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Generalized Jacobi spectral-Galerkin method for nonlinear Volterra integral equations with weakly singular kernels

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Abstract. We propose a generalized Jacobi spectral-Galerkin method for the nonlinear Volterra integral equations (VIEs) with weakly singular kernels. We establish the existence and uniqueness of the numerical solution, and characterize the convergence of the proposed method under reasonable assumptions on the nonlinearity. We also present numerical results which are consistent with the theoretical predictions.

Key Words: Generalized Jacobi spectral-Galerkin method, nonlinear Volterra integral equations with weakly singular kernels, convergence analysis.

AMS Subject Classifications: 45D05, 65L60, 41A30, 65L70

1 Introduction

This paper is concerned with the numerical solutions of the nonlinear Volterra integral equations with weakly singular kernels:

$$y(t)=f(t)+\mathcal{V}y(t):=f(t)+\int_0^t (t-s)^{-\mu}K(t,s)G(s,y(s))ds, \quad t\in I:=[0,T], \tag{1.1}$$

where $0<\mu<1$, $K\in C(D)$ with $D:=\{(t,s):\ 0\le s\le t\le T\}$, $f\in C(I)$ and G is a continuous function.

In recent years, there has been an increasing interest in studying VIEs. The main difficulties for dealing with weakly singular VIEs are: (i) the integral operator is non-local; (ii)

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the solutions are usually singular near $t=0$. Brunner [4] and Lubich [17] investigated the smoothness properties of the exact solutions of VIEs with weakly singular kernels. Various numerical approaches, using the piecewise polynomial collocation methods and the Runge-Kutta methods, have been proposed for approximating VIEs with weakly singular kernels [4, 5, 12, 23]. However, these numerical methods do not particularly deal with the above two difficulties.

Spectral methods are capable of providing exceedingly accurate numerical results with relatively less degree of freedoms, and have been widely used for scientific computation, see, e.g., [1,2,6,13,14,19,20]. Since the spectral methods are global methods, so they could be better suited for non-local problems. Recently, many kinds of spectral collocation methods are proposed for solving VIEs with smooth kernels. Li, Tang and Xu [16] introduced a time parallel method with spectral-subdomain enhancement for VIEs; Sheng, Wang and Guo [21] presented a multistep spectral collocation method for nonlinear VIEs; Wang and Sheng [22] also proposed a multistep spectral collocation method for nonlinear VIEs with delays.

To solve VIEs with weakly singular kernels, many attempts have been made to overcome the difficulties caused by the singularities of the solutions. Chen and Tang [9,10] proposed spectral collocation methods for weakly singular VIEs; Huang, Tang and Zhang [15] studied the supergeometric convergence of spectral collocation methods for weakly singular Volterra/Fredholm integral equations. These methods usually use orthogonal polynomials as basis functions. Another approach for solving weakly singular VIEs is to use the non polynomial singular functions (which reflect the singularities of the exact solutions) as basis functions. For example, Brunner [3] employed a non polynomial spline collocation method for VIEs with weakly singular kernels; Cao, Herdman and Xu [7] presented a non polynomial singularity preserving collocation method for VIEs with weakly singular kernels.

In this paper, we develop a non polynomial spectral-Galerkin method for VIEs with weakly singular kernels. More precisely, we construct a spectral-Galerkin method for weakly singular VIEs (1.1), using the generalized Jacobi functions as basis functions. This kind of basis functions have been used by Zayernouri & Karniadakis [24] and Chen, Shen & Wang [8] for approximating fractional differential equations. The main strategies and contributions are as follows.

- We propose a generalized Jacobi spectral-Galerkin method for nonlinear VIEs with weakly singular kernels. The basis functions can be tuned to match the singularities of the underlying solutions, and lead to an efficient implementation. The existing works (cf. [9, 10])

- We approximate the problem (1.1) directly without any variable transformations, as opposed to the approach in [9, 10] where a spectral-collection method is constructed for the transformed VIEs.

The rest of this paper is organized as follows. In Section 2, we present the generalized Jacobi spectral-Galerkin method for nonlinear VIEs (1.1). Some useful lemmas for the convergence analysis are provided in Section 3. The existence, uniqueness and convergence of the generalized Jacobi spectral-Galerkin method are given in Section 4. We present in Section 5 numerical experiments, which confirm the theoretical expectations. Some concluding remarks are given in the final section.

2 The generalized Jacobi spectral-Galerkin method

In this section, we shall propose a spectral-Galerkin method using generalized Jacobi functions as basis functions for problem (1.1). To this end, we first introduce the shifted Jacobi polynomials and the shifted generalized Jacobi functions on the interval I.

2.1 The shifted Jacobi polynomials on I.

For $\alpha,\beta>-1$, let $J_n^{(\alpha,\beta)}(x)$, $x\in\Lambda:=(-1,1)$ be the standard Jacobi polynomial of degree n, and denote the weight function $\chi^{(\alpha,\beta)}(x)=(1-x)^\alpha(1+x)^\beta$. The set of Jacobi polynomials is a complete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system, i.e.,

$$\int_\Lambda J_m^{(\alpha,\beta)}(x)J_n^{(\alpha,\beta)}(x)\chi^{(\alpha,\beta)}(x)dx=\gamma_m^{(\alpha,\beta)}\delta_{m,n}, \tag{2.1}$$

where $\delta_{m,n}$ is the Kronecker function, and

$$\gamma_m^{\alpha,\beta}=\begin{cases}\dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & m=0,\\ \dfrac{2^{\alpha+\beta+1}}{(2m+\alpha+\beta+1)}\dfrac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!\Gamma(m+\alpha+\beta+1)}, & m\geq 1.\end{cases} \tag{2.2}$$

In particular, $J_0^{(\alpha,\beta)}(x)=1$.

The shifted Jacobi polynomial of degree n is defined by

$$\widetilde{J}_n^{(\alpha,\beta)}(t)=J_n^{(\alpha,\beta)}(\frac{2t}{T}-1),\quad t\in I. \tag{2.3}$$

Clearly, the set of $\{\widetilde{J}_n^{(\alpha,\beta)}(t)\}_{n\geq 0}$ is a complete $L^2_{\chi_T^{(\alpha,\beta)}}(I)$-orthogonal system with the weight function $\chi_T^{(\alpha,\beta)}(t)=(T-t)^\alpha t^\beta$. In fact, by (2.1) and (2.3) we know that

$$\int_I \widetilde{J}_m^{(\alpha,\beta)}(t)\widetilde{J}_n^{(\alpha,\beta)}(t)\chi_T^{(\alpha,\beta)}(t)dt=\big(\frac{T}{2}\big)^{\alpha+\beta+1}\gamma_m^{(\alpha,\beta)}\delta_{m,n}. \tag{2.4}$$

For any integer $N\geq 0$, we denote by $\{x_j^{(\alpha,\beta)},\omega_j^{(\alpha,\beta)}\}_{j=0}^N$ the nodes and the corresponding Christoffel numbers of the standard Jacobi-Gauss interpolation on the interval Λ. Let

$\mathcal{P}_N(I)$ be the set of polynomials of degree at most N on the interval I, and $t_j^{(\alpha,\beta)}$ be the shifted Jacobi-Gauss quadrature nodes on the interval I,

$$t_j^{(\alpha,\beta)} = \frac{T}{2}(x_j^{(\alpha,\beta)}+1), \qquad 0 \le j \le N. \tag{2.5}$$

Due to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\phi(t) \in \mathcal{P}_{2N+1}(I)$,

$$\begin{aligned}\int_I \phi(t)\chi_T^{(\alpha,\beta)}(t)dt &= (\frac{T}{2})^{\alpha+\beta+1}\int_{-1}^1 \phi(\frac{T}{2}(x+1))\chi^{(\alpha,\beta)}(x)dx \\ &= (\frac{T}{2})^{\alpha+\beta+1}\sum_{j=0}^N \phi(\frac{T}{2}(x_j^{(\alpha,\beta)}+1))\omega_j^{(\alpha,\beta)} \\ &= (\frac{T}{2})^{\alpha+\beta+1}\sum_{j=0}^N \phi(t_j^{(\alpha,\beta)})\omega_j^{(\alpha,\beta)}.\end{aligned} \tag{2.6}$$

By (2.4) and (2.6), we further obtain that for any $0 \le m+n \le 2N+1$,

$$\sum_{j=0}^N \widetilde{J}_m^{(\alpha,\beta)}(t_j^{(\alpha,\beta)})\widetilde{J}_n^{(\alpha,\beta)}(t_j^{(\alpha,\beta)})\omega_j^{(\alpha,\beta)} = \gamma_m^{(\alpha,\beta)}\delta_{m,n}. \tag{2.7}$$

2.2 The shifted generalized Jacobi functions on I.

The shifted generalized Jacobi function of degree n is defined by (cf. [8])

$$P_n^{(\alpha,\beta)}(t) := t^\beta \widetilde{J}_n^{(\alpha,\beta)}(t), \qquad \alpha,\beta > -1, \qquad t \in I. \tag{2.8}$$

Let $\mathcal{F}_N^{(\beta)}(I)$ be the finite-dimensional fractional polynomial space (cf. [8])

$$\mathcal{F}_N^{(\beta)}(I) := \{t^\beta \psi(t) : \psi(t) \in \mathcal{P}_N(I)\} = \mathrm{span}\{P_n^{(\alpha,\beta)} : 0 \le n \le N\}. \tag{2.9}$$

Due to (2.4) and (2.8), it is clear that the set of $\{P_n^{(\alpha,\beta)}(t)\}_{n\ge 0}$ is a complete $L^2_{\chi_T^{(\alpha,-\beta)}}(I)$-orthogonal system with the weight function $\chi_T^{(\alpha,-\beta)}(t)$, namely,

$$\begin{aligned}\int_I P_m^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(t)\chi_T^{(\alpha,-\beta)}(t)dt &= \int_I t^{2\beta}\widetilde{J}_m^{(\alpha,\beta)}(t)\widetilde{J}_n^{(\alpha,\beta)}(t)\chi_T^{(\alpha,-\beta)}(t)dt \\ &= \int_I \widetilde{J}_m^{(\alpha,\beta)}(t)\widetilde{J}_n^{(\alpha,\beta)}(t)\chi_T^{(\alpha,\beta)}(t)dt = (\frac{T}{2})^{\alpha+\beta+1}\gamma_m^{(\alpha,\beta)}\delta_{m,n}.\end{aligned} \tag{2.10}$$

Because of (2.6), it follows that for any $\varphi(t) = t^{2\beta}\phi(t)$ with $\phi(t) \in \mathcal{P}_{2N+1}(I)$,

$$\begin{aligned}\int_I \varphi(t)\chi_T^{(\alpha,-\beta)}(t)dt &= \int_I \phi(t)\chi_T^{(\alpha,\beta)}(t)dt = (\frac{T}{2})^{\alpha+\beta+1}\sum_{j=0}^N \phi(t_j^{(\alpha,\beta)})\omega_j^{(\alpha,\beta)} \\ &= (\frac{T}{2})^{\alpha+\beta+1}\sum_{j=0}^N (t_j^{(\alpha,\beta)})^{-2\beta}\varphi(t_j^{(\alpha,\beta)})\omega_j^{(\alpha,\beta)}.\end{aligned} \tag{2.11}$$

Next, let $(u,v)_{\chi_T^{(\alpha,-\beta)}}$ and $\|v\|_{\chi_T^{(\alpha,-\beta)}}$ be the inner product and norm of space $L^2_{\chi_T^{(\alpha,-\beta)}}(I)$, respectively. We also introduce the following discrete inner product and norm on the interval I,

$$\begin{aligned}&\langle u,v\rangle_{\chi_T^{(\alpha,-\beta)}}=\Big(\frac{T}{2}\Big)^{\alpha+\beta+1}\sum_{j=0}^{N}(t_j^{(\alpha,\beta)})^{-2\beta}u(t_j^{(\alpha,\beta)})v(t_j^{(\alpha,\beta)})\omega_j^{(\alpha,\beta)},\\&\|v\|_{N,\chi_T^{(\alpha,-\beta)}}=\langle v,v\rangle^{\frac{1}{2}}_{\chi_T^{(\alpha,-\beta)}}.\end{aligned}\tag{2.12}$$

Thanks to (2.11), for any $\phi,\psi\in\mathcal{F}_N^{(\beta)}(I)$,

$$(\phi,\psi)_{\chi_T^{(\alpha,-\beta)}}=\langle\phi,\psi\rangle_{\chi_T^{(\alpha,-\beta)}},\qquad \|\phi\|_{\chi_T^{(\alpha,-\beta)}}=\|\phi\|_{N,\chi_T^{(\alpha,-\beta)}}.\tag{2.13}$$

2.3 The generalized Jacobi spectral-Galerkin method for problem (1.1)

To describe the spectral-Galerkin scheme for problem (1.1), we first transform the integral interval $[0,t]$ to I using the transformation:

$$s=\frac{t\tau}{T},\qquad \tau\in I.\tag{2.14}$$

Then the equation (1.1) becomes

$$y(t)=f(t)+\mathcal{V}y(t)=f(t)+\Big(\frac{t}{T}\Big)^{1-\mu}\int_I(T-\tau)^{-\mu}K\big(t,\frac{t\tau}{T}\big)G\big(\frac{t\tau}{T},y(\frac{t\tau}{T})\big)d\tau.\tag{2.15}$$

The generalized Jacobi spectral-Galerkin scheme is to seek $Y(t)\in\mathcal{F}_N^{(1-\mu)}(I)$, such that

$$(Y,\varphi)_{\chi_T^{(-\mu,\mu-1)}}=(f,\varphi)_{\chi_T^{(-\mu,\mu-1)}}+(\mathcal{V}Y,\varphi)_{\chi_T^{(-\mu,\mu-1)}},\qquad \forall\varphi\in\mathcal{F}_N^{(1-\mu)}(I).\tag{2.16}$$

We now describe a numerical implementation for (2.16). To this end, we set

$$Y(t)=\sum_{m=0}^{N}y_mP_m^{(-\mu,1-\mu)}(t).\tag{2.17}$$

Substituting (2.17) into (2.16) and taking $\varphi=P_n^{(-\mu,1-\mu)}(t)$, we obtain that for $0\le n\le N$,

$$\sum_{m=0}^{N}y_m(P_m^{(-\mu,1-\mu)},P_n^{(-\mu,1-\mu)})_{\chi_T^{(-\mu,\mu-1)}}=(f,P_n^{(-\mu,1-\mu)})_{\chi_T^{(-\mu,\mu-1)}}+(\mathcal{V}Y,P_n^{(-\mu,1-\mu)})_{\chi_T^{(-\mu,\mu-1)}}.\tag{2.18}$$

Set

$$\begin{aligned}&\mathbf{y}=(y_0,\cdots,y_N)^T,\qquad A=(a_{nm})_{0\le n,m\le N},\\&a_{nm}=(P_m^{(-\mu,1-\mu)},P_n^{(-\mu,1-\mu)})_{\chi_T^{(-\mu,\mu-1)}}=\Big(\frac{T}{2}\Big)^{2-2\mu}\gamma_m^{(-\mu,1-\mu)}\delta_{m,n},\\&f_n=(f,P_n^{(-\mu,1-\mu)})_{\chi_T^{(-\mu,\mu-1)}},\qquad \mathbf{f}=(f_0,\cdots,f_N)^T,\\&v_n(\mathbf{y})=(\mathcal{V}Y,P_n^{(-\mu,1-\mu)})_{\chi_T^{(-\mu,\mu-1)}},\qquad \mathbf{v}(\mathbf{y})=(v_0,\cdots,v_N)^T.\end{aligned}\tag{2.19}$$

Then, the system (2.18) becomes

$$A\mathbf{y}=\mathbf{f}+\mathbf{v}(\mathbf{y}). \tag{2.20}$$

In actual computation, we use the quadrature formula (2.12) to approximate the terms f_n and v_n, namely,

$$\begin{aligned} f_n &\approx \langle f, P_n^{(-\mu,1-\mu)}\rangle_{\chi_T^{(-\mu,\mu-1)}} \\ &= \Big(\frac{T}{2}\Big)^{2-2\mu} \sum_{j=0}^{N} (t_j^{(-\mu,1-\mu)})^{2\mu-2} f(t_j^{(-\mu,1-\mu)}) P_n^{(-\mu,1-\mu)}(t_j^{(-\mu,1-\mu)}) \omega_j^{(-\mu,1-\mu)}, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} v_n(\mathbf{y}) \approx & \frac{T^{2-2\mu}}{2^{3-3\mu}} \sum_{i,j=0}^{N} (t_i^{(-\mu,1-\mu)})^{\mu-1} K(t_i^{(-\mu,1-\mu)}, t_i^{(-\mu,1-\mu)} t_j^{(-\mu,0)}/T) \\ & \cdot G(t_i^{(-\mu,1-\mu)} t_j^{(-\mu,0)}/T, Y(t_i^{(-\mu,1-\mu)} t_j^{(-\mu,0)}/T)) P_n^{(-\mu,1-\mu)}(t_i^{(-\mu,1-\mu)}) \omega_i^{(-\mu,1-\mu)} \omega_j^{(-\mu,0)}. \end{aligned} \tag{2.22}$$

This is a (nonlinear) implicit scheme, which can be solved, for instance, by the Newton iterative method.

3 Some useful lemmas

In this section, we present some useful lemmas. For this purpose, we first recall the definitions of the fractional integrals and fractional derivatives in the sense of Riemann-Liouville (see, e.g., [11,18]).

Definition 3.1. (Fractional integrals and derivatives). For $\rho \in \mathbb{R}^+$, the left and right fractional integrals are respectively defined as

$${}_aI_x^{\rho}u(x)=\frac{1}{\Gamma(\rho)}\int_a^x \frac{u(y)}{(x-y)^{1-\rho}}dy, \quad x>a; \quad {}_xI_b^{\rho}u(x)=\frac{1}{\Gamma(\rho)}\int_x^b \frac{u(y)}{(y-x)^{1-\rho}}dy, \quad x<b, \tag{3.1}$$

where $\Gamma(\cdot)$ is the usual Gamma function.

For $s\in[k-1,k)$ with $k\in\mathbb{N}$, the left-sided Riemann-Liouville fractional derivative of order s is defined by

$${}_aD_x^{s}u(x)=\frac{1}{\Gamma(k-s)}\frac{d^k}{dx^k}\int_a^x \frac{u(y)}{(x-y)^{s-k+1}}dy, \quad x\in(a,b), \tag{3.2}$$

and the right-sided Riemann-Liouville fractional derivative of order s is defined by

$${}_xD_b^{s}u(x)=\frac{(-1)^k}{\Gamma(k-s)}\frac{d^k}{dx^k}\int_x^b \frac{u(y)}{(y-x)^{s-k+1}}dy, \quad x\in(a,b). \tag{3.3}$$

It is clear that for any $k\in\mathbb{N}_0$,

$$ {}_aD_x^k=D^k,\quad {}_xD_b^k=(-1)^kD^k,\quad \text{where}\quad D^k:=\frac{d^k}{dx^k}. $$

Thus, we can define the Riemann-Liouville fractional derivatives as

$$ {}_aD_x^s u(x)=D^k\,{}_aI_x^{k-s}u(x),\quad {}_xD_b^s=(-1)^kD^k\,{}_xI_b^{k-s}u(x). $$

According to Theorem 2.14 of [11], we have that for any absolutely integrable function u and real $s\geq 0$,

$$ {}_aD_x^s\,{}_aI_x^s u(x)=u(x),\quad {}_xD_b^s\,{}_xI_b^s u(x)=u(x),\quad x\in(a,b). \tag{3.4} $$

Next, let

$$ \widetilde{\mathcal{F}}_N^{(\beta)}(\Lambda):=\{(1+x)^\beta\psi(x):\psi(x)\in\mathcal{P}_N(\Lambda)\}=\text{span}\{(1+x)^\beta J_n^{(\alpha,\beta)}(x):0\leq n\leq N\}, $$

and

$$ \widetilde{\mathcal{B}}_{\alpha,\beta}^r(\Lambda):=\{u(x):u\in L^2_{\chi^{(\alpha,-\beta)}}(\Lambda),\ {}_{-1}D_x^{\beta+l}u\in L^2_{\chi^{(\alpha+\beta+l,l)}}(\Lambda)\quad\text{for}\quad 0\leq l\leq r\},\quad r\in\mathbb{N}_0. $$

Denote by c a generic positive constant independent of T, N and the solutions of $y(t)$ and $Y(t)$. According to Theorem 4.3 of [8], we have

Lemma 3.1. *Let $\alpha>-1$, $\beta>0$, for any $u\in\widetilde{\mathcal{B}}_{\alpha,\beta}^r(\Lambda)$ with integer $0\leq r\leq N$, we have*

$$ \|\widetilde{\pi}_N^{(\alpha,\beta)}u-u\|_{L^2_{\chi^{(\alpha,-\beta)}}(\Lambda)}\leq cN^{-(\beta+r)}\|{}_{-1}D_x^{\beta+r}u\|_{L^2_{\chi^{(\alpha+\beta+r,r)}}(\Lambda)}, \tag{3.5} $$

where $\widetilde{\pi}_N^{(\alpha,\beta)}$ is the standard $L^2_{\chi^{(\alpha,-\beta)}}(\Lambda)$-orthogonal projection upon $\widetilde{\mathcal{F}}_N^{(\beta)}(\Lambda)$, defined by

$$ \int_\Lambda(\widetilde{\pi}_N^{(\alpha,\beta)}u(x)-u(x))\psi(x)\chi^{(\alpha,-\beta)}(x)dx=0,\quad \forall\psi\in\widetilde{\mathcal{F}}_N^{(\beta)}(\Lambda). \tag{3.6} $$

Similarly, we define

$$ \begin{aligned} &\mathcal{B}_{\alpha,\beta}^r(I):=\{v(t):v\in L^2_{\chi_T^{(\alpha,-\beta)}}(I),\ {}_0D_t^{\beta+l}v\in L^2_{\chi_T^{(\alpha+\beta+l,l)}}(I)\quad\text{for}\quad 0\leq l\leq r\},\quad r\in\mathbb{N}_0,\\ &\mathcal{H}_{\alpha,\beta}^r(I):=\{v(t):v\in L^2_{\chi_T^{(\alpha,-\beta)}}(I),\ {}_0D_t^{\beta+l}v\in L^2_{\chi_T^{(\alpha,-\beta)}}(I)\quad\text{for}\quad 0\leq l\leq r\},\quad r\in\mathbb{N}_0. \end{aligned} $$

Denote by $\pi_N^{(\alpha,\beta)}$ the $L^2_{\chi_T^{(\alpha,-\beta)}}(I)$-orthogonal projection upon $\mathcal{F}_N^{(\beta)}(I)$,

$$ (\pi_N^{(\alpha,\beta)}v-v,\phi)_{\chi_T^{(\alpha,-\beta)}}=0,\quad \forall\phi\in\mathcal{F}_N^{(\beta)}(I). \tag{3.7} $$

By Lemma 3.1, we obtain the following results.

Lemma 3.2. *Let $\alpha>-1$, $\beta>0$. For any $v\in\mathcal{B}^r_{\alpha,\beta}(I)$ with integer $0\le r\le N$, we have*

$$\|\pi_N^{(\alpha,\beta)}v-v\|_{\chi_T^{(\alpha,-\beta)}}\le cT^{-\beta}N^{-(\beta+r)}\|{}_0D_t^{\beta+r}v\|_{\chi_T^{(\alpha+\beta+r,r)}}. \tag{3.8}$$

In particular, if $v\in\mathcal{H}^r_{\alpha,\beta}(I)$, then

$$\|\pi_N^{(\alpha,\beta)}v-v\|_{\chi_T^{(\alpha,-\beta)}}\le cT^{r}N^{-(\beta+r)}\|{}_0D_t^{\beta+r}v\|_{\chi_T^{(\alpha,-\beta)}}. \tag{3.9}$$

Proof. Set $u(x):=v(t)\Big|_{t=\frac{T}{2}(x+1)}$. Since $\pi_N^{(\alpha,\beta)}v(t)\Big|_{t=\frac{T}{2}(x+1)}$ and $\widetilde{\pi}_N^{(\alpha,\beta)}u(x)$ belong to $\widetilde{\mathcal{F}}_N^{(\beta)}(\Lambda)$ in the variable x, and hence by the definitions (3.6) and (3.7),

$$\pi_N^{(\alpha,\beta)}v(t)\Big|_{t=\frac{T}{2}(x+1)}=\widetilde{\pi}_N^{(\alpha,\beta)}u(x). \tag{3.10}$$

The above with (3.5) and (3.2) yields

$$\begin{aligned}\|\pi_N^{(\alpha,\beta)}v-v\|^2_{\chi_T^{(\alpha,-\beta)}}&=\Big(\frac{T}{2}\Big)^{\alpha-\beta+1}\int_\Lambda(\widetilde{\pi}_N^{(\alpha,\beta)}u(x)-u(x))^2(1-x)^\alpha(1+x)^{-\beta}dx\\&\le cT^{\alpha-\beta+1}N^{-2(\beta+r)}\int_\Lambda\big({}_{-1}D_x^{\beta+r}u(x)\big)^2(1-x)^{\alpha+\beta+r}(1+x)^r dx\\&\le cT^{-2\beta}N^{-2(\beta+r)}\int_I\big({}_0D_t^{\beta+r}v(t)\big)^2(T-t)^{\alpha+\beta+r}t^r dt.\end{aligned} \tag{3.11}$$

This leads to the result (3.8). Furthermore,

$$\int_I\big({}_0D_t^{\beta+r}v(t)\big)^2(T-t)^{\alpha+\beta+r}t^r dt\le(\frac{T}{2})^{2(\beta+r)}\int_I\big({}_0D_t^{\beta+r}v(t)\big)^2(T-t)^{\alpha}t^{-\beta}dt.$$

This leads to the result (3.9). □

4 Existence, uniqueness and error estimate

In this section, we first verify the existence and uniqueness of the solution of (2.16), and then we analyze and characterize the convergence of scheme (2.16) under reasonable assumptions on the nonlinearity.

Theorem 4.1. *Assume that $K(t,s)\in C(D)$ and G satisfies the following Lipschitz condition:*

$$|G(s,y_1)-G(s,y_2)|\le\gamma|y_1-y_2|,\quad \gamma\ge 0. \tag{4.1}$$

Then, for T sufficiently small such that

$$cT^{2-2\mu}\le\beta<1, \tag{4.2}$$

the equation (2.16) *possesses a unique solution.*

Proof. We first prove the existence. Consider the following iteration process:

$$(Y^{(m)},\varphi)_{\chi_T^{(-\mu,\mu-1)}}=(f,\varphi)_{\chi_T^{(-\mu,\mu-1)}}+(\mathcal{V}Y^{(m-1)},\varphi)_{\chi_T^{(-\mu,\mu-1)}},\qquad \forall\varphi\in\mathcal{F}_N^{(1-\mu)}(I). \tag{4.3}$$

According to the definition (3.7) of the projection operator $\pi_N^{(-\mu,1-\mu)}$, we know from (4.3) that

$$Y^{(m)}=\pi_N^{(-\mu,1-\mu)}\big(f+\mathcal{V}Y^{(m-1)}\big). \tag{4.4}$$

Next, let $\widetilde{Y}^{(m)}=Y^{(m)}-Y^{(m-1)}$. Then, by (4.4) we get

$$\widetilde{Y}^{(m)}=\pi_N^{(-\mu,1-\mu)}\big(\mathcal{V}Y^{(m-1)}-\mathcal{V}Y^{(m-2)}\big).$$

This, along with the projection theorem, implies

$$\begin{aligned}\|\widetilde{Y}^{(m)}\|^2_{\chi_T^{(-\mu,\mu-1)}}&=\|\pi_N^{(-\mu,1-\mu)}\big(\mathcal{V}Y^{(m-1)}-\mathcal{V}Y^{(m-2)}\big)\|^2_{\chi_T^{(-\mu,\mu-1)}}\\&\le\|\mathcal{V}Y^{(m-1)}-\mathcal{V}Y^{(m-2)}\|^2_{\chi_T^{(-\mu,\mu-1)}}.\end{aligned} \tag{4.5}$$

Further, by (4.5), (4.1), (2.1), the Cauchy-Schwarz inequality and using the tranformation $t=s+\frac{(\tau+1)(T-s)}{2}$, we get that

$$\begin{aligned}&\|\widetilde{Y}^{(m)}\|^2_{\chi_T^{(-\mu,\mu-1)}}\\&\le\int_I\Big(\int_0^t(t-s)^{-\mu}K(t,s)\big(G(s,Y^{(m-1)}(s))-G(s,Y^{(m-2)}(s))\big)ds\Big)^2(T-t)^{-\mu}t^{\mu-1}dt\\&\le c\int_I\Big[\int_0^t(t-s)^{-\mu}ds\int_0^t(t-s)^{-\mu}\big(\widetilde{Y}^{(m-1)}(s)\big)^2ds\Big](T-t)^{-\mu}t^{\mu-1}dt\\&\le c\int_I(T-t)^{-\mu}\int_0^t(t-s)^{-\mu}\big(\widetilde{Y}^{(m-1)}(s)\big)^2dsdt\\&\le c\int_I\big(\widetilde{Y}^{(m-1)}(s)\big)^2\int_s^T(T-t)^{-\mu}(t-s)^{-\mu}dtds\\&\le c\int_I\big(\widetilde{Y}^{(m-1)}(s)\big)^2\int_{-1}^1(1-\tau)^{-\mu}(1+\tau)^{-\mu}(T-s)^{1-2\mu}d\tau ds\\&\le c\int_I\big(\widetilde{Y}^{(m-1)}(s)\big)^2(T-s)^{1-2\mu}ds\le cT^{2-2\mu}\int_I\big(\widetilde{Y}^{(m-1)}(s)\big)^2(T-s)^{-\mu}s^{\mu-1}ds\\&=cT^{2-2\mu}\|\widetilde{Y}^{(m-1)}\|^2_{\chi_T^{(-\mu,\mu-1)}}.\end{aligned} \tag{4.6}$$

Thus, if $cT^{2-2\mu}\le\beta<1$, then $\|\widetilde{Y}^{(m)}\|_{\chi_T^{(-\mu,\mu-1)}}\to0$ as $m\to\infty$. This implies the existence of solution of (2.16). It is easy to prove the uniqueness of solution of (2.16). □

Theorem 4.2. *Assume that $K(t,s)\in C(D)$, $y\in\mathcal{B}^r_{-\mu,1-\mu}(I)$ with integer $0\le r\le N$, G fulfills the Lipschitz condition* (4.1) *and T is sufficiently small satisfying the condition* (4.2). *Then, there holds*

$$\|y-Y\|_{\chi_T^{(-\mu,\mu-1)}}\le cT^{\mu-1}N^{\mu-r-1}\|{}_0D_t^{-\mu+r+1}y\|_{\chi_T^{(1-2\mu+r,r)}}. \tag{4.7}$$

In particular, if $y\in\mathcal{H}^r_{-\mu,1-\mu}(I)$, then

$$\|y-Y\|_{\chi_T^{(-\mu,\mu-1)}}\le cT^rN^{\mu-r-1}\|{}_0D_t^{-\mu+r+1}y\|_{\chi_T^{(-\mu,\mu-1)}}. \tag{4.8}$$

Proof. By (2.16) we know

$$Y=\pi_N^{(-\mu,1-\mu)}\big(f+\mathcal{V}Y\big). \tag{4.9}$$

Subtracting (4.9) from (1.1) yields

$$y-Y=f-\pi_N^{(-\mu,1-\mu)}f+\mathcal{V}y-\pi_N^{(-\mu,1-\mu)}\mathcal{V}Y.$$

Using (1.1) again, we obtain

$$f-\pi_N^{(-\mu,1-\mu)}f=y-\pi_N^{(-\mu,1-\mu)}y+\pi_N^{(-\mu,1-\mu)}\mathcal{V}y-\mathcal{V}y.$$

A combination of the previous two equalities leads to

$$y-Y=y-\pi_N^{(-\mu,1-\mu)}y+\pi_N^{(-\mu,1-\mu)}\big(\mathcal{V}y-\mathcal{V}Y\big). \tag{4.10}$$

This, together with the projection theorem, gives

$$\begin{aligned}\|y-Y\|_{\chi_T^{(-\mu,\mu-1)}}&\le\|y-\pi_N^{(-\mu,1-\mu)}y\|_{\chi_T^{(-\mu,\mu-1)}}+\|\pi_N^{(-\mu,1-\mu)}\big(\mathcal{V}y-\mathcal{V}Y\big)\|_{\chi_T^{(-\mu,\mu-1)}}\\&\le\|y-\pi_N^{(-\mu,1-\mu)}y\|_{\chi_T^{(-\mu,\mu-1)}}+\|\mathcal{V}y-\mathcal{V}Y\|_{\chi_T^{(-\mu,\mu-1)}}.\end{aligned} \tag{4.11}$$

Next, by an argument similar to (4.6) we deduce that

$$\|\mathcal{V}y-\mathcal{V}Y\|^2_{\chi_T^{(-\mu,\mu-1)}}\le cT^{2-2\mu}\|y-Y\|^2_{\chi_T^{(-\mu,\mu-1)}}. \tag{4.12}$$

Moreover, by (3.8) we get that for $y\in\mathcal{B}^r_{-\mu,1-\mu}(I)$ with integer $0\le r\le N$,

$$\|y-\pi_N^{(-\mu,1-\mu)}y\|_{\chi_T^{(-\mu,\mu-1)}}\le cT^{\mu-1}N^{\mu-r-1}\|{}_0D_t^{-\mu+r+1}y\|_{\chi_T^{(1-2\mu+r,r)}}. \tag{4.13}$$

Therefore, by (4.11) - (4.13) and (4.2), we obtain the desired result (4.7). Finally, by (3.9) and a similar argument, we derive the result (4.8). $\square$

Remark 4.1. The Lipschitz condition (4.1) appears to be necessary for our convergence analysis. However, some numerical experiments below show that, even if the Lipschitz condition is not satisfied, the scheme is still convergent.

5 Numerical Results

In this section, we present some numerical results to illustrate the efficiency of the generalized Jacobi spectral-Galerkin method.

5.1 Linear problem

Consider first the linear VIE with weakly singular kernel (cf. [9]):

$$y(t)=f(t)-\int_0^t (t-s)^{-0.35}y(s)ds, \quad t\in[0,6], \tag{5.1}$$

where $f(t)=t^{3.6}+t^{4.25}B(4.6,0.65)$ and $B(\cdot,\cdot)$ is the Beta function defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\left(\frac{1}{2}\right)^{x+y-1}\gamma_0^{(y-1,x-1)}.$$

The exact solution is $y(t)=t^{3.6}$. In Figure 5.1, we list the discrete $L^2_{\chi_T^{(-\mu,\mu-1)}}(I)$-errors and the maximum errors of (5.1), they indicate the algebraic convergence. In fact, a direct computation shows that $y\in\mathcal{B}^r_{-\mu,1-\mu}(I)$ with $\mu=0.35$ and $r=6$. Hence, according to (4.7), we can expect a convergence rate for the $L^2_{\chi_T^{(-\mu,\mu-1)}}(I)$-norm to be of the order $r-\mu+1=6.65$. The observed convergence rate for the $L^2_{\chi_T^{(-\mu,\mu-1)}}(I)$-norm plotted in Figure 5.1 is about 8.3.

In Table 5.1 below, we compare the maximum errors of our algorithm with that of the collocation method suggested in [9] (see Table 1 of [9]). We observe that our method provides more accurate numerical results.

Figure 5.1: The numerical errors of (5.1).

Figure 5.2: The numerical errors of (5.2).

Table 5.1: A comparison of $L^\infty(I)$-errors for (5.1).

N	2	4	6	8
Ref. [9]	6.7887e+01	2.4594e-01	1.3307e-02	1.9500e-03
Our method	6.5755e+00	3.1740e-03	1.1957e-04	1.4431e-05
N	10	12	14	16
Ref. [9]	4.4826e-04	1.3478e-04	4.8583e-05	1.9980e-05
Our method	2.8175e-06	7.2733e-07	2.2795e-07	8.2480e-08

Consider next the linear VIE with weakly singular kernel:

$$y(t)=f(t)+\int_0^t (t-s)^{-1/3}y(s)ds, \quad t\in[0,1]. \tag{5.2}$$

We choose f such that the solution y of (5.2) is given by $y(t)=t^{2/3}\cos(t)$. It can be verified readily that $y\in\mathcal{B}^{\infty}_{-\mu,1-\mu}(I)$ with $\mu=1/3$, so Theorem 4.2 predicts that the errors of the generalized Jacobi spectral approximation will decrease faster than any algebraic rate.

In Figure 5.2, we list the discrete $L^2_{\chi_T^{(-\mu,\mu-1)}}(I)$-errors and the maximum errors of (5.2). We observe that the numerical errors decay exponentially as N increases.

5.2 Nonlinear problem

Consider first the nonlinear VIE with weakly singular kernel:

$$y(t)=f(t)+\int_0^t (t-s)^{-1/2}\exp\big(-s^{-1/2}y(s)/2\big)ds, \quad t\in[0,1]. \tag{5.3}$$

We choose f such that the solution y of (5.3) is given by $y(t)=t^{1/2}\ln(t+e)$. Clearly, the exact solution $y\in\mathcal{B}^{\infty}_{-1/2,1/2}(I)$. However, the Lipschitz condition (4.1) is not satisfied for problem (5.3).

In Figure 5.3, we plot the discrete $L^2_{\chi_T^{(-\mu,\mu-1)}}(I)$-errors with $\mu=1/2$ and the maximum errors of (5.3). It is shown that the numerical errors decay exponentially as N increases. This means that our algorithm is still valid for problem (5.3), even if the Lipschitz condition (4.1) is not satisfied.

Figure 5.3: The numerical errors of (5.3).

Figure 5.4: The numerical errors of (5.4).

Consider next the nonlinear VIE with weakly singular kernels:

$$y(t)=\sqrt{t}\exp(t)+\frac{4}{3}t^{3/2}-\int_0^t (t-s)^{-1/2}\exp(-2s)y^2(s)ds, \quad t\in[0,1], \tag{5.4}$$

with the exact solution $y(t)=\sqrt{t}\exp(t)$. Clearly, the exact solution $y\in\mathcal{B}^{\infty}_{-1/2,1/2}(I)$. Moreover, the Lipschitz condition (4.1) is not satisfied for problem (5.4).

In Figure 5.4, we list the discrete $L^2_{\chi_T^{(-\mu,\mu-1)}}(I)$-errors with $\mu=1/2$ and the maximum errors of (5.4). They also indicate that the numerical errors decay exponentially as N increases.

6 Concluding Remarks

In this paper, we proposed a generalized Jacobi spectral-Galerkin method for the nonlinear VIEs with weakly singular kernel. This method can be implemented efficiently. We showed the existence and uniqueness of the numerical solution and proved its convergence rate under reasonable assumptions on the nonlinearity. Numerical experiments demonstrate that the proposed method are very effective for dealing with linear and nonlinear VIEs.

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