear form $(\nabla \boldsymbol{w}_{\mathcal{N}}, \nabla \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}}$ and the continuity of the bilinear form $(\nabla \cdot \boldsymbol{v}_{\mathcal{N}}, q_{\mathcal{N}})_{\mathcal{N}}$ are evident, the well-posedness of the mixed formulation (2.13) relies on the so-called inf-sup condition [6]:

$$\inf_{q_{\mathcal{N}} \in M_{\mathcal{N}}} \sup_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} \frac{-(\nabla \cdot \boldsymbol{v}_{\mathcal{N}}, q_{\mathcal{N}})_{\mathcal{N}}}{\|\boldsymbol{v}_{\mathcal{N}}\|_{1,\Omega}\|q_{\mathcal{N}}\|_{0,\Omega}} \geq \beta_{\mathcal{N}} > 0, \tag{2.14}$$

where $\beta_{\mathcal{N}}$ is called the inf-sup constant. The next section is devoted to the estimation of this constant.

**3. Estimation of the inf-sup constant.** The main result in this section is what follows.

THEOREM 3.1.

$$\inf_{q_{\mathcal{N}} \in M_{\mathcal{N}}} \sup_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} \frac{-(\nabla \cdot \boldsymbol{v}_{\mathcal{N}}, q_{\mathcal{N}})_{\mathcal{N}}}{\|\boldsymbol{v}_{\mathcal{N}}\|_{1,\Omega}\|q_{\mathcal{N}}\|_{0,\Omega}} \gtrsim \frac{1}{M}. \tag{3.1}$$

*Remark* 3.1. It is surprising that the inf-sup constant is independent of $N$, since it is well known that the inf-sup constant of the Legendre–Legendre $P_N^2 - P_{N-2}$ method in $\Lambda^2$ is of order $N^{-\frac{1}{2}}$ (see, for instance, [3]), and we have found numerically that in the Legendre–Legendre case in $\Lambda^2$, the corresponding inf-sup constant behaves like $\max\{\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{N}}\}$, where $M, N$ are, respectively, the degrees of Legendre polynomials used in the $x$ and $y$ directions. However, our numerical results in section 5 indicate that the estimate (3.1) is sharp.

The proof of this result will be accomplished with a series of lemmas, which we present below. The confirmation of the result will be done by the numerical experiments carried out later.

LEMMA 3.1. *Given $q_{\mathcal{N}} \in M_{\mathcal{N}}$, the problem of finding $\psi_{\mathcal{N}} \in H_0^1(\Omega) \cap \hat{\mathbb{P}}_{M,N}(\Omega)$ such that*

$$(\Delta \psi_{\mathcal{N}}, r_{\mathcal{N}}) = -(q_{\mathcal{N}}, r_{\mathcal{N}}) \; \forall r_{\mathcal{N}} \in M_{\mathcal{N}} \tag{3.2}$$

*admits a unique solution satisfying*

$$\|\psi_{\mathcal{N}}\|_{2,\Omega} \lesssim M\|q_{\mathcal{N}}\|_{0,\Omega}. \tag{3.3}$$

*Proof.* Obviously, problem (3.2) defines a system with the number of unknowns equal to the number of equations, so the existence of such a function $\psi_{\mathcal{N}}$ is guaranteed by estimate (3.3), which we prove below.

By definition (3.2), we have

$$(\Delta \psi_{\mathcal{N}}, r_1(x)r_2(y)) = -(q_{\mathcal{N}}, r_1(x)r_2(y)), \forall r_1 \in \hat{\mathbb{P}}_{M-1}\left(R^+\right), \forall r_2 \in \mathbb{P}_{N-2}(\Lambda).$$

This implies

$$\begin{aligned}\int_0^\infty q_{\mathcal{N}}(x,y)r_1(x)dx &= -\pi_{N-2}^y \int_0^\infty \Delta\psi_{\mathcal{N}}(x,y)r_1(x)dx \\ &= -\int_0^\infty \pi_{N-2}^y(\Delta\psi_{\mathcal{N}}(x,y))r_1(x)dx \, \forall r_1 \in \hat{\mathbb{P}}_{M-1}\left(R^+\right)\end{aligned}$$

and consequently

$$\begin{aligned} q_{\mathcal{N}}(x,y) &= -\hat{\pi}^x_{M-1}\circ\pi^y_{N-2}(\Delta\psi_{\mathcal{N}}(x,y)) = -\hat{\pi}^x_{M-1}\circ\pi^y_{N-2}\left(\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2}+\frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\right)\\ &= -\hat{\pi}^x_{M-1}\circ\pi^y_{N-2}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2} - \hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\\ &= -\pi^y_{N-2}\circ\hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2} - \hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}. \end{aligned}$$

Hence,

$$\begin{aligned} \|q_{\mathcal{N}}\|^2_{0,\Omega} &= \left\|\hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\right\|^2_{0,\Omega} + \left\|\pi^y_{N-2}\circ\hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2}\right\|^2_{0,\Omega}\\ &\quad + 2\int_\Omega \hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\ \pi^y_{N-2}\circ\hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2}\\ &= \left\|\hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\right\|^2_{0,\Omega} + \left\|\pi^y_{N-2}\circ\hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2}\right\|^2_{0,\Omega} + 2\int_\Omega \frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\ \hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2}. \end{aligned} \tag{3.4}$$

Observing that

$$\hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2} = \frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2} - \frac{1}{4}\left(I-\hat{\pi}^x_{M-1}\right)\psi_{\mathcal{N}}, \tag{3.5}$$

where $I$ denotes the identity operator, then the last term in (3.4) can be rewritten as

$$\int_\Omega \frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\ \hat{\pi}^x_{M-1}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2} = \int_\Omega \frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2} - \frac{1}{4}\int_\Omega \frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\left(I-\hat{\pi}^x_{M-1}\right)\psi_{\mathcal{N}}.$$

For the first term on the right-hand side, we have, by integration by parts,

$$\int_\Omega \frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x^2} = \int_\Omega \frac{\partial^2\psi_{\mathcal{N}}}{\partial x\partial y}\frac{\partial^2\psi_{\mathcal{N}}}{\partial x\partial y}.$$

To estimate the second term, we write

$$\psi_{\mathcal{N}} = \sum_{m=0}^{M}\alpha_m(y)\mathcal{L}_m(x)e^{-x/2} \in H^1_0(\Omega)\cap\hat{\mathbb{P}}_{M,N}(\Omega). \tag{3.6}$$

Then, we have with $\alpha_m(y)\in\mathbb{P}^0_N(\Lambda), m=0,1,\dots,M$, and by using the orthogonality of the Laguerre polynomials,

$$\begin{aligned} -\frac{1}{4}\int_\Omega \frac{\partial^2\psi_{\mathcal{N}}}{\partial y^2}\left(I-\hat{\pi}^x_{M-1}\right)\psi_{\mathcal{N}} &= -\frac{1}{4}\int_\Omega\left[\sum_{m=0}^{M}\alpha''_m(y)\mathcal{L}_m(x)e^{-x/2}\right]\alpha_M(y)\mathcal{L}_M(x)e^{-x/2}\\ &= -\frac{1}{4}\sum_{m=0}^{M}\int_0^\infty\!\left(\int_{-1}^1\alpha''_m(y)\alpha_M(y)dy\right)\mathcal{L}_m(x)\mathcal{L}_M(x)e^{-x}dx\\ &= -\frac{1}{4}\int_{-1}^1\alpha''_M(y)\alpha_M(y)dy\\ &= \frac{1}{4}\int_{-1}^1\alpha'_M(y)\alpha'_M(y)dy. \end{aligned}$$

Combining the above estimates leads to

$$\int_\Omega \frac{\partial^2 \psi_\mathcal{N}}{\partial y^2}\ \hat{\pi}^x_{M-1}\frac{\partial^2 \psi_\mathcal{N}}{\partial x^2} = \int_\Omega \frac{\partial^2 \psi_\mathcal{N}}{\partial x \partial y}\frac{\partial^2 \psi_\mathcal{N}}{\partial x \partial y} + \frac{1}{4}\int_{-1}^{1} \alpha'_M(y)\alpha'_M(y)dy. \tag{3.7}$$

Hence, by using (3.7) and Lemma 2.3 in (3.4), we obtain

$$\begin{aligned} \|q_\mathcal{N}\|^2_{0,\Omega} &= \left\|\hat{\pi}^x_{M-1}\frac{\partial^2 \psi_\mathcal{N}}{\partial y^2}\right\|^2_{0,\Omega} + \left\|\pi^y_{N-2}\circ\hat{\pi}^x_{M-1}\frac{\partial^2 \psi_\mathcal{N}}{\partial x^2}\right\|^2_{0,\Omega} \\ &\quad + 2\int_\Omega \left(\frac{\partial^2 \psi_\mathcal{N}}{\partial x \partial y}\right)^2 dxdy + \frac{1}{2}\int_{-1}^{1}(\alpha'_M(y))^2 dy \\ &\gtrsim M^{-1}\left\|\frac{\partial^2 \psi_\mathcal{N}}{\partial y^2}\right\|^2_{0,\Omega} + \left\|\frac{\partial^2 \psi_\mathcal{N}}{\partial x \partial y}\right\|^2_{0,\Omega}. \end{aligned} \tag{3.8}$$

On the other hand, from the inverse inequality in the $x$ direction (cf. Lemma 2.1) and the Poincare inequality in the $y$ direction, we have

$$\left\|\frac{\partial^2 \psi}{\partial x^2}\right\|_{0,\Omega} \lesssim M\left\|\frac{\partial \psi}{\partial x}\right\|_{0,\Omega} \lesssim M\left\|\frac{\partial^2 \psi}{\partial x \partial y}\right\|_{0,\Omega} \quad \forall \psi \in H^1_0(\Omega)\cap\hat{\mathbb{P}}_{M,N}(\Omega).$$

Using the above inequality and Lemma 2.4 in (3.8) gives

$$\begin{aligned} \|q_\mathcal{N}\|^2_{0,\Omega} &\gtrsim M^{-1}\left\|\frac{\partial^2 \psi_\mathcal{N}}{\partial y^2}\right\|^2_{0,\Omega} + M^{-2}\left\|\frac{\partial^2 \psi_\mathcal{N}}{\partial x^2}\right\|^2_{0,\Omega} \\ &\gtrsim M^{-2}\|\psi_\mathcal{N}\|^2_{2,\Omega}. \end{aligned}$$

This leads to (3.3). □

LEMMA 3.2. *For all $q_\mathcal{N} \in M_\mathcal{N}$, there exists $\boldsymbol{z}_\mathcal{N} \in (\hat{\mathbb{P}}_{M,N}(\Omega)\cap H^1_0(\Omega)) \times (\hat{\mathbb{P}}_{M,N+1}(\Omega)\cap H^1_0(\Omega))$ such that*

$$(\nabla\cdot \boldsymbol{z}_\mathcal{N}, r_\mathcal{N}) = -(q_\mathcal{N}, r_\mathcal{N})\ \forall r_\mathcal{N}\in M_\mathcal{N}$$

*and*

$$\|\boldsymbol{z}_\mathcal{N}\|_{1,\Omega} \lesssim M\|q_\mathcal{N}\|_{0,\Omega}.$$

*Proof.* For any $q_\mathcal{N} \in M_\mathcal{N}$, let $\psi_\mathcal{N}$ be defined by (3.2) and $\boldsymbol{w}_\mathcal{N} = \nabla\psi_\mathcal{N}$. Then, we have $\boldsymbol{w}_\mathcal{N} \in \hat{\mathbb{P}}_{M,N}(\Omega)^2$ satisfying

$$\begin{cases} (\nabla\cdot \boldsymbol{w}_\mathcal{N}, r_\mathcal{N}) = -(q_\mathcal{N}, r_\mathcal{N})\ \forall r_\mathcal{N}\in M_\mathcal{N}, \\ \boldsymbol{w}_\mathcal{N}\cdot\boldsymbol{\tau} = 0, \\ \|\boldsymbol{w}_\mathcal{N}\|_{1,\Omega} \lesssim M\|q_\mathcal{N}\|_{0,\Omega}, \end{cases} \tag{3.9}$$

where $\boldsymbol{\tau}$ is the unit tangent vector along $\partial\Omega$.

We now construct a lifting function $\phi_\mathcal{N} \in \hat{\mathbb{P}}_{M,N+1}(\Omega)$ such that

$$\begin{cases} \dfrac{\partial \phi_\mathcal{N}}{\partial \boldsymbol{\tau}} = -\boldsymbol{w}_\mathcal{N}\cdot\boldsymbol{n} \ \text{ on } \Gamma, \\ \dfrac{\partial \phi_\mathcal{N}}{\partial \boldsymbol{n}} = 0 \ \text{ on } \Gamma, \\ \|\phi_\mathcal{N}\|_{2,\Omega} \lesssim \|\boldsymbol{w}_\mathcal{N}\|_{1,\Omega}, \end{cases} \tag{3.10}$$

where $\boldsymbol{n}$ is the outward normal to $\partial\Omega$. To this end, we define three functions on the boundaries $\Gamma_1 = \{(x,1), 0 \le x < \infty\}$, $\Gamma_2 = \{(0,y), -1 \le y \le 1\}$, $\Gamma_3 = \{(x,-1), 0 \le x < \infty\}$, respectively, as follows:

$$\begin{aligned} b^1_{\mathcal{N}}(x,y) &= -\int_{\infty}^{x} (\boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n})(\sigma, y) d\sigma, \\ b^2_{\mathcal{N}}(x,y) &= b^1_{\mathcal{N}}(0,1) - \int_{1}^{y} (\boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n})(x, \sigma) d\sigma, \\ b^3_{\mathcal{N}}(x,y) &= b^2_{\mathcal{N}}(0,-1) - \int_{0}^{x} (\boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n})(\sigma, y) d\sigma. \end{aligned}$$

Then, it can be easily verified that $b^j_{\mathcal{N}}$ $(j = 1,2,3)$ satisfy the following continuity conditions:

$$\begin{aligned} b^1_{\mathcal{N}}(0,1) &= b^2_{\mathcal{N}}(0,1), \\ b^2_{\mathcal{N}}(0,-1) &= b^3_{\mathcal{N}}(0,-1), \\ \frac{\partial b^1_{\mathcal{N}}}{\partial \boldsymbol{\tau}}(0,1) &= -\boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n}(0,1) = 0 = \frac{\partial b^2_{\mathcal{N}}}{\partial \boldsymbol{\tau}}(0,1), \\ \frac{\partial b^2_{\mathcal{N}}}{\partial \boldsymbol{\tau}}(0,-1) &= -\boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n}(0,-1) = 0 = \frac{\partial b^3_{\mathcal{N}}}{\partial \boldsymbol{\tau}}(0,-1), \\ \frac{\partial^2 b^1_{\mathcal{N}}}{\partial x \partial y}(0,1) &= \frac{\partial^2 b^2_{\mathcal{N}}}{\partial x \partial y}(0,1) = 0, \\ \frac{\partial^2 b^2_{\mathcal{N}}}{\partial x \partial y}(0,-1) &= \frac{\partial^2 b^3_{\mathcal{N}}}{\partial x \partial y}(0,-1) = 0. \end{aligned}$$

The above conditions, together with the fact that $b^j_{\mathcal{N}} \in \hat{\mathbb{P}}_{M,N+1}(\Omega)$ $(j = 1,2,3)$, guarantee that there exists a $\phi_{\mathcal{N}} \in \hat{\mathbb{P}}_{M,N+1}(\Omega)$ satisfying (see [2])

$$\begin{cases} \phi_{\mathcal{N}} = b^j_{\mathcal{N}} \ \text{ on } \Gamma_j, \ j = 1,2,3, \\ \dfrac{\partial \phi_{\mathcal{N}}}{\partial \boldsymbol{n}} = 0 \ \text{ on } \Gamma, \end{cases} \tag{3.11}$$

and

$$\|\phi_{\mathcal{N}}\|_{2,\Omega} \lesssim \sum_{j=1}^{3} \|b^j_{\mathcal{N}}\|_{3/2,\Gamma} \lesssim \|\boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n}\|_{1/2,\Gamma} \lesssim \|\boldsymbol{w}_{\mathcal{N}}\|_{1,\Omega}.$$

Moreover, it is seen that the first equality of (3.11) implies

$$\frac{\partial \phi_{\mathcal{N}}}{\partial \boldsymbol{\tau}} = -\boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n} \ \text{ on } \Gamma.$$

This completes the construction of the lifting function $\phi_{\mathcal{N}}$. Now, let

$$\boldsymbol{z}_{\mathcal{N}} = \boldsymbol{w}_{\mathcal{N}} + \mathrm{rot}\phi_{\mathcal{N}},$$

then $\boldsymbol{z}_{\mathcal{N}} \in \hat{\mathbb{P}}_{M,N}(\Omega) \times \hat{\mathbb{P}}_{M,N+1}(\Omega)$ and

$$\begin{aligned} \boldsymbol{z}_{\mathcal{N}} \cdot \boldsymbol{n}|_{\Gamma} &= \boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n} + \mathrm{rot}\phi_{\mathcal{N}} \cdot \boldsymbol{n} = \boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{n} + \frac{\partial \phi_{\mathcal{N}}}{\partial \boldsymbol{\tau}} = 0, \\ \boldsymbol{z}_{\mathcal{N}} \cdot \boldsymbol{\tau}|_{\Gamma} &= \boldsymbol{w}_{\mathcal{N}} \cdot \boldsymbol{\tau} + \mathrm{rot}\phi_{\mathcal{N}} \cdot \boldsymbol{\tau} = \frac{\partial \phi_{\mathcal{N}}}{\partial \boldsymbol{n}} = 0, \end{aligned}$$

which means $\boldsymbol{z}_{\mathcal{N}} \in (\hat{\mathbb{P}}_{M,N}(\Omega) \cap H_0^1(\Omega)) \times (\hat{\mathbb{P}}_{M,N+1}(\Omega) \cap H_0^1(\Omega))$. Moreover, we have

$$(\nabla \cdot \boldsymbol{z}_{\mathcal{N}}, r_{\mathcal{N}}) = (\nabla \cdot \boldsymbol{w}_{\mathcal{N}}, r_{\mathcal{N}}) + (\nabla \cdot \mathrm{rot}\phi_{\mathcal{N}}, r_{\mathcal{N}}) = -(q_{\mathcal{N}}, r_{\mathcal{N}}) \; \forall r_{\mathcal{N}} \in M_{\mathcal{N}},$$

and, by (3.9) and the last inequality of (3.10),

$$\begin{aligned} \|\boldsymbol{z}_{\mathcal{N}}\|_{1,\Omega} &= \|\boldsymbol{w}_{\mathcal{N}} + \mathrm{rot}\phi_{\mathcal{N}}\|_{1,\Omega} \le \|\boldsymbol{w}_{\mathcal{N}}\|_{1,\Omega} + \|\mathrm{rot}\phi_{\mathcal{N}}\|_{1,\Omega} \\ &\lesssim \|\boldsymbol{w}_{\mathcal{N}}\|_{1,\Omega} \lesssim M\|q_{\mathcal{N}}\|_{0,\Omega}. \end{aligned} \tag{3.12}$$

The proof is complete. □

LEMMA 3.3. *For all $q_{\mathcal{N}} \in M_{\mathcal{N}}$, there exists $\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}$ such that*

$$(\nabla \cdot \boldsymbol{v}_{\mathcal{N}}, r_{\mathcal{N}}) = -(q_{\mathcal{N}}, r_{\mathcal{N}}) \; \forall r_{\mathcal{N}} \in M_{\mathcal{N}}$$

*and*

$$\|\boldsymbol{v}_{\mathcal{N}}\|_{1,\Omega} \lesssim M\|q_{\mathcal{N}}\|_{0,\Omega}.$$

*Proof.* For given $q_{\mathcal{N}}$, let $\boldsymbol{z}_{\mathcal{N}} := (z_{\mathcal{N}}^{(1)}, z_{\mathcal{N}}^{(2)}) \in (\hat{\mathbb{P}}_{M,N}(\Omega) \cap H_0^1(\Omega)) \times (\hat{\mathbb{P}}_{M,N+1}(\Omega) \cap H_0^1(\Omega))$ be a function associated to $q_{\mathcal{N}}$ in Lemma 3.2. Then, the second component of $\boldsymbol{z}_{\mathcal{N}}$ can be written under form

$$z_{\mathcal{N}}^{(2)} = \sum_{i=2}^{N+1} \alpha_i(x) e^{-x/2}(L_i(y) - L_{i-2}(y)),$$

with $\alpha_i(x) \in \mathbb{P}_M(R^+) \cap H_0^1(R^+), i = 2, \ldots, M+1$. We decompose $z_{\mathcal{N}}^{(2)}$ into

$$z_{\mathcal{N}}^{(2)} = \tilde{z}_{\mathcal{N}}^{(2)} + \bar{z}_{\mathcal{N}}^{(2)},$$

with

$$\begin{aligned} \tilde{z}_{\mathcal{N}}^{(2)} &= \sum_{i=2}^{N} \alpha_i(x) e^{-x/2}(L_i(y) - L_{i-2}(y)), \\ \bar{z}_{\mathcal{N}}^{(2)} &= \alpha_{N+1}(x) e^{-x/2}(L_{N+1}(y) - L_{N-1}(y)), \end{aligned}$$

and let

$$\boldsymbol{v}_{\mathcal{N}} = \left(z_{\mathcal{N}}^{(1)}, \tilde{z}_{\mathcal{N}}^{(2)}\right).$$

Then, it is seen that $\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}$, moreover, for all $r_{\mathcal{N}} \in M_{\mathcal{N}}$, by using the orthogonality of the Legendre polynomials, we have

$$\begin{aligned} (\nabla \cdot \boldsymbol{v}_{\mathcal{N}}, r_{\mathcal{N}}) &= \left(\partial_x z_{\mathcal{N}}^{(1)} + \partial_y \tilde{z}_{\mathcal{N}}^{(2)}, r_{\mathcal{N}}\right) \\ &= \left(\partial_x z_{\mathcal{N}}^{(1)} + \partial_y \tilde{z}_{\mathcal{N}}^{(2)}, r_{\mathcal{N}}\right) - \left(\alpha_{N+1}(x) e^{-x/2}(L_{N+1}(y) - L_{N-1}(y)), \partial_y r_{\mathcal{N}}\right) \\ &= \left(\partial_x z_{\mathcal{N}}^{(1)} + \partial_y \tilde{z}_{\mathcal{N}}^{(2)}, r_{\mathcal{N}}\right) + \left(\alpha_{N+1}(x) e^{-x/2}\partial_y(L_{N+1}(y) - L_{N-1}(y)), r_{\mathcal{N}}\right) \\ &= \left(\partial_x z_{\mathcal{N}}^{(1)} + \partial_y z_{\mathcal{N}}^{(2)}, r_{\mathcal{N}}\right) \\ &= (\nabla \cdot \boldsymbol{z}_{\mathcal{N}}, r_{\mathcal{N}}) \\ &= -(q_{\mathcal{N}}, r_{\mathcal{N}}). \end{aligned}$$

It remains to prove $\|\boldsymbol{v}_{\mathcal{N}}\|_{1,\Omega} \lesssim \|\boldsymbol{z}_{\mathcal{N}}\|_{1,\Omega}$. Since $\tilde{z}_{\mathcal{N}}^{(2)} = z_{\mathcal{N}}^{(2)} - \bar{z}_{\mathcal{N}}^{(2)}$, we need only to prove $\|\bar{z}_{\mathcal{N}}^{(2)}\|_{1,\Omega} \lesssim \|z_{\mathcal{N}}^{(2)}\|_{1,\Omega}$. First, we have

$$\begin{aligned}
\partial_x \bar{z}_{\mathcal{N}}^{(2)} &= \partial_x \left(\alpha_{N+1}(x)e^{-x/2}\right)(L_{N+1}(y) - L_{N-1}(y)),\\
\partial_x z_{\mathcal{N}}^{(2)} &= \sum_{i=2}^{N+1} \partial_x \left(\alpha_i(x)e^{-x/2}\right)(L_i(y) - L_{i-2}(y))\\
&= \partial_x \left(\alpha_{N+1}(x)e^{-x/2}\right) L_{N+1}(y) + \partial_x \left(\alpha_N(x)e^{-x/2}\right) L_N(y) + \cdots,
\end{aligned}$$

thus

$$\begin{aligned}
\left\|\partial_x z_{\mathcal{N}}^{(2)}\right\|_{0,\Omega}^2 &= \int_{-1}^{1} \left[\int_0^\infty \left(\partial_x \left(\alpha_{N+1}(x)e^{-x/2}\right)\right)^2 dy\right] L_{N+1}(y)^2 dx + \cdots\\
&\gtrsim \left\|\partial_x \left(\alpha_{N+1}(x)e^{-x/2}\right)\right\|_{0,R^+}^2 \frac{1}{N+1+1/2}\\
&\geq \frac{1}{6}\left|\alpha_{N+1}(x)e^{-x/2}\right|_{1,R^+}^2 \left(\frac{2}{2N+3} + \frac{2}{2N-1}\right)\\
&= \frac{1}{6}\left|\alpha_{N+1}(x)e^{-x/2}\right|_{1,R^+}^2 \left(\|L_{N+1}\|_{0,\Lambda}^2 + \|L_{N-1}\|_{0,\Lambda}^2\right)\\
&\gtrsim \left\|\partial_x \bar{z}_{\mathcal{N}}^{(2)}\right\|_{0,\Omega}^2.
\end{aligned}$$

Similarly, we have

$$\left\|z_{\mathcal{N}}^{(2)}\right\|_{0,\Omega}^2 \gtrsim \left\|\bar{z}_{\mathcal{N}}^{(2)}\right\|_{0,\Omega}^2.$$

Second, from

$$\begin{aligned}
\partial_y \bar{z}_{\mathcal{N}}^{(2)} &= \alpha_{N+1}(x)e^{-x/2}(L'_{N+1}(y) - L'_{N-1}(y)) = \alpha_{N+1}(x)e^{-x/2}(2N+1)L_N(y),\\
\partial_y z_{\mathcal{N}}^{(2)} &= \sum_{i=2}^{N+1} \alpha_i(x)e^{-x/2}(L'_i(y) - L'_{i-2}(y)) = \sum_{i=2}^{N+1} \alpha_i(x)e^{-x/2}(2i-1)L_{i-1}(y),
\end{aligned}$$

we derive that

$$\begin{aligned}
\left\|\partial_y z_{\mathcal{N}}^{(2)}\right\|_{0,\Omega}^2 &= \sum_{i=2}^{N+1} \left\|\alpha_i(x)e^{-x/2}\right\|_{0,R^+}^2 (2i-1)^2 \|L_{i-1}\|_{0,\Lambda}^2\\
&\geq \left\|\alpha_{N+1}(x)e^{-x/2}\right\|_{0,R^+}^2 (2N+1)^2 \|L_N\|_{0,\Lambda}^2\\
&= \left\|\partial_y \bar{z}_{\mathcal{N}}^{(2)}\right\|_{0,\Omega}^2.
\end{aligned}$$

Combining all above estimations together gives

$$\left\|z_{\mathcal{N}}^{(2)}\right\|_{1,\Omega}^2 \gtrsim \left\|\bar{z}_{\mathcal{N}}^{(2)}\right\|_{1,\Omega}^2,$$

which yields

$$\left\|v_{\mathcal{N}}^{(2)}\right\|_{1,\Omega} = \left\|\tilde{z}_{\mathcal{N}}^{(2)}\right\|_{1,\Omega} = \left\|z_{\mathcal{N}}^{(2)} - \bar{z}_{\mathcal{N}}^{(2)}\right\|_{1,\Omega} \lesssim \left\|z_{\mathcal{N}}^{(2)}\right\|_{1,\Omega}.$$

This gives

$$\|\boldsymbol{v}_\mathcal{N}\|_{1,\Omega} \lesssim \|\boldsymbol{z}_\mathcal{N}\|_{1,\Omega} \lesssim M\|q_\mathcal{N}\|_{0,\Omega}. \qquad \square$$

*Proof of Theorem* 3.1. For all $q_\mathcal{N} \in M_\mathcal{N}$, let $\boldsymbol{v}_\mathcal{N} \in X_\mathcal{N}$ be the associated function given in Lemma 3.3, then

$$\begin{aligned}\frac{-(\nabla\cdot\boldsymbol{v}_\mathcal{N}, q_\mathcal{N})_\mathcal{N}}{\|\boldsymbol{v}_\mathcal{N}\|_{1,\Omega}\|q_\mathcal{N}\|_{0,\Omega}} &= \frac{-(\nabla\cdot\boldsymbol{v}_\mathcal{N}, q_\mathcal{N})}{\|\boldsymbol{v}_\mathcal{N}\|_{1,\Omega}\|q_\mathcal{N}\|_{0,\Omega}} = \frac{(q_\mathcal{N}, q_\mathcal{N})}{\|\boldsymbol{v}_\mathcal{N}\|_{1,\Omega}\|q_\mathcal{N}\|_{0,\Omega}}\\ &= \frac{\|q_\mathcal{N}\|_{0,\Omega}}{\|\boldsymbol{v}_\mathcal{N}\|_{1,\Omega}} \gtrsim \frac{1}{M}.\end{aligned}$$

This means (3.1) holds. $\square$

COROLLARY 3.1. *For all $\boldsymbol{f} \in C^0(\Omega)^2$, problem* (2.13) *admits a unique solution $(\boldsymbol{u}_\mathcal{N}, p_\mathcal{N})$ satisfying*

$$\|\boldsymbol{u}_\mathcal{N}\|_{1,\Omega} + \frac{1}{M}\|p_\mathcal{N}\|_{0,\Omega} \lesssim \|f\|_{L^\infty(\Omega)}. \tag{3.13}$$

*Proof.* First, it can be checked that, for each $\boldsymbol{u}_\mathcal{N} \in X_\mathcal{N}, \boldsymbol{v}_\mathcal{N} \in X_\mathcal{N}$,

$$\begin{aligned}|(\nabla\boldsymbol{u}_\mathcal{N}, \nabla\boldsymbol{v}_\mathcal{N})_\mathcal{N}| &\le 3\ \|\boldsymbol{u}_\mathcal{N}\|_1\|\boldsymbol{v}_\mathcal{N}\|_1,\\ (\nabla\boldsymbol{v}_\mathcal{N}, \nabla\boldsymbol{v}_\mathcal{N})_\mathcal{N} &\ge |\boldsymbol{v}_N|_1^2 \ge \frac{1}{3}\|\boldsymbol{v}_\mathcal{N}\|_1^2.\end{aligned}$$

Then, thanks to the above inequalities and (3.1), the well-posedness of problem (2.13) and stability estimate (3.13) are straightforward consequences of the abstract inf-sup theory (cf. [1, 6]). $\square$

## 4. Error estimation.

We start with some notations and definitions which are needed in the following error analysis. Denote $\omega_r(x) = x^r e^{-x}$, $\hat\omega_r(x) = x^r$, and, in particular, we set $\omega(x) = \omega_0(x)$, $\hat\omega(x) = \hat\omega_0(x)$. Then, for any non-negative integer $r$, we define two Banach spaces

$$\begin{aligned}\hat A^r(R^+) &:= \{v; v \text{ is measurable on } R^+ \text{ and } \|v\|_{\hat A^r,R^+} < \infty\},\\ A^r(R^+) &:= \{v; v \text{ is measurable on } R^+ \text{ and } \|v\|_{A^r,R^+} < \infty\},\end{aligned}$$

equipped, respectively, with the following norms:

$$\|v\|_{\hat A^r,R^+} = \left(\sum_{k=0}^r |v|^2_{\hat A^k,R^+}\right)^{\frac12}, \text{ with } |v|_{\hat A^k,R^+} = \left\|\partial_x^k v\right\|_{\omega_k,R^+} \ \forall v \in \hat A^r\left(R^+\right),$$

$$\|v\|_{A^r,R^+} = \left(\sum_{k=0}^r |v|^2_{A^k,R^+}\right)^{\frac12}, \text{ with } |v|_{A^k,R^+} = \left\|\partial_x^k v\right\|_{\hat\omega_k,R^+} \ \forall v \in A^r\left(R^+\right).$$

We now recall several approximation results. Let $\pi_M^x$ and $\hat\pi_M^x$ be the projection operators defined in section 2.

LEMMA 4.1 (cf. [19]). *For any $v \in \hat A^r(R^+)$ and integer $r \ge s \ge 0$,*

$$\|v - \pi_M^x v\|_{\hat A^s,R^+} \lesssim M^{(s-r)/2}|v|_{\hat A^r,R^+}. \tag{4.1}$$

A direct consequence of Lemma 4.1 is that, for any integer $r \ge 0$, $v \in A^r(R^+)$,

$$\|v - \hat\pi_M^x v\|_{0,R^+} \lesssim M^{-r/2}\left|e^{x/2}v\right|_{\hat A^r,R^+}. \tag{4.2}$$

Denote $W_M = \{v \in \mathbb{P}_M(R^+); v(0) = 0\}$, and let $\pi_{1,M}^{x,0} : H_{0,\omega}^1(R^+) \to W_M$ be the $H_{0,\omega}^1(R^+)$-orthogonal projection operator defined by

$$\int_0^\infty \left(\pi_{1,M}^{x,0} v\right)' \phi_M' \omega dx = \int_0^\infty v' \phi_M' \omega dx \quad \forall \phi_M \in W_M.$$

LEMMA 4.2. *If $v \in H_{0,\omega}^1(R^+)$, $\partial_x v \in \hat{A}^{r-1}(R^+)$, and integer $r \geq 1$, then*

$$\left\|v - \pi_{1,M}^{x,0} v\right\|_{1,\omega,R^+} \lesssim M^{\frac{1}{2}-\frac{r}{2}} |\partial_x v|_{\hat{A}^{r-1},R^+}.$$

*Proof.* Given $v \in H_{0,\omega}^1(R^+)$, let $v_M(z) = \int_0^z \pi_{M-1}^x \partial_x v(x) dx \ \forall z \in R^+$, then $v_M \in W_M$ and $\partial_x v_M(x) = \pi_{M-1}^x(\partial_x v(x))$. Hence, by Lemma 2.2 of [11] and Lemma 4.1,

$$\begin{aligned} \|v - \pi_{1,M}^{x,0} v\|_{1,\omega,R^+} &\leq \|v - v_M\|_{1,\omega,R^+} \lesssim |v - v_M|_{1,\omega,R^+} \\ &= \|\partial_x v - \pi_{M-1}^x(\partial_x v)\|_{\omega,R^+} \lesssim M^{\frac{1}{2}-\frac{r}{2}} |\partial_x v|_{\hat{A}^{r-1},R^+}. \end{aligned}$$ ∎

Now we set $\hat{W}_M = \{ve^{-x/2}; v \in W_M\}$ and define the projection operator $\hat{\pi}_{1,M}^{x,0}$ from $H_0^1(R^+)$ into $\hat{W}_M$ by

$$\hat{\pi}_{1,M}^{x,0} v(x) := e^{-x/2} \pi_{1,M}^{x,0}\left(v(x) e^{x/2}\right) \quad \forall v \in H_0^1\left(R^+\right).$$

Then, it follows from Lemma 4.2 that, for $r \geq 1$,

(4.3)
$$\left\|v - \hat{\pi}_{1,M}^{x,0} v\right\|_{1,R^+} = \left\|ve^{x/2} - \pi_{1,M}^{x,0}\left(e^{x/2} v(x)\right)\right\|_{1,\omega,R^+} \lesssim M^{\frac{1}{2}-\frac{r}{2}} \left|\partial_x\left(e^{x/2} v(x)\right)\right|_{\hat{A}^{r-1},R^+}.$$

For $r \geq 1$, we introduce the space, suitable for analyzing the approximation properties of the Laguerre interpolation (cf. [19]),

$$B^r(R^+) := \{v; v \text{ is measurable on } R^+ \text{ and } \|v\|_{B^r,R^+} < \infty\},$$

with norm

$$\|v\|_{B^r,R^+} = \left(\sum_{k=1}^r \left\|x^{(r-1)/2}(x+1)^{1/2} \partial_x^k v\right\|_{0,R^+}^2\right)^{1/2}.$$

Let $I_M^x$ be the Laguerre–Gauss–Radau interpolation, and define $\hat{I}_M^x v(x) = e^{-x/2} I_M^x (e^{x/2} v(x))$; the following result is proved in [19].

LEMMA 4.3. *For any $v \in B^r(R^+)$, and $0 \leq \mu \leq 1 \leq r$,*

$$\begin{aligned} \left\|v - \hat{I}_M^x v\right\|_{\mu,R^+} &\lesssim (\ln M)^{1/2} M^{\mu+1/2-r/2}\left(\left|e^{x/2} v\right|_{\hat{A}^r,R^+} + \left|\partial_x\left(e^{x/2} v\right)\right|_{\hat{A}^{r-1},R^+}\right) \\ &\lesssim (\ln M)^{1/2} M^{\mu+1/2-r/2} \|v\|_{B^r,R^+}. \end{aligned}$$

Let $\pi_{1,N}^{y,0} : H_0^1(\Lambda) \to \mathbb{P}_N^0(\Lambda)$ be the $H_0^1(\Lambda)$-orthogonal projector defined by

$$\int_{-1}^1 \partial_y(v - \pi_{1,N}^{y,0} v) \partial_y \phi dy = 0 \quad \forall \phi \in \mathbb{P}_N^0(\Lambda).$$

Then, it follows from [3] that, for all $s \geq 1$ and all $v \in H_0^1(\Lambda) \cap H^s(\Lambda)$, it holds that

$$\left\| v - \pi_{1,N}^{y,0} v \right\|_{k,\Lambda} \lesssim N^{k-s} \|v\|_{s,\Lambda}, \quad k = 0, 1. \tag{4.4}$$

We denote by $L^2(\Lambda, A^r(R^+))$ the space of the measurable functions $v : \Lambda \to A^r(R^+)$ such that

$$\|v\|_{A^r;0} := \left\{ \int_\Lambda \|v(\cdot, y)\|_{A^r,R^+}^2 dy \right\}^{\frac{1}{2}} < \infty.$$

Moreover, for any nonnegative integer $s$, we define

$$H^s\left(\Lambda, L^2\left(R^+\right)\right) := \left\{ v; \frac{\partial^j v}{\partial y^j} \in L^2\left(\Lambda, L^2\left(R^+\right)\right), 0 \leq j \leq s \right\}.$$

The norm of this space is given by

$$\|v\|_{0;s} = \left\{ \sum_{j=0}^{s} \left\| \frac{\partial^j v}{\partial y^j} \right\|_{0,\Omega}^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{j=0}^{s} \left\| \frac{\partial^j v}{\partial y^j} \right\|_{0;0}^2 \right\}^{\frac{1}{2}}.$$

Now, for any nonnegative integer $r$ and $s$, we define

$$A^{r;s}(\Omega) := H^s\left(\Lambda, L^2\left(R^+\right)\right) \cap L^2\left(\Lambda, A^r\left(R^+\right)\right),$$

with the following norm:

$$\|v\|_{A^{r;s}} = \left\{ \|v\|_{A^r;0}^2 + \|v\|_{0;s}^2 \right\}^{\frac{1}{2}} \qquad \forall v \in A^{r;s}(\Omega).$$

We also define

$$\begin{aligned} \bar{B}^{r;s}(\Omega) &:= H^s\left(\Lambda, L^2\left(R^+\right)\right) \cap H^1\left(\Lambda, B^{r-1}\left(R^+\right)\right) \cap L^2\left(\Lambda, A^r\left(R^+\right)\right), \\ Y^{m;n}(\Omega) &:= H^n\left(\Lambda, L^2\left(R^+\right)\right) \cap H^1\left(\Lambda, A^{m-1}\left(R^+\right)\right) \cap H^{n-1}\left(\Lambda, H^1\left(R^+\right)\right) \\ &\quad \cap L^2\left(\Lambda, A^m\left(R^+\right)\right), \end{aligned}$$

equipped, respectively, with the following norms:

$$\begin{aligned} \|v\|_{\bar{B}^{r;s}} &= \left( \|v\|_{0;s}^2 + \|v\|_{B^{r-1};1}^2 + \|v\|_{A^r;0}^2 \right)^{\frac{1}{2}}, \\ \|v\|_{Y^{m;n}} &= \left( \|v\|_{0;n}^2 + \|v\|_{A^{m-1};1}^2 + \|v\|_{1;n-1}^2 + \|v\|_{A^m;0}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

THEOREM 4.1. *If the solution* $(\boldsymbol{u}, p)$ *of problem* (2.12) *satisfies* $\boldsymbol{u} \in H_0^1(\Omega)^2 \cap Y^{m;n}(\Omega)^2 \cap C(\Omega)$, $p \in A^{m-1;n-1}(\Omega) \cap C(\Omega)$, $m \geq 1, n \geq 1$ *and* $\boldsymbol{f} \in \bar{B}^{r;s}(\Omega)^2 \cap C(\Omega)$, $r \geq 1, s \geq 1$, *then the solution* $(\boldsymbol{u}_\mathcal{N}, p_\mathcal{N})$ *of* (2.13) *admits the following error estimates:*

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_\mathcal{N}\|_{1,\Omega} \; &\lesssim \; \left(M^{\frac{1}{2}-\frac{m}{2}} + N^{1-n}\right)\left(M\|\boldsymbol{u}\|_{Y^{m;n}} + \|p\|_{A^{m-1;n-1}}\right) \\ &\quad + \left((\ln M)^{\frac{1}{2}} M^{1-\frac{r}{2}} + N^{-s}\right)\|\boldsymbol{f}\|_{\bar{B}^{r;s}}, \\ \|p - p_\mathcal{N}\|_{0,\Omega} \; &\lesssim \; M\left[\left(M^{\frac{1}{2}-\frac{m}{2}} + N^{1-n}\right)\left(M\|\boldsymbol{u}\|_{Y^{m;n}} + \|p\|_{A^{m-1;n-1}}\right)\right. \\ &\quad \left. + \left((\ln M)^{\frac{1}{2}} M^{1-\frac{r}{2}} + N^{-s}\right)\|\boldsymbol{f}\|_{\bar{B}^{r;s}}\right]. \end{aligned}$$

*Proof.* Let

$$V_{\mathcal{N}} := \{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}; (p_{\mathcal{N}}, \nabla \cdot \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}} = 0 \; \forall p_{\mathcal{N}} \in M_{\mathcal{N}}\}.$$

Then, for all $\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}, \boldsymbol{w}_{\mathcal{N}} \in V_{\mathcal{N}}$,

$$\begin{aligned}(\nabla \boldsymbol{w}_{\mathcal{N}}, \nabla \boldsymbol{v}_{\mathcal{N}}) - (\nabla \boldsymbol{w}_{\mathcal{N}}, \nabla \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}} &= (\nabla(\boldsymbol{w}_{\mathcal{N}} - \boldsymbol{u}), \nabla \boldsymbol{v}_{\mathcal{N}}) + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_{\mathcal{N}}) \\ &\quad - (\nabla \boldsymbol{w}_{\mathcal{N}}, \nabla \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}} \\ &\lesssim \left(|\boldsymbol{u} - \boldsymbol{w}_{\mathcal{N}}|_{1,\Omega} + \left|\boldsymbol{u} - \hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N-1}\boldsymbol{u}\right|_{1,\Omega}\right)|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega} \\ &\quad + \left|\boldsymbol{w}_{\mathcal{N}} - \hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N-1}\boldsymbol{u}\right|_{1,\Omega}|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega} \\ &\lesssim \left(|\boldsymbol{u} - \boldsymbol{w}_{\mathcal{N}}|_{1,\Omega} + \left|\boldsymbol{u} - \hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N-1}\boldsymbol{u}\right|_{1,\Omega}\right)|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}.\end{aligned}$$

This result, together with Theorem IV.2.5 and Remark IV.2.7 of [3], leads to

$$\begin{aligned}|\boldsymbol{u} - \boldsymbol{u}_{\mathcal{N}}|_{1,\Omega} &\lesssim \inf_{\boldsymbol{w}_{\mathcal{N}} \in V_{\mathcal{N}}} \left(|\boldsymbol{u} - \boldsymbol{w}_{\mathcal{N}}|_{1,\Omega} + \sup_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} \frac{(\nabla \boldsymbol{w}_{\mathcal{N}}, \nabla \boldsymbol{v}_{\mathcal{N}}) - (\nabla \boldsymbol{w}_{\mathcal{N}}, \nabla \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}}}{|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}}\right) \\ &\quad + \inf_{q_{\mathcal{N}} \in M_{\mathcal{N}}} \|p - q_{\mathcal{N}}\|_{0,\Omega} + \sup_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} \frac{(\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}} - (\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})}{|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}} \\ &\lesssim \inf_{\boldsymbol{w}_{\mathcal{N}} \in V_{\mathcal{N}}} |\boldsymbol{u} - \boldsymbol{w}_{\mathcal{N}}|_{1,\Omega} + \left|\boldsymbol{u} - \hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N-1}\boldsymbol{u}\right|_{1,\Omega} + \inf_{q_{\mathcal{N}} \in M_{\mathcal{N}}} \|p - q_{\mathcal{N}}\|_{0,\Omega} \\ &\quad + \sup_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} \frac{(\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}} - (\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})}{|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}} \\ &\lesssim M \inf_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} |\boldsymbol{u} - \boldsymbol{v}_{\mathcal{N}}|_{1,\Omega} + \left|\boldsymbol{u} - \hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N-1}\boldsymbol{u}\right|_{1,\Omega} \\ &\quad + \inf_{q_{\mathcal{N}} \in M_{\mathcal{N}}} \|p - q_{\mathcal{N}}\|_{0,\Omega} + \sup_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} \frac{(\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}} - (\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})}{|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}}.\end{aligned}$$

And,

$$\begin{aligned}\|p - p_{\mathcal{N}}\|_{0,\Omega} \lesssim M \Bigg[&\left|\boldsymbol{u} - \hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N-1}\boldsymbol{u}\right|_{1,\Omega} + \inf_{q_{\mathcal{N}} \in M_{\mathcal{N}}} \|p - q_{\mathcal{N}}\|_{0,\Omega} \\ &+ M \inf_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} |\boldsymbol{u} - \boldsymbol{v}_{\mathcal{N}}|_{1,\Omega} + \sup_{\boldsymbol{v}_{\mathcal{N}} \in X_{\mathcal{N}}} \frac{(\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}} - (\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})}{|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}}\Bigg].\end{aligned}$$

Now, for all $\boldsymbol{f}_{M,N-1} \in \hat{\mathbb{P}}_{M,N-1}(\Omega)^2$, we have

$$(\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}} - (\boldsymbol{f}, \boldsymbol{v}_{\mathcal{N}}) \lesssim \left(\left\|\boldsymbol{f} - I^y_N \hat{I}^x_M \boldsymbol{f}\right\|_{0,\Omega} + \|\boldsymbol{f} - \boldsymbol{f}_{M,N-1}\|_{0,\Omega}\right)|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}.$$

We know from the interpolation results of $I^y_N$ and $\hat{I}^x_M$ that

$$\begin{aligned}\left\|\boldsymbol{f} - I^y_N \hat{I}^x_M \boldsymbol{f}\right\|_{0,\Omega} &\le \|\boldsymbol{f} - I^y_N \boldsymbol{f}\|_{0;0} + \left\|I^y_N(\boldsymbol{f} - \hat{I}^x_M \boldsymbol{f})\right\|_{0;0} \\ &\lesssim N^{-s}\|\boldsymbol{f}\|_{0;s} + \left\|\boldsymbol{f} - \hat{I}^x_M \boldsymbol{f}\right\|_{0;1} \\ &\lesssim N^{-s}\|\boldsymbol{f}\|_{0;s} + (\ln M)^{\frac{1}{2}} M^{1-\frac{r}{2}} \|\boldsymbol{f}\|_{B^{r-1};1}\ .\end{aligned}$$

Furthermore,

$$\|\boldsymbol{f} - \boldsymbol{f}_{M,N-1}\|_{0,\Omega} \le \left\|\boldsymbol{f} - \pi^y_{N-1} \circ \hat{\pi}^x_M \boldsymbol{f}\right\|_{0;0} \lesssim N^{-s}\|\boldsymbol{f}\|_{0,s} + M^{-\frac{r}{2}}\|\boldsymbol{f}\|_{A^r;0}\ .$$

Combining the above two inequalities, we get

$$(\boldsymbol{f},\boldsymbol{v}_{\mathcal{N}})_{\mathcal{N}}-(\boldsymbol{f},\boldsymbol{v}_{\mathcal{N}})\lesssim\Big((\ln M)^{\frac{1}{2}}M^{1-\frac{r}{2}}+N^{-s}\Big)\|\boldsymbol{f}\|_{\bar{B}^{r;s}}|\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}.$$

Now we estimate $\inf_{\boldsymbol{v}_{\mathcal{N}}\in X_{\mathcal{N}}}|\boldsymbol{u}-\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}$. Since

$$|\boldsymbol{u}-\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}=\left\|\frac{\partial}{\partial x}(\boldsymbol{u}-\boldsymbol{v}_{\mathcal{N}})\right\|_{0,\Omega}+\left\|\frac{\partial}{\partial y}(\boldsymbol{u}-\boldsymbol{v}_{\mathcal{N}})\right\|_{0,\Omega},$$

by choosing $\boldsymbol{v}_{\mathcal{N}}=\hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N}\boldsymbol{u}$, we know from the approximation results of $\hat{\pi}^{x,0}_{1,M}$ and $\pi^{y,0}_{1,N}$ that

$$\begin{aligned}\left\|\frac{\partial}{\partial x}(\boldsymbol{u}-\boldsymbol{v}_{\mathcal{N}})\right\|_{0,\Omega}&\le\left\|\frac{\partial}{\partial x}\big(\boldsymbol{u}-\hat{\pi}^{x,0}_{1,M}\boldsymbol{u}\big)\right\|_{0,\Omega}+\left\|\frac{\partial}{\partial x}\hat{\pi}^{x,0}_{1,M}\big(\boldsymbol{u}-\pi^{y,0}_{1,N}\boldsymbol{u}\big)\right\|_{0,\Omega}\\&\lesssim M^{\frac{1}{2}-\frac{m}{2}}\big|\partial_x\big(e^{x/2}\boldsymbol{u}\big)\big|_{\hat{A}^{m-1};0}+N^{1-n}\left\|\frac{\partial}{\partial x}\hat{\pi}^{x,0}_{1,M}\boldsymbol{u}\right\|_{0;n-1}\\&\lesssim M^{\frac{1}{2}-\frac{m}{2}}\|\boldsymbol{u}\|_{A^m;0}+N^{1-n}\left\|\frac{\partial}{\partial x}\big(e^{x/2}\boldsymbol{u}\big)\right\|_{\omega_0;n-1}\\&\lesssim M^{\frac{1}{2}-\frac{m}{2}}\|\boldsymbol{u}\|_{A^m;0}+N^{1-n}\|\boldsymbol{u}\|_{1;n-1};\end{aligned}$$

$$\begin{aligned}\left\|\frac{\partial}{\partial y}(\boldsymbol{u}-\boldsymbol{v}_{\mathcal{N}})\right\|_{0,\Omega}&\le\left\|\frac{\partial}{\partial y}\big(\boldsymbol{u}-\pi^{y,0}_{1,N}\boldsymbol{u}\big)\right\|_{0,\Omega}+\left\|\frac{\partial}{\partial y}\pi^{y,0}_{1,N}\big(\boldsymbol{u}-\hat{\pi}^{x,0}_{1,M}\boldsymbol{u}\big)\right\|_{0,\Omega}\\&\lesssim N^{1-n}\|\boldsymbol{u}\|_{0;n}+M^{\frac{1}{2}-\frac{m}{2}}\left|e^{x/2}\frac{\partial}{\partial y}\pi^{y,0}_{1,N-1}\boldsymbol{u}\right|_{\hat{A}^{m-1};0}\\&\lesssim N^{1-n}\|\boldsymbol{u}\|_{0;n}+M^{\frac{1}{2}-\frac{m}{2}}\|\boldsymbol{u}\|_{A^{m-1};1}.\end{aligned}$$

Combining the above two results leads to

$$\inf_{\boldsymbol{v}_{\mathcal{N}}\in X_{\mathcal{N}}}|\boldsymbol{u}-\boldsymbol{v}_{\mathcal{N}}|_{1,\Omega}\le\big|\boldsymbol{u}-\hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N}\boldsymbol{u}\big|_{1,\Omega}\lesssim\big(M^{\frac{1}{2}-\frac{m}{2}}+N^{1-n}\big)\|\boldsymbol{u}\|_{Y^{m;n}}.$$

Similarly, we have

$$\big|\boldsymbol{u}-\hat{\pi}^{x,0}_{1,M}\pi^{y,0}_{1,N-1}\boldsymbol{u}\big|_{1,\Omega}\lesssim\big(M^{\frac{1}{2}-\frac{m}{2}}+N^{1-n}\big)\|\boldsymbol{u}\|_{Y^{m;n}}.$$

Now it remains to estimate $\inf_{q_{\mathcal{N}}\in M_{\mathcal{N}}}\|p-q_{\mathcal{N}}\|_{0,\Omega}$. By using the known properties of the projectors $\hat{\pi}^x_M$ and $\pi^y_N$ in [16], it follows that

$$\begin{aligned}\inf_{q_{\mathcal{N}}\in M_{\mathcal{N}}}\|p-q_{\mathcal{N}}\|_{0,\Omega}&\le\|p-\pi^y_{N-2}\circ\hat{\pi}^x_{M-1}p\|_{0;0}\\&\lesssim(N-2)^{1-n}\|p\|_{0;n-1}+\|p-\hat{\pi}^x_{M-1}p\|_{0;0}\\&\lesssim N^{1-n}\|p\|_{0;n-1}+M^{\frac{1}{2}-\frac{m}{2}}\|p\|_{A^{m-1};0}\\&\lesssim\big(M^{\frac{1}{2}-\frac{m}{2}}+N^{1-n}\big)\|p\|_{A^{m-1;n-1}}.\end{aligned}$$

As a direct consequence of the above estimates, we finally obtain

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_{\mathcal{N}}\|_{1,\Omega}\lesssim&\big(M^{\frac{1}{2}-\frac{m}{2}}+N^{1-n}\big)\big(M\|\boldsymbol{u}\|_{Y^{m;n}}+\|p\|_{A^{m-1;n-1}}\big)\\&+\big((\ln M)^{\frac{1}{2}}M^{1-\frac{r}{2}}+N^{-s}\big)\|\boldsymbol{f}\|_{\bar{B}^{r;s}},\\\|p-p_{\mathcal{N}}\|_{0,\Omega}\lesssim&M\Big[\big(M^{\frac{1}{2}-\frac{m}{2}}+N^{1-n}\big)\big(M\|\boldsymbol{u}\|_{Y^{m;n}}+\|p\|_{A^{m-1;n-1}}\big)\\&+\big((\ln M)^{\frac{1}{2}}M^{1-\frac{r}{2}}+N^{-s}\big)\|\boldsymbol{f}\|_{\bar{B}^{r;s}}\Big].\qquad\square\end{aligned}$$

**5. Numerical results and discussions.** We start with some implementation details. Let $\boldsymbol{u}_\mathcal{N} = (u^1_\mathcal{N}, u^2_\mathcal{N})^t$, and we write

$$u^r_\mathcal{N}(x,y) = \sum_{j=1}^{N-1}\sum_{i=1}^{M} u^r_\mathcal{N}\big(\hat{\xi}_i, \xi_j\big)\hat{h}_i(x)h_j(y), \quad r = 1, 2,$$

where $h_j \in \mathbb{P}_N(\Lambda)\,(0 \le j \le N)$ are the Legendre–Gauss–Lobatto interpolants satisfying $h_j(\xi_q) = \delta_{qj}$, while $\hat{h}_i \in \hat{\mathbb{P}}_M(R^+)\,(0 \le i \le M)$ are the Laguerre–Gauss–Radau interpolants satisfying $\hat{h}_i(\hat{\xi}_q) = \delta_{qi}$. We use $\underline{\boldsymbol{u}}_\mathcal{N}$ to denote the vector consisting of the values of $\boldsymbol{u}_\mathcal{N}$ at the nodes $(\hat{\xi}_i, \xi_j)_{1\le i\le M, 1\le j\le N-1}$.

Similarly, we write

$$p_\mathcal{N}(x,y) = \sum_{j=1}^{N-1}\sum_{i=1}^{M} p_\mathcal{N}\big(\hat{\zeta}_i, \zeta_j\big)\hat{\ell}_i(x)\ell_j(y),$$

where $(\hat{\zeta}_i)_{1\le i\le M}$ and $(\zeta_j)_{1\le j\le N-1}$ are, respectively, the Laguerre–Gauss and Legendre–Gauss points, and $\ell_j \in \mathbb{P}_{N-2}(\Lambda)$ $(1 \le j \le N-1)$ are the Legendre–Gauss interpolants satisfying $\ell_j(\xi_q) = \delta_{qj}$, while $\hat{\ell}_i \in \hat{\mathbb{P}}_{M-1}(R^+)\,(1 \le i \le M)$ are the Laguerre–Gauss interpolants satisfying $\hat{\ell}_i(\hat{\zeta}_q) = \delta_{qi}$. We use $\underline{p}_\mathcal{N}$ to denote the vector consisting of the values of $p_\mathcal{N}$ at the nodes $(\hat{\zeta}_i, \zeta_j)_{1\le i\le M, 1\le j\le N-1}$.

Inserting the expansions of $\boldsymbol{u}_\mathcal{N}$ and $p_\mathcal{N}$ into (2.13), the resulting set of algebraic equations can be written under a matrix form:

$$\boldsymbol{A}_\mathcal{N}\underline{\boldsymbol{u}}_\mathcal{N} + \boldsymbol{D}_\mathcal{N}\underline{p}_\mathcal{N} = \boldsymbol{B}_\mathcal{N}\underline{\boldsymbol{f}}_\mathcal{N}, \tag{5.1}$$

$$\boldsymbol{D}^T_\mathcal{N}\underline{\boldsymbol{u}}_\mathcal{N} = 0, \tag{5.2}$$

where $\underline{\boldsymbol{f}}_\mathcal{N}$ is a vector representation of the $\boldsymbol{f}$ at the nodes $(\hat{\xi}_i, \xi_j)$. The matrices $\boldsymbol{A}_\mathcal{N}$, $\boldsymbol{D}_\mathcal{N}$, and $\boldsymbol{B}_\mathcal{N}$ are block-diagonal matrices with 2 blocks each. The blocks of $\boldsymbol{A}_\mathcal{N}$ are the discrete Laplace operators, and those of $\boldsymbol{D}_\mathcal{N}$ are associated to the different components of the discrete gradient operators, while blocks of $\boldsymbol{B}_\mathcal{N}$ are the mass matrices with respect to each component of $\boldsymbol{f}$.

Eliminating $\underline{\boldsymbol{u}}_\mathcal{N}$ from (5.1)–(5.2), we obtain

$$\underbrace{\boldsymbol{D}^T_\mathcal{N}\boldsymbol{A}^{-1}_\mathcal{N}\boldsymbol{D}_\mathcal{N}}_{\boldsymbol{S}_\mathcal{N}}\ \underline{p}_\mathcal{N} = \boldsymbol{D}^T_\mathcal{N}\boldsymbol{A}^{-1}_\mathcal{N}\boldsymbol{B}_\mathcal{N}\ \underline{\boldsymbol{f}}_\mathcal{N}. \tag{5.3}$$

The matrix $\boldsymbol{S}_\mathcal{N} := \boldsymbol{D}^T_\mathcal{N}\boldsymbol{A}^{-1}_\mathcal{N}\boldsymbol{D}_\mathcal{N}$ is usually referred as the Uzawa matrix. A typical procedure for solving (5.1)–(5.2) is to first solve $\underline{p}_\mathcal{N}$ from (5.3) and then solve $\underline{\boldsymbol{u}}_\mathcal{N}$ from the Poisson equation (5.1) with known $\underline{p}_\mathcal{N}$.

The Uzawa matrix is of dimension $M \times (N-1)$, full, symmetric, and semidefinite. A usual procedure is to use a preconditioned conjugate gradient procedure with the Gauss mass matrix $\tilde{\boldsymbol{B}}_\mathcal{N}$ as a preconditioner [3, 7, 9, 14]. Each outer iteration requires the inversion of two Laplace operators ($\boldsymbol{A}_\mathcal{N}$ matrix), which can be carried out by the fast diagonalization method (see [13]). Hence, the efficiency of the method is dictated by the condition number $\kappa_\mathcal{N}$ of $\tilde{\boldsymbol{B}}^{-1}_\mathcal{N}\boldsymbol{S}_\mathcal{N}$. Another important consequence of the inf-sup constant is that $\kappa_\mathcal{N} = \frac{1}{\beta^2_\mathcal{N}}$ [14].

The first computational investigation is concerned with the sharpness of the lower bound on the inf-sup constant derived in section 3. In the left of Figure 1, we plot

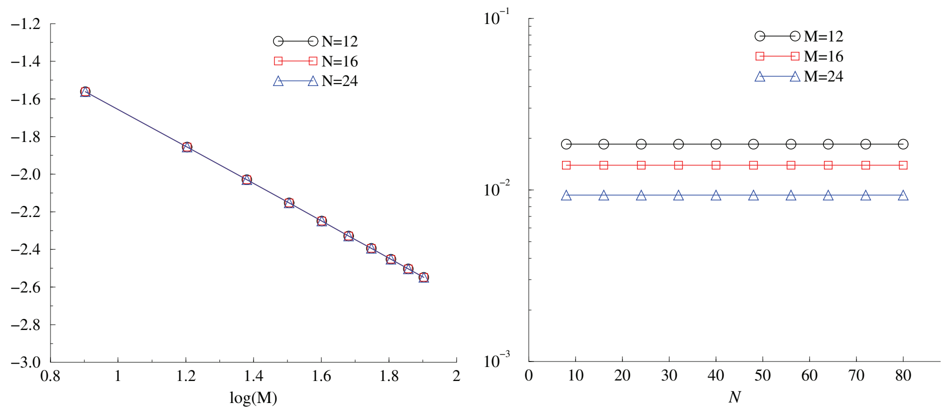



FIG. 1. *Left: inf-sup constant* $\beta_{\mathcal{N}}$ *vs.* $M$ *in* log-log *scale; right: inf-sup constant* $\beta_{\mathcal{N}}$ *vs.* $N$.

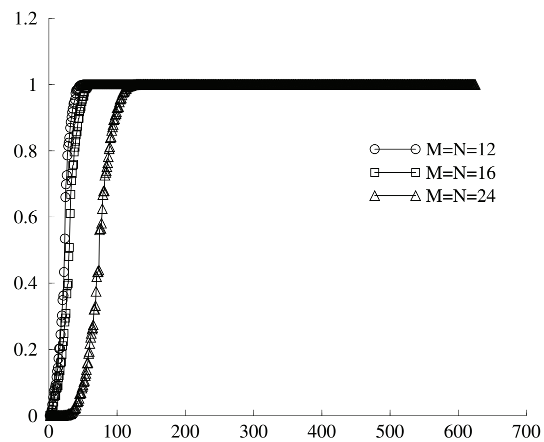



FIG. 2. *Spectra of the Uzawa operator for three different values of* $N$ *with* $M = N$.

the variations of $\beta_{\mathcal{N}}$ versus (vs.) $M$ in log-log scale for several $N$. We observe that $\beta_{\mathcal{N}}$ is independent of $N$ while it decays as $\frac{1}{M}$. In the right of Figure 1, we plot the variations of $\beta_{\mathcal{N}}$ vs. $N$ for several $M$. We observe that $\beta_{\mathcal{N}}$ remains to be constant as we vary $N$ with $M$ fixed. These results are fully consistent with Theorem 3.1, indicating that our estimate for the inf-sup constant is sharp.

In view of inverting the Uzawa operator, the knowledge of the eigenvalues' distribution of the matrix $\tilde{\boldsymbol{B}}_{\mathcal{N}}^{-1}\boldsymbol{S}_{\mathcal{N}}$ may help to design adapted preconditioners for (5.3). The efficiency of the iterating methods depends on how the preconditioners affect the eigenvalues of $\boldsymbol{S}_{\mathcal{N}}$. In Figure 2 we plot all of the eigenvalues of $\tilde{\boldsymbol{B}}_{\mathcal{N}}^{-1}\boldsymbol{S}_{\mathcal{N}}$ for some values of $M = N \in \{12, 16, 24\}$.

The first feature of the spectra is the similarity of their distribution for different values of $N$ with $M = N$. Another interesting aspect is a strong concentration of the eigenvalues around the largest value 1. It is known that this type of clustering is very advantageous for the conjugate gradient iteration since the contribution of the eigenspaces associated with a given multiple eigenvalue is resolved in only one iteration.

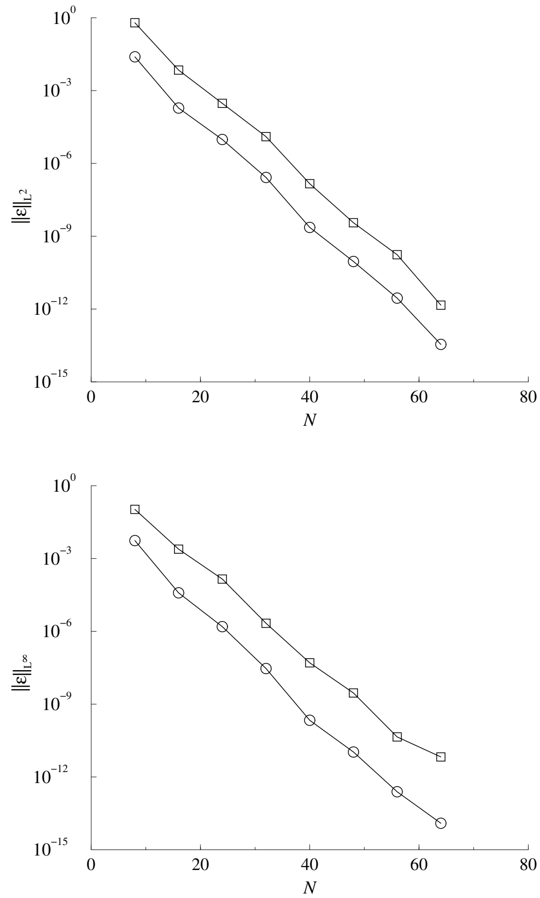



FIG. 3. *The velocity* ($\circ$) *and pressure* ($\square$) *errors as a function of* $N$ *with* $M = N$*: left, in* $L^2$ *norm; right, in* $L^\infty$ *norm.*

We now present some numerical tests to validate the error estimates. We consider the Stokes problem with the following analytical solution:

$$\boldsymbol{u} = \begin{pmatrix} \sin(x)\cos(y)e^{-x} \\ (\sin(x) - \cos(x))\sin(y)e^{-x} \end{pmatrix}, \qquad p = \cos(x)\cos(y)e^{-x}.$$

In Figure 3, we plot, in a semilogarithmic scale, the $L^2$-velocity and the $L^2$-pressure errors (top figure), and the $L^\infty$-velocity and the $L^\infty$-pressure errors (bottom figure) with respect to $N$ with $M = N$. We observe that the errors converge exponentially, which is a typical behavior for spectral methods with analytical solutions.

Finally, in order to justify the use of compatible discrete velocity and pressure spaces, we show via a simple test that the equal-order velocity-pressure approximation $\mathbb{P}_{M,N}(\Omega)^2 \times \mathbb{P}_{M,N}(\Omega)$ is ill-posed. In Figure 4, we present the velocity and the pressure errors in the $L^2$-norm as a function of $N$ with $M = N$. Obviously the pressure fails to converge when the polynomial degree increases. The reason for this failure is that there are spurious pressure modes in the pressure space, similar to the well-known case of the Legendre–Legendre $P_N^2 - P_N$ method for the Stokes problem in a rectangular domain.

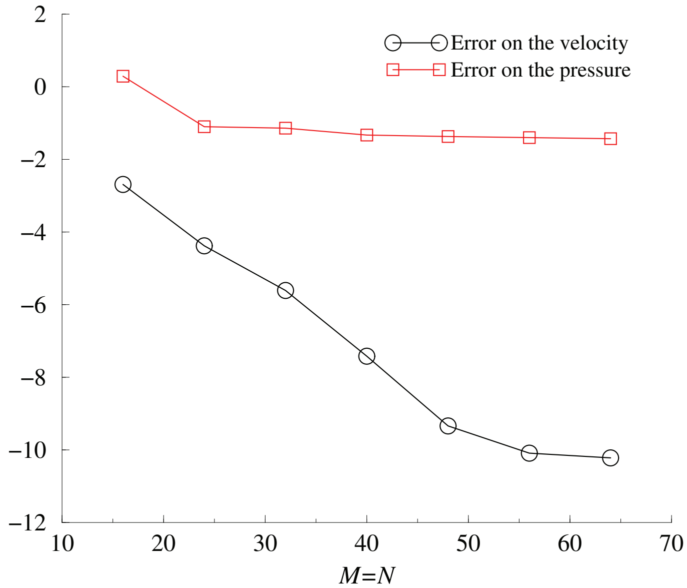



FIG. 4. *The velocity and pressure errors as a function of* $N(M = N)$ *by the incompatible* $\mathbb{P}_{M,N}(\Omega)^2 \times \mathbb{P}_{M,N}(\Omega)$ *method.*

In summary, we have presented a mixed Laguerre–Legendre spectral method for the Stokes problem on a semi-infinite channel. We established the well-posedness of this method by deriving a lower bound on the inf-sup constant and presented numerical results which indicated that the derived lower bound is sharp. We have also derived error estimates by using the inf-sup condition and the Laguerre and Legendre approximation properties.

## REFERENCES


[1] I. Babuška, *The finite element method with Lagrangian multipliers*, Numer. Math., 20 (1972/73), pp. 179–192.

[2] C. Bernardi, M. Dauge, and Y. Maday, *Polynomial in the Sobolev World*, preprint, Laboratoire Jacques-Louis Lions, Paris, 2003, http://www.ann.jussieu.fr/publications/2003/R03038.html.

[3] C. Bernardi and Y. Maday, *Approximations Spectrales de Problèmes aux Limites Elliptiques*, Springer-Verlag, Paris, 1992.

[4] C. Bernardi and Y. Maday, *Spectral method*, in Handb. Numer. Anal. 5 (Part 2), P. G. Ciarlet and L. L. Lions, eds., North-Holland, Amsterdam, 1997.

[5] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[6] F. Brezzi, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 8 (1974), pp. 129–151.

[7] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin, 1987.

[8] O. Coulaud, D. Funaro, and O. Kavian, *Laguerre spectral approximation of elliptic problems in exterior domains*, Comput. Methods Appl. Mech. Engrg., 80 (1990), pp. 451–458.

[9] M. O. Deville, P. F. Fischer, and E. H. Mund, *High-order Methods for Incompressible Fluid Flow*, Camb. Monogr. Appl. Comput. Math. 9, Cambridge University Press, Cambridge, 2002.

[10] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1986.

[11] B. Guo and J. Shen, *Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval*, Numer. Math., 86 (2000), pp. 635–654.
[12] V. Iranzo and A. Falqués, *Some spectral approximations for differential equations in unbounded domains*, Comput. Methods Appl. Mech. Engrg., 98 (1992), pp. 105–126.
[13] R. E. Lynch, J. R. Rice, and D. H. Thomas, *Direct solution of partial differential equations by tensor product methods*, Numer. Math., 6 (1964), pp. 185–199.
[14] Y. Maday, D. Meiron, A. T. Patera, and E. M. Rønquist, *Analysis of iterative methods for the steady and unsteady Stokes problem: Application to spectral element discretizations*, SIAM J. Sci. Comput., 14 (1993), pp. 310–337.
[15] H.-P. Ma and B.-Y. Guo, *Composite Legendre-Laguerre pseudospectral approximation in unbounded domains*, IMA J. Numer. Anal., 21 (2001), pp. 587–602.
[16] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer Ser. Comput. Math. 23, Springer-Verlag, Berlin, 1994.
[17] R. L. Sani and P. M. Gresho, *Résumé and remarks on the open boundary condition minisymposium*, Internat. J. Numer. Methods Fluids, 18 (1994), pp. 983–1008.
[18] J. Shen, *A new fast Chebyshev-Fourier algorithm for the Poisson-type equations in polar geometries*, Appl. Numer. Math., 33 (2000), pp. 183–190.
[19] L. Wang and B. Guo, *Modified Laguerre pseudospectral method refined by multidomain Legendre pseudospectral approximation*, J. Comput. Appl. Math., 190 (2006), pp. 304–324.