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Legendre and Chebyshev dual-Petrov–Galerkin methods for Hyperbolic equations

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Dedicated to Professor Ivo Babuska on the occasion of his 80th birthday.

Abstract

A Legendre and Chebyshev dual-Petrov–Galerkin method for hyperbolic equations is introduced and analyzed. The dual-Petrov–Galerkin method is based on a natural variational formulation for hyperbolic equations. Consequently, it enjoys some advantages which are not available for methods based on other formulations. More precisely, it is shown that (i) the dual-Petrov–Galerkin method is always stable without any restriction on the coefficients; (ii) it leads to sharper error estimates which are made possible by using the optimal approximation results developed here with respect to some generalized Jacobi polynomials; (iii) one can build an optimal preconditioner for an implicit time discretization of general hyperbolic equations.

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1. Introduction

We consider in this paper Legendre and Chebyshev approximations of the linear hyperbolic equation

$$\partial_t u + \partial_x(au) + bu = f, \quad |x| < 1, \;\; 0 < t \leqslant T, \tag{1.1}$$

with given initial data and appropriate non-periodic boundary conditions.

There exist a large body of literature on using spectral methods for solving hyperbolic systems (cf. [10,4,3,9] and the references therein). We refer in particular to the recent review paper by Gottlieb and Hesthaven [9] for a up-to-date account on this subject. Previous work can be essentially classified into four different approaches: collocation, Galerkin, tau (cf. [10]) and penalty (cf. [9]). In this paper, we shall take a different point of view by proposing a dual-Petrov–Galerkin method.

The dual-Petrov–Galerkin method was recently introduced by Shen [22] for solving third and higher odd-order differential equations. The key idea is to choose the trial functions satisfying the underlying boundary conditions, and the test functions satisfying the "dual" boundary conditions. This approach enjoys a number of appealing advantages: (i) it leads to a strongly coercive bilinear form despite the fact that the leading-order differential operator is not elliptic and non-symmetric. (ii) it leads to a well-conditioned linear system, sparse for problems with constant-coefficients,

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which can be efficiently solved; (iii) it leads to optimal error estimates.

The purpose of this paper is to present a Legendre and Chebyshev dual-Petrov–Galerkin method for hyperbolic equations, and to investigate whether their advantages for third and higher-order equations would carry over to first-order equations. The following three issues will be addressed:

(1) *Stability*: We shall show that the dual-Petrov–Galerkin method is always stable without any sign restriction on the coefficients a and b.
(2) *Error analysis*: We shall develop new approximation results based on the special basis functions which can be regarded as generalized Jacobi polynomials with index $\alpha \leqslant -1$ and/or $\beta \leqslant -1$. We shall then use these new approximation results to derive sharper error estimates for the dual-Petrov–Galerkin method.
(3) *Efficiency*: We shall discuss some implementation details of the dual-Petrov–Galerkin method in frequency space as well as in physical space. In particular, we shall show that when working in frequency space, the sparse matrix for a problem with constant coefficients can be used as an optimal (independent of number of modes) preconditioner for the full matrix associated with a large class of variable coefficients.

The paper is organized as follows. In Section 2, we introduce the spectral and pseudo-spectral dual-Petrov–Galerkin methods in a general setting and prove their stability. In Section 3, we discuss some of the implementation details. Then, in Section 4, we develop sharp approximation results based on special basis functions which are mutually orthogonal in weighted (generalized) Jacobi spaces. We then use these new approximation results to derive, in Section 5, error estimates for the Legendre and Chebyshev spectral dual-Petrov–Galerkin methods. We conclude with several remarks.

We now introduce some notations which will be used throughout the paper.

Let $\chi(x)$ be a weight function in $I=(-1,1)$, which is not necessary in $L^1(I)$. We denote by $H^r_\chi(I)(r=0,1,\ldots)$ the weighted Sobolev spaces whose inner products, norms and semi-norms are $(\cdot,\cdot)_{r,\chi}, \|\cdot\|_{r,\chi}$ and $|\cdot|_{r,\chi}$, respectively. In particular, the norm and inner product of $L^2_\chi(I)=H^0_\chi(I)$ are denoted by $\|\cdot\|_\chi$ and $(\cdot,\cdot)_\chi$, respectively. The subscript χ will be omitted from the notations in case of $\chi(x)\equiv 1$.

We denote by $\omega^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$ the Jacobi weight function. In particular, we use $\omega(x)$ to denote respectively the Legendre ($\omega(x)\equiv 1$) or Chebyshev ($\omega(x)=(1-x^2)^{-1/2}$) weight function.

For any non-negative integer N, we denote by P_N the set of all algebraic polynomials of degree $\leqslant N$. We shall use c to denote a generic positive constant independent of any ... the expression $A \lesssim B$ to mean

2. The dual-Petrov–Galerkin method and its stability

2.1. An illustrative example

To illustrate the attractive properties of the dual-Petrov–Galerkin method, we first consider the following model equation:

$$\partial_t u + a\partial_x u = f, \quad (x,t)\in I\times(0,T],$$
$$u(-1,t)=0, \quad t\in[0,T]; \quad u(x,0)=u_0(x),\ x\in\bar{I}, \tag{2.1}$$

where a is a positive constant.

2.1.1. Variational formulation

Define the "dual" approximation spaces:

$$V_N=\{u\in P_N : u(-1)=0\}, \quad V_N^*=\{v\in P_N : v(1)=0\}. \tag{2.2}$$

The Legendre or Chebyshev dual-Petrov–Galerkin method for (2.1) is

$$\begin{cases} \text{Find } u_N(\cdot,t)\in V_N \text{ such that for all } t\in(0,T], \\ (\partial_t u_N, v_N)_\omega + a(\partial_x u_N, v_N)_\omega = (f, v_N)_\omega, \quad \forall v_N\in V_N^*, \end{cases} \tag{2.3}$$

with $u_N|_{t=0}=u_{0,N}$ being a suitable approximation to u_0, and $\omega(x)$ being either the Legendre or Chebyshev weight function.

Note that for any $v_N\in V_N$, we have $v_N\frac{1-x}{1+x}\in V_N^*$. Hence, by setting $\omega_0(x)=\omega(x)\frac{1-x}{1+x}$, we can rewrite the dual-Petrov–Galerkin formulation (2.3) in the equivalent (weighted) Galerkin formulation:

$$\begin{cases} \text{Find } u_N(\cdot,t)\in V_N \text{ such that for all } t\in(0,T], \\ (\partial_t u_N, v_N)_{\omega_0} + a(\partial_x u_N, v_N)_{\omega_0} = (f, v_N)_{\omega_0}, \quad \forall v_N\in V_N, \end{cases} \tag{2.4}$$

with $u_N|_{t=0}=u_{0,N}$.

2.1.2. Stability

The key to stability is the following identities which can be derived directly from an integration by parts:

$$\begin{aligned} (v_x,v)_{\omega_0} &= \int_{-1}^1 v^2(x)\frac{1}{(1+x)^2}\,\mathrm{d}x, \quad \forall v\in V_N \quad (\text{for } \omega(x)=1), \\ (v_x,v)_{\omega_0} &= \int_{-1}^1 v^2(x)\frac{2-x}{2\sqrt{(1+x)^5(1-x)}}\,\mathrm{d}x, \\ &\forall v\in V_N \quad (\text{for } \omega(x)=(1-x^2)^{-1/2}). \end{aligned} \tag{2.5}$$

Hence, taking $v_N=u_N$ in (2.4) leads to that for $\omega(x)=1$,

$$\begin{aligned} &\frac{1}{2}\partial_t\int_{-1}^1 u_N^2\omega_0(x)\,\mathrm{d}x + a\int_{-1}^1 u_N^2\frac{1}{(1+x)^2}\,\mathrm{d}x \\ &\quad=\int_{-1}^1 f u_N\frac{1-x}{1+x}\,\mathrm{d}x \\ &\quad\leqslant \frac{a}{2}\int_{-1}^1 u_N^2\frac{1}{(1+x)^2}\,\mathrm{d}x + \frac{1}{2a}\int_{-1}^1 f^2(1-x)^2\,\mathrm{d}x, \end{aligned}$$

and for $\omega(x) = \frac{1}{\sqrt{1-x^2}}$,

$$\frac{1}{2}\partial_t \int_{-1}^{1} u_N^2 \omega_0(x)\,\mathrm{d}x + \frac{a}{2}\int_{-1}^{1} u_N^2 \frac{1}{\sqrt{(1+x)^5(1-x)}}\,\mathrm{d}x$$
$$\leqslant \int_{-1}^{1} f u_N \sqrt{\frac{1-x}{(1+x)^3}}\mathrm{d}x$$
$$\leqslant \frac{a}{2}\int_{-1}^{1} u_N^2 \frac{1}{\sqrt{(1+x)^5(1-x)}}\,\mathrm{d}x + \frac{1}{2a}\int_{-1}^{1} f^2 \sqrt{\frac{(1-x)^3}{1+x}}\mathrm{d}x.$$

The stability of the scheme follows immediately from the above and the Gronwall inequality.

2.2. General setup

We introduce in this subsection a general setup and some notations to be used throughout this paper.

Without loss of generality, we conventionally assume that the variable coefficients a and b in (1.1) satisfy:

(1) $a(\pm 1, t)$ does not change sign in $[0, T]$;[2]
(2) the functions $a, a_x, b \in L^\infty(I \times [0, T])$.

To impose boundary conditions, we denote

$$\Gamma := \partial I = \{-1, 1\}, \quad \Gamma^- := \{x \in \Gamma : xa(x,t) < 0\},$$
$$\Gamma^+ := \Gamma \setminus \Gamma^-. \tag{2.6}$$

The problem of interest is of the form

$$\partial_t u(x,t) + \partial_x(a(x,t)u(x,t)) + b(x,t)u(x,t) = f(x,t),$$
$$(x,t) \in I \times (0,T], u(x,t) = g(t), \quad (x,t) \in \Gamma^- \times [0,T];$$
$$u(x,0) = u_0(x), \quad x \in \bar{I}. \tag{2.7}$$

More precisely, the boundary conditions are as follows:

$$\begin{aligned}
&(\mathrm{i})_{\mathrm{B}}\ u(\pm 1,t) = g_\pm(t), \quad \text{if } a(-1,t) > 0, a(1,t) < 0;\\
&(\mathrm{ii})_{\mathrm{B}}\ u(-1,t) = g_-(t), \quad \text{if } a(-1,t) > 0, a(1,t) \geqslant 0;\\
&(\mathrm{iii})_{\mathrm{B}}\ u(1,t) = g_+(t), \quad \text{if } a(-1,t) \leqslant 0, a(1,t) < 0;\\
&(\mathrm{iv})_{\mathrm{B}}\ \text{no BC}, \quad \text{if } a(-1,t) \leqslant 0, a(1,t) \geqslant 0.
\end{aligned} \tag{2.8}$$

Since the non-homogeneous boundary conditions can be easily homogenized by subtracting a simple linear function from the exact solution, we shall only consider, without loss of generality, the case $g_\pm(t) = 0$.

To formulate the dual Petrov–Galerkin formulation uniformly for the four cases, we use the notations

$$\hat{\alpha} = \begin{cases} -1, & \text{if } 1 \in \Gamma^-, \\ 1, & \text{if } 1 \in \Gamma^+, \end{cases} \quad \hat{\beta} = \begin{cases} -1, & \text{if } -1 \in \Gamma^-, \\ 1, & \text{if } -1 \in \Gamma^+ \end{cases} \tag{2.9}$$

and define the weight functions

$$\omega_0(x) = \omega(x)\omega^{\hat{\alpha},\hat{\beta}}(x), \quad \omega_1(x) = (1-x^2)^{-1}\omega_0(x),$$
$$\omega_2(x) = (1-x^2)\omega_0(x). \tag{2.10}$$

More precisely, corresponding to each boundary condition $(\mathrm{i})_{\mathrm{B}}$–$(\mathrm{iv})_{\mathrm{B}}$ in (2.8), we have

$$(\mathrm{i})_{\mathrm{B}}\ \hat{\alpha} = \hat{\beta} = -1; \quad (\mathrm{ii})_{\mathrm{B}}\ \hat{\alpha} = 1, \hat{\beta} = -1;$$
$$(\mathrm{iii})_{\mathrm{B}}\ \hat{\alpha} = -1, \hat{\beta} = 1; \quad (\mathrm{iv})_{\mathrm{B}}\ \hat{\alpha} = \hat{\beta} = 1. \tag{2.11}$$

Hereafter, the conditions or expressions labeled by $(\mathrm{i})_{\mathrm{B}}$ correspond to the boundary condition $(\mathrm{i})_{\mathrm{B}}$ with $g_\pm(t) = 0$ in (2.8), and likewise for $(\mathrm{ii})_{\mathrm{B}}$–$(\mathrm{iv})_{\mathrm{B}}$.

For each of the boundary conditions $(\mathrm{i})_{\mathrm{B}}$–$(\mathrm{iv})_{\mathrm{B}}$, we define the "dual" approximation spaces:

$$\begin{aligned}
&V_N = \{u \in P_N : u(\pm 1) = 0\}, \quad V_N^* = P_{N-2}, \quad \text{for } (\mathrm{i})_{\mathrm{B}};\\
&V_N = \{u \in P_N : u(-1) = 0\}, \quad V_N^* = \{v \in P_N : v(1) = 0\}, \quad \text{for } (\mathrm{ii})_{\mathrm{B}};\\
&V_N = \{u \in P_N : u(1) = 0\}, \quad V_N^* = \{v \in P_N : v(-1) = 0\}, \quad \text{for } (\mathrm{iii})_{\mathrm{B}};\\
&V_N = P_{N-2}, \quad V_N^* = \{v \in P_N : v(\pm 1) = 0\}, \quad \text{for } (\mathrm{iv})_{\mathrm{B}}.
\end{aligned} \tag{2.12}$$

One verifies readily that $\dim(V_N) = \dim(V_N^*)$, and $v_N\omega^{\hat{\alpha},\hat{\beta}} \in V_N^*$ for all $v_N \in V_N$.

2.3. Spectral approximation

With the above setup, we are ready to formulate the approximation schemes. The Legendre or Chebyshev spectral dual-Petrov–Galerkin approximation to (2.7) is

$$\begin{cases} \text{Find } u_N(\cdot,t) \in V_N \text{ such that for all } t \in (0,T], \\ (\partial_t u_N, v_N)_\omega + (\partial_x(au_N), v_N)_\omega + (bu_N, v_N)_\omega \\ \quad = (f, v_N)_\omega, \quad \forall v_N \in V_N^*, \end{cases} \tag{2.13}$$

with $u_N|_{t=0} = u_{0,N}$ being a suitable approximation of u_0 (to be specified later).

Since for any $v_N \in V_N$, we have $v_N\omega^{\hat{\alpha},\hat{\beta}} \in V_N^*$, the scheme (2.13) is equivalent to the following weighted Galerkin formulation (notice that $\omega_0 = \omega\omega^{\hat{\alpha},\hat{\beta}}$):

$$\begin{cases} \text{Find } u_N(\cdot,t) \in V_N \text{ such that for all } t \in (0,T], \\ (\partial_t u_N, v_N)_{\omega_0} + (\partial_x(au_N), v_N)_{\omega_0} + (bu_N, v_N)_{\omega_0} \\ \quad = (f, v_N)_{\omega_0}, \quad \forall v_N \in V_N, \end{cases} \tag{2.14}$$

with $u_N|_{t=0} = u_{0,N}$. It will become clear that the dual-Petrov–Galerkin scheme (2.13) is more suitable for implementation, while the weighted Galerkin formulation (2.14) is more convenient for stability and error analysis.

The following coercivity property is essential for the well-posedness of the problems (2.7) and (2.14).

Lemma 2.1. *Let*

$$A(u,v) = (\partial_x(au), v)_{\omega_0} + (bu, v)_{\omega_0}. \tag{2.15}$$

If $u \in L^2_{\omega_1}(I)$ and $u_x \in L^2_{\omega_0}(I)$, then there exist three real numbers λ_i $(i = 0,1,2)$ with $\lambda_1, \lambda_2 > 0$ such that for $t \in (0,T]$,

$$\lambda_0\|u\|^2_{\omega_0} + \lambda_1\|u\|^2_{\omega_1} \leqslant A(u,u) \leqslant \lambda_2\|u\|^2_{\omega_1}, \tag{2.16}$$

where the weight functions ω_0 and ω_1 are defined in (2.10).

...me $a(\pm 1, t)$ change sign, the variational ...ccordingly.

Proof. We first claim that if $u \in L^2_{\omega_1}(I)$ and $u_x \in L^2_{\omega_0}(I)$, then $u^2\omega_0 \in C(\bar{I})$. Indeed, for any $x_1, x_2 \in [-1,1]$, we have from the definitions of ω_0 and ω_1 that

$$\begin{aligned}|u^2(x_2)\omega_0(x_2) - u^2(x_1)\omega_0(x_1)| &= \left|\int_{x_1}^{x_2} \partial_x(u^2\omega_0)\,dx\right| \leqslant 2\int_{x_1}^{x_2} |\partial_x u||u|\omega_0\,dx + \int_{x_1}^{x_2} |u|^2|\partial_x\omega_0|\,dx \\ &\lesssim \int_{x_1}^{x_2} (|\partial_x u|^2 + |u|^2)\omega_0\,dx + \int_{x_1}^{x_2} |u|^2\omega_1\,dx \\ &\lesssim \int_{x_1}^{x_2} |\partial_x u|^2\omega_0\,dx + \int_{x_1}^{x_2} |u|^2\omega_1\,dx.\end{aligned}$$

Here, we used the inequalities: $\omega_0 \lesssim \omega_1$ and $|\partial_x\omega_0| \lesssim \omega_1$. Hence, letting $x_2 \to x_1$, we find that $u^2\omega_0 \in C(\bar{I})$. Thanks to this fact and $u \in L^2_{\omega_1}(I)$, i.e.,

$$\int_{-1}^{1} \frac{u^2(x)\omega_0(x)}{1-x^2}\,dx < +\infty,$$

we have that $u^2(x)\omega_0(x) \to 0$ as $|x| \to 1$. Integration by parts yields that

$$\begin{aligned}A(u,u) &= \int_{-1}^{1} (\partial_x a + b)u^2\omega_0\,dx + \frac{1}{2}\int_{-1}^{1} a\partial_x(u^2)\omega_0\,dx \\ &= \int_{-1}^{1} su^2\omega_0\,dx,\end{aligned} \tag{2.17}$$

where

$$s(x,t) = \frac{1}{2}\partial_x a + b - \frac{1}{2}a\omega_0^{-1}\partial_x\omega_0. \tag{2.18}$$

More precisely, we have from (2.10) that in the Legendre case ($\omega = 1$):

$$s(x,t) = \begin{cases} \frac{1}{2}\partial_x a + b - \frac{xa}{1-x^2}, & \text{for (i)}; \\ \frac{1}{2}\partial_x a + b + \frac{a}{1-x^2}, & \text{for (ii)}; \\ \frac{1}{2}\partial_x a + b - \frac{a}{1-x^2}, & \text{for (iii)}; \\ \frac{1}{2}\partial_x a + b + \frac{xa}{1-x^2}, & \text{for (iv)}, \end{cases} \tag{2.19}$$

and in the Chebyshev case $\left(\omega = \frac{1}{\sqrt{1-x^2}}\right)$:

$$s(x,t) = \begin{cases} \frac{1}{2}\partial_x a + b - \frac{3}{2}\frac{xa}{1-x^2}, & \text{for (i)}; \\ \frac{1}{2}\partial_x a + b + \frac{1}{2}\frac{(2-x)a}{1-x^2}, & \text{for (ii)}; \\ \frac{1}{2}\partial_x a + b - \frac{1}{2}\frac{(2+x)a}{1-x^2}, & \text{for (iii)}; \\ \frac{1}{2}\partial_x a + b + \frac{1}{2}\frac{xa}{1-x^2}, & \text{for (iv)}. \end{cases} \tag{2.20}$$

Next, by an argument similar to the proof of Lemma 1.1 in [4], we prove that there exist three constants λ_i $(i = 0,1,2)$ with $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_0 + \frac{\lambda_1}{1-x^2} \leqslant s(x,t) \leqslant \frac{\lambda_2}{1-x^2}, \quad (x,t) \in I \times [0,T]. \tag{2.21}$$

Indeed, the inequality at the right-hand side in the above relation is obvious so we only need to prove the one at the left-hand side. Let us consider for instance the case (i) ... inuity of $a(x,t)$ and the condi- ... way from zero on $[0,T]$, we

know that there exists $\delta \in (0,1)$ such that $-\frac{xa}{1-x^2} \geqslant \frac{\lambda_1}{1-x^2}$ with $\lambda_1 > 0$ and $x \in I \setminus [-\delta, \delta]$. Then, since $\partial_x a$ and b are bounded in I, and $-\frac{xa}{1-x^2}$ is bounded in $[-\delta,\delta]$, we infer (2.21). The other cases can be proved in a similar way. Finally, the desired result follows from (2.17) and (2.21). □

Remark 2.1. We may even require that the constant $\lambda_0 \geqslant 0$. Indeed, if $\lambda_0 < 0$, the change of variable $u \to e^{\lambda_0 t}u$ in (2.7) leads to the same equation with $b(x)$ replaced by $b(x) - \lambda_0$ (cf. [4]). Hence, the transformed equation will satisfy $s(x,t) \geqslant \frac{\lambda_1}{1-x^2}, x \in I$.

We note that the existence and uniqueness of solutions for (2.7) in the weighted Sobolev spaces that we consider here have been established recently in [13]. We provide below an *a priori* estimate which is an immediate consequence of Lemma 2.1:

Theorem 2.1. *Let u and u_N be respectively the solutions of (2.7) and (2.14). If $u_0 \in L^2_{\omega_0}(I)$ and $f \in L^2(0,T;L^2_{\omega_2}(I))$, then we have*

$$\|u\|_{L^\infty(0,T;L^2_{\omega_0}(I))} + \lambda_1\|u\|_{L^2(0,T;L^2_{\omega_1}(I))} \lesssim \|u_0\|_{\omega_0} + \|f\|_{L^2(0,T;L^2_{\omega_2}(I))} \tag{2.22}$$

and

$$\|u_N\|_{L^\infty(0,T;L^2_{\omega_0}(I))} + \lambda_1\|u_N\|_{L^2(0,T;L^2_{\omega_1}(I))} \lesssim \|u_{0,N}\|_{\omega_0} + \|f\|_{L^2(0,T;L^2_{\omega_2}(I))}, \tag{2.23}$$

where the weights ω_i $(i = 0,1,2)$ are defined in (2.10).

Proof. Taking the inner product of the first equation of (2.7) with $\omega_0 u$, and using the fact: $\omega_0^2 = \omega_1\omega_2$, we derive from the Cauchy–Schwarz inequality that

$$\frac{1}{2}\partial_t\|u\|^2_{\omega_0} + A(u,u) = (f,u)_{\omega_0} \leqslant \frac{\lambda_1}{2}\|u\|^2_{\omega_1} + \frac{1}{2\lambda_1}\|f\|^2_{\omega_2}.$$

Hence, by Lemma 2.1,

$$\frac{1}{2}\partial_t\|u\|^2_{\omega_0} + \frac{\lambda_1}{2}\|u\|^2_{\omega_1} \leqslant |\lambda_0|\|u\|^2_{\omega_0} + \frac{1}{2\lambda_1}\|f\|^2_{\omega_2}. \tag{2.24}$$

Consequently, using the Gronwall inequality leads to (2.22).

Following exactly the same procedure, one can prove (2.23). □

Remark 2.2. The definition of $\hat{\alpha}$ and $\hat{\beta}$ is different from e^- and e^+ defined in (1.3) of [4]. Hence, although the approach in this section is similar to that in [4], our variational formulation, and therefore our scheme as well as its stability and convergence properties, is different from theirs. In particular, Theorem 2.1 holds without assuming that the coefficients a and b satisfy any coercivity condition such as (1.2) in [4]. We note that by assuming $a(x) > 0$, a different set of stability conditions involving the weight and $a(x)$ is derived recently in [16].

2.4. Pseudo-spectral approximation

In practical implementations, the continuous inner product $(\cdot,\cdot)_\omega$ should be replaced by a discrete inner product $\langle\cdot,\cdot\rangle_{N,\omega}$ (pseudo-spectral) which is based on a suitable Gaussian-type quadrature.

Let $\langle\cdot,\cdot\rangle_{N,\omega}$ be the discrete inner product associated, respectively, with the Gauss–Lobatto, Gauss–Radau (with $x_0=-1$), Gauss–Radau (with $x_N=1$) and Gauss quadrature for the four different boundary conditions $(\mathrm{i})_{\mathrm{B}}$, $(\mathrm{ii})_{\mathrm{B}}$, $(\mathrm{iii})_{\mathrm{B}}$ and $(\mathrm{iv})_{\mathrm{B}}$. Let $I_N : C(I)\to P_N$ be the corresponding interpolation operator. We recall that (cf. [3])

$$\langle u,v\rangle_{N,\omega}=(u,v)_\omega,\quad \forall uv\in P_{2N+\delta}, \tag{2.25}$$

where $\delta=1,0,-1$ for Gauss, Gauss–Radau and Gauss–Lobatto, respectively.

It is well-known that one needs to use the skew-symmetric form in a pseudo-spectral approximation to ensure the *theoretical* stability for general situations (see [17,4,9]).

We denote

$$A_N(u_N,v_N)=\frac{1}{2}\langle a\partial_x u_N+\partial_x I_N(au_N),v_N\rangle_{N,\omega}+\left\langle\left(\frac{1}{2}\partial_x a+b\right)u_N,v_N\right\rangle_{N,\omega}. \tag{2.26}$$

The skew-symmetric Legendre or Chebyshev pseudo-spectral dual-Petrov–Galerkin approximation to (2.7) is:

$$\begin{cases}\text{Find } u_N(\cdot,t)\in V_N \text{ such that for all } 0<t\leqslant T,\\ \langle\partial_t u_N,v_N\rangle_{N,\omega}+A_N(u_N,v_N)=\langle f,v_N\rangle_{N,\omega},\quad \forall v_N\in V_N^*.\end{cases} \tag{2.27}$$

Lemma 2.2. *Let λ_i $(i=0,1,2)$ be the same as in Lemma 2.1. Then,*

$$\lambda_0\|u_N\|^2_{\omega_0}+\lambda_1\|u_N\|^2_{\omega_1}\leqslant A_N(u_N,u_N\omega^{\hat\alpha,\hat\beta})\leqslant\lambda_2\|u_N\|^2_{\omega_1},\quad\forall u_N\in V_N, \tag{2.28}$$

where $\omega^{\hat\alpha,\hat\beta}$, ω_0 and ω_1 correspond to any of the boundary conditions in (2.8).

Proof. The proof is essentially the same as that of Lemma 2.1. Thanks to (2.25), we only need to show

$$\left\langle\left(\lambda_0+\frac{\lambda_1}{1-x^2}\right)u_N,u_N\omega^{\hat\alpha,\hat\beta}\right\rangle_{N,\omega}\leqslant A_N(u_N,u_N\omega^{\hat\alpha,\hat\beta})\leqslant\left\langle\frac{\lambda_2}{1-x^2}u_N,u_N\omega^{\hat\alpha,\hat\beta}\right\rangle_{N,\omega}. \tag{2.29}$$

Indeed, we derive from (2.25) and integration by parts that

$$\langle a\partial_x u_N,u_N\omega^{\hat\alpha,\hat\beta}\rangle_{N,\omega}=\langle\partial_x u_N,I_N(au_N)\omega^{\hat\alpha,\hat\beta}\rangle_{N,\omega}$$

$$\ldots(au_N)\omega^{\hat\alpha,\hat\beta})_\omega$$

$$=-(u_N,\partial_x I_N(au_N)\omega_0+I_N(au_N)\partial_x\omega_0)$$
$$=-\langle\partial_x I_N(au_N),u_N\omega^{\hat\alpha,\hat\beta}\rangle_{N,\omega}-\langle a\omega_0^{-1}\partial_x\omega_0 u_N,u_N\omega^{\hat\alpha,\hat\beta}\rangle_{N,\omega},$$

which along with (2.26) implies that

$$A_N(u_N,u_N\omega^{\hat\alpha,\hat\beta})=\left\langle\left(\frac{1}{2}\partial_x a+b-\frac{1}{2}a\omega_0^{-1}\partial_x\omega_0\right)u_N,v_N\right\rangle_{N,\omega}. \tag{2.30}$$

Hence, (2.29) is a directly consequence of (2.19). □

Thanks to the above lemma, we have immediately the following stability result:

Theorem 2.2. *If $u_{0,N}\in L^2_{\omega_0}(I)$ and $f\in L^2(0,T;L^2_{\omega_2}(I))$, then we have*

$$\|u_N\|_{L^\infty(0,T;L^2_{\omega_0}(I))}+\lambda_1\|u_N\|_{L^2(0,T;L^2_{\omega_1}(I))}\lesssim\|u_{0,N}\|_{\omega_0}+\|I_N f\|_{L^2(0,T;L^2_{\omega_2}(I))}. \tag{2.31}$$

3. Implementations

In this section, we discuss some of the implementation details of the dual-spectral-Galerkin method. We note that the slightly more costly skew-symmetric form may not be necessary since the standard form is often numerically stable, at least for well-resolved problems (cf. [11,8,9]). Hence, we present two implementations below, one uses the skew-symmetric form (2.27) with basis functions in frequency space, and the other is the standard pseudo-spectral form with basis functions in physical spaces.

3.1. Implementations in frequency space

To simplify the presentation, we shall only provide details for the second boundary condition in (2.8). The other cases can be treated in a similar fashion.

As demonstrated in [20,21], it is advantageous to use basis functions which are compact combinations of the Legendre and Chebyshev polynomials. Therefore, we set in the Legendre case

$$\phi_k(x)=L_k(x)+L_{k+1}(x),\quad \psi_k(x)=L_k(x)-L_{k+1}(x)$$

and in the Chebyshev case

$$\phi_k(x)=(1+x)T_k(x),\quad \psi_k(x)=(1-x)T_k(x),$$

where $L_k(x)$ and $T_k(x)$ are respectively the Legendre and Chebyshev polynomials of degree k. Then, we have for the second case in (2.12),

$$V_N=\mathrm{span}\{\phi_0,\phi_1,\ldots,\phi_{N-1}\},\quad V_N^*=\mathrm{span}\{\psi_0,\psi_1,\ldots,\psi_{N-1}\}. \tag{3.1}$$

Therefore, setting

$$
\begin{aligned}
& f_j = \langle f, \psi_j \rangle_{N,\omega}, \quad \tilde{\boldsymbol{f}} = (f_0, f_1, \ldots, f_{N-1})^t, \\
& u_N = \sum_{k=0}^{N-1} \tilde{u}_k \phi_k, \quad \tilde{\boldsymbol{u}} = (\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{N-1})^t, \\
& m_{jk} = \langle \phi_k, \psi_j \rangle_{N,\omega}, \quad M = (m_{jk})_{j,k=0,1,\ldots,N-1}, \\
& s_{jk} = \frac{1}{2} \langle a\phi_k' + \partial_x I_N(a\phi_k), \psi_j \rangle_{N,\omega}, \quad S = (s_{jk})_{j,k=0,1,\ldots,N-1}, \\
& q_{jk} = \left\langle \left(\frac{1}{2} \partial_x a + b \right) \phi_k, \psi_j \right\rangle_{N,\omega}, \quad Q = (q_{jk})_{j,k=0,1,\ldots,N-1},
\end{aligned}
\tag{3.2}
$$

we find that (2.27) becomes the following system of ODEs:

$$
M\tilde{\boldsymbol{u}}_t + (S+Q)\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{f}}. \tag{3.3}
$$

We recall that the discrete inner product $\langle \cdot, \cdot \rangle_{N,\omega}$ here is associated with the Gauss–Radau interpolation nodes with $x_0 = -1$.

To avoid severe restrictions on time step associated with explicit time discretizations of spectral methods, we shall consider implicit schemes for integrating (3.3). After discretizing (3.3) by a suitable implicit scheme, we need to solve at each time step the following linear system:

$$
(\alpha M + S + Q)\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{g}}, \tag{3.4}
$$

where $\alpha = O(\frac{1}{\Delta t})$ is a constant.

It follows from the orthogonality of Legendre and Chebyshev polynomials that $m_{jk} = 0$ for $|j-k| > 1$. Hence, the mass matrix M is (non-symmetric) tridiagonal and its entries can be easily determined.

On the other hand, the matrices S and Q are usually full, unless a and b are simple, low-order polynomials in x.

3.1.1. Case 1: $a(x,t) \equiv \bar{a}$ and $b(x,t) \equiv \bar{b}$ are two constants

In this simple case, we have from (4.6) and (A.13) that for $\omega(x) \equiv 1$,

$$
\begin{aligned}
& s_{jk} = \bar{a} \langle \phi_k', \psi_j \rangle_{N,\omega} = \bar{a} (\phi_k', \psi_j)_\omega = 2\bar{a}\delta_{jk}, \\
& q_{jk} = \bar{b} \langle \phi_k, \psi_j \rangle_{N,\omega} = 0, \quad \forall |j-k| > 1,
\end{aligned}
\tag{3.5}
$$

while for $\omega(x) = \frac{1}{\sqrt{1-x^2}}$,

$$
\begin{aligned}
& s_{jk} = \bar{a} \langle \phi_k', \psi_j \rangle_{N,\omega} = \bar{a} (\phi_k', \psi_j)_\omega = 0, \quad \forall |j-k| > 1, \\
& q_{jk} = \bar{b} \langle \phi_k, \psi_j \rangle_{N,\omega} = 0, \quad \forall |j-k| > 2.
\end{aligned}
\tag{3.6}
$$

Let us denote

$$
\begin{aligned}
& s_{jk}^0 = \langle \phi_k', \psi_j \rangle_{N,\omega}, \quad S_0 = (s_{jk}^0)_{j,k=0,1,\ldots,N-1}, \\
& q_{jk}^0 = \langle \phi_k, \psi_j \rangle_{N,\omega}, \quad Q_0 = (q_{jk}^0)_{j,k=0,1,\ldots,N-1}.
\end{aligned}
\tag{3.7}
$$

Then, we can rewrite (3.4) as

$$
(\alpha M + \bar{a} S_0 + \bar{b} Q_0)\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{g}}, \tag{3.8}
$$

which can be efficiently inverted.

As we demonstrate below, the linear system in this sim- … ffective preconditioner for the

3.1.2. Case 2: variable coefficients

As observed above, for general variable coefficients a and b, the matrices S and Q are full. Hence, a direct inversion of (3.4) is not advisable. However, we shall use Lemma 2.2 to show that (3.4) can be solved effectively by using a preconditioned iterative method.

Since $\phi_k \omega^{1,-1} \in V_N^*$, there exists a unique set of $\{h_{kj}\}$ such that

$$
\phi_k \omega^{1,-1} = \sum_{j=0}^{N-1} h_{kj} \psi_j, \quad k = 0, 1, \ldots, N-1. \tag{3.9}
$$

We denote $H = (h_{kj})_{k,j=0,1,\ldots N-1}$ and for $\boldsymbol{v} = (v_0, v_1, \ldots, v_{N-1})^t$, we define $\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{l^2} := \sum_{j=0}^{N-1} v_j^2$ which is the inner product in l^2.

Let u_N, $\{\tilde{u}_j\}$ and $\tilde{\boldsymbol{u}}$ be the same as before, we have

$$
\begin{aligned}
\langle HS_0 \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle_{l^2} &= \sum_{k,j,l=0}^{N-1} \tilde{u}_k h_{kj} s_{jl}^0 \tilde{u}_l \\
&= \sum_{k,j,l=0}^{N-1} \tilde{u}_k h_{kj} \langle \phi_l', \psi_j \rangle_{N,\omega} \tilde{u}_l \\
&= \left\langle \sum_{l=0}^{N-1} \tilde{u}_l \phi_l', \sum_{k,j=0}^{N-1} \tilde{u}_k h_{kj} \psi_j \right\rangle_{N,\omega} \\
&= \left\langle \sum_{l=0}^{N-1} \tilde{u}_l \phi_l', \sum_{k=0}^{N-1} \tilde{u}_k \phi_k \omega^{1,-1} \right\rangle_{N,\omega} \\
&= \langle \partial_x u_N, \omega^{1,-1} u_N \rangle_{N,\omega} = (\partial_x u_N, u_N)_{\omega_0}.
\end{aligned}
\tag{3.10}
$$

Therefore, by (2.5),

$$
\begin{aligned}
& \langle HS_0 \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle_{l^2} = \|u_N\|_{\omega_1}^2, \quad \text{for } \omega \equiv 1; \\
& \frac{1}{2} \|u_N\|_{\omega_1}^2 \leqslant \langle HS_0 \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle_{l^2} \leqslant \frac{3}{2} \|u_N\|_{\omega_1}^2, \quad \text{for } \omega = \frac{1}{\sqrt{1-x^2}}.
\end{aligned}
\tag{3.11}
$$

Similarly,

$$
\begin{aligned}
& \langle H(\alpha M + S + Q)\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle_{l^2} \\
& \quad = \sum_{k,j,l=0}^{N-1} \tilde{u}_k h_{kj} (\alpha m_{jl} + s_{jl} + q_{jl}) \tilde{u}_l \\
& \quad = \left\langle \sum_{j=0}^{N-1} \tilde{u}_j \left[\frac{1}{2} \left(a\phi_j' + \partial_x I_N(a\phi_j) \right) + \left(\alpha + \frac{1}{2} \partial_x a + b \right) \phi_j \right], \sum_{k,j=0}^{N-1} \tilde{u}_k h_{kj} \psi_j \right\rangle_{N,\omega} \\
& \quad = \left\langle \sum_{j=0}^{N-1} \tilde{u}_j \left[\frac{1}{2} \left(a\phi_j' + \partial_x I_N(a\phi_j) \right) \left(\alpha + \frac{1}{2} \partial_x a + b \right) \phi_j \right], \sum_{k=0}^{N-1} \tilde{u}_k \phi_k \omega^{1,-1} \right\rangle_{N,\omega} \\
& \quad = \left\langle \frac{1}{2} (a\partial_x u_N + \partial_x I_N(a u_N)) + \left(\alpha + \frac{1}{2} \partial_x a + b \right) u_N, u_N \omega^{1,-1} \right\rangle_{N,\omega}.
\end{aligned}
\tag{3.12}
$$

Hence, thanks to Lemma 2.2,

$$
\begin{aligned}
(\alpha + \lambda_0) \|u_N\|_{\omega_0}^2 + \lambda_1 \|u_N\|_{\omega_1}^2 &\leqslant \langle H(\alpha M + S + Q)\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle_{l^2} \\
&\leqslant \alpha \|u_N\|_{\omega_0}^2 + \lambda_2 \|u_N\|_{\omega_1}^2.
\end{aligned}
$$

Assuming $\alpha + \lambda_0 \geqslant 0$ (which holds for most cases since $\alpha = O(\frac{1}{\Delta t}) \gg 1$, see Remark 2.1 otherwise), we derive from the above, (3.11) and the fact $\omega_0 \leqslant \omega_1$ that

Table 1
number of BCG iterations needed with $(\beta M + S_0)^{-1}$ as preconditioner

N	8	16	32	64	128	256
Example I with $\alpha = 1, \beta = 0$	6	7	7	7	7	7
Example I with $\alpha = \beta = 100$	4	5	7	11	12	12
Example II with $\alpha = 1, \beta = 0$	8	16	33	68	125	260
Example I with $\alpha = \beta = 100$	3	5	6	8	8	8

$$\frac{1}{2}\lambda_1\langle HS_0\tilde{\boldsymbol{u}},\tilde{\boldsymbol{u}}\rangle_{l^2} \leqslant \langle H(\alpha M + S + Q)\tilde{\boldsymbol{u}},\tilde{\boldsymbol{u}}\rangle_{l^2} \leqslant \frac{3}{2}(\alpha+\lambda_2)\langle HS_0\tilde{\boldsymbol{u}},\tilde{\boldsymbol{u}}\rangle_{l^2}. \tag{3.13}$$

The relation (3.13) indicates that a good preconditioner for $\alpha M + S + Q$ is S_0^{-1}. In fact, a more robust preconditioner is $(\beta M + S_0)^{-1}$ for some $\beta \geqslant 0$ with $\beta \sim \alpha$ for $\alpha \gg 1$.

In Table 1, we list the number of BCG iterations needed to achieve six-digit accuracy for solving (3.4) in the Legendre case for the following two test examples using $(\beta M + S_0)^{-1}$ as preconditioner:

Example I. $a(x) = 2 + \sin(2\pi x)$, $b(x) = -2\pi\cos(2\pi x)$. ($a(x)$ does not change sign in $(-1, 1)$).

Example II. $a(x) = 1 + 2\sin(2\pi x)$, $b(x) = -4\pi\cos(2\pi x)$. ($a(x)$ changes sign in $(-1, 1)$).

Note that in the Legendre case, we have $S_0 = 2I$. Hence, no preconditioner is needed in this case if we take $\beta = 0$.

We observe from Table 1 that (i) for Example I where $a(x)$ does not change sign, the preconditioner is very robust for both $\alpha = 1$ and $\alpha = 100$, and (ii) for Example II where $a(x)$ changes sign, the preconditioner is only robust when α is sufficiently large and $\beta \sim \alpha$. This is consistent with (3.13) where it is assumed that $\alpha + \lambda \geqslant 0$.

We note however that although the preconditioner built in the frequency space is quite robust with respect to N, it may not be robust with large variations of a and b. For the latter case, it may be advantageous to use an implementation in physical space which we shall discuss below.

3.2. Implementations in physical space

Let $\{x_j\}_{j=0}^{N}$ (with $x_0 = -1$) be the set of Legendre or Chebyshev Gauss–Radau points. We set $x_{N+1} = 1$. Hence, replacing $(\cdot,\cdot)_\omega$ by the discrete inner product $\langle\cdot,\cdot\rangle_{N,\omega}$ in (2.13), the standard pseudo-spectral dual-Petrov–Galerkin method is:

$$\begin{cases} \text{Find } u_N(\cdot,t) \in V_N \text{ such that for all } t \in (0,T], \\ \langle \partial_t u_N, v_N\rangle_{N,\omega} + \langle \partial_x(a u_N) + b u_N, v_N\rangle_{N,\omega} \\ \quad = \langle f, v_N\rangle_{N,\omega}, \quad \forall v_N \in V_N^*. \end{cases} \tag{3.14}$$

Let $\hat{\phi}_j(x) \in P_N$ be the Lagrange polynomial associated with $\{x_j\}_{j=0}^{N}$ such that $\hat{\phi}_j(x_k) = \delta_{kj}$ for $j,k = 0,1,\dots,N$, and let $\hat{\psi}_j(x) \in P_N$ be the Lagrange polynomial associated with $\{x_j\}_{j=1}^{N+1}$ such that $\hat{\psi}_j(x_k) = \delta_{kj}$ for $j,k = 1,2,\dots,N+1$.

$$V_N = \mathrm{span}\{\hat{\phi}_1,\hat{\phi}_2,\dots,\hat{\phi}_N\}, \quad V_N^* = \mathrm{span}\{\hat{\psi}_1,\hat{\psi}_2,\dots,\hat{\psi}_N\}, \tag{3.15}$$

and we have

$$\begin{aligned} w_{jk} &:= \langle\hat{\phi}_k,\hat{\psi}_j\rangle_{N,\omega} = \delta_{jk}\rho_j, \\ s_{jk} &:= \langle\partial_x(a\hat{\phi}_k),\hat{\psi}_j\rangle_{N,\omega} \\ &= (a(x_j,t)\hat{\phi}_k'(x_j) + \partial_x a(x_k,t)\delta_{kj})\rho_j + a(-1,t)\hat{\phi}_k'(-1)\hat{\psi}_j(-1)\rho_0, \\ q_{jk} &:= \langle b\hat{\phi}_k,\hat{\phi}_j\rangle_{N,\omega} = b(x_k,t)\delta_{kj}\rho_j, \\ \langle f,\psi_j\rangle_{N,\omega} &= f(x_j,t)\rho_j + f(-1,t)\hat{\psi}_j(-1)\rho_0. \end{aligned} \tag{3.16}$$

Let us denote

$$\begin{aligned} &W = \mathrm{diag}(\rho_1,\rho_2,\dots,\rho_N), \quad S = (s_{jk})_{j,k=1,2,\dots,N}, \\ &Q = (q_{jk})_{j,k=1,2,\dots,N}, \\ &u_N = \sum_{k=1}^{N} u_N(x_j,t)\hat{\phi}_k(x), \quad \boldsymbol{u} = (u(x_1,t),u(x_2,t),\dots,u(x_N,t))^t, \\ &\boldsymbol{g} = (\hat{\psi}_1(-1),\hat{\psi}_2(-1),\dots,\hat{\psi}_N(-1))^t, \\ &\boldsymbol{f} = (f(x_1,t),f(x_2,t),\dots,f(x_N,t))^t. \end{aligned} \tag{3.17}$$

Then, (3.14) becomes

$$W\boldsymbol{u}_t + (S+Q)\boldsymbol{u} = W\boldsymbol{f} + f(-1,t)\rho_0\boldsymbol{g}. \tag{3.18}$$

In a pointwise form, the above equation, after inverting the diagonal matrix W, can be written as:

$$\begin{cases} \text{Find } u_N \in P_N \quad \text{for all } t \in (0,T] \text{ such that} \\ \partial_t u_N(x_j,t) + \partial_x(a(x,t)u_N(x,t))|_{x=x_j} + b(x_j,t)u_N(x_j,t) \\ \quad = f(x_j,t) + \psi_j(-1)\frac{\rho_0}{\rho_j}(f(-1,t) - a(-1,t)u_N'(-1,t)), \\ \quad 1 \leqslant j \leqslant N, \\ u_N(-1,t) = 0. \end{cases} \tag{3.19}$$

Hence, the dual-Petrov–Galerkin method does not correspond to a *pure* collocation method, instead, it is a *pseudo* collocation scheme with an additional *boundary residual term* on the right-hand side.

In Fig. 1, we plot the eigenvalue distribution of $W^{-1}S$ for various N, where W and S are the matrices defined in (3.17) with $a(x) = 1$. It is clear that for all N, the real parts of the eigenvalues are always positive, indicating the good stability of our dual-Petrov formulation.

Remark 3.1. One may solve (3.18) using a suitable explicit scheme, which will be subjected to a usual CFL constraint (see, for instance, [10,12]). On the other hand, since (3.18) was derived from a proper variational formulation with a coercive bilinear form, it may be possible, as in the case of elliptic equations (cf. [18,5,19]), to build an optimal finite element preconditioner which is robust with respect to both the number of points and the large variations of coefficients a and b. However, this subject and a detailed study on the robustness of the preconditioning in frequency space is

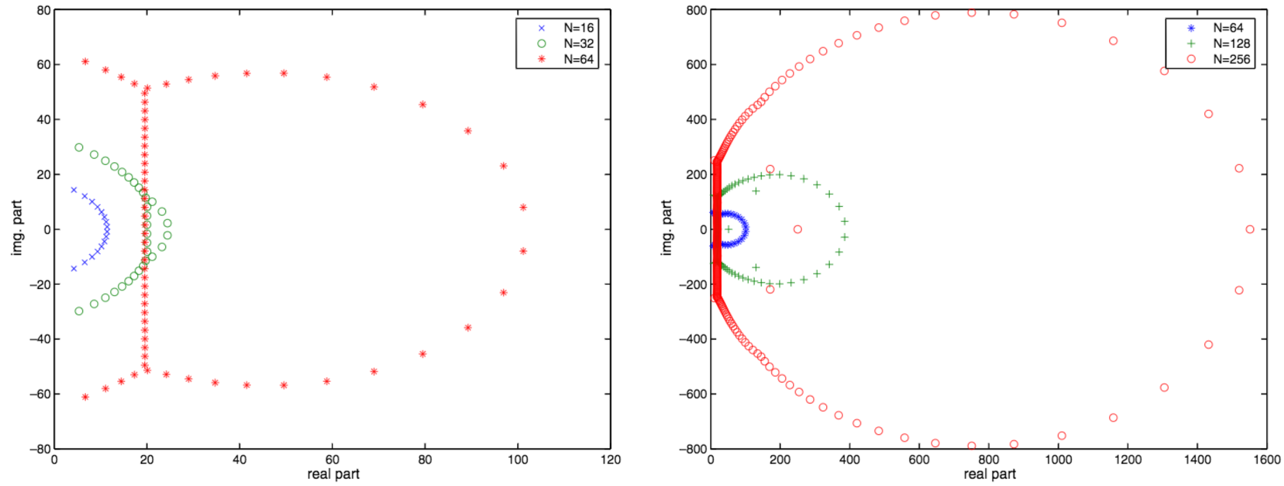

Fig. 1. Eigenvalue distribution of $W^{-1}S$ with $a(x) = 1$.

beyond the scope of this paper and will be investigated elsewhere.

4. Error estimates

In this section, we shall present some optimal Legendre and Chebyshev approximation results measured in strong norms, and perform the error analysis for the proposed dual-Petrov–Galerkin schemes.

4.1. Legendre case ($\omega = 1$)

We first introduce the basis functions for the dual spaces in (2.12). Let

$$\varphi_n(x) = L_{n+2}(x) - L_n(x), \quad \phi_n(x) = L_n(x) + L_{n+1}(x), \\ \psi_n(x) = L_n(x) - L_{n+1}(x). \tag{4.1}$$

Thanks to the fact: $L_k(\pm 1) = (\pm 1)^k$, one verifies that $\varphi_n(\pm 1) = \phi_n(-1) = \psi_n(1) = 0$. Let $J_k^{\alpha,\beta}(\alpha, \beta > -1)$ be the classical Jacobi polynomial of degree k (see Appendix A for its properties). The following identities hold (see Appendix B for the proof):

$$\varphi_n(x) = -\frac{2n+3}{2(n+1)}(1-x^2)J_n^{1,1}(x),$$

$$\partial_x \varphi_n(x) = (2n+3)L_{n+1}(x); \tag{4.2}$$

$$\phi_n(x) = (1+x)J_n^{0,1}(x), \quad \partial_x \phi_n(x) = (n+1)J_n^{1,0}(x); \tag{4.3}$$

$$\psi_n(x) = (1-x)J_n^{1,0}(x), \quad \partial_x \psi_n(x) = -(n+1)J_n^{0,1}(x). \tag{4.4}$$

The orthogonality of Jacobi polynomials (cf. (A.1)) implies that $\{\varphi_n\}$, $\{\phi_n\}$ and $\{\psi_n\}$ are mutually orthogonal in $L^2_{\omega^{-1,-1}}(I)$, $L^2_{\omega^{0,-1}}(I)$ and $L^2_{\omega^{-1,0}}(I)$, respectively. Hence, $\{\varphi_n\}$, $\{\phi_n\}$ and $\{\psi_n\}$ can be viewed as extensions of the classical Jacobi polynomials to the cases with parameters ... $0, -1)$ and $(\alpha, \beta) = (-1, 0)$, ... that

$$\int_I \partial_x \varphi_n(x) L_{m+1}(x)\,dx = -\int_I \varphi_n(x)\partial_x L_{m+1}(x)\,dx = 2\delta_{n,m}, \tag{4.5}$$

$$\int_I \partial_x \phi_n(x)\psi_m(x)\,dx = -\int_I \phi_n(x)\partial_x \psi_m(x)\,dx = 2\delta_{n,m}. \tag{4.6}$$

Besides, as shown in Appendix B, we have

$$\int_I \partial_x^l \varphi_m(x)\partial_x^l \varphi_n(x)\omega^{l-1,l-1}(x)\,dx = \mu_{n,l}^{(1)}\delta_{m,n}, \quad n \geqslant l \geqslant 0; \tag{4.7}$$

$$\int_I \partial_x^l \phi_m(x)\partial_x^l \phi_n(x)\omega^{l,l-1}(x)\,dx = \mu_{n,l}^{(2)}\delta_{m,n}, \quad n \geqslant l \geqslant 0; \tag{4.8}$$

$$\int_I \partial_x^l \psi_m(x)\partial_x^l \psi_n(x)\omega^{l-1,l}(x)\,dx = \mu_{n,l}^{(2)}\delta_{m,n}, \quad n \geqslant l \geqslant 0, \tag{4.9}$$

with

$$\mu_{n,l}^{(1)} = 2(2n+3)\frac{\Gamma(n+l+1)}{\Gamma(n-l+3)}, \quad \mu_{n,l}^{(2)} = 2\frac{\Gamma(n+l+1)}{\Gamma(n-l+2)}. \tag{4.10}$$

4.1.1. Legendre approximations

We first notice that the polynomial space V_N in (2.12) for cases $\mathrm{i_B}$–$\mathrm{iv_B}$ are respectively identical to

$(\mathrm{i})_\mathrm{B}$ $V_N^{-1,-1} = \mathrm{span}\{\varphi_k : 0 \leqslant k \leqslant N-2\}$,
$(\mathrm{ii})_\mathrm{B}$ $V_N^{0,-1} = \mathrm{span}\{\phi_k : 0 \leqslant k \leqslant N-1\}$,
$(\mathrm{iii})_\mathrm{B}$ $V_N^{-1,0} = \mathrm{span}\{\psi_k : 0 \leqslant k \leqslant N-1\}$,
$(\mathrm{iv})_\mathrm{B}$ $V_N^{0,0} = \mathrm{span}\{L_k : 0 \leqslant k \leqslant N-2\}$.

For each pair (α, β) listed below,

$$(\mathrm{i})_\mathrm{B}\ \alpha = \beta = -1, \quad (\mathrm{ii})_\mathrm{B}\ \alpha = 0, \beta = -1, \\ (\mathrm{iii})_\mathrm{B}\ \alpha = -1, \beta = 0, \quad (\mathrm{iv})_\mathrm{B}\ \alpha = \beta = 0, \tag{4.11}$$

we define the weighted Sobolev space

$$B^r_{\alpha,\beta}(I) = \{u : \partial^l_x u \in L^2_{\omega^{\alpha+l,\beta+l}}(I), 0 \leqslant l \leqslant r\}, \quad r \in \mathbb{N}, \tag{4.12}$$

equipped with the norm and semi-norm

$$\|u\|_{B^r_{\alpha,\beta}} = \left(\sum_{l=0}^{r} \|\partial^l_x u\|^2_{\omega^{\alpha+l,\beta+l}}\right)^{\frac{1}{2}}, \quad |u|_{B^r_{\alpha,\beta}} = \|\partial^r_x u\|_{\omega^{\alpha+r,\beta+r}}.$$

Consider the orthogonal projection $\pi^{\alpha,\beta}_N : L^2_{\omega^{\alpha,\beta}}(I) \to V^{\alpha,\beta}_N$ defined by

$$(\pi^{\alpha,\beta}_N u - u, v_N)_{\omega^{\alpha,\beta}} = 0, \quad \forall v_N \in V^{\alpha,\beta}_N. \tag{4.13}$$

Let us first present a very special property of $\pi^{\alpha,\beta}_N$.

Lemma 4.1. *For each pair* (α, β) *in* (4.11), *let* $(\hat\alpha, \hat\beta)$ *be the corresponding pair defined in* (2.11). *Then*

$$\left(\partial_x\left(\pi^{\alpha,\beta}_N u - u\right), v_N \omega^{\hat\alpha,\hat\beta}\right) = 0, \quad \forall v_N \in V^{\alpha,\beta}_N, \tag{4.14}$$

and

$$\left(\partial_x\left(\pi^{\alpha,\beta}_N u - u\right), v^*_N\right) = 0, \quad \forall v^*_N \in V^*_N. \tag{4.15}$$

Proof. For each case of $(\mathrm{i})_{\mathrm{B}}$–$(\mathrm{iv})_{\mathrm{B}}$, we notice that

$$(\pi^{\alpha,\beta}_N u - u) v_N \omega^{\hat\alpha,\hat\beta}\Big|_{-1}^{1} = 0, \quad \omega^{\alpha,\beta}\partial_x(v_N \omega^{\hat\alpha,\hat\beta}) \in V^{\alpha,\beta}_N.$$

Hence, (4.14) is a direct consequence of an integration by parts and the definition (4.13). Since any $v^*_N \in V^*_N$ can be expressed as $v^*_N = v_N \omega^{\hat\alpha,\hat\beta}$ with $v_N \in V^{\alpha,\beta}_N$, (4.15) follows from (4.14). □

We are now in position to state the main approximation properties of these orthogonal projection operators.

Theorem 4.1. *For each pair of* (α, β) *in* (4.11), *and for any* $u \in B^r_{\alpha,\beta}(I)$ *with* $r \in \mathbb{N}$,

$$\left\|\partial^m_x\left(\pi^{\alpha,\beta}_N u - u\right)\right\|_{\omega^{\alpha+m,\beta+m}} \lesssim N^{m-r}\|\partial^r_x u\|_{\omega^{\alpha+r,\beta+r}}, \quad 0 \leqslant m \leqslant r. \tag{4.16}$$

Proof. The result with $\alpha = \beta = 0$ is well-known (see, for instance, [7,2]). A more general result (for all α and β which are integers) was presented in [15] but the proof was omitted due to space limitation. For the readers' convenience, we provide below a detailed proof for (4.16) with (i) $\alpha = \beta = -1$. The other cases can be proved similarly.

For any $u \in L^2_{\omega^{-1,-1}}(I)$, we write

$$u(x) = \sum_{n=0}^{\infty} \hat u_n \varphi_n(x), \quad \text{with } \hat u_n = \frac{1}{\mu^{(1)}_{n,0}}(u, \varphi_n)_{\omega^{-1,-1}}. \tag{4.17}$$

So formally we have from (B.1) and (A.1) that

$$[illegible]{}^{l-1,l-1}_{l+2} = \sum_{n=l-1}^{\infty} \mu^{(1)}_{n,l}\hat u_n^2. \tag{4.18}$$

On the other hand,

$$\pi^{-1,-1}_N u(x) - u(x) = -\sum_{n=N-1}^{\infty} \hat u_n \varphi_n(x).$$

Hence, by (4.2) and (4.18),

$$\left\|\partial^m_x\left(\pi^{-1,-1}_N u - u\right)\right\|^2_{\omega^{l-1,l-1}} = \sum_{n=N-1}^{\infty} \hat u_n^2 \mu^{(1)}_{n,m} \leqslant C_{m,r}\sum_{n=N-1}^{\infty} \mu^{(1)}_{n,r}\hat u_n^2$$
$$\leqslant C_{m,r}\|\partial^r_x u\|^2_{\omega^{r-1,r-1}}$$

where by (4.10) and the Stirling formula (see [6]),

$$C_{m,r} = \max_{n \geqslant N-1} \frac{\mu^{(1)}_{n,m}}{\mu^{(1)}_{n,r}} \lesssim N^{2(m-r)}. \quad \square$$

4.1.2. Convergence of (2.13) with $\omega \equiv 1$

Let u and u_N be the solutions of (2.7) and (2.13) (with $\omega \equiv 1$), respectively, and set

$$\hat e_N = \pi^{\alpha,\beta}_N u - u_N, \quad e_N = u - u_N = (u - \pi^{\alpha,\beta}_N u) + \hat e_N.$$

Theorem 4.2. *Let* $u_N|_{t=0} = u_{0,N} = \pi^{\alpha,\beta}_N u_0$. *For each of the pair* (α, β) *in* (4.11), *assuming* $u \in L^2(0, T; L^2_{\omega_1}(I)) \cap L^\infty(0, T; B^r_{\alpha,\beta}(I))$ *and* $\partial_t u \in L^2(0, T; B^{r-1}_{\alpha,\beta}(I))$ *with integer* $r \geqslant 1$, *we have*

$$\|u_N - u\|_{L^\infty(0,T;L^2_{\omega_0}(I))}$$
$$\lesssim N^{1-r}\left(\|\partial_t\partial^{r-1}_x u\|_{L^2(0,T;L^2_{\omega^{\alpha+r-1,\beta+r-1}}(I))} + \|\partial^r_x u\|_{L^\infty(0,T;L^2_{\omega^{\alpha+r,\beta+r}}(I))}\right), \tag{4.19}$$

where ω_0 *and* ω_1 *are given in* (2.10).

Proof. By (2.7) and (2.14),

$$(\partial_t \hat e_N, v_N)_{\omega_0} + A(\hat e_N, v_N) = \left(\partial_t\left(\pi^{\alpha,\beta}_N u - u\right), v_N\right)_{\omega_0}$$
$$+ \left(\partial_x\left(a\left(\pi^{\alpha,\beta}_N u - u\right)\right), v_N\right)_{\omega_0}$$
$$+ \left(b(\pi^{\alpha,\beta}_N u - u), v_N\right)_{\omega_0}, \quad \forall v_N \in V_N. \tag{4.20}$$

Taking $v_N = \hat e_N$ in (4.20) and using (2.23) (note: $\hat e_N(0) = 0$), we find that

$$\|\hat e_N\|_{L^\infty(0,T;L^2_{\omega_0}(I))} + \lambda_1\|\hat e_N\|_{L^2(0,T;L^2_{\omega_1}(I))} \lesssim G_1 + G_2 \tag{4.21}$$

with

$$G_1 = \left\|\partial_t\left(\pi^{\alpha,\beta}_N u - u\right)\right\|_{L^2(0,T;L^2_{\omega_2}(I))},$$
$$G_2 = \left\|\partial_x\left(a\left(\pi^{\alpha,\beta}_N u - u\right)\right) + b\left(\pi^{\alpha,\beta}_N u - u\right)\right\|_{L^2(0,T;L^2_{\omega_2}(I))}.$$

By (4.16) with $m = 0, 1$, and the definition of $\omega_2 (\leqslant \omega^{\alpha,\beta})$ in (2.10),

$$G_1 \lesssim \|\partial_t(\pi_N^{\alpha,\beta}u - u)\|_{L^2(0,T;L^2_{\omega^{\alpha,\beta}}(I))}$$
$$\lesssim N^{1-r}\|\partial_t\partial_x^{r-1}u\|_{L^2(0,T;L^2_{\omega^{\alpha+r-1,\beta+r-1}}(I))},$$

and using the fact $a, a_x, b \in L^\infty(I \times (0,T])$,

$$G_2 \lesssim \|\pi_N^{\alpha,\beta}u - u\|_{L^2(0,T;B^1_{\alpha,\beta}(I))} \lesssim N^{1-r}\|\partial_x^r u\|_{L^2(0,T;L^2_{\omega^{\alpha+r,\beta+r}}(I))}.$$

Hence, plugging the estimates of G_1 and G_2 into (4.21) leads to

$$\|\hat{e}_N\|_{L^\infty(0,T;L^2_{\omega_0}(I))} + \lambda_1\|\hat{e}_N\|_{L^2(0,T;L^2_{\omega_1}(I))}$$
$$\lesssim N^{1-r}\Big(\|\partial_t\partial_x^{r-1}u\|_{L^2(0,T;L^2_{\omega^{\alpha+r-1,\beta+r-1}}(I))} + \|\partial_x^r u\|_{L^2(0,T;L^2_{\omega^{\alpha+r,\beta+r}}(I))}\Big). \tag{4.22}$$

Since in the Legendre case, $\omega_0 = \omega^{\hat{\alpha},\hat{\beta}} \leqslant \omega^{\alpha,\beta}$ (cf. (2.11) and (4.11)), using (4.16) (with $m = 0$) and (4.22) yields that

$$\|u_N - u\|_{L^\infty(0,T;L^2_{\omega_0}(I))}$$
$$\leqslant \|\pi_N^{\alpha,\beta}u - u\|_{L^\infty(0,T;L^2_{\omega_0}(I))} + \|\hat{e}_N\|_{L^\infty(0,T;L^2_{\omega_0}(I))}$$
$$\leqslant \|\pi_N^{\alpha,\beta}u - u\|_{L^\infty(0,T;L^2_{\omega^{\alpha,\beta}}(I))} + \|\hat{e}_N\|_{L^\infty(0,T;L^2_{\omega_0}(I))}$$
$$\lesssim N^{1-r}\Big(\|\partial_t\partial_x^{r-1}u\|_{L^2(0,T;L^2_{\omega^{\alpha+r-1,\beta+r-1}}(I))} + \|\partial_x^r u\|_{L^\infty(0,T;L^2_{\omega^{\alpha+r,\beta+r}}(I))}\Big).$$

This completes the proof. □

Remark 4.1. If $a(x)$ is a constant, the above result can be slightly improved. Indeed, when taking $v_N = \hat{e}_N$ in (4.20) in this case, thanks to Lemma 4.1, we find that the term corresponding to G_2 becomes $G_2 = \|b(\pi_N^{\alpha,\beta}u - u)\|_{L^2(0,T;L^2_{\omega_2}(I))}$. Therefore,

$$G_2 \leqslant \max_{x\in\bar{I}}|b(x)|\|\pi_N^{\alpha,\beta}u - u\|_{L^2(0,T;L^2_{\omega_2}(I))}$$
$$\lesssim N^{-r}\|\partial_x^r u\|_{L^2(0,T;L^2_{\omega^{\alpha+r,\beta+r}}(I))},$$

and

$$G_1 \lesssim N^{-r}\|\partial_t\partial_x^r u\|_{L^2(0,T;L^2_{\omega^{\alpha+r,\beta+r}}(I))}.$$

Hence, we have

$$\|u_N - u\|_{L^\infty(0,T;L^2_{\omega_0}(I))}$$
$$\lesssim N^{-r}\Big(\|\partial_x^r\partial_t u\|_{L^2(0,T;L^2_{\omega^{\alpha+r,\beta+r}}(I))} + \|\partial_x^r u\|_{L^2(0,T;L^2_{\omega^{\alpha+r,\beta+r}}(I))}\Big).$$

4.2. Chebyshev case ($\omega = (1 - x^2)^{-1/2}$)

In this subsection, we perform the error analysis of the Chebyshev dual-Petrov–Galerkin scheme (2.13) with V_N/V_N^* given by $(\text{i})_{\text{B}}$–$(\text{iv})_{\text{B}}$ in (2.12).

4.2.1. Chebyshev approximations

First, we establish an embedding result associated with the weights given by (2.10)–(2.11):

$$\omega_0(x) = \omega(x)\omega^{\hat{\alpha},\hat{\beta}}(x) = \omega^{-3/2,-3/2}, \omega^{1/2,-3/2}, \omega^{-3/2,1/2}, \omega^{1/2,1/2},$$
$$\omega_1(x) = (1 - x^2)^{-1}\omega_0(x) = \omega^{-5/2,-5/2}, \omega^{-1/2,-5/2}, \omega^{-5/2,-1/2}, \omega^{-1/2,-1/2}, \tag{4.23}$$

 ively.

Lemma 4.2

$$\|u\|_{\omega_1} \lesssim \|u\|_{1,\omega}, \quad \forall u \in L^2_{\omega_1}(I) \cap H^1_\omega(I), \tag{4.24}$$

and

$$\|u\|_{\omega_0} \lesssim \|u\|_{1,\omega}, \quad \forall u \in L^2_{\omega_0}(I) \cap H^1_\omega(I). \tag{4.25}$$

Proof. We first prove (4.24). The case $(\text{iv})_{\text{B}}$ (i.e., $\omega_1 = \omega^{-1/2,-1/2} = \omega$) is well-known (cf. [3]). For the case $(\text{i})_{\text{B}}$ (i.e., $\omega_1 = \omega^{-5/2,-5/2}$), we recall the Hardy inequality

$$\int_0^1\left(\frac{1}{1-x}\int_x^1\psi(y)\,\mathrm{d}y\right)^2(1-x)^d\,\mathrm{d}x$$
$$\leqslant \frac{4}{1-d}\int_0^1\psi^2(x)(1-x)^d\,\mathrm{d}x, \tag{4.26}$$

and

$$\int_{-1}^0\left(\frac{1}{1+x}\int_{-1}^x\psi(y)\,\mathrm{d}y\right)^2(1+x)^d\,\mathrm{d}x$$
$$\leqslant \frac{4}{1-d}\int_{-1}^0\psi^2(x)(1+x)^d\,\mathrm{d}x, \tag{4.27}$$

which hold for any measurable function $\phi(x)$, and real number $d < 1$. Taking $\psi = \partial_x u$ and $d = -\frac{1}{2}$ in (4.26) leads to

$$\int_0^1 u^2(x)(1-x^2)^{-5/2}\,\mathrm{d}x \lesssim \int_0^1 u^2(x)(1-x)^{-5/2}$$
$$\lesssim \int_0^1(\partial_x u)^2(1-x)^{-1/2}\,\mathrm{d}x$$
$$\lesssim \int_0^1(\partial_x u)^2(1-x^2)^{-1/2}\,\mathrm{d}x. \tag{4.28}$$

A similar inequality holds on the subinterval $[-1,0]$ by using (4.27). A combination of them yields

$$\|u\|^2_{\omega_1} \lesssim \|\partial_x u\|^2_\omega \lesssim \|u\|^2_{1,\omega}.$$

Since the proofs for the cases $(\text{ii})_{\text{B}}$ and $(\text{iii})_{\text{B}}$ are essentially the same, we will only consider the case $(\text{iii})_{\text{B}}$. Thanks to (4.28), we have

$$\int_0^1 u^2(x)(1-x)^{-5/2}(1+x)^{-1/2}\,\mathrm{d}x \lesssim \int_0^1 u^2(x)(1-x)^{-5/2}$$
$$\lesssim \int_0^1(\partial_x u)^2(1-x)^{-1/2}\,\mathrm{d}x$$
$$\lesssim \int_0^1(\partial_x u)^2(1-x^2)^{-1/2}\,\mathrm{d}x. \tag{4.29}$$

On the other hand,

$$\int_{-1}^0 u^2(x)(1-x)^{-5/2}(1+x)^{-1/2}\,\mathrm{d}x \lesssim \int_{-1}^0 u^2(x)(1-x^2)^{-1/2}\,\mathrm{d}x.$$

A combination of them leads to the desired result.

The inequality (4.25) can be proved in a similar fashion. □

As in the Legendre case, we need to derive some Chebyshev approximation results in suitable weighted Sobolev spaces. Let us define the subspaces of $H^1_\omega(I)$:

$$_0H^1_\omega(I) = \{u \in H^1_\omega(I) : u(-1) = 0\},$$
$$^0H^1_\omega(I) = \{u \in H^1_\omega(I) : u(1) = 0\}$$

and $H^1_{0,\omega}(I) = {}_0H^1_\omega(I) \cap {}^0H^1_\omega(I)$. Below, we shall use V to denote $H^1_{0,\omega}(I)$, ${}_0H^1_\omega(I)$, ${}^0H^1_\omega(I)$ and $H^1_\omega(I)$ for the cases $(\mathrm{i})_{\mathrm{B}}$, $(\mathrm{ii})_{\mathrm{B}}$, $(\mathrm{iii})_{\mathrm{B}}$ and $(\mathrm{iv})_{\mathrm{B}}$, respectively.

Define the orthogonal projection: $\pi^1_{N,\omega} : V \to V_N$ (defined in (2.12) for each case) by

$$\Big(\partial_x(\pi^1_{N,\omega}u - u), \partial_x v_N\Big)_\omega + (\pi^1_{N,\omega}u - u, v_N)_\omega = 0, \quad \forall v_N \in V_N. \tag{4.30}$$

Lemma 4.3. *For any $u \in V \cap B^r_{-3/2,-3/2}(I)$ with integer $r \geqslant 1$, we have that for each case of* $(\mathrm{i})_{\mathrm{B}}$–$(\mathrm{iv})_{\mathrm{B}}$,

$$\|\pi^1_{N,\omega}u - u\|_{\mu,\omega} \lesssim N^{\mu-r}\|\partial^r_x u\|_{\omega^{r-3/2,r-3/2}}, \quad \mu = 0, 1. \tag{4.31}$$

If, in addition, $u \in L^2_\chi(I)$ with $\chi = \omega_0$ or ω_1 given in (4.23), *then*

$$\|\pi^1_{N,\omega}u - u\|_\chi \lesssim N^{1-r}\|\partial^r_x u\|_{\omega^{r-3/2,r-3/2}}. \tag{4.32}$$

Proof. The estimate (4.31) for $(\mathrm{i})_{\mathrm{B}}$ $H^1_{0,\omega}$-orthogonal projection, and (iv) H^1_ω-orthogonal projection, can be found, for instance in [3], an improvement with the semi-norm in the upper bound can be found in [14]. The other two cases of (4.31) are stated in Theorem 3.2 of [14]. Then, a combination of Lemma 4.24 and (4.31) implies (4.32). □

4.2.2. Convergence of (2.13) with $\omega = (1-x^2)^{-1/2}$

Theorem 4.3. *Let $u_N(0) = u_{0,N} = \pi^1_{N,\omega}u_0$. If $u \in L^2(0,T; L^2_{\omega_1}(I)) \cap L^\infty(0,T; B^r_{-3/2,-3/2}(I))$ and $\partial_t u \in L^2(0,T; B^{r-1}_{-3/2,-3/2}(I))$ with integer $r \geqslant 2$, then*

$$\|u - u_N\|_{L^\infty(0,T;L^2_{\omega_0}(I))} + \lambda_1\|u - u_N\|_{L^2(0,T;L^2_{\omega_1}(I))} \lesssim N^{1-r}\Big(\|\partial_t\partial^{r-1}_x u\|_{L^2(0,T;L^2_{\omega^{r-5/2,r-5/2}}(I))} + \|\partial^r_x u\|_{L^\infty(0,T;L^2_{\omega^{r-3/2,r-3/2}}(I))}\Big). \tag{4.33}$$

where λ_1, ω_0 and ω_1 are the same as in Theorem 2.1 (also see (4.23)).

Proof. Set $\hat{e}_N = \pi^1_{N,\omega}u - u_N$ and $e_N = u - u_N = (u - \pi^1_{N,\omega}u) + \hat{e}_N$. By an argument similar to the derivation of (4.21), we have

$$\ldots;L^2_{\omega_1}(I)) \lesssim W_1 + W_2, \tag{4.34}$$

where

$$W_1 = \|\partial_t(\pi^1_{N,\omega}u - u)\|_{L^2(0,T;L^2_{\omega_2}(I))},$$
$$W_2 = \|\partial_x(a(\pi^1_{N,\omega}u - u)) + b(\pi^1_{N,\omega}u - u)\|_{L^2(0,T;L^2_{\omega_2}(I))}$$

and

$$\omega_2 = \omega^{-1/2,-1/2}, \omega^{3/2,-1/2}, \omega^{-1/2,3/2}, \omega^{3/2,3/2},$$

for the cases $(\mathrm{i})_{\mathrm{B}}$–$(\mathrm{iv})_{\mathrm{B}}$, respectively. Since $\omega_2 \lesssim \omega$, we derive from (4.31) that

$$W_1 \lesssim N^{1-r}\|\partial_t\partial^{r-1}_x u\|_{L^2(0,T;L^2_{\omega^{r-5/2,r-5/2}}(I))}, \quad r \geqslant 2$$

and

$$W_2 \lesssim N^{1-r}\|\partial^r_x u\|_{L^2(0,T;L^2_{\omega^{r-3/2,r-3/2}}(I))}, \quad r \geqslant 1.$$

On the other hand, by (4.32), we have that

$$\|\pi^1_{N,\omega}u - u\|_{L^\infty(0,T;L^2_{\omega_0}(I))} + \lambda_1\|\pi^1_{N,\omega}u - u\|_{L^2(0,T;L^2_{\omega_1}(I))} \lesssim N^{1-r}\Big(\|\partial^r_x u\|_{L^\infty(0,T;L^2_{\omega^{r-3/2,r-3/2}}(I))} + \|\partial^r_x u\|_{L^2(0,T;L^2_{\omega^{r-3/2,r-3/2}}(I))}\Big) \lesssim N^{1-r}\|\partial^r_x u\|_{L^\infty(0,T;L^2_{\omega^{r-3/2,r-3/2}}(I))}. \tag{4.35}$$

Hence, using a triangle inequality and the above estimates leads to the desired result. □

5. Concluding remarks

We presented in this paper a Legendre and Chebyshev dual-Petrov–Galerkin method for hyperbolic equations.

The dual-Petrov–Galerkin method is based on a natural variational formulation for hyperbolic equations. A distinctive feature of this variational formulation is that the associated bilinear form for general hyperbolic equations is *coercive*. An immediate consequence of this property is that the dual-Petrov–Galerkin method is always stable without any restriction on the coefficients. Another consequence is that one can build robust preconditioners as in the elliptic equations. In fact, by working in the frequency space, we were able to build an optimal (in the sense that for a given accuracy, the required iteration number is independent of the number of modes) preconditioner, which is the sparse matrix associated with an equation with suitable constant coefficients, for the linear system of an implicit time discretization of general hyperbolic equations.

This paper is our first effort in developing robust spectral algorithms for hyperbolic equations/systems. In future works, we shall investigate whether the dual-Petrov–Galerkin framework can be extended to effectively handle hyperbolic systems, and whether one can build more robust preconditioners using a suitable finite element approximation of the dual-Petrov–Galerkin formulation. We shall also investigate the numerical and theoretical issues of the dual-Petrov–Galerkin method for nonlinear hyperbolic equations.

Appendix A. Properties of Jacobi polynomials

We collect below some relevant formulas of the Jacobi polynomials used in this paper. The Jacobi polynomials $J_n^{\alpha,\beta}(x)(\alpha,\beta>-1)$ are orthogonal with respect to the Jacobi weight $\omega^{\alpha,\beta}=(1-x)^{\alpha}(1+x)^{\beta}$, i.e.,

$$\int_I J_n^{\alpha,\beta}(x)J_m^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)\,\mathrm{d}x=\gamma_n^{\alpha,\beta}\delta_{mn}, \tag{A.1}$$

with

$$\gamma_n^{\alpha,\beta}=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{A.2}$$

There hold the following recursive formulas (cf. Szegö [23] and Askey [1]):

$$J_{n-1}^{\alpha,\beta}(x)=J_n^{\alpha,\beta-1}(x)-J_n^{\alpha-1,\beta}(x),\quad \alpha,\beta>0; \tag{A.3}$$

$$J_n^{\alpha,\beta}(x)=\frac{1}{n+\alpha+\beta}\{(n+\beta)J_n^{\alpha,\beta-1}(x)+(n+\alpha)J_n^{\alpha-1,\beta}(x)\},\quad \alpha,\beta>0; \tag{A.4}$$

$$(1-x)J_n^{\alpha+1,\beta}(x)=\frac{2}{2n+\alpha+\beta+2}\{(n+\alpha+1)J_n^{\alpha,\beta}(x)-(n+1)J_{n+1}^{\alpha,\beta}(x)\}; \tag{A.5}$$

$$(1+x)J_n^{\alpha,\beta+1}(x)=\frac{2}{2n+\alpha+\beta+2}\{(n+\beta+1)J_n^{\alpha,\beta}(x)+(n+1)J_{n+1}^{\alpha,\beta}(x)\}; \tag{A.6}$$

$$\partial_x J_n^{\alpha,\beta}(x)=\frac{1}{2}(n+\alpha+\beta+1)J_{n-1}^{\alpha+1,\beta+1}(x),\quad n\geqslant 1. \tag{A.7}$$

The Legendre polynomials: $L_n(x):=J_n^{0,0}(x), n\geqslant 0$, satisfy

$$(2n+1)L_n(x)=\partial_x L_{n+1}(x)-\partial_x L_{n-1}(x),\quad n\geqslant 1; \tag{A.8}$$

$$(1-x^2)\partial_x L_n(x)=\frac{n(n+1)}{2n+1}(L_{n-1}(x)-L_{n+1}(x)),\quad n\geqslant 1. \tag{A.9}$$

The Chebyshev polynomials are defined by

$$T_n(x)=\frac{J_n^{-\frac{1}{2},-\frac{1}{2}}(x)}{J_n^{-\frac{1}{2},-\frac{1}{2}}(1)}=\cos(n\arccos(x)),\quad n\geqslant 0. \tag{A.10}$$

We have that

$$2xT_n(x)=T_{n-1}(x)+T_{n+1}(x),\quad n\geqslant 1; \tag{A.11}$$

$$(1-x^2)\partial_x T_n(x)=\frac{n}{2}(T_{n-1}(x)-T_{n+1}(x)),\quad n\geqslant 1. \tag{A.12}$$

As a consequence,

$$(1-x)\partial_x((1+x)T_n(x))=\frac{n-1}{2}T_{n-1}(x)+T_n(x)-\frac{n+1}{2}T_{n+1}(x). \tag{A.13}$$

Appendix B. The proofs of (4.2)–(4.4) and (4.7)–(4.9)

Using (A.7), (A.8) and (A.9), we obtain (4.2). Then, the ... directly from (A.6). Next, we ... and (4.1) that

$$\begin{aligned}\partial_x\phi_n(x)&\overset{(A.7)}{=}\frac{1}{2}(n+2)J_n^{1,1}(x)+\frac{1}{2}(n+1)J_{n-1}^{1,1}(x)\\&\overset{(A.3)}{=}\frac{1}{2}(n+2)J_n^{1,1}(x)+\frac{1}{2}(n+1)\left(J_n^{1,0}(x)-J_n^{0,1}(x)\right)\\&=\frac{1}{2}\left((n+2)J_n^{1,1}(x)-(n+1)J_n^{0,1}(x)\right)+\frac{1}{2}(n+1)J_n^{1,0}(x)\\&\overset{(A.4)}{=}(n+1)J_n^{1,0}(x).\end{aligned}$$

Similarly, we can prove (4.4).

Using (A.7), (4.2) and an induction argument, we find that

$$\partial_x^l\varphi_n(x)=(2n+3)\partial_x^{l-1}L_{n+1}(x)=\kappa_{n,l}J_{n-l+2}^{l-,l-1}(x) \tag{B.1}$$

with

$$\kappa_{n,l}=\frac{(2n+3)\Gamma(n+l+1)}{2^{l-1}\Gamma(n+2)}.$$

Therefore, (4.7) follows from the orthogonality (A.1) with

$$\mu_{n,l}^{(1)}=\kappa_{n,l}^2\gamma_{n-l+2}^{l-1,l-1}=2(2n+3)\frac{\Gamma(n+l+1)}{(\Gamma(n-l+3))}.$$

Similarly, we can prove (4.8) and (4.9).

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