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MÜNTZ–GALERKIN METHODS AND APPLICATIONS TO MIXED DIRICHLET–NEUMANN BOUNDARY VALUE PROBLEMS*

JIE SHEN† AND YINGWEI WANG†

Abstract. Solutions for many problems of interest exhibit singular behaviors at domain corners or points where the boundary condition changes type. For these types of problems, direct spectral methods with the usual polynomial basis functions do not lead to a satisfactory convergence rate. We develop in this paper a Müntz–Galerkin method which is based on specially tuned Müntz polynomials to deal with the singular behaviors of the underlying problems. By exploring the relations between Jacobi polynomials and Müntz polynomials, we develop efficient implementation procedures for the Müntz–Galerkin method, and provide optimal error estimates. As an example of applications, we consider the Poisson equation with mixed Dirichlet–Neumann boundary conditions, whose solution behaves like $O(r^{1/2})$ near the singular point, and demonstrate that the Müntz–Galerkin method greatly improves the rates of convergence of the usual spectral method.

Key words. spectral-Galerkin method, Müntz polynomial, error estimate, singular solution

AMS subject classifications. 65N35, 65F05, 65F30

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1. Introduction.

Spectral methods have been extensively used in numerical solutions of differential equations, function approximations, and other variational problems [7, 4, 9, 21, 22]. For usual spectral methods based on polynomials, their convergence rate is only limited by the smoothness of the problem, so they are particularly effective for problems with smooth solutions. However, many problems of interest exhibit singular behaviors at domain corners or points where the boundary condition changes type, so direct application of spectral methods to these types of problems does not yield a satisfactory convergence rate.

Within the finite element framework, methods to deal with singular solutions can be classified into two categories: (i) one is based on local adaptivity [17] and (ii) the other is the so called *extended or generalized finite element method* [2, 6] in which one adds, to the usual local polynomial basis, special shape functions that capture local singular properties, such as jumps, kinks, and singularities, etc.

For many singular problems, it is often possible to determine their singular expansion near a singular point in the form $\sum_{k=0}^{\infty} c_k x^{\lambda_k}$, where $\{\lambda_k\}_{k=0}^{\infty}$ is an increasing sequence, and also called a Müntz sequence [1]. We develop in this paper a *Müntz–Galerkin method* in which Müntz polynomials [1], instead of the usual polynomials, are employed to form the approximation space. However, the Müntz polynomials themselves are not suitable as basis functions due to their poor conditioning. We shall explore relations between Jacobi polynomials and Müntz polynomials to develop efficient implementation procedures for the Müntz–Galerkin method as well as derive optimal error estimates. As examples of applications, we shall use the Müntz–Galerkin method to solve the Poisson equation with mixed Dirichlet–Neumann boundary conditions, whose solution behaves like $O(r^{1/2})$ near the singular points, and demonstrate

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†Department of Mathematics, Purdue University, West Lafayette, IN 47907 (shen7@purdue.edu, wywshtj@gmail.com).

that the rates of convergence are greatly improved compared to the classical spectral methods.

The rest of the paper is organized as follows. In section 2, we motivate our work by analyzing the singular behaviors of the Poisson equation with mixed Dirichlet–Neumann boundary conditions. We introduce the Müntz polynomials and Müntz–Jacobi functions, and give the error estimates for Müntz–Jacobi approximations in section 3. In section 4, we propose a Müntz–Galerkin method for a model problem, and develop an efficient implementation procedure as well as optimal approximation results. Several examples are presented in section 5 to illustrate the performance of the proposed algorithms. Some concluding remarks are given in section 6.

2. Singularities of the mixed Dirichlet–Neumann BVPs.

In this section, we recall some well-known results on the solution of the two-dimensional Laplace equation

$$\Delta u = 0, \quad (x, y) \in \Omega, \tag{2.1}$$

with mixed Dirichlet–Neumann boundary conditions. For more details, we refer the readers to [8, 13, 15] and the references therein.

Let (r, θ) be the polar coordinates with $(x, y) = (r\cos(\theta), r\cos(\theta))$. The Laplacian operator in polar coordinates takes the form $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$. Using the separation of variables with respect to the polar coordinates (r, θ), one can prove the following result.

Lemma 2.1. *The general solution of the Laplace equation* (2.1) *in polar coordinates is*

$$u(r, \theta) = (cr^{\alpha} + dr^{-\alpha})(a\cos(\alpha\theta) + b\sin(\alpha\theta)), \tag{2.2}$$

where $\alpha > 0$, a, b, c, d *are real constants.*

Let Ω be the half-circle domain shown in Figure 1. Since the origin is on $\partial\Omega$, the parameter d in (2.2) should be 0, i.e., the general solution in this case is

$$u(r, \theta) = r^{\alpha}(a\cos(\alpha\theta) + b\sin(\alpha\theta)), \quad \alpha > 0. \tag{2.3}$$

At the bottom of Ω, the outward normal vector is $\mathbf{n} = (0, -1)^t$, and the normal flux is

$$q(r, \theta) := \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial u}{\partial \theta} = \alpha r^{\alpha}(-a\sin(\alpha\theta) + b\cos(\alpha\theta)). \tag{2.4}$$

We consider two kinds of mixed boundary conditions on the bottom of Ω, shown in Figure 1. One is

$$\frac{\partial u}{\partial \mathbf{n}} \,|_{\{x<0\}\cap\{y=0\}} = 0, \quad u\,|_{\{x>0\}\cap\{y=0\}} = 0, \tag{2.5}$$

which is referred as "N-D" boundary conditions (Neumann boundary condition on the left half-line of the x axis, and Dirichlet boundary condition on the right half-line of the x axis). The other is

$$u\,|_{\{x<0\}\cap\{y=0\}} = 0, \quad \frac{\partial u}{\partial \mathbf{n}} \,|_{\{x>0\}\cap\{y=0\}} = 0, \tag{2.6}$$

which is referred as "D-N" boundary conditions (Dirichlet boundary condition on the left half-line of the x axis, and Neumann condition on the right half-line of the x axis).

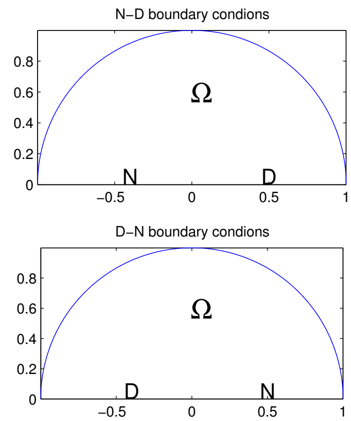


Fig. 1. *Mixed boundary conditions: top is N-D and bottom is D-N.*

Theorem 2.2 (Asymptotic expansion in the vicinity of one singular point). *The singular parts of the general solutions of Laplacian equation* (2.1) *with N-D and D-N boundary conditions are, respectively,*

$$u^{ND}(r,\theta) = \sum_{k=0}^{\infty} b_k r^{k+1/2} \sin(k+1/2)\theta, \tag{2.7}$$

$$u^{DN}(r,\theta) = \sum_{k=0}^{\infty} a_k r^{k+1/2} \cos(k+1/2)\theta. \tag{2.8}$$

Let $u(r,\theta) = v(r)w(\theta)$ be the solution to the problem $-\Delta u = f$ with either N-D or D-N boundary conditions. According to the expansions in (2.7) and (2.8), $v(r)$ should have two parts, the singular part v_S from the general solution of homogeneous equation (2.1), and the regular part v_R from the particular solution associated with the right-hand side f, i.e.,

$$v(r) = v_R(r) + v_S(r).$$

The regular part $v_R(r)$ can be well approximated by the set of classical polynomials $\{1, r, r^2, \ldots\}$, but the polynomial expansion of the singular part $v_S(r)$ will converge very slowly. From the expansions in (2.7) and (2.8), it is apparent that we should seek approximate solutions (in the r-direction) in the space spanned by $\{r^{\frac{1}{2}k}\}$ which includes both the singular basis and the regular basis.

More generally, if the singular part expansion takes the form $u_S(r) = \sum_{k=0}^{\infty} b_k r^{k+p/q}$, $p < q$ are two integers, or, as found in solving some fractional PDEs, $u_S(r) = \sum_{k,j=0}^{\infty} b_{kj} r^{(p/q)k+j}$, we can use an expansion of the form $u(r) = \sum_{k=0}^{\infty} a_k r^{k/q}$, which can also cover both the singular part and regular part. The sequence $\{r^{k/q}\}$ is just a special sequence of Müntz polynomials that we shall consider in the next section.

3. Müntz polynomials and Müntz–Jacobi functions. Assume that the solutions to some singular problems have the following expansion

$$\sum_{k=0}^{\infty} c_k x^{\lambda_k} \text{ with } \lambda_0 < \lambda_1 < \lambda_2 < \cdots. \tag{3.1}$$

It is then natural for us to look for their approximations in the space $M_N = \text{span}\{x^{\lambda_k} : 0 \le k \le N\}$. However, it is obvious that the functions $\{x^{\lambda_k}\}_{k=0}^{\infty}$ themselves are not suitable as basis functions. The goal of this section is to construct a sequence of Müntz–Jacobi functions such that (i) they are mutually orthogonal and (ii) they are easy to evaluate. Besides, the error estimates for Müntz–Jacobi approximations are carried out at the end of this section.

3.1. Müntz polynomials. We define a *Müntz sequence* as an increasing sequence of distinct real numbers

$$\Lambda := \{\lambda_k\}_{k=0}^{\infty}, \quad \lambda_0 < \lambda_1 < \lambda_2 \cdots,$$

and we call a system of the form $(x^{\lambda_0}, x^{\lambda_1}, \dots)$ a *Müntz system* with the corresponding *Müntz space* associated with Λ:

$$M(\Lambda) := \bigcup_{n=0}^{\infty} M_n(\Lambda) = \text{span}\{x^{\lambda_n} : n = 0, 1, \dots\}, \quad x \in [0,1],$$

where $M_n(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ for each $n = 0, 1, \dots$.

The celebrated Müntz theorem shows the relation between the density of the Müntz polynomials $\sum_{k=0}^{n} c_k x^{\lambda_k}$ in $C([0,1])$ and their growth of the exponents $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$.

THEOREM 3.1 (see [1]). *If $\lambda_{k_0} = 0$ for some $k_0 \ge 0$, the Müntz space associated with the Müntz sequence Λ is a dense subset of $C([0,1])$ if and only if*

$$\sum_{k=k_0}^{\infty} \frac{1}{\lambda_k} = \infty.$$

We define the nth *Müntz–Legendre* polynomial associated with Λ by (cf. [23])

$$L_n(x;\Lambda) := \frac{1}{2\pi i} \int_{\Gamma} \prod_{k=0}^{n-1} \frac{t + \lambda_k + 1}{t - \lambda_k} \frac{x^t}{t - \lambda_n} \mathrm{d}t, \tag{3.2}$$

where Γ is a simple contour surrounding all zeros of the denominator in the integrand.

If the Müntz sequence Λ satisfies the condition

$$\lambda_n > -\frac{1}{2}, \quad n = 0, 1, \dots, \tag{3.3}$$

a straightforward application of the residue theorem shows that the Müntz–Legendre polynomial $L_n(x;\Lambda) \in M_n(\Lambda)$. More precisely, for each $n = 0, 1, 2, \dots$, we have

$$L_n(x;\Lambda) = \sum_{k=0}^{n} c_{k,n} x^{\lambda_k}, \quad c_{k,n} = \frac{\prod_{j=0}^{n-1}(\lambda_k + \lambda_j + 1)}{\prod_{j=0, j\ne k}^{n-1}(\lambda_k - \lambda_j)}. \tag{3.4}$$

It is shown (cf. Theorem 2.4 in [3]) that the Müntz–Legendre polynomials are orthogonal in $L^2[0,1]$ with respect to the Legendre weight $\omega = 1$, i.e.,

$$\int_0^1 L_n(x;\Lambda) L_m(x;\Lambda) \mathrm{d}x = \frac{\delta_{nm}}{2\lambda_n + 1}. \tag{3.5}$$

We can also define *Müntz–Jacobi* polynomials. Consider first the Jacobi index $(0, \beta)$. Let $\omega^{\beta}(x) = x^{\beta}$ with $\beta > -1$, then the nth Müntz–Jacobi polynomial with index $(0, \beta)$ associated with Λ is defined by

$$L_n^{\beta}(x;\Lambda) := \frac{x^{-\beta/2}}{2\pi i}\int_{\Gamma}\prod_{k=0}^{n-1}\frac{t+\lambda_k+\beta/2+1}{t-\lambda_k-\beta/2}\frac{x^t}{t-\lambda_n-\beta/2}\mathrm{d}t, \tag{3.6}$$

where the contour Γ encloses all zeros of the denominator of the integrand. We can show that $L_n^{\beta}(x;\Lambda) \in M_n(\Lambda)$, and the following orthogonality relation holds:

$$\int_0^1 L_n^{\beta}(x;\Lambda)L_m^{\beta}(x;\Lambda)x^{\beta}\mathrm{d}x = \frac{\delta_{nm}}{2\lambda_n+\beta+1}. \tag{3.7}$$

Similarly, we can define Müntz–Jacobi polynomials ${}^{\alpha}L_n(x;\Lambda)$ with index $(\alpha, 0)$.

While Müntz–Legendre polynomials are mutually orthogonal, but their direct evaluation is numerically poorly conditioned, since the coefficients $c_{k,n}$ defined in (3.4) quickly become unmanageably large, although the summation of them is always equal to 1. For example, for $\Lambda = \{n/2\}$, the tenth Müntz–Legendre polynomial is

$$\begin{aligned} L_{10}(t;\Lambda) &= 11 - 660t^{1/2} + 12870t - 120120t^{3/2} + 630630t^2 - 2018016t^{5/2} \\ &\quad + 4084080t^3 - 5250960t^{5/2} + 4157010t^4 - 1847560t^{7/2} + 352716t^5. \end{aligned}$$

The troubles associated with poor conditioning have been addressed in [5, 16].

3.2. Müntz–Jacobi functions. In this paper, we consider special Müntz sequences in the following form:

$$\Lambda(\alpha) = \{\lambda_k = \alpha k\}_{k=0}^{\infty}, \tag{3.8}$$

where $\alpha > 0$ is a constant, and construct stable algorithms to deal with the Müntz–Legendre polynomials $\{L_k(x;\Lambda(\rho))\}_{k=0}^{\infty}$. Solutions of many interesting problems (see some examples in the subsequent sections), have singularities which can be characterized by such Müntz sequences. Other type of Müntz sequences require different treatments that we will address in a subsequent work.

For obvious reasons, we shall work on the interval $I = (-1, 1)$. Then, we can define the left and right *Müntz spaces* as follows:

$$\begin{aligned} M_n^L(\alpha) &:= \mathrm{span}\{1, (1+x)^{\alpha}, \ldots, (1+x)^{\alpha n}\}, \quad M^L(\alpha) = \cup_{n=0}^{\infty} M_n^L(\alpha), \\ M_n^R(\alpha) &:= \mathrm{span}\{1, (1-x)^{\alpha}, \ldots, (1-x)^{\alpha n}\}, \quad M^R(\alpha) = \cup_{n=0}^{\infty} M_n^R(\alpha). \end{aligned} \tag{3.9}$$

We consider first the case with singularity at the left endpoint, and define the following one-to-one mapping $I \to I$:

$$x = x(y) := 2^{1-1/\alpha}(1+y)^{1/\alpha} - 1, \tag{3.10}$$

$$y = y(x) := 2^{1-\alpha}(1+x)^{\alpha} - 1. \tag{3.11}$$

We define the Müntz–Jacobi function with index $(0, 1/\alpha - 1)$ by

$$\hat{J}_k^{0,1/\alpha-1}(x) := J_k^{0,1/\alpha-1}(y(x)) \quad \forall k = 0, 1, \ldots, \tag{3.12}$$

where $J_k^{0,1/\alpha-1}(\cdot)$ is the Jacobi polynomial with index $(0, 1/\alpha - 1)$ and $y(x)$ is defined in (3.11).

Similarly, for the case with singularity at the right endpoint, we use the following mappings:

$$\bar{x} = \bar{x}(\bar{y}) := 1 - 2^{1-1/\alpha}(1-\bar{y})^{1/\alpha}, \tag{3.13}$$

$$\bar{y} = \bar{y}(\bar{x}) := 1 - 2^{1-\alpha}(1-\bar{x})^{\alpha}, \tag{3.14}$$

to define the Müntz–Jacobi function with index $(1/\alpha - 1, 0)$ by

$$\hat{J}_k^{1/\alpha-1,0}(\bar{x}) := J_k^{1/\alpha-1,0}(\bar{y}(\bar{x})) \quad \forall k = 0, 1, \dots. \tag{3.15}$$

Hereafter, the pair of functions $u(x)$ and $U(y)$ are related by

$$u(x) \equiv U(y(x)). \tag{3.16}$$

THEOREM 3.2. *Let $\alpha \in (0,1)$. The Müntz–Jacobi functions $\{\hat{J}_n^{0,1/\alpha-1}(x)\}$ are mutually orthogonal and form a complete orthogonal system in $L^2(I)$. Furthermore,*

$$M_N^L(\alpha) = span\left\{\hat{J}_n^{0,1/\alpha-1} : 0 \le n \le N\right\}. \tag{3.17}$$

The Müntz–Jacobi functions $\{\hat{J}_n^{1/\alpha-1,0}(x)\}$ are mutually orthogonal and form a complete orthogonal system in $L^2(I)$. Furthermore,

$$M_N^R(\alpha) = span\left\{\hat{J}_n^{1/\alpha-1,0} : 0 \le n \le N\right\}. \tag{3.18}$$

Proof. We shall only prove the left case as the proof for the right case is similar. From (A.1), we know that

$$\int_{-1}^{1} J_m^{0,1/\alpha-1}(y) J_n^{0,1/\alpha-1}(y)(1+y)^{1/\alpha-1}\mathrm{d}y = \frac{2^{1/\alpha}}{2n+1/\alpha}\delta_{mn}. \tag{3.19}$$

From the definition of $\{\hat{J}_k^{0,1/\alpha-1}(x)\}_{k=0}$ in (3.12), and

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2^{1-1/\alpha}}{\alpha}(1+y)^{1/\alpha-1}, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \alpha 2^{1-\alpha}(1+x)^{\alpha-1}, \tag{3.20}$$

we derive

$$\begin{aligned}\int_{-1}^{1} \hat{J}_n^{0,1/\alpha-1}(x)\, \hat{J}_m^{0,1/\alpha-1}(x)\mathrm{d}x =& \frac{2^{1-1/\alpha}}{\alpha}\int_{-1}^{1} J_n^{0,1/\alpha-1}(y)\, J_m^{0,1/\alpha-1}(y)\,(1+y)^{1/\alpha-1}\mathrm{d}y,\\ =& \frac{2}{2n\alpha+1}\delta_{mn},\end{aligned} \tag{3.21}$$

which implies the orthogonality of $\{\hat{J}_k^{0,1/\alpha-1}(x)\}_{k=0}$ in $L^2(I)$.

For any $u(\cdot) \in L^2(I)$, we have $U(\cdot) \in L^2_{\omega_\alpha}(I)$. Thus, by the completeness of Jacobi polynomials $\{J_k^{0,1/\alpha-1}(y)\}_{k=0}$, we have the following unique expansion

$$u(x) = U(y) = \sum_{k=0}^{\infty} u_k J_k^{0,1/\alpha-1}(y) = \sum_{k=0}^{\infty} u_k \hat{J}_k^{0,1/\alpha-1}(x), \tag{3.22}$$

where $u_k = \frac{\langle U, J_k^{0,1/\alpha-1}\rangle_{\omega_\alpha}}{\|J_k^{0,1/\alpha-1}\|^2_{L^2_{\omega_\alpha}}}$, $\omega_\alpha = \omega^{0,1/\alpha-1}(y) = (1+y)^{1/\alpha-1}$.

Finally, (3.17) is a direct consequence of the facts that $\hat{J}_n^{0,1/\alpha-1}(x) \in M_N^L(\alpha)$ for $0 \le n \le N$ and they are mutually orthogonal. ∎

In addition, since we will frequently use the relation (3.20) in the error estimates, we rewrite it as

$$\frac{\mathrm{d}x}{\mathrm{d}y} = c_\alpha \omega_\alpha, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{c_\alpha \omega_\alpha}, \quad \text{where } c_\alpha = \frac{2^{1-1/\alpha}}{\alpha}. \tag{3.23}$$

3.3. Approximation results for Müntz–Jacobi functions. In this section, we develop the estimates for the weighted L^2 and H^1 projection errors of the Müntz–Jacobi function approximation.

First of all, recall the usual Jacobi weight $\omega^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta,\ \alpha,\beta > -1$. We introduce a new weight function based on the relation between x and y defined in (3.11):

$$\tilde{\omega}^{a,b}(x) := (1-y(x))^a(1+y(x))^b \quad \forall a,b \in \mathbb{R}. \tag{3.24}$$

Thanks to (3.23), it is easy to know that

$$\|u\|^2_{L^2} = \int_{-1}^{1} |u(x)|^2 \mathrm{d}x = c_\alpha \int_{-1}^{1} |U(y)|^2 \omega_\alpha \mathrm{d}y = c_\alpha \|U\|^2_{\omega^{0,1/\alpha-1}}, \tag{3.25}$$

$$\begin{aligned} \|\partial_x u\|^2_{\tilde{\omega}^{1,2/\alpha-1}} &= \int_{-1}^{1} |\partial_x u(x)|^2 (1-y(x))(1+y(x))^{2/\alpha-1}\mathrm{d}x \\ &= \frac{1}{c_\alpha}\int_{-1}^{1} |\partial_y U(y)|^2 (1-y)(1+y)^{1/\alpha}\mathrm{d}y = \frac{1}{c_\alpha}\|\partial_y U\|^2_{\omega^{1,1/\alpha}}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} \|\partial_x u\|^2_{\omega^{0,2-2\alpha}} &= \int_{-1}^{1} |\partial_x u(x)|^2 (1+x)^{2-2\alpha}\mathrm{d}x \\ &= \alpha 2^{(1-1/\alpha)(1-2\alpha)} \int_{-1}^{1} |\partial_y U(y)|^2 (1+y)^{1/\alpha-1}\mathrm{d}y \\ &= \tilde{c}_\alpha \|\partial_y U\|^2_{\omega^{0,1/\alpha-1}}, \end{aligned} \tag{3.27}$$

where $\tilde{c}_\alpha = \alpha 2^{(1-1/\alpha)(1-2\alpha)}$.

Next, we introduce the following differential operators according to the relation between the function pair (u, U) shown in (3.16):

$$D_x u := \frac{\mathrm{d}U}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}y}\partial_x u, \tag{3.28}$$

$$D_x^2 u := \frac{\mathrm{d}^2U}{\mathrm{d}y^2} = \frac{\mathrm{d}x}{\mathrm{d}y}\partial_x\left(\frac{\mathrm{d}x}{\mathrm{d}y}\partial_x u\right), \tag{3.29}$$

$$\dots$$

$$D_x^k u := \frac{\mathrm{d}^kU}{\mathrm{d}y^k} = \frac{\mathrm{d}x}{\mathrm{d}y}\partial_x\left(\frac{\mathrm{d}x}{\mathrm{d}y}\partial_x\left(\dots\left(\frac{\mathrm{d}x}{\mathrm{d}y}\partial_x u\right)\dots\right)\right), \quad k = 0,1,2,\dots. \tag{3.30}$$

Then we have the following relations

$$\|U\|^2_{\omega^{0,1/\alpha-1}} = \int_{-1}^{1} |U(y)|^2(1+y)^{1/\alpha-1}\mathrm{d}y = \frac{1}{c_\alpha}\int_{-1}^{1}|u(x)|^2\mathrm{d}x = \frac{1}{c_\alpha}\|u\|^2_{\tilde{\omega}^{0,0}}, \tag{3.31}$$

$$\|\partial_y U\|^2_{\omega^{1,1/\alpha}} = \int_{-1}^{1} |\partial_y U(y)|^2(1-y)(1+y)(1+y)^{1/\alpha-1}\mathrm{d}y = \frac{1}{c_\alpha}\|D_x u\|^2_{\tilde{\omega}^{1,1}}, \tag{3.32}$$

$$\dots$$

$$\|\partial_y^k U\|^2_{\omega^{k,k+1/\alpha-1}} = \frac{1}{c_\alpha}\|D_x^k u\|^2_{\tilde{\omega}^{k,k}}, \quad k = 0,1,2,\dots. \tag{3.33}$$

Let $\mathcal{P}_N$ be the set of all polynomials of degree at most N, and define the Müntz approximation space as

$$\mathcal{V}_N := \{p : p(x) = P(y(x)),\ \forall P \in \mathcal{P}_N\}. \tag{3.34}$$

By Theorem 3.2, we know that

$$\mathcal{P}_N(I) = \text{span}\{J_k^{0,1/\alpha-1}(y)\}_{k=0}^N, \quad \mathcal{V}_N(I) = \text{span}\{\hat{J}_k^{0,1/\alpha-1}(x)\}_{k=0}^N.$$

Consider the weighted L^2 spaces

$$L^2_{\omega^{0,1/\alpha-1}}(I) = \{U : \|U\|_{\omega^{0,1/\alpha-1}} < \infty\},$$
$$\tilde{L}^2_{\omega^{0,1/\alpha-1}}(I) = \left\{u : u(x) = U(y(x)),\ U(y) \in L^2_{\omega^{0,1/\alpha-1}}(I)\right\}.$$

Let $\Pi_N : L^2_{\omega^{0,1/\alpha-1}}(I) \to \mathcal{P}_N$ be the orthogonal projection defined by

$$(\Pi_N U - U, P)_{\omega^{0,1/\alpha-1}} = 0 \quad \forall P \in \mathcal{P}_N. \tag{3.35}$$

We define $\pi_N : \tilde{L}^2_{\omega^{0,1/\alpha-1}}(I) \to \mathcal{P}_N$ by

$$(\pi_N u)(x) := (\Pi_N U)(y(x)). \tag{3.36}$$

Then we have $\pi_N u \in \mathcal{V}_N$ and by (3.25),

$$(\pi_N u - u, p)_{L^2} = c_\alpha \left(\Pi_N U - U, P\right)_{\omega^{0,1/\alpha-1}} = 0 \quad \forall p \in \mathcal{V}_N. \tag{3.37}$$

Recall the error estimate for orthogonal projection Π_N defined in (3.35) (cf. Theorem 3.35 in [22]): for $0 \le l \le m$,

$$\|\partial_y^l \left(\Pi_N U - U\right)\|_{\omega^{l,l+1/\alpha-1}} \lesssim N^{l-m} \|\partial_y^m U\|_{\omega^{m,m+1/\alpha-1}} \tag{3.38}$$

for any $U \in B^m_{\omega^{0,1/\alpha-1}}(I) := \{U : \partial_y^k U \in L^2_{\omega^{k,k+1/\alpha-1}}, 0 \le k \le m\}$. To perform the error estimates for the projection π_N defined in (3.36), we define the mapped space

$$\tilde{B}^m_{\omega^{0,1/\alpha-1}}(I) = \left\{u : \|u\|_{\tilde{B}^m_{\omega^{0,1/\alpha-1}}} < \infty\right\} \tag{3.39}$$

equipped with the norm and seminorm

$$\|u\|_{\tilde{B}^m_{\omega^{0,1/\alpha-1}}} = \left(\sum_{k=0}^m \|D_x^k u\|^2_{\tilde{\omega}^{k,k}}\right)^{1/2}, \quad |u|_{\tilde{B}^m_{\omega^{0,1/\alpha-1}}} = \|D_x^m u\|_{\tilde{\omega}^{m,m}}.$$

THEOREM 3.3. *Let $u \in \tilde{B}^m_{\omega^{0,1/\alpha-1}}(I) \cap \tilde{L}^2_{\omega^{0,1/\alpha-1}}(I)$ and $\mu = 0, 1$. Then for the L_2 projection π_N defined in* (3.37)*, we have the following estimates:*

$$\|\partial_x^\mu (\pi_N u - u)\|_{\tilde{\omega}^{\mu,s(\mu)}} \lesssim N^{\mu-m} |u|_{\tilde{B}^m_{\omega^{0,1/\alpha-1}}}, \tag{3.40}$$

where $s(\mu) = \begin{cases} 0, & \mu = 0, \\ 2/\alpha - 1, & \mu = 1. \end{cases}$

Proof. By (3.25) and (3.26), we have

$$\|\pi_N u - u\|_{L^2} = \sqrt{c_\alpha}\|\Pi_N U - U\|_{\omega^{0,1/\alpha-1}}, \tag{3.41}$$

$$\|\partial_x (\pi_N u - u)\|_{\tilde{\omega}^{1,2/\alpha-1}} = \frac{1}{\sqrt{c_\alpha}}\|\partial_y (\Pi_N U - U)\|_{\omega^{1,1/\alpha}}. \tag{3.42}$$

Thus, using the results (3.38) with $l = 0, 1$, we obtain (3.40) from (3.41)–(3.42) and the definitions in (3.33) and (3.39). ∎

Consider the weighted H^1 spaces

$$H^1_{\omega^{0,1/\alpha-1}}(I) = \left\{U : \|U\|_{H^1_{\omega^{0,1/\alpha-1}}} < \infty\right\},$$
$$\tilde{H}^1_{\omega^{0,1/\alpha-1}}(I) = \left\{u : u(x) = U(y(x)),\ H^1_{\omega^{0,1/\alpha-1}}(I)\right\},$$

where the weighted H^1 norm in $H^1_{\omega^{0,1/\alpha-1}}(I)$ is defined by

$$\|U\|^2_{H^1_{\omega^{0,1/\alpha-1}}} = \|\partial_y U\|^2_{\omega^{0,1/\alpha-1}} + \|U\|^2_{\omega^{0,1/\alpha-1}}.$$

By (3.25) and (3.27), the weighted H^1 norm in $\tilde{H}^1_{\omega^{0,1/\alpha-1}}(I)$ should be

$$\begin{aligned}\|u\|^2_{\tilde{H}^1_{\omega^{0,1/\alpha-1}}} &= \|\partial_x u\|^2_{\omega^{0,2-2\alpha}} + \|u\|^2_{L^2}\\ &= \tilde{c}_\alpha\|\partial_y U\|^2_{\omega^{0,1/\alpha-1}} + c_\alpha\|U\|^2_{\omega^{0,1/\alpha-1}}.\end{aligned}$$

It is easy to know that

$$\sqrt{\min(c_\alpha, \tilde{c}_\alpha)}\|U\|_{H^1_{\omega^{0,1/\alpha-1}}} \le \|u\|_{\tilde{H}^1_{\omega^{0,1/\alpha-1}}} \le \sqrt{\max(c_\alpha, \tilde{c}_\alpha)}\|U\|_{H^1_{\omega^{0,1/\alpha-1}}}. \tag{3.43}$$

Let us denote the inner product in $H^1_{\omega^{0,1/\alpha-1}}(I)$ by

$$a_{\omega^{0,1/\alpha-1}}(U, V) = (U', V')_{\omega^{0,1/\alpha-1}} + (U, V)_{\omega^{0,1/\alpha-1}},$$

and define the orthogonal projection $\Pi^1_N : H^1_{\omega^{0,1/\alpha-1}}(I) \to \mathcal{P}_N$ by

$$a_{\omega^{0,1/\alpha-1}}(\Pi^1_N U - U, V) = 0 \quad \forall V \in \mathcal{P}_N. \tag{3.44}$$

Similarly, we can define $\pi^1_N : \tilde{H}^1_{\omega^{0,1/\alpha-1}}(I) \to \mathcal{P}_N$ by

$$(\pi^1_N u)(x) := (\Pi^1_N U)(y(x)). \tag{3.45}$$

Recall the error estimate for orthogonal projection Π^1_N defined in (3.44) (cf. Theorem 3.36 in [22]):

$$\|\Pi^1_N U - U\|_{H^1_{\omega^{0,1/\alpha-1}}} \lesssim N^{1-m}\|\partial^m_y U\|_{\omega^{m-1,m+1/\alpha-2}} \tag{3.46}$$

for any U satisfying $\partial_y U \in B^{m-1}_{\omega^{0,1/\alpha-1}}(I)$. By (3.33), (3.43), and (3.46), we can prove the following theorem.

THEOREM 3.4. *Let u satisfy $u \in \tilde{H}^1_{\omega^{0,1/\alpha-1}}(I)$ and $D_x u \in \tilde{B}^{m-1}_{\omega^{0,1/\alpha-1}}(I)$. Then for the H^1 projection π^1_N defined in* (3.45)*, we have the following estimate*

$$\|\pi^1_N u - u\|_{\tilde{H}^1_{\omega^{0,1/\alpha-1}}} \lesssim N^{1-m}\|D^m_x u\|_{\tilde{\omega}^{m-1,m-1}}. \tag{3.47}$$

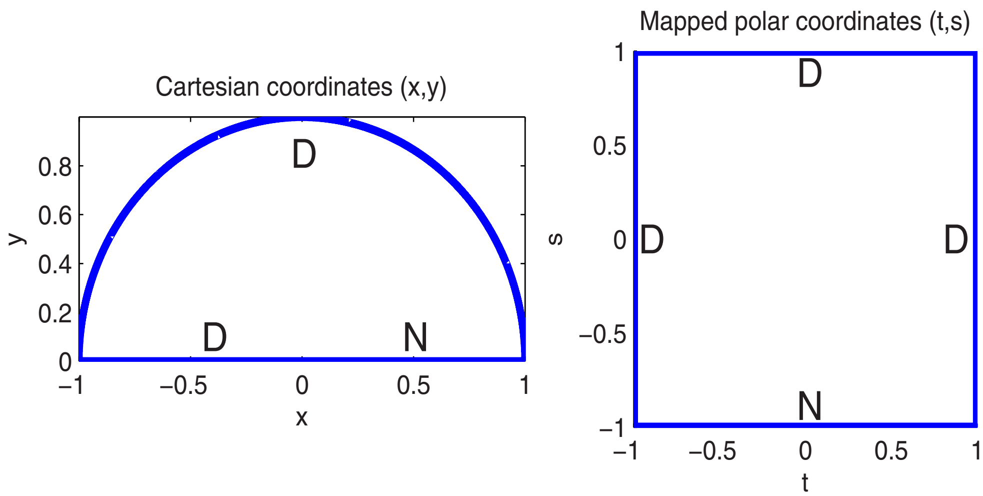


FIG. 2. *The original and transformed domains.*

4. Müntz–Galerkin method.

In this section, we shall develop a Müntz–Galerkin method using the Müntz–Jacobi functions for solving the problem $-\Delta u = f$ with Ω being the upper half of the unit disk with D-N boundary condition on the bottom of Ω; cf. the left part of Figure 2.

4.1. Galerkin formulation.

In polar coordinates, this problem reads

$$\begin{cases} -\frac{1}{r}(ru_r)_r - \frac{1}{r^2}u_{\theta\theta} = f, & (r,\theta) \in (0,1)\times(0,\pi), \\ u\,|_{\{r=1,0<\theta<\pi\}\cup\{0<r<1,\theta=\pi\}} = 0, & \frac{\partial u}{\partial \mathbf{n}}\big|_{\{0<r<1,\theta=0\}} = 0. \end{cases} \tag{4.1}$$

Since the original domain is a half-disk, we cannot apply a Fourier transform in the θ-direction to reduce the above problem to a sequence of one-dimensional problems as in [20]. In order to apply a spectral method, we make the following transform,

$$\begin{cases} t = 2r - 1 \in (-1,1), \\ s = \frac{2}{\pi}\theta - 1 \in (-1,1), \end{cases} \tag{4.2}$$

and denote $\tilde{u}(t,s) = u\left((t+1)/2, (s+1)\pi/2\right), \tilde{f}(t,s) = f\left((t+1)/2, (s+1)\pi/2\right)$. Then the problem (4.1) becomes

$$\begin{cases} -\partial_t\left((t+1)\partial_t\tilde{u}\right) - \frac{4}{\pi^2}\frac{1}{t+1}\partial_{ss}\tilde{u} = \frac{t+1}{4}\tilde{f}, & (t,s) \in \tilde{\Omega} = (-1,1)^2, \\ \tilde{u}(t=\pm1, -1<s<1) = 0, & \\ \frac{\partial\tilde{u}}{\partial s}(-1<t<1,\ s=-1) = 0, & \tilde{u}(-1<t<1, s=1) = 0. \end{cases} \tag{4.3}$$

The original and transformed domains are shown in Figure 2.

Let $(u,v) = \int_{-1}^{1}\int_{-1}^{1} uv\ dtds$. For a given finite-dimensional space V_N, the Galerkin approximation for problem (4.3) is to find $\tilde{u}_N \in V_N$ such that

$$((t+1)\partial_t\tilde{u}_N,\ \partial_t v) + \frac{4}{\pi^2}\left(\frac{1}{t+1}\partial_s\tilde{u}_N,\ \partial_s v\right) = \left(\frac{t+1}{4}\tilde{f},\ v\right) \quad \forall v \in V_N. \tag{4.4}$$

Let $\phi_k(t)$ and $\psi_j(s)$ be two sequences of one-dimensional basis functions satisfying the boundary conditions in (4.3), respectively. Setting $V_N = \text{span}\{\phi_k(t)\}_{k=0}^N \otimes$ span

$\{\psi_j(s)\}_{j=0}^N$, and writing

$$\tilde{u}_N(t,s) = \sum_{k=0}^{N}\sum_{j=0}^{N} u_{kj}\phi_k(t)\psi_j(s) \tag{4.5}$$

in (4.4), we find that the problem (4.4) reduces to the following linear system

$$SU\tilde{M}^T + \frac{4}{\pi^2}MU\tilde{S}^T = F, \tag{4.6}$$

where $U = (u_{k,j})_{0\le k,j\le N}$ and

$$\begin{aligned} &S = (s_{kj}),\ s_{kj} = \left((t+1)\phi_j'(t), \phi_k'(t)\right),\ \tilde{S} = (\tilde{s}_{kj}),\ \tilde{s}_{kj} = (\psi_j'(s), \psi_k'(s)),\\ &M = (m_{kj}),\ m_{kj} = \left(\frac{1}{t+1}\phi_j(t), \phi_k(t)\right),\ \tilde{M}_{kj} = (\tilde{m}_{kj}),\ \tilde{m}_{kj} = (\psi_j(s), \psi_k(s)),\\ &F = (f_{kj}),\ f_{k,j} = \left(\frac{t+1}{4}\tilde{f}(t,s), \phi_k(t)\psi_j(s)\right). \end{aligned} \tag{4.7}$$

In the classical Legendre–Galerkin method [20], we use polynomial basis functions satisfying boundary conditions $\phi_k(-1) = \phi_k(1) = 0$ and $\psi_j'(-1) = \psi_j(1) = 0$:

$$\phi_k(t) = L_k(t) - L_{k+2}(t), \tag{4.8}$$

$$\psi_j(s) = L_j(s) - \frac{2j+3}{(j+2)^2}L_{j+1}(s) - \frac{(j+1)^2}{(j+2)^2}L_{j+2}(s), \tag{4.9}$$

where L_n is the nth degree Legendre polynomial. The stiffness and mass matrices S, $\tilde{S}$, M, $\tilde{M}$ corresponding to the above basis sets are all sparse, so the linear system (4.6) can be efficiently solved [20].

However, according to Theorem 2.2, the polynomial approximation will lead to a poor convergence rate due to the singularity of the solution $\tilde{u}(t,s)$ in the t direction near the left endpoint $t=-1$. Instead, we should use Müntz–Jacobi functions with $\alpha = \frac{1}{2}$ to replace $\phi_k(t)$ in the t-direction.

4.2. Müntz–Jacobi basis functions. As in the classical Legendre–Galerkin method, we should construct basis functions satisfying required boundary conditions using *compact linear combinations* [19] of Müntz–Jacobi functions. Namely, we can determine the coefficient pair (a_k, b_k) such that for each k

$$\phi_k^{0,1/\alpha-1}(y) := J_k^{0,1/\alpha-1}(y) + a_kJ_{k+1}^{0,1/\alpha-1}(y) + b_kJ_{k+2}^{0,1/\alpha-1}(y) \tag{4.10}$$

satisfies $\phi^{0,1/\alpha-1}(\pm1) = 0$. In fact, using (A.7) and (A.8), we find

$$a_k = -\frac{(1/\alpha-1)(2k+1/\alpha+2)}{(k+1/\alpha)(2k+1/\alpha+3)},\quad b_k = -\frac{(k+2)(2k+1/\alpha+1)}{(k+1/\alpha)(2k+1/\alpha+3)}.$$

We then obtain our desired basis functions in x:

$$\begin{aligned} \hat{\phi}_k^{0,1/\alpha-1}(x) &:= \phi_k^{0,1/\alpha-1}(y(x))\\ &= \hat{J}_k^{0,1/\alpha-1}(x) + a_k\hat{J}_{k+1}^{0,1/\alpha-1}(x) + b_k\hat{J}_{k+2}^{0,1/\alpha-1}(x), \end{aligned} \tag{4.11}$$

where $y(x)$ is defined in (3.11).

Case (I)*:* $\alpha = 1/2$. In particular, taking $\alpha = 1/2$ in (4.10) and (4.11) yields the following basis functions

$$\phi_k^{0,1}(y) = J_k^{0,1}(y) - \frac{2}{2k+5} J_{k+1}^{0,1}(y) - \frac{2k+3}{2k+5} J_{k+2}^{0,1}(y), \tag{4.12}$$

$$\hat{\phi}_k^{0,1}(x) = \hat{J}_k^{0,1}(x) - \frac{2}{2k+5} \hat{J}_{k+1}^{0,1}(x) - \frac{2k+3}{2k+5} \hat{J}_{k+2}^{0,1}(x). \tag{4.13}$$

It remains to compute the stiffness and mass matrices associated with $\hat{\phi}_k^{0,1}(x)$, namely,

$$\begin{aligned} \hat{S}^{0,1} &= (\hat{s}_{kj}^{0,1}),\ \hat{s}_{kj}^{0,1} = \left((t+1)\partial_t \hat{\phi}_j^{0,1}(t), \partial_t \hat{\phi}_k^{0,1}(t)\right), \\ \hat{M}^{0,1} &= (\hat{m}_{kj}^{0,1}),\ \hat{m}_{kj}^{0,1} = \left(\frac{1}{t+1}\hat{\phi}_j^{0,1}(t), \hat{\phi}_k^{0,1}(t)\right). \end{aligned} \tag{4.14}$$

Unlike in the polynomial case where the stiffness and mass matrices are sparse and can be computed with ease, the matrices in (4.14) are dense. Direct computations using Gaussian quadratures are costly and may suffer loss of accuracy [14]. Below, we shall find explicit relations between these two sets of stiffness and mass matrices, which allow us to compute $S^{0,1}$ and $M^{0,1}$ efficiently and accurately.

It is obvious that $J_n^{0,1}(y) \equiv \hat{J}_n^{0,1}(x)$ and $\phi_k^{0,1}(y) \equiv \hat{\phi}_k^{0,1}(x)$ under the mappings (3.10) and (3.11). The next theorem shows the relation between $\phi_k^{0,1}(y)$ defined in (4.12) and $\phi_k(y)$ defined in (4.8) .

Theorem 4.1. *The basis functions* $\phi_k^{0,1}(y)$ *defined in* (4.12) *can be written as a linear combination of* $\{\phi_i(y) := L_i(y) - L_{i+2}(y)\}_{i=0}^k$*, i.e.,*

$$\phi_k^{0,1}(y) = \frac{2k+3}{(k+1)(k+2)(k+3)} \sum_{i=0}^{k} (-1)^{k-i}(i+1)(i+2)\phi_i(y). \tag{4.15}$$

Consequently, let S *and* M *(resp.,* $\hat{S}^{0,1}$ *and* $\hat{M}^{0,1}$*) be the stiffness and mass matrices in* (4.7) *(resp.,* (4.14)*) associated with* $\phi_k = L_k - L_{k+2}$ *(resp.,* $\hat{\phi}_k^{0,1}$*); we have*

$$\hat{M}^{0,1} = 2HMH^T, \quad \hat{S}^{0,1} = \frac{1}{2}HSH^T, \tag{4.16}$$

where $H = (h_{k,j})_{k,j=0}^N$ *is a lower triangular matrix with nonzero entries*

$$h_{k,j} = \frac{(-1)^{k-j}(2k+3)(j+1)(j+2)}{(k+1)(k+2)(k+3)}, \quad 0 \le j \le k \le N. \tag{4.17}$$

Proof. Let $\alpha = \beta = 0$ in (A.4). Noticing that $J_n^{0,0} = L_n$, we have the relation between $J_n^{0,1}(y)$ and $L_n(y)$:

$$(1+y)J_n^{0,1}(y) = (L_n(y) + L_{n+1}(y)). \tag{4.18}$$

Let $\alpha = 0, \beta = 1$ in (A.3), then we have

$$(1-y)J_n^{1,1}(y) = \frac{2(n+1)}{2n+3}(J_n^{0,1}(y) - J_{n+1}^{0,1}(y)). \tag{4.19}$$

By (4.18) and (4.19), we know that

$$(1-y)^2 J_n^{1,1}(y) = \frac{2(n+1)}{2n+3}(L_n(y) - L_{n+2}(y)), \tag{4.20}$$

which implies that we can rewrite ϕ_k as

$$\phi_k(y) = L_k(y) - L_{k+2}(y) = \frac{2k+3}{2(k+1)}(1-y^2)J_k^{1,1}(y). \tag{4.21}$$

On the other hand, we can also rewrite $\phi_k^{0,1}$ as

$$\begin{aligned}
\phi_k^{0,1}(y) =& J_k^{0,1}(y) - \frac{2}{2k+5}J_{k+1}^{0,1}(y) - \frac{2k+3}{2k+5}J_{k+2}^{0,1}(y) \\
\overset{(4.18)}{=\!=}& \frac{1}{1+y}\left[L_k(y) + \frac{2k+3}{2k+5}L_{k+1}(y) - L_{k+2}(y) - \frac{2k+3}{2k+5}L_{k+3}(y)\right] \\
\overset{(4.8)}{=\!=}& \frac{1}{1+y}\left[\phi_k(y) + \frac{2k+3}{2k+5}\phi_{k+1}(y)\right] \\
\overset{(4.21)}{=\!=}& \frac{2k+3}{2}(1-y)\left[\frac{1}{k+1}J_k^{1,1}(y) + \frac{1}{k+2}J_{k+1}^{1,1}(y)\right]. \qquad (4.22)
\end{aligned}$$

Consider a new function

$$\varphi_k(y) := \frac{1}{1+y}\left[\frac{1}{k+1}J_k^{1,1}(y) + \frac{1}{k+2}J_{k+1}^{1,1}(y)\right]. \tag{4.23}$$

Plugging $\alpha = \beta = 1$ in (A.2) yields the three-term recurrence relation

$$\frac{1}{k+2}J_{k+1}^{1,1} = \frac{2k+3}{(k+1)(k+3)}yJ_k^{1,1} - \frac{1}{k+3}J_{k-1}^{1,1}, \quad k \geq 1. \tag{4.24}$$

It follows that

$$\begin{aligned}
\frac{1}{k+1}J_k^{1,1} + \frac{1}{k+2}J_{k+1}^{1,1} &= \left(\frac{(2k+3)y}{(k+1)(k+3)} + \frac{1}{k+1}\right)J_k^{1,1} - \frac{1}{k+3}J_{k-1}^{1,1} \\
&= \left(\frac{(2k+3)(1+y)-k}{(k+1)(k+3)}\right)J_k^{1,1} - \frac{1}{k+3}J_{k-1}^{1,1} \\
&= \frac{(2k+3)(1+y)}{(k+1)(k+3)}J_k^{1,1} - \frac{k}{k+3}\left[\frac{1}{k+1}J_k^{1,1} + \frac{1}{k}J_{k-1}^{1,1}\right]. \qquad (4.25)
\end{aligned}$$

Plugging (4.25) into (4.23) yields

$$\varphi_k(y) = \frac{(2k+3)}{(k+1)(k+3)}J_k^{1,1}(y) - \frac{k}{k+3}\varphi_{k-1}(y). \tag{4.26}$$

From the recursive relation (4.26), we obtain the expansion

$$\varphi_k(y) = \frac{1}{(k+1)(k+2)(k+3)}\sum_{i=0}^{k}(-1)^{k-i}(i+2)(2i+3)J_i^{1,1}(y). \tag{4.27}$$

Therefore,

$$\begin{aligned}
\phi_k^{0,1}(y) =& \frac{2k+3}{2}(1-y^2)\varphi_k(y), \\
\overset{(4.27)}{=\!=}& \frac{(2k+3)(1-y^2)}{2(k+1)(k+2)(k+3)}\sum_{i=0}^{k}(-1)^{k-i}(i+2)(2i+3)J_i^{1,1}(y), \qquad (4.28) \\
\overset{(4.21)}{=\!=}& \frac{2k+3}{(k+1)(k+2)(k+3)}\sum_{i=0}^{k}(-1)^{k-i}(i+1)(i+2)\phi_i(y). \qquad (4.29)
\end{aligned}$$

It remains to show (4.16).

Thanks to (4.29), we immediately obtain the relation

$$M^{0,1} = HMH^T, \quad S^{0,1} = HSH^T, \tag{4.30}$$

where

$$\begin{aligned} &S^{0,1} = (s_{kj}^{0,1}),\ s_{kj}^{0,1} = \Big((1+y)\partial_y \phi_j^{0,1}(y), \partial_y \phi_k^{0,1}(y)\Big), \\ &M^{0,1} = (p_{kj}^{0,1}),\ m_{kj}^{0,1} = \left(\frac{1}{1+y}\phi_j^{0,1}(y), \phi_k(y)\right). \end{aligned} \tag{4.31}$$

So we only need to relate $\hat{M}^{0,1}, \hat{S}^{0,1}$ to $M^{0,1}, S^{0,1}$.

We derive from the mapping defined in (3.10) and (3.11) with $\alpha = 1/2$ that

$$\begin{aligned} \hat{s}_{kj}^{0,1} &= \int_{-1}^{1} (1+x)\partial_x \hat{\phi}_j^{0,1}(x)\partial_x \hat{\phi}_k^{0,1}(x)\mathrm{d}x \\ &= \int_{-1}^{1} \frac{1+y}{2}\partial_y \phi_j^{0,1}(y)\partial_y \phi_k^{0,1}(y)\mathrm{d}y = \frac{1}{2}s_{k,j}^{0,1}, \end{aligned}$$

which shows that $\hat{S}^{0,1} = \frac{1}{2}S^{0,1}$. Similarly, we can show that $\hat{M}^{0,1} = 2M^{0,1}$. Therefore, we obtain (4.16) from (4.30). □

Finally, the linear system associated with the Müntz–Galerkin method for problem (4.4) is

$$\hat{S}^{0,1}U\tilde{M}^T + \frac{4}{\pi^2}\hat{M}^{0,1}U\tilde{S}^T = F, \tag{4.32}$$

which can be efficiently solved by using the matrix diagonalization method [19] in $O(N^3)$ operations.

Case (II)*: $\alpha = 1/q$, where q is a positive integer.* In this case, taking $\alpha = 1/q$ in (4.10) and (4.11) yields the following basis functions

$$\phi_k^{0,q-1}(y) = J_k^{0,q-1}(y) + a_k^q J_{k+1}^{0,q-1}(y) + b_k^q J_{k+2}^{0,q-1}(y), \tag{4.33}$$

$$\hat{\phi}_k^{0,q-1}(x) = \hat{J}_k^{0,q-1}(x) + a_k^q \hat{J}_{k+1}^{0,q-1}(x) + b_k^q \hat{J}_{k+2}^{0,q-1}(x), \tag{4.34}$$

where

$$a_k^q = -\frac{(q-1)(2k+q+2)}{(k+q)(2k+q+3)}, \quad b_k^q = -\frac{(k+2)(2k+q+1)}{(k+q)(2k+q+3)}.$$

Furthermore, it is easy to check that

$$\phi_k^{0,q-1}(y) = \frac{(2k+q+1)(2k+q+2)}{4(k+1)(k+q)}(1-y^2)J_k^{1,q}(y). \tag{4.35}$$

In order to obtain a relation similar to (4.15), we need the following

$$J_k^{1,q} = \sum_{j=0}^{k} d_{k,j}^q J_j^{1,1}. \tag{4.36}$$

Actually, by the demotion relation (A.6), we can find a series of $\{d_{k,j}^{i\to(i+1)}\}_{i=1}^{q-1}$ such that

$$J_k^{1,i+1} = \sum_{j=0}^{k} d_{k,j}^{i\to(i+1)} J_j^{1,i}.$$

Then $(d^q_{k,j})$ can be obtained by the product of $\{d^{i\to(i+1)}_{k,j}\}_{i=1}^{q-1}$. Plugging (4.36) into (4.35) yields

$$\begin{aligned}\phi_k^{0,q-1}(y) &= \frac{(2k+q+1)(2k+q+2)}{4(k+1)(k+q)}(1-y^2)\sum_{j=0}^{k} d^q_{k,j} J_j^{1,1}(y)\\ &\overset{(4.21)}{=\!=} \frac{(2k+q+1)(2k+q+2)}{2(k+1)(k+q)}\sum_{j=0}^{k}\frac{j+1}{2j+3} d^q_{k,j}\phi_j(y).\end{aligned} \tag{4.37}$$

Consider the stiffness and mass matrices with respect to the basis set $\{\hat\phi_k^{0,q-1}\}$ defined in (4.34):

$$\begin{aligned}\hat S^{0,q-1} &= (\hat s^{0,q-1}_{kj}),\ \hat s^{0,q-1}_{kj} = \left((1+x)\partial_x\hat\phi_j^{0,q-1}(x), \partial_x\hat\phi_k^{0,q-1}(x)\right),\\ \hat M^{0,q-1} &= (\hat p^{0,q-1}_{kj}),\ \hat m^{0,q-1}_{kj} = \left(\frac{1}{1+x}\hat\phi_j^{0,q-1}(x), \hat\phi_k^{0,q-1}(x)\right).\end{aligned} \tag{4.38}$$

Then we can find the relations between $\hat S^{0,q-1}, \hat M^{0,q-1}$ defined above and M, S defined in (4.7) as follows:

$$\hat M^{0,q-1} = qH_q M H_q^T, \quad \hat S^{0,q-1} = \frac{1}{q}H_q S H_q^T, \tag{4.39}$$

where $H_q = (h^q_{k,j})_{k,j=0}^N$ is a lower triangular matrix with nonzero entries

$$h^q_{k,j} = \frac{(2k+q+1)(2k+q+2)(j+1)}{2(k+1)(k+q)(2j+3)} d^q_{k,j}, \quad 0 \le j \le k \le N. \tag{4.40}$$

4.3. Error estimates. Since in the Müntz–Galerkin method (4.4), the s-direction is treated with the usual polynomials so its error estimate is well known [22], we only need to consider applying the Müntz–Galerkin method to the one-dimensional problem:

$$\begin{cases} -\partial_x((1+x)\partial_x u) + \dfrac{1}{1+x}u = f, \quad x \in I = (-1,1),\\ u(\pm1) = 0,\end{cases} \tag{4.41}$$

where f is a given function.

We define $X = \{u : \ |||u|||_E < \infty\}$ with

$$|||u|||_E := \sqrt{a(u,u)} = \left(\|u'\|^2_{\omega^{0,1}} + \|u\|^2_{\omega^{0,-1}}\right)^{1/2}. \tag{4.42}$$

Then, a weak formulation of (4.41) is to find $u \in X$ such that

$$a(u,v) := ((1+x)u', v') + \left((1+x)^{-1}u, v\right) = (f,v) \quad \forall v \in X. \tag{4.43}$$

It is easy to verify that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive in X, so (4.43) is well posed.

Let $\mathcal{P}_N$ be the set of all polynomials of degree at most N, and define

$$\mathcal{P}_N^0 := \{u(y) \in \mathcal{P}_N : u(\pm1) = 0\}. \tag{4.44}$$

We define the Müntz–Jacobi approximation space

$$\mathcal{V}_N^0 := \{\phi : \phi(x) = \Psi(y(x)),\ \forall \Psi \in \mathcal{P}_N^0\}. \tag{4.45}$$

Then, the Müntz–Galerkin approximation to (4.41) is to find $u_N \in \mathcal{V}_N^0$ such that

$$a(u_N, v_N) := (\partial_x u_N, \partial_x v_N)_{\omega^{0,1}} + (u_N, v_N)_{\omega^{0,-1}} = (f, v_N) \quad \forall v_N \in \mathcal{V}_N^0. \tag{4.46}$$

Let us recall the definition of generalized Jacobi polynomials $\{J_n^{-1,-1}\}_{n=0}^{\infty}$ [9, 10]:

$$J_n^{-1,-1}(y) := (1-y^2) J_{n-2}^{1,1}(y) \quad \forall n \in \mathbb{N}. \tag{4.47}$$

On the other hand, it is known that the Legendre basis $\{\phi_k(y)\}_{k=0}^{\infty}$ with Dirichlet boundary conditions satisfy the following:

$$\phi_k(y) := L_k(y) - L_{k+2}(y) = \frac{2k+3}{2(k+1)}(1-y^2) J_k^{1,1}(y) = \frac{2k+3}{2(k+1)} J_{k+2}^{-1,-1}(y). \tag{4.48}$$

Furthermore, for the Jacobi basis functions $\{\phi_k^{0,1/\alpha-1}\}$ defined in (4.10), we have

$$\begin{aligned}\phi_k^{0,1/\alpha-1}(y) &= \frac{(2k+1/\alpha+1)(2k+1/\alpha+2)}{4(k+1)(k+1/\alpha)}(1-y^2) J_k^{1,1/\alpha}(y)\\ &= \frac{(2k+1/\alpha+1)(2k+1/\alpha+2)}{4(k+1)(k+1/\alpha)}(1-y^2) \sum_{j=0}^{k} d_j J_j^{1,1}(y)\\ &= \sum_{j=0}^{k} \tilde{d}_{j,k} J_{j+2}^{-1,-1}(y),\end{aligned} \tag{4.49}$$

where the coefficients $\{d_j\}$ are called *connection coefficients* [24] between two Jacobi families $\{J_k^{1,1/\alpha}\}$ and $\{J_k^{1,1}\}$. Now it is clear that

$$\begin{aligned}\mathcal{P}_N^0(I) &= \operatorname{span}\{\phi_k^{0,1/\alpha-1}(y)\}_{k=0}^{N-2} = \operatorname{span}\{J_k^{-1,-1}(y)\}_{k=2}^{N},\\ \mathcal{V}_N^0(I) &= \operatorname{span}\{\hat{\phi}_k^{0,1/\alpha-1}(x)\}_{k=0}^{N-2}.\end{aligned}$$

Consider the weighted L^2 spaces

$$\begin{aligned}L^2_{\omega^{-1,-1}}(I) &= \{U : \|U\|_{\omega^{-1,-1}} < \infty\},\\ \tilde{L}^2_{\omega^{-1,-1}}(I) &= \left\{u : u(x) = U(y(x)),\ U(y) \in L^2_{\omega^{-1,-1}}(I)\right\}.\end{aligned}$$

Let $\Pi_N^0 : L^2_{\omega^{-1,-1}}(I) \to \mathcal{P}_N^0$ be the orthogonal projection defined by

$$(\Pi_N^0 U - U, \Phi)_{\omega^{-1,-1}} = 0 \quad \forall \Phi \in \mathcal{P}_N^0, \tag{4.50}$$

where the weight $\omega^{-1,-1}(y) = (1-y^2)^{-1}$. We define $\pi_N^0 : \tilde{L}^2_{\omega^{-1,-1}}(I) \to \mathcal{V}_N^0$,

$$(\pi_N^0 u)(x) := (\Pi_N^0 U)(y(x)). \tag{4.51}$$

Then we easily derive by definition that

$$\left(\pi_N^0 u - u, \phi\right)_{\tilde{\omega}^{-1,-1/\alpha}} = c_\alpha \left(\Pi_N^0 U - U, \Phi\right)_{\omega^{-1,-1}} = 0 \quad \forall \phi \in \mathcal{V}_N^0, \tag{4.52}$$

where $\tilde{\omega}^{-1,-1/\alpha}$ is defined in (3.24) by setting $a = -1, b = -1/\alpha$, and $c_\alpha = \frac{2^{1-1/\alpha}}{\alpha}$ is a constant only dependant on α.

To describe the error of the Müntz–Jacobi approximation for the above projections Π_N^0 and π_N^0, we find that the weighted norms of functions $D_x^k u$ and $\partial_y^k U$ defined in (3.30) are related by

$$\begin{aligned}\|U\|^2_{\omega^{-1,-1}} &= \int_{-1}^{1} |U(y)|^2(1+y)^{-1/\alpha}(1-y)^{-1}(1+y)^{1/\alpha-1}\mathrm{d}y\\ &= \frac{1}{c_\alpha}\int_{-1}^{1}|u(x)|^2\tilde{\omega}^{-1,-1/\alpha}\mathrm{d}x = \frac{1}{c_\alpha}\|u\|^2_{\tilde{\omega}^{-1,-1/\alpha}},\end{aligned} \tag{4.53}$$

$$\begin{aligned}\|\partial_y U\|^2_{\omega^{0,0}} &= \int_{-1}^{1}|\partial_y U(y)|^2(1+y)^{1-1/\alpha}(1+y)^{1/\alpha-1}\mathrm{d}y\\ &= \frac{1}{c_\alpha}\|D_x u\|^2_{\tilde{\omega}^{0,-1/\alpha+1}},\end{aligned} \tag{4.54}$$

$$\cdots$$

$$\|\partial_y^k U\|^2_{\omega^{-1+k,-1+k}} = \frac{1}{c_\alpha}\|D_x^k u\|^2_{\tilde{\omega}^{-1+k,-1/\alpha+k}}. \tag{4.55}$$

Recall the error estimate for orthogonal projection Π_N^0 defined in (4.50) (cf. Theorem 6.1 in [22]): for $0 \le l \le m$,

$$\|\partial_y^l\left(\Pi_N^0 U - U\right)\|_{\omega^{-1+l,-1+l}} \lesssim N^{l-m}\|\partial_y^m U\|_{\omega^{-1+m,-1+m}} \tag{4.56}$$

for any $U \in B^m_{\omega^{-1,-1}}(I) := \{U : \partial_y^k U \in L^2_{\omega^{-1+k,-1+k}}, 0 \le k \le m\}$.

Next, we define the mapped space

$$\tilde{B}^m_{\omega^{-1,-1/\alpha}}(I) = \left\{u : \|u\|_{\tilde{B}^m_{\omega^{-1,-1/\alpha}}} < \infty\right\}, \tag{4.57}$$

equipped with the norm and seminorm

$$\|u\|_{\tilde{B}^m_{\omega^{-1,-1/\alpha}}}\left(\sum_{k=0}^{m}\|D_x^k u\|^2_{\tilde{\omega}^{-1+k,-1/\alpha+k}}\right)^{1/2}, \quad |u|_{\tilde{B}^m_{\omega^{-1,-1/\alpha}}} = \|D_x^m u\|_{\tilde{\omega}^{-1+m,-1/\alpha+m}}.$$

We now present our main approximation result for Müntz–Jacobi approximations.

THEOREM 4.2. *Let $u \in \tilde{B}^m_{\omega^{-1,-1/\alpha}}(I) \cap \tilde{L}^2_{\omega^{-1,-1}}(I)$ and $\mu = 0, 1$. Then we have*

$$\|\partial_x^\mu(\pi_N^0 u - u)\|_{\omega^{0,\hat{s}(\mu)}} \lesssim N^{\mu-m}|u|_{\tilde{B}^m_{\omega^{-1,-1/\alpha}}}, \tag{4.58}$$

where $\hat{s}(\mu) = \begin{cases} -1, & \mu = 0,\\ 1, & \mu = 1.\end{cases}$

Proof. First of all, for any u and U satisfying the relation (3.16), we have

$$\|u\|^2_{\omega^{0,-1}} = \int_{-1}^{1}|u(x)|^2(1+x)^{-1}\mathrm{d}x = \frac{1}{\alpha}\int_{-1}^{1}|U(y)|^2(1+y)^{-1}\mathrm{d}y = \frac{1}{\alpha}\|U\|^2_{\omega^{0,-1}}, \tag{4.59}$$

$$\|\partial_x u\|^2_{\omega^{0,1}} = \int_{-1}^{1}|\partial_x u(x)|^2(1+x)\mathrm{d}x = \alpha\int_{-1}^{1}|\partial_y U(y)|^2(1+y)\mathrm{d}y = \alpha\|\partial_y U\|^2_{\omega^{0,1}}. \tag{4.60}$$

Thanks to (4.59) and (4.60), we have the following estimates

$$\|\pi_N^0 u - u\|_{\omega^{0,-1}} = \frac{1}{\sqrt{\alpha}}\|\Pi_N^0 U - U\|_{\omega^{0,-1}} \le \frac{1}{\sqrt{\alpha}}\|\Pi_N^0 U - U\|_{\omega^{-1,-1}}, \tag{4.61}$$

$$\|\partial_x\left(\pi_N^0 u - u\right)\|_{\omega^{0,1}} = \sqrt{\alpha}\|\partial_y\left(\Pi_N^0 U - U\right)\|_{\omega^{0,1}} \le \sqrt{\alpha}\|\partial_y\left(\Pi_N^0 U - U\right)\|_{\omega^{0,0}}. \tag{4.62}$$

Thus, using the result (4.56) with $l = 0, 1$, we obtain (4.58) from (4.61)–(4.62) and the definitions in (4.55) and (4.57). ∎

We can now state error estimates for our Müntz–Galerkin method.

THEOREM 4.3. *Let u and u_N be the solutions of* (4.43) *and* (4.46)*, respectively. If $u \in \tilde{B}^m_{\omega^{-1,-1/\alpha}}(I) \cap \tilde{L}^2_{\omega^{-1,-1}}(I)$, we have*

$$|||u - u_N|||_E \lesssim N^{1-m} |u|_{\tilde{B}^m_{\omega^{-1,-1/\alpha}}}. \tag{4.63}$$

Proof. By (4.43) and (4.46), we have

$$a(u - u_N, v_N) = 0 \quad \forall v_N \in \mathcal{V}_N^0. \tag{4.64}$$

It implies that

$$\begin{aligned} |||u - u_N|||_E^2 &= a(u - u_N, u - u_N) = a(u - u_N, u - \pi_N^0 u) \\ &\le |||u - u_N|||_E |||u - \pi_N^0 u|||_E. \end{aligned}$$

The desired result (4.63) follows from the above and (4.58). ∎

5. Numerical experiments. In this section, we present several numerical experiments to illustrate the efficiency and the accuracy of our Müntz–Galerkin method.

5.1. Model problem. Consider the one dimensional problem

$$\begin{cases} -(ru')' + \dfrac{u}{r} = f(r), \quad r \in (0, 1), \\ u(0) = u(1) = 0 \end{cases} \tag{5.1}$$

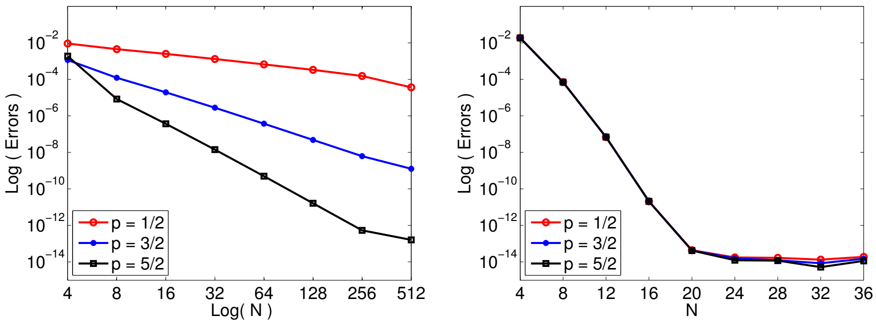
with the exact solution given by $u(r) = r^p - r + \sin(\pi r)$, where p is a parameter to be chosen.

We choose two categories for the values of p: (i) $p = 1/2, 3/2, 5/2$; (ii) $p = 1/3, 4/3, 7/3$. We first solved the above problems by the classical Legendre–Galerkin method with basis functions defined in (4.8) [20]. The errors with energy norm in log-log scale are shown in the left parts of Figures 3 and 4 We observe that the convergence rates are algebraic due to the singular term r^p in the solution. We then solved the same problems by the Müntz–Galerkin method (4.46) using the basis function (4.11) with $\alpha = 1/2$ for $p = 1/2, 3/2, 5/2$ and $\alpha = 1/3$ for $p = 1/3, 4/3, 7/3$. The errors with energy norm in semilog scale are plotted on the right parts of Figures 3 and 4. We observe that the errors converge exponentially. These results are consistent with the error estimates in Theorem 4.3.

5.2. Semicircular membrane with a free half-edge. Next, we consider the so-called *Motz's problem* [18] which is the following Poisson equation in a semicircular domain with N-D mixed boundary condition on the bottom (see Figure 5):

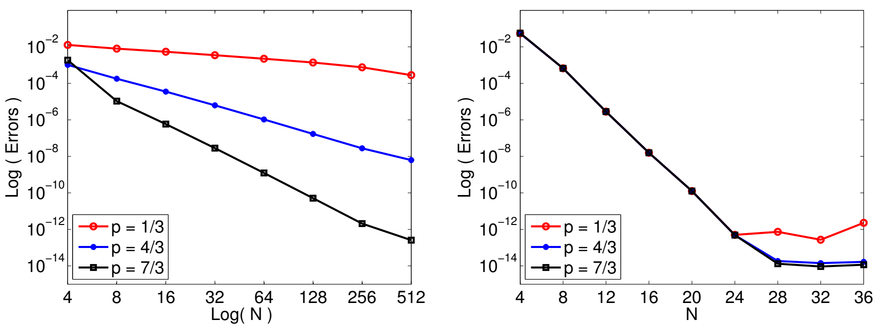
$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1 \cup \Gamma_3, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_2, \end{cases} \tag{5.2}$$

where $\Gamma_1 = \{(r, \theta) : r = 1, 0 \le \theta \le 2\pi\}$, $\Gamma_2 = \{(r, \theta) : 0 \le r < 1, \theta = 0\}$, $\Gamma_3 = \{(r, \theta) : 0 < r < 1, \theta = 2\pi\}$. It is shown in Theorem 2.2 that the exact solution in the vicinity of the origin takes the form (2.7).



(i) Legendre–Galerkin method (ii) Müntz–Galerkin method ($\alpha = 1/2$)

FIG. 3. *Convergence rates for problem* (5.1) *with* $p = 1/2, 3/2, 5/2$.



(i) Legendre–Galerkin method (ii) Müntz–Galerkin method ($\alpha = 1/3$)

FIG. 4. *Convergence rates for problem* (5.1) *with* $p = 1/3, 4/3, 7/3$.

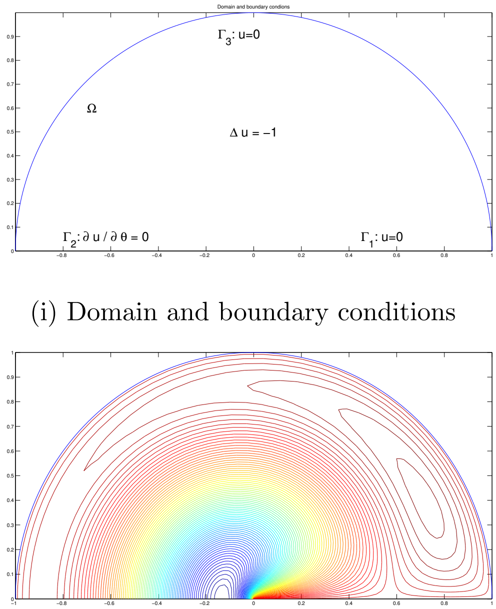
We solve this problem using two methods:

1. Legendre–Galerkin method: using classical Legendre basis in (4.8) and (4.9);
2. Müntz–Galerkin method: using the Legendre basis (4.9) in the θ direction and the Müntz basis (4.13) in the r direction to treat the singularity at $r = 0$.

The comparison of convergence rates in the L^2-norm is shown in Figure 6. The rates of convergence are greatly improved (from second order to fifth order) since the main singularity at the origin is taken care of in our proposed Müntz–Galerkin method, but we did not attempt to treat the mild singularities at the two corners.

5.3. The eigenvalue problem. As the last example, we focus on the spectral problems of a Laplacian operator with mixed Dirichlet–Neumann boundary conditions on the half-unit disc:

(5.3)
$$\begin{cases} -\Delta u = \dfrac{4\lambda}{(1+r^2)^2} u & \text{in } \Omega = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi\}, \\ \dfrac{\partial u}{\partial n}\Big|_{\partial_1\Omega} = 0 & \text{on } \partial_1\Omega = \{(r,0) : r \in (0,1)\} \cup \left\{(1,\theta) : \left|\theta - \dfrac{\pi}{2}\right| < \dfrac{\pi}{4}\right\}, \\ u|_{\partial_2\Omega} = 0 & \text{on } \partial_2\Omega = \{(r,\pi) : r \in (0,1)\} \cup \left\{(1,\theta) : \left|\theta - \dfrac{\pi}{2}\right| > \dfrac{\pi}{4}\right\}. \end{cases}$$



(i) Domain and boundary conditions

(ii) Numerical solution

FIG. 5. *Semicircular membrane with a free half-edge.*

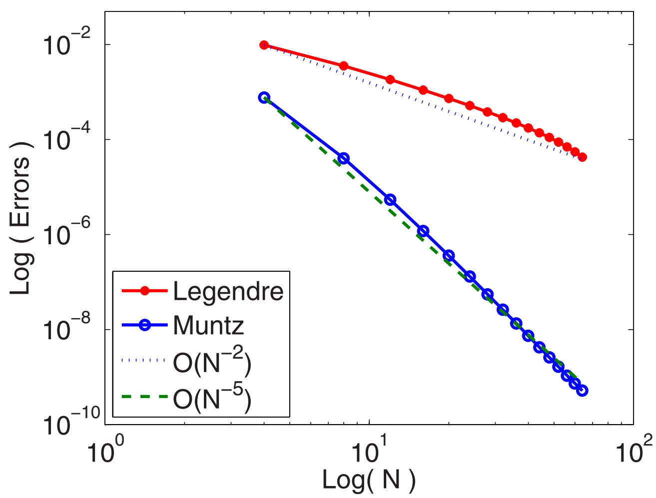
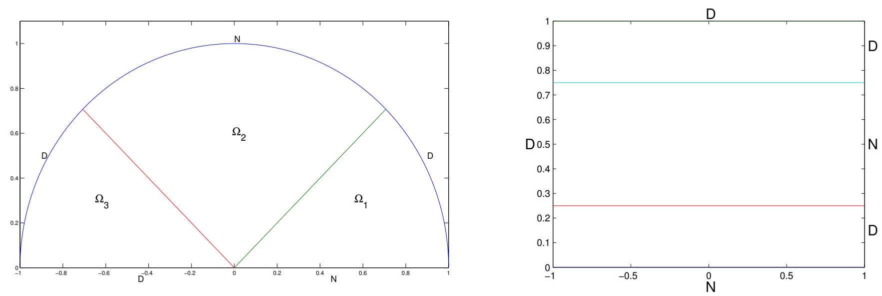


FIG. 6. *Convergence rates: Legendre–Galerkin method and Müntz–Galerkin method.*

This problem is related to the isospectrality question for mixed Dirichlet–Neumann boundary conditions, called the *Zaremba problem* [25]. Jakobson et al. [11, 12] proposed a conjecture about the first eigenvalue Λ_1 of the problem (5.3). Since a rigorous proof of this conjecture is still not available, attempts [11] have been made to numerically verify it.



(i) Domain and boundary conditions (ii) Transformed domain

Fig. 7. *Boundary conditions and computational domain of the problem* (5.3).

Table 1
First eigenvalues obtained by FEMs.

DOF	P1 adaptive conforming		P1 adaptive nonconforming	
	Λ_1	$\Delta\Lambda_1$	Λ_1	$\Delta\Lambda_1$
169	2.45590105457		2.13042989031	
625	2.36301118569	9.2890e-02	2.20743747322	7.7008e-02
2401	2.32089716556	4.2114e-02	2.24561396752	3.8176e-02
9409	2.30111238184	1.9785e-02	2.26440630518	1.8792e-02
37249	2.29161462311	9.4978e-03	2.27364314423	9.2368e-03

Using the transformation given in (4.2), the problem (5.3) becomes

(5.4)
$$\begin{cases} -\partial_t\left((t+1)\partial_t\tilde{u}\right) - \dfrac{1}{\pi^2}\dfrac{1}{t+1}\partial_{ss}\tilde{u} = \lambda\dfrac{16(t+1)}{(4+(t+1)^2)^2}\tilde{u} & \text{in } \tilde{\Omega} = [-1,1]^2, \\ \partial_s\tilde{u}(t,0) = 0, & t\in(-1,1), \\ \partial_t\tilde{u}(1,s) = 0, & s\in(1/4,3/4), \\ \tilde{u}(t,1) = \tilde{u}(-1,s) = 0, & t\in(-1,1),\ s\in(0,1), \\ \tilde{u}(1,s) = 0, & s\in(0,1/4)\cup(3/4,1), \end{cases}$$

where $\tilde{u}(t,s) = u\left((t+1)/2, \pi s\right)$. The original and transformed domains are shown in Figure 7.

We solve the above problem using the Müntz spectral-element method. More precisely, we divide the transformed domain into three subdomains, and use the Müntz basis sets in the t direction, and the Legendre basis set in each subinterval $(0,1/4), (1/4,3/4)$ and $(3/4)$ plus two nodal basis centered at the $s = 1/4$ and $3/4$ in the s direction.

The approximate first eigenvalues obtained by P1 adaptive conforming and nonconforming finite element methods in [11] and our Müntz spectral-element method are shown in Tables 1 and 2, respectively. Here, we denote DOF as the degree of freedoms, N as the maximal degree of polynomials, Λ_1 and $\Delta\Lambda_1$ as the approximate first eigenvalue and difference between the two successive values of Λ_1, respectively.

It is well known that for eigenvalue problems, the conforming finite element method always gives upper bounds while the nonconforming one here gives lower bounds. Based on the numerical results shown in Table 1, we can conclude that the first eigenvalue should be located in the interval $2.2736 < \Lambda_1 < 2.2916$, which means

TABLE 2
First eigenvalues obtained by Müntz spectral-element method.

N	DOF	Λ_1	$\Delta\Lambda_1$
8	208	2.297300191344	
16	800	2.281650971945	1.5649e-02
32	3136	2.276933351716	1.2834e-03
64	12416	2.275629083928	1.3745e-04
128	49408	2.275285624415	1.5959e-05
192	110976	2.275225763943	3.3971e-06
232	161936	2.275218759615	2.5856e-07

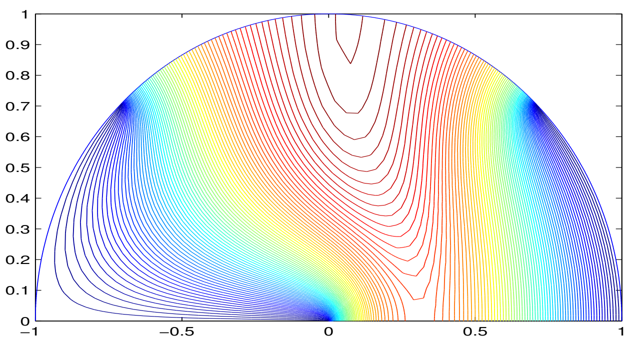


FIG. 8. *Contour of the first eigenfunction of problem* (5.3).

the number of significant digits obtained by these finite element methods is only 2. However, the results in Table 2 indicate that we can achieve 6 significant digits with a smaller DOF by using the Müntz spectral-element method. Note also that, with the same DOFs, our Müntz spectral-element method is also much faster than adaptive finite element methods (FEMs), since we use a direct matrix diagonalization method on each subdomain with a total cost being a small multiple of $\mathrm{DOF}^{3/2}$. It demonstrates that for this problem the Müntz–Galerkin method is much more efficient than adaptive FEMs.

In Figure 8, we plot the first eigenfunction obtained by the Müntz spectral-element method.

6. Concluding remarks. We developed in this paper the Müntz–Galerkin methods for problems with singular solutions for which the direct spectral method with the usual polynomial basis functions does not lead to a satisfactory convergence rate. Assuming that we have a singular expansion for the solution near a singular point in the form $O(r^\alpha)$, our Müntz–Galerkin method is based on Müntz polynomials defined from the singular expansion. To overcome the poor conditioning of the Müntz polynomials, we explored relations between Jacobi polynomials and Müntz polynomials, and developed efficient implementation procedures for the Müntz–Galerkin method. We also developed a framework to analyze the approximation errors of Müntz polynomials and derived the optimal error estimates for the Müntz–Galerkin method. As examples of applications, we employed the Müntz–Galerkin method to solve the Poisson equation with mixed Dirichlet–Neumann boundary conditions, and showed that the Müntz–Galerkin method leads to much improved rates of convergence compared to classical spectral methods.

Note that the rates of convergence by our Müntz spectral method for the last two problems in section 5 are not exponential, since we only treated the main singularity at the origin, but did not make special treatments for the weaker singularities at $(r,\theta) = (1,0)$, $(1,\pi)$ for the problem in subsection 5.2 and at $(1,\pi/4)$ and $(1,3\pi/4)$ for the problem in subsection 5.3. How to efficiently treat all these singularities simultaneously is still under investigation.

We showed the explicit formula for the Müntz–Jacobi basis functions as well as the efficient algorithm to compute the corresponding stiffness and mass matrices only for a special Müntz sequence $\Lambda(\alpha)$, where $\alpha = 1/q$ and q is an integer. For the case with arbitrary Müntz sequence, one could employ the quadrature technique developed in [14] to calculate the inner products. We shall consider singular problems characterized by other typical Müntz sequences in the future.

Appendix A. Some useful formulas for Jacobi polynomials. The Jacobi polynomials, denoted by $J_n^{\alpha,\beta}(x)$, are orthogonal with respect to the Jacobi weight $\omega^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$ over the interval $I := (-1,1)$, namely,

$$\int_{-1}^{1} J_n^{\alpha,\beta}(x) J_m^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)\mathrm{d}x = \gamma_n^{\alpha,\beta}\delta_{mn}, \tag{A.1}$$

where the δ_{mn} is the Kronecker function and

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

Here $\Gamma(\cdot)$ is the gamma function.

The Jacobi polynomials are generated by the three-term recurrence relation

$$J_{n+1}^{\alpha,\beta}(x) = (a_n^{\alpha,\beta}x - b_n^{\alpha,\beta})J_n^{\alpha,\beta}(x) - c_n^{\alpha,\beta}J_{n-1}^{\alpha,\beta}(x), \quad n \ge 1, \tag{A.2}$$

with

$$J_0^{\alpha,\beta}(x) = 1, \quad J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta),$$

where

$$a_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)},$$
$$b_n^{\alpha,\beta} = \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)},$$
$$c_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}.$$

This relation allows us to evaluate the Jacobi polynomials at any given abscissa $x \in [-1,1]$, and it is the starting point to derive the following useful properties.

1. *Index shifting:*

$$J_n^{\alpha+1,\beta} = \frac{2}{2n+\alpha+\beta+2}\frac{(n+\alpha+1)J_n^{\alpha,\beta} - (n+1)J_{n+1}^{\alpha,\beta}}{1-x}, \tag{A.3}$$

$$J_n^{\alpha,\beta+1} = \frac{2}{2n+\alpha+\beta+2}\frac{(n+\beta+1)J_n^{\alpha,\beta} + (n+1)J_{n+1}^{\alpha,\beta}}{1+x}. \tag{A.4}$$

2. *Demotion relation:*

$$J_n^{\alpha+1,\beta}(x) = \sum_{l=0}^{n} \kappa_{nl}^{(\alpha,\beta)\to(\alpha+1,\beta)} J_l^{\alpha,\beta}(x), \tag{A.5}$$

$$J_n^{\alpha,\beta+1}(x) = \sum_{l=0}^{n} \kappa_{nl}^{(\alpha,\beta)\to(\alpha,\beta+1)} J_l^{\alpha,\beta}(x), \tag{A.6}$$

where

$$\kappa_{nl}^{(\alpha,\beta)\to(\alpha+1,\beta)} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \times \frac{(2l+\alpha+\beta+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\beta+1)},$$
$$\kappa_{nl}^{(\alpha,\beta)\to(\alpha,\beta+1)} = (-1)^{n+l}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+2)} \times \frac{(2l+\alpha+\beta+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)}.$$

3. *Boundary values:*

$$J_n^{\alpha,\beta}(-1) = (-1)^n \frac{\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)}, \tag{A.7}$$

$$J_n^{\alpha,\beta}(1) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}. \tag{A.8}$$

4. *Derivatives:*

$$\partial_x J_n^{\alpha,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1) J_{n-1}^{\alpha+1,\beta+1}(x). \tag{A.9}$$

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