article

Jie Shen and Changtao Sheng

Spectral methods for fractional differential equations using generalized Jacobi functions

Abstract: We present essential properties of the generalized Jacobi functions (GJFs) and their application to construct efficient and accurate spectral methods for a class of fractional differential equations. In particular, it is shown that GJFs allow us to effortlessly compute the stiffness matrices, and resolve the leading singular term for a general class of fractional differential equations.

Keywords: fractional differential equations; generalized Jacobi functions; spectral method; Petrov-Galerkin method

AMS: 65N35, 65E05, 65M70, 41A05, 41A10

1 Introduction

Fractional differential equations (FDEs) have attracted considerable attention in recent years due to their ability to model certain processes which can not be adequately described by usual partial differential equations. Two main difficulties for dealing with FDEs are (i) fractional derivatives are non-local operators; (ii) fractional derivatives involve singular kernel/weight functions, and the solutions of FDEs are usually weakly singular near the boundaries and at initial time. Hence, a straightforward extension of usual polynomial based numerical methods for FDEs are not effective as it usually involves dense matrices even for the simplest FDEs and suffers from low convergence rate due to the weak singularity. Thus, one needs to develop non-standard and non-polynomial based numerical methods to effectively deal with the difficulties associated with FDEs.

Jie Shen, Department of Mathematics, Purdue University; School of Mathematical Sciences and Fujian Provincial Key Laboratory on Mathematical Modeling & High Performance Scientific Computing, Xiamen University. This work is supported in part by AFOSR FA9550-16-1-0102 and NSF DMS-1620262, DMS-1720442.
Changtao Sheng, School of Mathematical Sciences and Fujian Provincial Key Laboratory on Mathematical Modeling & High Performance Scientific Computing, Xiamen University.

In this paper, we shall focus on some recently developed spectral methods using the generalized Jacobi functions (GJFs) for solving FDEs. Some pioneer work on using spectral methods for solving FDEs are carried out [17, 18]. However, the methods in these papers use the usual polynomial approximations which do not particularly address the two difficulties mentioned above. A breakthrough is made in [32] in which the authors introduced the so called poly-fractonomials, which are eigenfunctions of some fractional Sturm-Liouville operator. So using them as basis functions greatly simplifies the computation of fractional derivatives and leads to sparse matrices for some simple model equations. Furthermore, one can choose suitable parameters in poly-fractonomials so that its leading term is in fact the leading singular term of the corresponding FDE. Thus, the spectral method using poly-fractonomials can also resolve the leading singularity in a FDE. It turns out that the poly-fractonomials introduced in [32] coincide, within certain parameter range, with the generalized Jacobi functions introduced in [10]. In [5], the authors reexamined the generalized Jacobi functions in the context of FDEs and derived optimal approximation results with norms suitable for fractional derivatives. In particular, it is shown in [5] that well constructed spectral methods using GJFs for some typical FDEs can lead to exponential convergence despite the fact that their solutions are weakly singular.

Thanks to the aforementioned remarkable properties of the poly-fractonomials/GJFs with regards to fractional derivatives and fractional differential equations, there is now a significant number of recent work on using the poly-fractonomials/GJFs for different kind of FDEs and singular integral equations, including, for examples, spectral-Galerkin or spectral Petrov-Galerkin methods [31, 11, 36, 23, 21, 28, 15]; spectral collocation methods [35, 36, 15, 12, 14, 34, 13]; DG spectral-element methods [33, 16].

The aim of this paper is not to provide an exhaustive review of all developments with regards to spectral methods for FDEs, rather, it aims to present essential properties of the GJFs, including relations to the fractional derivatives and their approximation results in properly weighted Sobolev spaces, and how one can use them to construct efficient and accurate spectral methods for FDEs.

In order to illustrate the idea and advantage of spectral methods using GJFs, we shall consider the following FDE:

$$\begin{cases} \gamma\, {}^{C}D_t^{\alpha}u(x,t) - p\,{}^{R}D_x^{\beta}u(x,t) - (1-p)\; {}_{x}^{R}D^{\beta}\,u(x,t) = f(x,t), \\ u(x,t)|_{\partial\Lambda} = 0, \quad \forall t \in I := (0,T), \\ u(x,0) = u_0(x), \quad \forall x \in \Lambda = (a,b), \end{cases} \tag{1.1}$$

where $0 < \alpha < 1, 1 < \beta < 2$, $p \in [0,1]$; ${}^{C}D_t^{\alpha}$, ${}^{R}D_x^{\beta}$ and ${}_{x}^{R}D^{\beta}$ are the Caputo, left- and right- Riemann Liouville fractional derivative operator, respectively. More precisely, we shall consider, successively, the following four cases: (i) $\gamma = 0$, $p = 1$: an initial-value problem and a boundary-value problem with one-sided fractional derivative; (ii) $\gamma = 0$, $p = 1/2$: Riesz FDE; (iii) $\gamma = 0$, $p \neq 0, 1/2, 1$: a boundary-value problem with general two-sided fractional derivatives; and (iv) $\gamma \neq 0$ and $p = 1/2$: a space-time fractional diffusion equation.

The rest of the paper is organized as follows. In the next section, we introduce the GJFs corresponding to the one-sided fractional derivative, Riesz derivative and two-sided fractional derivatives with different coefficients, summarize their special properties, particularly with regards to fractional derivatives, and present approximation results using these GJFs. In Section 3, we construct successively efficient spectral methods using GJFs for the four cases above, and derive their error estimates. We provide some concluding remarks in the last section.

Due to the space constraint, we shall not present numerical results in this paper. However, theoretical results presented in this paper have been validated by ample numerical experiments in the correspondingly cited papers.

2 Generalized Jacobi functions and their approximation properties

In this section, we collect some basic relations and approximation properties of GJFs with respect to fractional derivatives from [21, 23, 22]. These results will play essential roles in developing efficient algorithms for FDEs and in deriving the corresponding error estimates in the next section.

2.1 Fractional integrals/derivatives

We inotrduce below the definitions of fractional integrals/derivatives. Let $a < b$ and denote $\Lambda = (a, b)$.

Definition 2.1 (One-sided fractional integrals and derivatives [24, 7]). *For $\rho \in \mathbb{R}^+$, the left and right fractional integrals are respectively defined as*

$$\begin{aligned} I_x^{\rho} v(x) &= \frac{1}{\Gamma(\rho)} \int_a^x \frac{v(y)}{(x-y)^{1-\rho}} dy, \quad x \in \Lambda, \\ {}_x I^{\rho} v(x) &= \frac{1}{\Gamma(\rho)} \int_x^b \frac{v(y)}{(y-x)^{1-\rho}} dy, \quad x \in \Lambda, \end{aligned} \tag{2.1}$$

where $\Gamma(\cdot)$ *is the usual Gamma function.*

For $s \in [k-1, k)$ *with* $k \in \mathbb{N}$, *the left-sided Riemann-Liouville fractional derivative of order* s *is defined by*

$$ {}^{R}D_x^s v(x) = \frac{1}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_a^x \frac{v(y)}{(x-y)^{s-k+1}} dy, \quad x \in \Lambda, \tag{2.2}$$

and the right-sided Riemann-Liouville fractional derivative of order s *is defined by*

$$ {}_x^R D^s v(x) = \frac{(-1)^k}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_x^b \frac{v(y)}{(y-x)^{s-k+1}} dy, \quad x \in \Lambda. \tag{2.3}$$

For $s \in [k-1, k)$ *with* $k \in \mathbb{N}$, *the left-sided Caputo fractional derivatives of order* s *is defined by*

$$ {}^{C}D_x^s v(x) := \frac{1}{\Gamma(k-s)} \int_a^x \frac{v^{(k)}(y)}{(x-y)^{s-k+1}} dy, \quad x \in \Lambda, \tag{2.4}$$

and the right-sided Caputo fractional derivatives of order s *is defined by*

$$ {}_x^C D^s v(x) := \frac{(-1)^k}{\Gamma(k-s)} \int_x^b \frac{v^{(k)}(y)}{(y-x)^{s-k+1}} dy, \quad x \in \Lambda. \tag{2.5}$$

According to [7, Thm. 2.14], we have that for any absolutely integrable function v, and real $s \geq 0$;

$$ {}^{R}D_x^s I_x^s v(x) = v(x), \qquad {}_x^R D^s {}_x I^s v(x) = v(x), \quad a.e. \quad in\ \Lambda. \tag{2.6}$$

The following lemma shows the relationship between the Riemann-Liouville and Caputo fractional derivatives (see, e.g., [7, 24]).

Lemma 2.1. *For* $s \in [k-1, k)$ *with* $k \in \mathbb{N}$, *we have*

$$ {}^{R}D_t^s u(t) = {}^{C}D_t^s u(t) + \sum_{j=0}^{k-1} \frac{u^{(j)}(a)}{\Gamma(1+j-s)} (t-a)^{j-s}. \tag{2.7}$$

Definition 2.2. (Riesz fractional integrals and derivatives [25]) For $\rho \in [0, 1)$, the Riesz fractional integral of order ρ is defined as

$$ I^\rho v(x) := \frac{1}{2\cos(\pi\rho/2)} (I_x^\rho + {}_x I^\rho) v(x), \quad x \in \Lambda. \tag{2.8}$$

For $\rho \in [2k-1, 2k)$ with $k \in \mathbb{N}$, the Riesz fractional derivative (RFD) of order ρ is defined by:

$$ D^\rho v(x) := D^{2k} I^{2k-\rho} v(x). \tag{2.9}$$

The following results [8, 17] play fundamental roles in the analysis of FDEs.

Lemma 2.2. *For all* $0 < \alpha < 2$ *and* $\alpha \neq 1$, *we have*

$$({}^R D_t^{\alpha} v, w)_\Lambda = \big({}^R D_t^{\frac{\alpha}{2}} v, {}_t^R D^{\frac{\alpha}{2}} w\big)_\Lambda, \quad \forall v, w \in H_0^{\alpha/2}(\Lambda); \tag{2.10}$$

Lemma 2.3. *For all* $\alpha > 0$ *and that* $\alpha - 1/2$ *is not an integer, we have*

$$\big({}^R D_t^{\frac{\alpha}{2}} v, {}_t^R D^{\frac{\alpha}{2}} v\big)_\Lambda \cong \|v\|^2_{H_0^{\alpha/2}(\Lambda)}, \quad \forall v \in H_0^{\alpha/2}(\Lambda). \tag{2.11}$$

In the sequel, we use c to denote a generic constant.

2.2 GJFs for one-sided fractional derivatives

We start by considering the one-sided fractional derivatives. Unless otherwise specified, we set $\Lambda = (-1, 1)$.

2.2.1 Jacobi polynomials and Bateman's fractional integral formula

According to [30, (4.21.2)], the classical Jacobi polynomials with parameters $\alpha, \beta \in \mathbb{R}$ can be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\Big) \tag{2.12}$$

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function, and the rising factorial in the Pochhammer symbol, for $a \in \mathbb{R}$ and $j \in \mathbb{N}_0$, is defined by

$$(a)_0 = 1; \quad (a)_j := a(a+1)\cdots(a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}, \;\text{ for }\; j \geq 1. \tag{2.13}$$

For $\alpha, \beta > -1$, the classical Jacobi polynomials are orthogonal with respect to the Jacobi weight function: $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$, namely,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_{n'}^{(\alpha,\beta)}(x) \omega^{(\alpha,\beta)}(x) dx = \gamma_n^{(\alpha,\beta)} \delta_{nn'}, \tag{2.14}$$

where $\delta_{nn'}$ is the Dirac Delta symbol, and the normalization constant is given by

$$\gamma_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}. \tag{2.15}$$

The following fractional integral formula of hypergeometric functions due to Bateman [3] (also see [2, P. 313]) plays an important role in the computation of fractional integrals/derivatives: for real $c, \rho \geq 0$,

$$ {}_2F_1(a, b; c+\rho; x) = \frac{\Gamma(c+\rho)}{\Gamma(c)\Gamma(\rho)} x^{1-(c+\rho)} \int_0^x t^{c-1}(x-t)^{\rho-1} {}_2F_1(a, b; c; t)\, dt, \quad |x| < 1. \tag{2.16} $$

One derives easily from (2.12) and (2.16) the following results (cf. [30, P. 96]):

Lemma 2.4. *Let $\rho \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*
(i) For $\alpha > -1$ and $\beta \in \mathbb{R}$,

$$ (1-x)^{\alpha+\rho} \frac{P_n^{(\alpha+\rho,\beta-\rho)}(x)}{P_n^{(\alpha+\rho,\beta-\rho)}(1)} = \frac{\Gamma(\alpha+\rho+1)}{\Gamma(\alpha+1)\Gamma(\rho)} \int_x^1 \frac{(1-y)^\alpha}{(y-x)^{1-\rho}} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}\, dy. \tag{2.17} $$

(ii) For $\alpha \in \mathbb{R}$ and $\beta > -1$,

$$ (1+x)^{\beta+\rho} \frac{P_n^{(\alpha-\rho,\beta+\rho)}(x)}{P_n^{(\beta+\rho,\alpha-\rho)}(1)} = \frac{\Gamma(\beta+\rho+1)}{\Gamma(\beta+1)\Gamma(\rho)} \int_{-1}^x \frac{(1+y)^\beta}{(x-y)^{1-\rho}} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\beta,\alpha)}(1)}\, dy. \tag{2.18} $$

Using the notation in Definition 2.1, we can rewrite the formulas in Lemma 2.4 as follows.

Lemma 2.5. *Let $\rho \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*

– *For $\alpha > -1$ and $\beta \in \mathbb{R}$,*

$$ {}_xI^\rho \big\{(1-x)^\alpha P_n^{(\alpha,\beta)}(x)\big\} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\rho+1)} (1-x)^{\alpha+\rho} P_n^{(\alpha+\rho,\beta-\rho)}(x). \tag{2.19} $$

– *For $\alpha \in \mathbb{R}$ and $\beta > -1$,*

$$ I_x^\rho \big\{(1+x)^\beta P_n^{(\alpha,\beta)}(x)\big\} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\rho+1)} (1+x)^{\beta+\rho} P_n^{(\alpha-\rho,\beta+\rho)}(x). \tag{2.20} $$

Thanks to (2.6), we obtain from Lemma 2.5 the following useful "inverse" rules.

Lemma 2.6. *Let $s \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*

– *For $\alpha > -1$ and $\beta \in \mathbb{R}$,*

$$ {}_x^R D^s \big\{(1-x)^{\alpha+s} P_n^{(\alpha+s,\beta-s)}(x)\big\} = \frac{\Gamma(n+\alpha+s+1)}{\Gamma(n+\alpha+1)} (1-x)^\alpha P_n^{(\alpha,\beta)}(x). \tag{2.21} $$

– *For $\alpha \in \mathbb{R}$ and $\beta > -1$,*

$$ {}^{R}D_x^s\big\{(1+x)^{\beta+s}P_n^{(\alpha-s,\beta+s)}(x)\big\} = \frac{\Gamma(n+\beta+s+1)}{\Gamma(n+\beta+1)}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x). \tag{2.22}$$

The above lemmas are remarkable in the sense that fractional integrals/derivatives of functions in the form of $(1\pm x)^{\alpha}P_n^{(\alpha,\beta)}$ can be expressed in the same form with a set of different parameters (α,β). In other words, the **global** fractional integral/derivative operators become **local** operators in suitable "spectral" spaces. In particular, if $\alpha = 0$ in (2.21), the fractional derivative operator ${}_x^RD^s$ takes $(1-x)^sP_n^{(s,\beta-s)}(x)$ to the polynomial $P_n^{(0,\beta)}(x)$; conversely, if $\alpha+s = k \in \mathbb{N}_0$, ${}_x^RD^s$ takes the polynomial $(1-x)^kP_n^{(k,\beta-s)}(x)$ to $(1-x)^{k-s}P_n^{(k-s,\beta)}(x)$. Such remarkable properties are essential for constructing efficient spectral algorithms for FDEs.

Definition 2.3 (One-sided generalized Jacobi functions [5])**.** *Define*

$$ {}^{+}J_n^{(-\alpha,\beta)}(x) := (1-x)^{\alpha}P_n^{(\alpha,\beta)}(x), \quad \text{for } \alpha > -1, \ \beta \in \mathbb{R}, \tag{2.23}$$

and

$$ {}^{-}J_n^{(\alpha,-\beta)}(x) := (1+x)^{\beta}P_n^{(\alpha,\beta)}(x), \quad \text{for } \alpha \in \mathbb{R}, \ \beta > -1, \tag{2.24}$$

for all $x \in \Lambda$ and $n \in \mathbb{N}_0$.

2.2.2 Some important properties of GJFs

It follows straightforwardly from (2.14) and Definition 2.3 that for $\alpha,\beta > -1$,

$$\begin{aligned} &\int_{-1}^{1} {}^{+}J_n^{(-\alpha,\beta)}(x)\,{}^{+}J_{n'}^{(-\alpha,\beta)}(x)\,\omega^{(-\alpha,\beta)}(x)\,dx \\ &= \int_{-1}^{1} {}^{-}J_n^{(\alpha,-\beta)}(x)\,{}^{-}J_{n'}^{(\alpha,-\beta)}(x)\,\omega^{(\alpha,-\beta)}(x)\,dx = \gamma_n^{(\alpha,\beta)}\delta_{nn'}, \end{aligned} \tag{2.25}$$

Similarly, we have that for $\alpha > -1$ and $k \in \mathbb{N}$, and $n, n' \geq k$,,

$$\begin{aligned} &\int_{-1}^{1} {}^{+}J_n^{(-\alpha,-k)}(x)\,{}^{+}J_{n'}^{(-\alpha,-k)}(x)\,\omega^{(-\alpha,-k)}(x)\,dx \\ &= \int_{-1}^{1} {}^{-}J_n^{(-k,-\alpha)}(x)\,{}^{-}J_{n'}^{(-k,-\alpha)}(x)\,\omega^{(-k,-\alpha)}(x)\,dx = \gamma_n^{(\alpha,-k)}\delta_{nn'}, \end{aligned} \tag{2.26}$$

where $\gamma_n^{(\alpha,\beta)}$ and $\gamma_n^{(\alpha,-k)}$ are some constants which can be found in [5].

With the above definitions, we can rewrite Lemma 2.6 as

Theorem 2.1. *Let $s \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*

– *For $\alpha > s-1$ and $\beta \in \mathbb{R}$,*

$$ {}_x^R D^s \{ {}^+J_n^{(-\alpha,\beta)}(x) \} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-s+1)} {}^+J_n^{(-\alpha+s,\beta+s)}(x). \tag{2.27} $$

– *For $\alpha \in \mathbb{R}$ and $\beta > s-1$,*

$$ {}^R D_x^s \{ {}^-J_n^{(\alpha,-\beta)}(x) \} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta-s+1)} {}^-J_n^{(\alpha+s,-\beta+s)}(x). \tag{2.28} $$

The analysis of GJFs essentially relies on the orthogonality of fractional derivatives of GJFs. Recall the derivative formula of the classical Jacobi polynomials (see, e.g., [29, P. 72]): for $\alpha, \beta > -1$ and $n \ge l$,

$$ D^l P_n^{(\alpha,\beta)}(x) = \kappa_{n,l}^{(\alpha,\beta)} P_{n-l}^{(\alpha+l,\beta+l)}(x), \tag{2.29} $$

where $\kappa_{n,l}^{(\alpha,\beta)}$ is some constant. Noting that ${}_x^R D^{s+l} = (-1)^l D^l {}_x^R D^s$ and ${}^R D_x^{s+l} = D^l {}^R D_x^s$ for $s \in \mathbb{R}^+$ and $l \in \mathbb{N}$, we derive from (2.14) and (2.27)-(2.29) the following orthogonality relations:

– For $\alpha > 0$, $\alpha + \beta > -1$, and $n, n' \ge l \ge 0$,

$$ \int_{-1}^{1} {}_x^R D^{\alpha+l}\, {}^+J_n^{(-\alpha,\beta)}(x)\ {}_x^R D^{\alpha+l}\, {}^+J_{n'}^{(-\alpha,\beta)}(x)\, \omega^{(l,\alpha+\beta+l)}(x)\, dx = h_{n,l}^{(\alpha,\beta)} \delta_{nn'}, \tag{2.30} $$

where

$$ h_{n,l}^{\alpha,\beta} := \frac{2^{\alpha+\beta+1}\Gamma^2(n+\alpha+1)\Gamma(n+\alpha+\beta+l+1)}{(2n+\alpha+\beta+1)n!(n-l)!\Gamma(n+\alpha+\beta+1)}. \tag{2.31} $$

– For $\alpha + \beta > -1$, $\beta > 0$, and $n, n' \ge l \ge 0$,

$$ \int_{-1}^{1} {}^R D_x^{\beta+l}\, {}^-J_n^{(\alpha,-\beta)}(x)\ {}^R D_x^{\beta+l}\, {}^-J_{n'}^{(\alpha,-\beta)}(x)\, \omega^{(\alpha+\beta+l,l)}(x)\, dx = h_{n,l}^{(\beta,\alpha)} \delta_{nn'}. \tag{2.32} $$

2.2.3 Approximation by the one-sided GJFs

We show below that approximation by GJFs can lead to truly spectral convergence for functions in properly weighted Sobolev spaces involving fractional derivatives.

For simplicity of presentation, we only provide the results $\left\{{}^{-}J_n^{(\alpha,-\beta)}\right\}$. Similar results can be established for $\left\{{}^{+}J_n^{(-\alpha,\beta)}\right\}$ [5].

Let $\mathcal{P}_N$ be the set of all algebraic (real-valued) polynomials of degree at most N. Let $\varpi(x) > 0$, $x \in \Lambda$, be a generic weight function. The weighted space $L^2_\varpi(\Lambda)$ is defined as in Adams [1] with the inner product and norm

$$(u,v)_\varpi = \int_\Lambda u(x)v(x)\varpi(x)dx, \quad \|u\|_\varpi = (u,u)_\varpi^{1/2}.$$

If $\varpi \equiv 1$, we omit the weight function in the notation.

We define the finite-dimensional fractional-polynomial space:

$${}^{-}\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda) = \{\phi = (1+x)^\beta \psi : \psi \in \mathcal{P}_N\} = \mathrm{span}\{{}^{-}J_n^{(\alpha,-\beta)} : 0 \le n \le N\},$$

By the orthogonality (2.25), we can expand any $u \in L^2_{\omega^{(\alpha,-\beta)}}$ as

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n^{(\alpha,-\beta)}\, {}^{-}J_n^{(\alpha,-\beta)}(x), \tag{2.33}$$

where

$$\hat{u}_n^{(\alpha,-\beta)} = \frac{1}{\gamma_n^{(\alpha,\beta)}} \int_{-1}^{1} u\, {}^{-}J_n^{(\alpha,-\beta)} \omega^{(\alpha,-\beta)} dx,$$

and there holds the Parseval identity:

$$\|u\|^2_{\omega^{(\alpha,-\beta)}} = \sum_{n=0}^{\infty} \gamma_n^{(\alpha,\beta)} |\hat{u}_n^{(\alpha,-\beta)}|^2. \tag{2.34}$$

Consider the $L^2_{\omega^{(\alpha,-\beta)}}$-orthogonal projection onto ${}^{-}\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda)$:

$$\left({}^{-}\pi_N^{(\alpha,-\beta)}u - u, v_N\right)_{\omega^{(\alpha,-\beta)}} = 0, \quad \forall v_N \in {}^{-}\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda). \tag{2.35}$$

Then, it is easy to derive from (2.28) that for any $l \in \mathbb{N}_0$, we have

$$\left({}^{R}D_x^{\beta+l}({}^{-}\pi_N^{(\alpha,-\beta)}u - u), D^l w_N\right)_{\omega^{(\alpha+\beta+l,l)}} = 0, \quad \forall w_N \in \mathcal{P}_N(\Lambda). \tag{2.36}$$

To describe the projection error, we define

$${}^{-}\mathcal{B}^m_{\alpha,\beta}(\Lambda) := \left\{u \in L^2_{\omega^{(\alpha,-\beta)}}(\Lambda) \;:\; {}^{R}D_x^{\beta+l}u \in L^2_{\omega^{(\alpha+\beta+l,l)}}(\Lambda) \;\text{ for }\; 0 \le l \le m\right\}. \tag{2.37}$$

By (2.32) and (2.33), we have that for $(\alpha,\beta) \in {}^{-}\Sigma^{\alpha,\beta} := \left\{(\alpha,\beta) \;:\; \beta > 0, \;\; \alpha > -1\right\}$ and $l \in \mathbb{N}_0$

$$\|{}^{R}D_x^{\beta+l}u\|^2_{\omega^{(\alpha+\beta+l,l)}} = \sum_{n=l}^{\infty} h_{n,l}^{(\beta,\alpha)} |\hat{u}_n^{(\alpha,-\beta)}|^2. \tag{2.38}$$

Theorem 2.2. *Let* $(\alpha,\beta)\in {}^-\Sigma^{\alpha,\beta}$, *and* $u\in {}^-\mathcal{B}^m_{\alpha,\beta}(\Lambda)$.

– *For* $0\le l\le m$, *we have*

$$\big\|{}^R D_x^{\beta+l}({}^-\pi_N^{(\alpha,-\beta)}u-u)\big\|_{\omega^{(\alpha+\beta+l,l)}}\le cN^{l-m}\,\big\|{}^R D_x^{\beta+m}u\big\|_{\omega^{(\alpha+\beta+m,m)}}. \tag{2.39}$$

– *For* $0\le m$, *we also have the* $L^2_{\omega^{(\alpha,-\beta)}}$*-estimate:*

$$\big\|{}^-\pi_N^{(\alpha,-\beta)}u-u\big\|_{\omega^{(\alpha,-\beta)}}\le cN^{-(\beta+m)}\big\|{}^R D_x^{\beta+m}u\big\|_{\omega^{(\alpha+\beta+m,m)}}. \tag{2.40}$$

Proof. By (2.33), (2.35) and (2.38), we have

$$\begin{aligned}\big\|{}^R D_x^{\beta+l}({}^-\pi_N^{(\alpha,-\beta)}u-u)\big\|^2_{\omega^{(\alpha+\beta+l,l)}}&=\sum_{n=N+1}^{\infty}h_{n,l}^{(\beta,\alpha)}|\hat u_n^{(\alpha,-\beta)}|^2\\ &=\sum_{n=N+1}^{\infty}\frac{h_{n,l}^{(\beta,\alpha)}}{h_{n,m}^{(\beta,\alpha)}}h_{n,m}^{(\beta,\alpha)}|\hat u_n^{(\alpha,-\beta)}|^2\le\frac{h_{N+1,l}^{(\beta,\alpha)}}{h_{N+1,m}^{(\beta,\alpha)}}\big\|{}^R D_x^{\beta+m}u\big\|^2_{\omega^{(\alpha+\beta+m,m)}}.\end{aligned} \tag{2.41}$$

We now estimate the constant factor. By (2.13), (2.31) and a direct calculation, we find that for $0\le l\le m\le N$,

$$\begin{aligned}\frac{h_{N+1,l}^{(\beta,\alpha)}}{h_{N+1,m}^{(\beta,\alpha)}}&=\frac{\Gamma(N+\alpha+\beta+l+2)(N-m+1)!}{\Gamma(N+\alpha+\beta+m+2)(N-l+1)!}\\ &=\frac{1}{(N+\alpha+\beta+2+l)\cdots(N+\alpha+\beta+1+m)}\frac{(N-m+1)!}{(N-l+1)!}\\ &\le N^{l-m}\frac{(N-m+1)!}{(N-l+1)!},\end{aligned} \tag{2.42}$$

where we used the fact: $\alpha+\beta>-1$. Thus, we obtain (2.39) from (2.41), (2.42) and the property of the Gamma function.

The $L^2_{\omega^{(\alpha,-\beta)}}$-estimates can be obtained by using the same argument. We sketch the derivation below. By (2.34) and (2.38),

$$\begin{aligned}\big\|{}^-\pi_N^{(\alpha,-\beta)}u-u\big\|^2_{\omega^{(\alpha,-\beta)}}&=\sum_{n=N+1}^{\infty}\gamma_n^{(\alpha,\beta)}|\hat u_n^{(\alpha,-\beta)}|^2\\ &=\sum_{n=N+1}^{\infty}\frac{\gamma_n^{(\alpha,\beta)}}{h_{n,m}^{(\beta,\alpha)}}h_{n,m}^{(\beta,\alpha)}|\hat u_n^{(\alpha,-\beta)}|^2\le\frac{\gamma_{N+1}^{(\alpha,\beta)}}{h_{N+1,m}^{(\beta,\alpha)}}\big\|{}^R D_x^{\beta+m}u\big\|^2_{\omega^{(\alpha+\beta+m,m)}}.\end{aligned} \tag{2.43}$$

Working out the constants by (2.15) and (2.31), we use the property of the Gamma function again to get that

$$\frac{\gamma_{N+1}^{(\alpha,\beta)}}{h_{N+1,m}^{(\beta,\alpha)}}=\frac{\Gamma(N+\alpha+2)\Gamma(N+m+2)(N-m+1)!}{\Gamma(N+\beta+2)\Gamma(N+\alpha+\beta+m+2)(N+m+1)!}\le cN^{-2(\beta+m)}. \tag{2.44}$$

□

Remark 2.1. Note that the error estimates, in the above theorem and in subsequent theorems, depend on the smoothness of fractional derivatives of the function, instead of the usual smoothness. To better understand the above results, we consider a typical solution of a one-sided fractional differential equation

$$u(x) = (1-x)^{\beta} g(x), \quad \beta \in \mathbb{R}^{+}, \ x \in \Lambda, \tag{2.45}$$

where g is a smooth function, and compare the GJF approximation with the Legendre approximation. Recall the best L^2-approximation of u by its orthogonal projection $\pi_N^L u$ (see, e.g., [29, Ch. 3]):

$$\|\pi_N^L u - u\| \le cN^{1-m} \|D^m u\|_{\omega^{(m,m)}}.$$

If β is not an integer, a direct calculation shows that the righthand side is only bounded for $m < 1 + 2\beta - \epsilon$. On the other hand, using the explicit formulas for fractional integral/derivative of $(1+x)^{\beta}$ and the Leibniz' formula (see [7, Ch. 2]), we find that ${}^{R}D_x^{\beta+m} u$ is integrable for any $m \in \mathbb{N}_0$, so the convergence by GJF approximation is faster than any algebraic rate.

Remark 2.2. The results in Theorem 2.2 can be extended to some other $(\alpha, \beta) \notin {}^{-}\Sigma^{\alpha,\beta}$, we refer to [5] for more detail.

2.3 GJFs for Riesz derivatives

The Riesz derivatives include both the left- and right-sided fractional derivatives so the one-sided GJFs defined above are not suitable. Instead, we define a new class of GJFs:

$$\mathcal{J}_n^{-\mu,-\nu}(x) = (1-x)^{\mu}(1+x)^{\nu} P_n^{(\mu,\nu)}(x), \quad \mu, \nu > -1. \tag{2.46}$$

It can be derived from (2.14) that the general Jacobi functions $\mathcal{J}_n^{-\mu,-\nu}(x)$ are mutually orthogonal:

$$\int_{-1}^{1} \mathcal{J}_n^{-\mu,-\nu}(x) \mathcal{J}_m^{-\mu,-\nu}(x) \omega^{(-\mu,-\nu)}(x) = \gamma_n^{(\mu,\nu)} \delta_{mn}, \tag{2.47}$$

and

$$\mathbb{F}_N^{-\mu,-\nu}(\Lambda) := \left\{ \mathcal{J}_n^{-\mu,-\nu}(x) : n = 0, 1, \cdots, N \right\}. \tag{2.48}$$

For Riesz derivatives, we shall use $\mathcal{J}_n^{-\alpha,-\alpha}(x)$ which satisfied the following:

Theorem 2.3. *If $s \in (2k-1, 2k)$ with $k \in \mathbb{N}$, then*

$$I^{2k-s} \mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = (-1)^k \frac{\Gamma(m+s+1-2k)}{2^{-2k} m!} P_{m+2k}^{(\frac{s}{2}-2k, \frac{s}{2}-2k)}(x), \tag{2.49}$$

and for $j = 0, 1, \cdots, 2k-1$,

$$D^{s-j}\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = 2^j(-1)^k\frac{\Gamma(m-j+1+s)}{m!}P_{m+j}^{(\frac{s}{2}-j,\frac{s}{2}-j)}(x). \tag{2.50}$$

We point out in particular that with $j = 0$ in (2.50), we have

Corollary 2.1. *If* $s \in (2k-1, 2k)$ *with* $k \in \mathbb{N}$, *then*

$$D^s\mathcal{J}_m^{-\frac{s}{2},-\frac{s}{2}}(x) = (-1)^k\frac{\Gamma(m+1+s)}{m!}P_m^{(\frac{s}{2},\frac{s}{2})}(x). \tag{2.51}$$

Let $s \in (2k-1, 2k)$ with $k \in \mathbb{N}$, and $l \in \mathbb{N}_0$, we denote for any $m \in \mathbb{N}_0$

$$\mathcal{B}_s^m(\Lambda) := \{u \in L^2_{\omega^{(-\frac{s}{2},-\frac{s}{2})}}(\Lambda):\ D^{s-k+l}u \in L^2_{\omega^{(\frac{s}{2}-k+l,\frac{s}{2}-k+l)}}(\Lambda),\ \text{for } 0 \le l \le m\}. \tag{2.52}$$

By the orthogonality (2.47), we can expand any $u \in L^2_{\omega^{(-\alpha,-\alpha)}}(\Lambda)$ as

$$u(x) = \sum_{n=0}^{\infty}\hat{u}_n^{(-\alpha,-\alpha)}\mathcal{J}_n^{-\alpha,-\alpha}(x), \tag{2.53}$$

and there holds the Parseval identity:

$$\|u\|^2_{\omega^{(-\alpha,-\alpha)}} = \sum_{n=0}^{\infty}\gamma_n^{(\alpha,\alpha)}|\hat{u}_n^{(-\alpha,-\alpha)}|^2. \tag{2.54}$$

Moreover, for any $2\alpha \in (2k-1, 2k)$ with $k \in \mathbb{N}$, we have

$$\|D^{2\alpha-k+l}u\|^2_{\omega^{(\alpha-k+l,\alpha-k+l)}} = \sum_{n=l}^{\infty}h_{n,l}^{(\alpha,\alpha)}|\hat{u}_n^{(-\alpha,-\alpha)}|^2. \tag{2.55}$$

Theorem 2.4. *Assume* $2\alpha \in (2k-1, 2k)$ *and* $k \in \mathbb{N}$. *Let* $u \in \mathcal{B}^m_{2\alpha}(\Lambda)$. *We have*

$$\|D^{2\alpha-k+l}(\pi_N^{(-\alpha,-\alpha)}u - u)\|_{\omega^{(\alpha-k+l,\alpha-k+l)}} \le cN^{l-m}\|D^{2\alpha-k+m}u\|_{\omega^{(\alpha-k+m,\alpha-k+m)}}, \tag{2.56}$$

and

$$\|\pi_N^{(-\alpha,-\alpha)}u - u\|_{\omega^{(-\alpha,-\alpha)}} \le cN^{k-(2\alpha+m)}\|D^{2\alpha-k+m}u\|_{\omega^{(\alpha-k+m,\alpha-k+m)}}. \tag{2.57}$$

Proof. By (2.68) (with $\mu = \nu = \alpha$) and (2.55), we have

$$\begin{aligned}\left\|D^{2\alpha-k+l}(\pi_N^{(-\alpha,-\alpha)}u - u)\right\|^2_{\omega^{(\alpha-k+l,\alpha-k+l)}} &= \sum_{n=N+1}^{\infty}h_{n,l}^{(\alpha,\alpha)}|\hat{u}_n^{(-\alpha,-\alpha)}|^2\\ = \sum_{n=N+1}^{\infty}\frac{h_{n,l}^{(\alpha,\alpha)}}{h_{n,m}^{(\alpha,\alpha)}}h_{n,m}^{(\alpha,\alpha)}|\hat{u}_n^{(-\alpha,-\alpha)}|^2 &\le \frac{h_{N+1,l}^{(\alpha,\alpha)}}{h_{N+1,m}^{(\alpha,\alpha)}}\left\|D^{2\alpha-k+m}u\right\|_{\omega^{(\alpha-k+m,\alpha-k+m)}}.\end{aligned} \tag{2.58}$$

We now estimate the constant factor. Similar to the proof in Theorem 2.2, we find that for $0 \le l \le m \le N$,

$$\frac{h_{N+1,l}^{(\alpha,\alpha)}}{h_{N+1,m}^{(\alpha,\alpha)}} \le N^{l-m} \frac{(N+k-m+1)!}{(N+k-l+1)!}. \tag{2.59}$$

Thus, we obtain (2.56) from (2.58), (2.59) and the property of the Gamma function.

The $L^2_{\omega^{(-\alpha,-\alpha)}}$-estimates can be obtained by using the same argument. We sketch the derivation below. By (2.54) and (2.55),

$$\begin{aligned} \left\|\pi_N^{(-\alpha,-\alpha)} u - u\right\|^2_{\omega^{(-\alpha,-\alpha)}} &= \sum_{n=N+1}^{\infty} \gamma_n^{(\alpha,\alpha)} |\hat{u}_n^{(-\alpha,-\alpha)}|^2 \\ = \sum_{n=N+1}^{\infty} \frac{\gamma_n^{(\alpha,\alpha)}}{h_{n,m}^{(\alpha,\alpha)}} h_{n,m}^{(\alpha,\alpha)} |\hat{u}_n^{(\alpha,\alpha)}|^2 &\le \frac{\gamma_{N+1}^{(\alpha,\alpha)}}{h_{N+1,m}^{(\alpha,\alpha)}} \left\|D^{2\alpha-k+m} u\right\|_{\omega^{(\alpha-k+m,\alpha-k+m)}}. \end{aligned}$$

Similarly, by (2.15) and (2.31), we obtain that

$$\begin{aligned} \frac{\gamma_{N+1}^{(\alpha,\alpha)}}{h_{N+1,m}^{(\alpha,\alpha)}} &= \frac{(N+1)!(N+k+m+1)!(N+k-m+1)!}{\Gamma(N+2\alpha+2)\Gamma(N+2\alpha-k+m+2)(N+k+m+1)!} \\ &\le N^{2k-(4\alpha+2m)}. \end{aligned}$$

This completes the proof. □

2.4 GJFs for two-sided fractional derivatives with different coefficients

For $1 < \beta < 2$, $0 < \mu, \nu < \beta$, $\mu + \nu = \beta$, $0 \le p \le 1$ and $p \ne 1/2$, we define the two-sided fractional integral operator

$$\mathcal{I}_p^{\mu,\nu,\rho} := C_{\beta,p}(p I_x^{\rho} + (1-p)_x I^{\rho}), \tag{2.60}$$

where

$$C_{\beta,p} := C(\beta,\mu,\nu) = \frac{\sin(\pi\mu) + \sin(\pi\nu)}{\sin(\pi\beta)}.$$

Then for $s \in (k-1,k)$, we define the two-sided fractional derivative operator

$$\mathcal{D}_p^{\mu,\nu,s} := \frac{d^k}{dx^k} \mathcal{I}_p^{\mu,\nu,k-s}. \tag{2.61}$$

It turns out that suitable basis functions for dealing with two-sided FDEs with different coefficients are of the form $\mathcal{J}_n^{-\mu,-\nu}(x)$ where (μ,ν) are determined as follows ([9, 22]):

Theorem 2.5. *Given (p,β) such that $0 \le p \le 1$ and $1 < \beta < 2$, let (μ,ν) be determined from*

$$\mu+\nu=\beta, \qquad p\sin(\pi\mu)=(1-p)\sin(\pi\nu), \tag{2.62}$$

then for $n=0,1,2,\cdots$, it holds that

$$\mathcal{I}_p^{\mu,\nu,2-\beta}\mathcal{J}_n^{-\mu,-\nu}(x)=4\frac{\Gamma(n+\beta-1)}{n!}P_{n+2}^{(\nu-2,\mu-2)}(x), \tag{2.63}$$

and for $k=1,2,\cdots,n+2$,

$$\mathcal{D}_p^{\mu,\nu,k+\beta-2}\mathcal{J}_n^{-\mu,-\nu}(x)=\frac{\Gamma(n+k+\beta-1)}{2^{k-2}n!}P_{n+2-k}^{(\nu-2+k,\mu-2+k)}(x). \tag{2.64}$$

In particular, for $k=2$,

$$\mathcal{D}_p^{\mu,\nu,\beta}\mathcal{J}_n^{-\mu,-\nu}(x)=\frac{\Gamma(n+\beta+1)}{n!}P_n^{(\nu,\mu)}(x). \tag{2.65}$$

For the sake of simplicity, we shall denote $\mathcal{I}_p^{\mu,\nu,\rho}$ and $\mathcal{D}_p^{\mu,\nu,\rho}$ by $\mathcal{I}_p^{\rho}$ and $\mathcal{D}_p^{\rho}$, respectively.

By virtue of (2.14), a consequent result of equation (2.64) is the orthogonality of $\mathcal{D}_p^{\beta+l}$ for $l=-1,0,1,\cdots,\min\{m,n\}$. If (μ,ν) and (p,β) satisfy the conditions of Theorem 2.5, then

$$\int_{-1}^{1}\mathcal{D}_p^{\beta+l}\mathcal{J}_m^{-\mu,-\nu}(x)\mathcal{D}_p^{\beta+l}\mathcal{J}_n^{-\mu,-\nu}(x)\omega^{(\nu+l,\mu+l)}(x)dx=0, \quad \forall n\ne m, \tag{2.66}$$

$$\int_{-1}^{1}\mathcal{D}_{1-p}^{\beta+l}\mathcal{J}_m^{-\mu,-\nu}(x)\mathcal{D}_{1-p}^{\beta+l}\mathcal{J}_n^{-\mu,-\nu}(x)\omega^{(\nu+l,\mu+l)}(x)dx=0, \quad \forall n\ne m. \tag{2.67}$$

We define

$$\left(\pi_N^{(-\mu,-\nu)}u-u,v_N\right)_{\omega^{(-\mu,-\nu)}}=0, \quad \forall v_N\in\mathbb{F}_N^{-\mu,-\nu}(\Lambda), \tag{2.68}$$

and for μ,ν satisfying (2.62), we denote

$$\tilde{\mathcal{B}}_{\beta,p}^{m}(\Lambda):=\{u\in L^2_{\omega^{(-\mu,-\nu)}}(\Lambda):\ \mathcal{D}_p^{\beta+l}u\in L^2_{\omega^{(\nu+l,\mu+l)}}(\Lambda),\ \text{for } -1\le l\le m\}. \tag{2.69}$$

Then, we have the approximation results for the projection errors.

Theorem 2.6. *Assume* $1 < \beta < 2$ *and let* $u \in \tilde{\mathcal{B}}^m_{\beta,p}(\Lambda)$ *with* $m \in \mathbb{N}$. *Then for a given* p, $0 \le p \le 1$, *if* $0 < \mu, \nu < \beta$, *and* μ, ν *satisfying* (2.62), *we have that, for* $-1 \le l \le m \le N$,

$$\|\mathcal{D}_p^{\beta+l}(\pi_N^{(-\mu,-\nu)}u - u)\|_{\omega^{(\nu+l,\mu+l)}} \le cN^{l-m}\|\mathcal{D}_p^{\beta+m}u\|_{\omega^{(\nu+m,\mu+m)}}, \tag{2.70}$$

and

$$\|\pi_N^{(-\mu,-\nu)}u - u\|_{\omega^{(-\mu,-\nu)}} \le cN^{-(\beta+m)}\|\mathcal{D}_p^{\beta+m}u\|_{\omega^{(\nu+m,\mu+m)}}. \tag{2.71}$$

Proof. The proof is similar to that of Theorem 2.4. □

3 Spectral methods for FDEs based on generalized Jacobi functions

In this section, we present spectral methods using the GJFs defined in the last section to solve several typical fractional differential equations.

3.1 Fractional differential equations with one-sided fractional derivative

We consider first a fractional initial value problem, followed by a fractional boundary value problem with one-sided fractional derivative.

3.1.1 A fractional initial value problem (FIVP)

As the first example, we consider the fractional initial value problem of order $s \in (0,1)$:

$${}^{C}\!D_t^s\, u(t) = f(t), \ \ t \in I := (0,T); \quad u(0) = u_0. \tag{3.1}$$

For the non-homogeneous initial conditions $u(0) = u_0$, we first decompose the solution $u(t)$ into two parts as

$$u(t) = u^h(t) + u_0, \tag{3.2}$$

with $u^h(0) = 0$. By definition, ${}^{C}\!D_t^s\, u_0 = 0$, we then derive from (2.7) that the equation (3.1) is equivalent to the following equation with Riemann-Liouville fractional derivative:

$${}^{R}\!D_t^s\, u^h(t) = f(t), \ \ t \in I := (0,T); \quad u^h(0) = 0. \tag{3.3}$$

A Petrov-Galerkin scheme for (3.3) is: find $u_N^h \in {}^{-}\mathcal{F}_N^{(-s,-s)}(I)$ (defined in (2.2.3)) such that

$$({}^R D_t^s\, u_N^h, v_N) = (f, v_N), \quad \forall\, v_N \in \mathcal{P}_N(I). \tag{3.4}$$

We expand $f(t)$ as

$$f(t) = \sum_{n=0}^{\infty} \tilde{f}_n\, \tilde{P}_n(t), \tag{3.5}$$

where $\tilde{P}_n(t) := P_n^{(0,0)}(x(t))$, $x(t) = (2t - T)/T$ is the shifted Legendre polynomial of degree n on I, and write

$$u_N^h(t) = \sum_{n=0}^{N} \tilde{u}_n^{(s)}\, {}^{-}\tilde{J}_n^{(-s,-s)}(t) \in {}^{-}\mathcal{F}_N^{(-s,-s)}(I). \tag{3.6}$$

Taking $v_N = \tilde{P}_l(t)$ in (3.4), we obtain from (2.28) and the orthogonality of Legendre polynomials that

$$\tilde{u}_n^{(s)} = \frac{n!}{\Gamma(n+s+1)} \tilde{f}_n, \quad 0 \le n \le N. \tag{3.7}$$

Therefore, we obtain the numerical solution u_N^h without solving any algebraic equation. Hence, the method is very efficient. As for the error estimate, we have the following result [5]:

Theorem 3.7. *Let u^h and u_N^h be the solution of* (3.3) *and* (3.4), *respectively. Then*

$$\|{}^R D_t^s (u^h - u_N^h)\| \le cN^{-m} \|f^{(m)}\|_{\omega^{(m-1,m-1)}}. \tag{3.8}$$

Proof. Let ${}^{-}\pi_N^{(-s,-s)} u^h$ be as defined in (2.35) for $0 < s < 1$. By (2.28), we have

$$\big({}^R D_t^s ({}^{-}\pi_N^{(-s,-s)} u^h - u^h), \psi\big) = 0, \quad \forall\, \psi \in \mathcal{P}_N.$$

Then by (3.3),

$$\big(f - {}^R D_t^s\ {}^{-}\pi_N^{(-s,-s)} u^h, \psi\big) = \big({}^R D_t^s u^h - {}^R D_t^s\ {}^{-}\pi_N^{(-s,-s)} u^h, \psi\big) = 0, \quad \forall\, \psi \in \mathcal{P}_N.$$

Let $\pi_N f$ be the L^2-orthogonal projection of f upon $\mathcal{P}_N$. We infer from the above that ${}^R D_t^s\ {}^{-}\pi_N^{(-s,-s)} u^h = \pi_N f = {}^R D_t^s\, u_N^h$. Therefore,

$$\begin{aligned} \|{}^R D_t^s (u^h - u_N^h)\| &= \big\|{}^R D_t^s (u^h - {}^{-}\pi_N^{(-s,-s)} u^h)\big\| \\ &\le \big\|{}^R D_t^s (u^h - {}^{-}\pi_N^{(-s,-s)} u^h)\big\| + \|\pi_N f - f\|. \end{aligned} \tag{3.9}$$

It follows from Theorem 2.2 (with $\alpha = -\beta = s$ and $0 < s < 1$), and the Legendre approximation results (see, e.g., [29, Ch. 3]) that

$$\|{}^R D_t^s (u^h - u_N^h)\| \le cN^{-m} \big(\|{}^R D_t^{s+m} u^h\|_{\omega^{(m,m)}} + \|f^{(m)}\|_{\omega^{(m-1,m-1)}}\big). \tag{3.10}$$

We deduce from (3.1) that

$$\big\|{}^{R}D_t^{s+m}u^h\big\|_{\omega^{(m,m)}} \le c\big\|f^{(m)}\big\|_{\omega^{(m-1,m-1)}}.$$

□

3.1.2 One sided fractional boundary value problems

Now we consider an one-sided fractional boundary value problem

$$ {}_x^RD^{\nu}\, u(x) = f(x), \;\; x \in \Lambda = (-1,1); \quad u(\pm 1) = 0, \tag{3.11}$$

where $\nu \in (1,2)$.

Let $s = \nu - 1$ and introduce the solution and test function spaces:

$$\begin{aligned} U &:= \big\{u \in L^2_{\omega^{(-s,-1)}}(\Lambda) \;:\; {}_x^RD^s\, u \in L^2_{\omega^{(0,s-1)}}(\Lambda)\big\};\\ V &:= \big\{v \in L^2_{\omega^{(-1,-s)}}(\Lambda) \;:\; Dv \in L^2_{\omega^{(0,1-s)}}(\Lambda)\big\}, \end{aligned} \tag{3.12}$$

equipped with the norms

$$\begin{aligned} \|u\|_U &= \big(\|u\|^2_{\omega^{(-s,-1)}} + \|{}_x^RD^s u\|^2_{\omega^{(0,s-1)}}\big)^{1/2};\\ \|v\|_V &= \big(\|v\|^2_{\omega^{(-1,-s)}} + \|Dv\|^2_{\omega^{(0,1-s)}}\big)^{1/2}. \end{aligned} \tag{3.13}$$

For $u \in U$ and $v \in V$, we write

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} \hat{u}_n\, {}^{+}J_n^{(-s,-1)}(x) = (1-x)^s(1+x)\sum_{n=1}^{\infty}\tilde{u}_n P_{n-1}^{(s,1)}(x),\\ v(x) &= \sum_{n=1}^{\infty} \hat{v}_n\, {}^{-}J_n^{(-1,-s)}(x) = (1-x)(1+x)^s\sum_{n=1}^{\infty}\tilde{v}_n P_{n-1}^{(1,s)}(x). \end{aligned} \tag{3.14}$$

With the above setup, we can build in the homogenous boundary conditions and also perform fractional integration by parts. Hence, a weak form of (3.11) is to find $u \in U$ such that

$$a(u,v) := ({}_x^RD^s u,\, Dv) = (f,v), \quad \forall\, v \in V. \tag{3.15}$$

Let ${}^{+}\mathcal{F}_N^{(-s,-1)}(\Lambda)$ and ${}^{-}\mathcal{F}_N^{(-1,-s)}(\Lambda)$ be the finite-dimensional spaces as defined in the previous section. Then the GJF-Petrov-Galerkin scheme for (3.15) is to find $u_N \in {}^{+}\mathcal{F}_N^{(-s,-1)}(\Lambda)$ such that

$$a(u_N, v_N) = ({}_x^RD^s u_N,\, Dv_N) = (f, v_N), \quad \forall\, v_N \in {}^{-}\mathcal{F}_N^{(-1,-s)}(\Lambda). \tag{3.16}$$

We derive from (2.27) and ${}^{R}D_x^1 = \frac{d}{dx}$ that

$$a({}^{+}J_n^{(-s,-1)}(x), {}^{-}J_m^{(-1,-s)}(x)) = C_{n,m}^{s}({}^{+}J_n^{(0,s-1)}(x), {}^{-}J_m^{(0,1-s)}(x)) = 0 \quad \forall n \neq m. \tag{3.17}$$

Hence, we can obtain u_N directly without solving any algebraic equation.

To characterise the regularity of u, we define for any $m \in \mathbb{N}_0$,

$${}^{+}\mathcal{B}_{\alpha,\beta}^{m}(\Lambda) := \big\{u \in L^2_{\omega^{(-\alpha,\beta)}}(\Lambda) \,:\, {}_{x}^{R}D^{\alpha+l}u \in L^2_{\omega^{(l,\alpha+\beta+l)}}(\Lambda) \ \text{ for } \ 0 \le l \le m\big\}. \tag{3.18}$$

Theorem 3.8. *Let $s \in (0,1)$, and let u and u_N be the solutions of* (3.15) *and* (3.16)*, respectively. If $u \in U \cap {}^{+}\mathcal{B}_{s,-1}^{m}(\Lambda)$ with $0 \le m \le N$, then we have the error estimates:*

$$\|u - u_N\|_U \le cN^{-m}\big\|{}_{x}^{R}D^{s+m}u\big\|_{\omega^{(m,s-1+m)}}. \tag{3.19}$$

In particular, if $f^{(m-1)} \in L^2_{\omega^{(m,s-1+m)}}(\Lambda)$ for $m \ge 1$, we have

$$\|u - u_N\|_U \le cN^{-m}\big\|f^{(m-1)}\big\|_{\omega^{(m,s-1+m)}}. \tag{3.20}$$

Here, c is a positive constant independent u, N and m.

Proof. We derive from (3.15) and (3.16) that

$$a(u - u_N, v_N) = 0 \quad \forall\, v_N \in {}^{-}\mathcal{F}_N^{(-1,-s)}(\Lambda),$$

which, along with (3.17), implies immediately that $u_N = {}^{+}\pi_N^{(-s,-1)}u$. Hence, the desired results follow from Theorem 4.1 in [5] and the fact ${}_{x}^{R}D^{\nu}u = {}_{x}^{R}D^{s+1}u = f$. $\square$

3.2 Fractional boundary value problems with Riesz derivatives

We consider first the so called Riesz fractional differential equations, followed by a more general case with a zeroth-order term.

3.2.1 Riesz fractional differential equations

We consider the Riesz fractional equation of order $2\alpha \in (2k-1, 2k)$ with $k \in \mathbb{N}$:

$$\begin{aligned}(-1)^k D^{2\alpha}u(x) =& f(x), \quad x \in \Lambda,\\ u^{(l)}(\pm 1) =& 0, \quad l = 0, 1, \cdots, k-1.\end{aligned} \tag{3.21}$$

A Petrov-Galerkin spectral method for (3.21) is: Find $u_N \in \mathbb{F}_N^{-\alpha,-\alpha}$ such that

$$(-1)^k (D^{2\alpha} u_N, v_N)_{\omega^{(\alpha,\alpha)}} = (f, v_N)_{\omega^{(\alpha,\alpha)}}, \quad \forall v_N \in P_N. \tag{3.22}$$

The solution to this discrete problem can be found directly as follows. Given

$$f(x) = \sum_{m=0}^{\infty} f_m P_m^{(\alpha,\alpha)}(x), \tag{3.23}$$

and write

$$u_N(x) = \sum_{n=0}^{N} \hat{u}_n \mathcal{J}_n^{-\alpha,-\alpha}(x). \tag{3.24}$$

Plugging the above in (3.22), using (2.51) and the orthogonality of $\{P_m^{(\alpha,\alpha)}\}$ in $L^2_{\omega^{(\alpha,\alpha)}}(\Lambda)$, we find

$$\hat{u}_n = f_n \Big/ \Big(2\cos(\pi\alpha) \frac{\Gamma(n+1+2\alpha)}{n!} \Big), \quad \forall\, 0 \le n \le N. \tag{3.25}$$

As for the error estimate, we have

Theorem 3.9. *Assuming $f^{(j)} \in L^2_{\omega^{\alpha+j,\alpha+j}}(\Lambda)$ for $0 \le j \le m$, we have*

$$\|u - u_N\|_{\omega^{(-\alpha,-\alpha)}} \le cN^{-2\alpha-m} \|f^{(m)}\|_{\omega^{(\alpha+m,\alpha+m)}}. \tag{3.26}$$

$$\|D^{2\alpha}(u - u_N)\|_{\omega^{(\alpha,\alpha)}} \le cN^{-m} \|f^{(m)}\|_{\omega^{(\alpha+m,\alpha+m)}}. \tag{3.27}$$

Proof. It is clear from (3.25) that $u_N = \pi_N^{(-\alpha,-\alpha)} u$. Hence, by (2.57), we have

$$\|u - u_N\|_{\omega^{(-\alpha,-\alpha)}} = \|u - \pi_N^{(-\alpha,-\alpha)} u\|_{\omega^{(-\alpha,-\alpha)}} \le cN^{-2\alpha-m} \|D^{2\alpha+m} u\|_{\omega^{(\alpha+m,\alpha+m)}}.$$

On the other hand, we derive from (2.56) with $k = l = 0$ that

$$\|D^{2\alpha} u - u_N\|_{\omega^{(\alpha,\alpha)}} = \|D^{2\alpha}(u - \pi_N^{(-\alpha,-\alpha)} u)\|_{\omega^{(\alpha,\alpha)}} \le cN^{-m} \|D^{2\alpha+m} u\|_{\omega^{(\alpha+m,\alpha+m)}}.$$

Since $D^{2\alpha+m} u = f^{(m)}$, we obtain the desired results from the above two inequalities. □

3.2.2 A more general case

In the previous examples, we have developed optimal spectral methods using GJFs in the sense that (i) the numerical solution can be determined directly without solving any algebraic equation; and (ii) the error converges faster than any algebraic rate as long as the righthand side function f is smooth, despite the fact that the solution is weakly singular at the endpoint(s).

However, it is not possible to construct such optimal spectral methods for more general FDEs. Nevertheless, using proper GJFs still allows us to (i) deal with fractional derivatives efficiently, and (ii) resolve the leading singular term in the solution. Consider for example

$$\begin{aligned} &\rho u(x) - D^{2\alpha}u(x) = f, \quad x \in \Lambda = (-1,1),\\ &u(\pm 1) = 0, \end{aligned} \tag{3.28}$$

where $\rho > 0$ and $2\alpha \in (1,2)$. A GJF spectral Galerkin approximation to (3.28) is: find $u_N \in \mathbb{F}_N^{-\alpha,-\alpha}(\Lambda)$ such that

$$a(u_N, v_N) := \rho(u_N, v_N) - (D^{2\alpha}u_N, v_N) = (f, v_N), \qquad \forall v_N \in \mathbb{F}_N^{-\alpha,-\alpha}(\Lambda). \tag{3.29}$$

Setting

$$u_N(x) = \sum_{n=0}^{N} \tilde{u}_n \mathcal{J}_n^{-\alpha,-\alpha}(x). \tag{3.30}$$

Setting

$$\begin{aligned} &u_N(x) = \sum_{n=0}^{N} \tilde{u}_n \mathcal{J}_n^{-\alpha,-\alpha}(x),\ U = (\tilde{u}_0, \tilde{u}_1, \cdots, \tilde{u}_N),\\ &m_{jk} = (\mathcal{J}_k^{-\alpha,-\alpha}(x), \mathcal{J}_j^{-\alpha,-\alpha}(x)), \quad s_{jk} = -(D^{2\alpha}\mathcal{J}_k^{-\alpha,-\alpha}(x), \mathcal{J}_j^{-\alpha,-\alpha}(x)),\\ &M = (m_{jk}),\ S = (s_{jk}),\ \tilde{f}_j = (f, \mathcal{J}_j^{-\alpha,-\alpha}(x)), \quad F = (\tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_N)^T. \end{aligned} \tag{3.31}$$

Then, (3.29) reduces to the following matrix system:

$$(\rho M + S)U = F.$$

We recall from (2.50), (2.14) that S is a diagonal matrix

$$s_{jk} = \frac{2^{2\alpha+1}\Gamma(k+\alpha+1)^2}{(k!)^2(2k+2\alpha+1)}\delta_{jk}. \tag{3.32}$$

The mass matrix M is full but its entries can be evaluated exactly (cf. [4]):

$$\begin{aligned} m_{jk} &= \int_{-1}^{1} (1-x^2)^\alpha P_j^{(\alpha,\alpha)}(x)(1-x^2)^\alpha P_k^{(\alpha,\alpha)}(x)dx\\ &= \frac{(-1)^{\frac{j-k}{2}}(j+k)!}{2^{j+k}j!k!} \frac{(-j-\alpha)_{\frac{j+k}{2}}(-k-\alpha)_{\frac{j+k}{2}}}{(2\alpha+\frac{3}{2})_{\frac{j+k}{2}}\frac{j+k}{2}!} \frac{\sqrt{\pi}\Gamma(2\alpha+1)}{\Gamma(2\alpha+\frac{3}{2})}, \end{aligned} \tag{3.33}$$

where the Pochhammer symbol $(a)_\nu = \frac{\Gamma(a+\nu)}{\Gamma(a)}$.

Lemma 3.7. *For all $u \in \{u \in L^2_{\omega^{-\alpha,-\alpha}}(\Lambda) : D^{2\alpha}u \in L^2(\Lambda)\}$, we have*

$$\|u\|^2_{\omega^{(-\alpha,-\alpha)}} \leq -(D^{2\alpha}u, u). \tag{3.34}$$

Proof. We write $u = \sum_{i=0}^{\infty} \tilde{u}_i \mathcal{J}_i^{-\alpha,-\alpha}(x)$, so that

$$\|u\|^2_{\omega^{(-\alpha,-\alpha)}} = \sum_{i=0}^{\infty} \tilde{u}_i^2 \gamma_i^{(\alpha,\alpha)}.$$

By Theorem 2.3 and (2.14), we have

$$\begin{aligned}
-(D^{2\alpha}u, u) &= (-\sum_{i=0}^{\infty} \tilde{u}_i D^{2\alpha} \mathcal{J}_i^{-\alpha,-\alpha}, \sum_{j=0}^{\infty} \tilde{u}_j \mathcal{J}_j^{-\alpha,-\alpha}) \\
&= (\sum_{i=0}^{\infty} \tilde{u}_i \frac{\Gamma(i+2\alpha+1)}{i!} P_i^{(\alpha,\alpha)}, \sum_{j=0}^{\infty} \tilde{u}_j P_j^{(\alpha,\alpha)})_{\omega^{(\alpha,\alpha)}} \\
&= \sum_{i=0}^{\infty} \frac{\Gamma(i+2\alpha+1)}{i!} \tilde{u}_i^2 \gamma_i^{(\alpha,\alpha)}.
\end{aligned}$$

We can easily to get the desired result (3.34) by comparing the above results. □

We define the energy norm associated with (3.28) by

$$\|u\|^2_{B^\alpha} = \rho(u, u) - (D^{2\alpha}u, u). \tag{3.35}$$

Theorem 3.10. *Assume $2\alpha \in (1, 2)$. Let u and u_N be the solution of* (3.28) *and* (3.29)*, then we have*

$$\|u - u_N\|_{B^\alpha} \leq N^{1-m} \|D^{2\alpha-1+m}u\|_{\omega^{(\alpha-1+m,\alpha-1+m)}}. \tag{3.36}$$

Proof. Let us denote $\widetilde{u}_N := \pi_N^{(-\alpha,-\alpha)}u$ and $e_N := \widetilde{u}_N - u_N$. We derive from (3.28) and (3.29) that

$$\begin{aligned}
a(e_N, v_N) &= \rho(e_N, v_N) - (D^{2\alpha}e_N, v_N) \\
&= \rho(\widetilde{u}_N - u, v_N) - (D^{2\alpha}(\widetilde{u}_N - u), v_N), \ \forall v \in \mathbb{F}_N^{-\alpha,-\alpha}(\Lambda).
\end{aligned}$$

Taking $v_N = e_N$ in the above, we derive from (3.35) and Cauchy-Schwarz inequality that

$$\begin{aligned}
\|e_N\|^2_{B^\alpha} &:= \rho(e_N, e_N) - (D^{2\alpha}e_N, e_N) \leq \rho \|\pi_N^{(-\alpha,-\alpha)}u - u\|_{\omega^{(\alpha,\alpha)}} \|e_N\|_{\omega^{(-\alpha,-\alpha)}} \\
&\quad + \|D^{2\alpha}(\pi_N^{(-\alpha,-\alpha)}u - u)\|_{\omega^{(\alpha,\alpha)}} \|e_N\|_{\omega^{(-\alpha,-\alpha)}}.
\end{aligned}$$

The above and Lemma 3.7 lead to

$$\|e_N\|_{B^\alpha} \leq \rho \|\pi_N^{(-\alpha,-\alpha)}u - u\|_{\omega^{(-\alpha,-\alpha)}} + \|D^{2\alpha}(\pi_N^{(-\alpha,-\alpha)}u - u)\|_{\omega^{(\alpha,\alpha)}}.$$

Then, by Theorem 2.4, the right hand side terms of (3.39) can be estimate as

$$\|\pi_N^{(-\alpha,-\alpha)}u - u\|_{\omega^{(-\alpha,-\alpha)}} \le cN^{1-(2\alpha+m)}\|D^{2\alpha-1+m}u\|_{\omega^{(\alpha-1+m,\alpha-1+m)}}, \quad (3.37)$$

and

$$\|D^{2\alpha}(\pi_N^{(-\alpha,-\alpha)}u - u)\|_{\omega^{(\alpha,\alpha)}} \le cN^{1-m}\|D^{2\alpha-1+m}u\|_{\omega^{(\alpha-1+m,\alpha-1+m)}}, \quad (3.38)$$

which implied that

$$\|e_N\|_{B^\alpha} \le cN^{1-m}\|D^{2\alpha-1+m}u\|_{\omega^{(\alpha-1+m,\alpha-1+m)}}. \quad (3.39)$$

On the other hand,

$$\begin{aligned}
&-(D^{2\alpha}(u-\pi_N^{(-\alpha,-\alpha)}u), u-\pi_N^{(-\alpha,-\alpha)}u) \\
&\le \|D^{2\alpha}(\pi_N^{(-\alpha,-\alpha)}u-u)\|_{\omega^{(\alpha,\alpha)}}\|u-\pi_N^{(-\alpha,-\alpha)}u\|_{\omega^{(-\alpha,-\alpha)}}.
\end{aligned}$$

Since

$$\|u-\pi_N^{(-\alpha,-\alpha)}u\| \le \|u-\pi_N^{(-\alpha,-\alpha)}u\|_{\omega^{(-\alpha,-\alpha)}}.$$

We find from the above that

$$\|u-\pi^{(-\alpha,-\alpha)}u\|_{B^\alpha} \le \rho\|u-\pi_N^{(-\alpha,-\alpha)}u\|_{\omega^{(-\alpha,-\alpha)}} + \|D^{2\alpha}(\pi_N^{(-\alpha,-\alpha)}u-u)\|_{\omega^{(\alpha,\alpha)}}.$$

Finally, since $u - u_N = u - \tilde{u}_N + e_N$, combining the above with (3.39), (3.37) and (3.38), we obtain the desired result. □

Remark 3.1. We emphasize that unlike in previous examples, here we can not easily bound the errors in terms of f. In particular, the smoothness of f does not imply that the righthand side of (3.45) is bounded for any m. Hence, only an algebraic convergence rate can be achieved in this case.

3.3 Two-sided fractional differential equations with different coefficients

We consider the two-sided fractional differential equation

$$\begin{aligned}
\mathcal{D}_p^\beta u(x) := C_{\beta,p}(p\,{}^R D_x^\beta u(x) + (1-p)\,{}_x^R D^\beta u(x)) = f(x), \quad x\in\Lambda,& \\
u(\pm 1) = 0,&
\end{aligned} \quad (3.40)$$

where $1<\beta<2$, $0\le p\le 1$, $C_{\beta,p}$ defined in (2.4), and $f(x)$ is a given function.

Let μ, ν satisfy (2.62) and $0 < \mu, \nu < \beta$. A Petrov-Galerkin spectral method for (3.40) is: Find $u_N \in \mathbb{F}_N^{-\mu,-\nu}$ such that

$$(\mathcal{D}_p^\beta u_N, v_N)_{\omega^{(\nu,\mu)}} = (f, v_N)_{\omega^{(\nu,\mu)}}, \quad \forall v_N \in \mathcal{P}_N. \tag{3.41}$$

The solution to this discrete problem can be found directly as follows. We expand $f(x)$ as

$$f(x) = \sum_{m=0}^{\infty} f_m P_m^{(\nu,\mu)}(x), \tag{3.42}$$

and write

$$u_N(x) = \sum_{n=0}^{N} \hat{u}_n \mathcal{J}_n^{-\mu,-\nu}(x). \tag{3.43}$$

Plugging (3.43) and (3.42) in (3.41), using (2.65) and the orthogonality of $\{P_m^{(\nu,\mu)}\}$ in $L^2_{\omega^{(\nu,\mu)}}(\Lambda)$, we find

$$\hat{u}_n = f_n \Big/ \left(\frac{\Gamma(n+1+\beta)}{n!} \right), \quad \forall\, 0 \le n \le N. \tag{3.44}$$

As for the error estimate, we have

Theorem 3.11. *Assuming $f^{(j)} \in L^2_{\omega^{(\nu+j,\mu+j)}}(\Lambda)$ for $0 \le j \le m$, let u and u_N be the solution of* (3.40) *and* (3.41)*, then we have*

$$\|u - u_N\|_{\omega^{(-\mu,-\nu)}} \le cN^{-\beta-m} \|f^{(m)}\|_{\omega^{(\nu+m,\mu+m)}}. \tag{3.45}$$

$$\|\mathcal{D}_p^\beta (u - u_N)\|_{\omega^{(\nu,\mu)}} \le cN^{-m} \|f^{(m)}\|_{\omega^{(\nu+m,\mu+m)}}. \tag{3.46}$$

Proof. It is clear from (3.44) that $u_N = \pi_N^{(-\mu,-\nu)} u$. Hence, by (2.71), we have

$$\|u - u_N\|_{\omega^{(-\mu,-\nu)}} = \|u - \pi_N^{(-\mu,-\nu)} u\|_{\omega^{(-\mu,-\nu)}} \le cN^{-\beta-m} \|\mathcal{D}_p^{\beta+m} u\|_{\omega^{(\nu+m,\mu+m)}}.$$

On the other hand, we derive from (2.70) with $l = 0$ that

$$\|\mathcal{D}_p^\beta (u - u_N)\|_{\omega^{(\nu,\mu)}} = \|\mathcal{D}_p^\beta (u - \pi_N^{(-\mu,-\nu)} u)\|_{\omega^{(\nu,\mu)}} \le cN^{-m} \|\mathcal{D}_p^{\beta+m} u\|_{\omega^{(\nu+m,\mu+m)}}.$$

Since $\mathcal{D}_p^{\beta+m} u = f^{(m)}$, we obtain the desired results from the above. □

3.4 Space-time fractional differential equations

As the last example, we consider

$$\begin{aligned}
&{}^C D_t^\alpha u(x,t) - D^{2\beta} u(x,t) = f(x,t), \quad \forall (x,t) \in Q = \Lambda \times I,\\
&u(x,t)|_{\partial\Lambda} = 0, \quad \forall t \in I := (0 < T),\\
&u(x,0) = u_0(x) \quad \forall x \in \Lambda,
\end{aligned} \tag{3.47}$$

where $\alpha \in (0,1)$, $2\beta \in (1,2)$. To deal with the non-homogeneous initial condition, we first decompose the solution $u(x,t)$ into two parts as

$$u(x,t) = u^h(x,t) + u_0(x), \tag{3.48}$$

with $u^h(x,0) = 0$. Hence, the equation (3.47) is equivalent to the following with Riemann-Liouville fractional derivative:

$$\begin{aligned} {}^R D_t^\alpha u^h(x,t) - D^{2\beta} u^h(x,t) &= g(x,t), \qquad \forall (x,t) \in Q, \\ u^h(x,0) &= 0, \qquad \forall x \in \Lambda, \\ u^h(x,t)|_{\partial\Lambda} &= 0, \qquad \forall t \in I, \end{aligned} \tag{3.49}$$

where

$$g(x,t) = f(x,t) + D^{2\beta} u_0(x).$$

We consider the following weak formulation of (3.49): find $u \in X$, such that

$$\mathcal{A}(u^h, v) := ({}^R D_t^\alpha u^h, v)_Q - (D^{2\beta} u^h, v)_Q = (g, v)_Q, \quad \forall v \in Y, \tag{3.50}$$

and describe a space-time spectral method for solving the above equation.

For the time variable, we shall use the shifted fractional polynomial space. Let $x(t) = 2t/T - 1$. We denote $\tilde{P}_n^{(\alpha,\beta)}(t) = P_n^{(\alpha,\beta)}(x(t))$, ${}^-\tilde{J}_n^{(-\alpha,-\alpha)}(t) := t^\alpha \tilde{P}_n^{(-\alpha,\alpha)}(t)$, and

$${}^-\mathcal{F}_N^{(-\alpha,-\alpha)}(I) = \text{span}\{{}^-\tilde{J}_n^{(-\alpha,-\alpha)}(t) : 0 \le n \le N\}. \tag{3.51}$$

For the space variable, we shall use $\mathbb{F}_M^{-\beta,-\beta}$ defined in (2.48). Then, the Petrov-Galerkin method for (3.49) is: find $u_L^h(x,t) := u_{MN}^h \in \mathbb{F}_M^{-\beta,-\beta} \otimes {}^-\mathcal{F}_N^{(-\alpha,-\alpha)}$ such that

$$\mathcal{A}(u_L^h, v) := ({}^R D_t^\alpha u_L^h, v)_Q - (D^{2\beta} u_L^h, v)_Q = (g, v)_Q, \quad \forall v \in \mathbb{F}_M^{-\beta,-\beta} \otimes \mathcal{P}_N. \tag{3.52}$$

We first describe an efficient algorithm for solving (3.52) similar to the one used in [23]. We write

$$u_L^h(x,t) = \sum_{m=0}^{M} \sum_{n=0}^{N} \tilde{u}_{mn}^h \mathcal{J}_m^{-\beta,-\beta}(x) {}^-\tilde{J}_n^{(-\alpha,-\alpha)}(t). \tag{3.53}$$

For the test function, we take $v_L = \mathcal{J}_p^{-\beta,-\beta}(x) L_q^{(\alpha)}(t)$ where $L_q^{(\alpha)}(t) := \kappa_{q,\alpha} \tilde{P}_q^{(0,0)}(t)$ with $\kappa_{q,\alpha} = \dfrac{q!(2q+1)}{T \cdot \Gamma(q+\alpha+1)}$. Substituting the above in into (3.52), we obtain

$$\begin{aligned} \sum_{m=0}^{M} \sum_{n=0}^{N} \tilde{u}_{mn}^h \Big\{ &(\mathcal{J}_m^{-\beta,-\beta}, \mathcal{J}_p^{-\beta,-\beta}) \, ({}^R D_t^\alpha {}^-\tilde{J}_n^{(-\alpha,-\alpha)}, L_q^{(\alpha)}) \\ &- (D^{2\beta} \mathcal{J}_m^{-\beta,-\beta}, \mathcal{J}_p^{-\beta,-\beta}) \, ({}^-\tilde{J}_n^{(-\alpha,-\alpha)}, L_q^{(\alpha)}) \Big\} \\ = (g, \mathcal{J}_p^{-\beta,-\beta} L_q^{(\alpha)})_Q. \end{aligned} \tag{3.54}$$

Denote

$$g_{pq} = (g, \mathcal{J}_p^{-\beta,-\beta}(x) L_q^{(\alpha)}(t))_Q, \quad G = (g_{pq})_{0\le p\le M, 0\le q\le N},$$
$$s_{pq}^t = \int_I {}^R D_t^{\alpha} {}^{-}\tilde{J}_q^{(-\alpha,-\alpha)}(t) L_p^{(\alpha)}(t)dt, \qquad m_{pq}^t = \int_I {}^{-}\tilde{J}_q^{(-\alpha,-\alpha)}(t) L_p^{(\alpha)}(t)dt,$$
$$U = (\tilde{u}_{mn}^h)_{0\le m\le M, 0\le n\le N}, \quad \boldsymbol{S}^t = (s_{pq}^t)_{0\le p,q\le N}, \qquad \boldsymbol{M}^t = (m_{pq}^t)_{0\le p,q\le N}.$$

Note that $\boldsymbol{S}^t$ is a diagonal matrix, $\boldsymbol{M}^t$ is not sparse but its entries can be accurately computed by Jacobi-Gauss quadrature with index $(0, \alpha)$.

Then, from (3.4), we find that (3.52) is equivalent to the following linear system:

$$\boldsymbol{M}^x U (\boldsymbol{S}^t)^T + \boldsymbol{S}^x U (\boldsymbol{M}^t)^T = G, \tag{3.55}$$

where $\boldsymbol{M}^x$ and $\boldsymbol{S}^x$ are the mass and stiffness matrix in the x-direction defined in (3.31). $\boldsymbol{S}^x$ is a diagonal matrix and $\boldsymbol{M}^x$ is full but symmetric. The linear system (3.55) can be solved efficiently by using the matrix diagonalization method [26]. Indeed, let $\boldsymbol{E} := (\bar{e}_0, \cdots, \bar{e}_N) = (e_{pq})_{p,q=0,\cdots,N}$ be the matrix formed by the orthonormal eigenvectors of the generalized eigenvalue problem $\boldsymbol{M}^x \bar{e}_j = \lambda_j \boldsymbol{S}^x \bar{e}_j$ and $\Lambda = \mathrm{diag}(\lambda_0, \cdots, \lambda_N)$, i.e.,

$$\boldsymbol{M}^x \boldsymbol{E} = \boldsymbol{S}^x \boldsymbol{E} \Lambda. \tag{3.56}$$

Setting $U = \boldsymbol{E} V$, and multiplying both sides of (3.55) by $(\boldsymbol{S}^x \boldsymbol{E})^{-1} = \boldsymbol{E}^T \boldsymbol{S}^x$, we arrive at

$$\Lambda V (\boldsymbol{S}^t)^T + V (\boldsymbol{M}^t)^T = H := \boldsymbol{E}^T \boldsymbol{S}^x G. \tag{3.57}$$

Hence, let $\mathbf{v}_m$ and $\mathbf{h}_m$ be the m-th row of V and H, respectively, the above matrix equation becomes:

$$(\lambda_m \boldsymbol{S}^t + \boldsymbol{M}^t)\mathbf{v}_m = \mathbf{h}_m, \qquad 0 \le m \le M, \tag{3.58}$$

which we solve directly with LU decomposition. Once we obtain V, we set $U = \boldsymbol{E} V$. Finally, we obtain the numerical solutions of (3.47) by $u_L = u_L^h + u_0$.

We now turn to the error estimate. Thanks to (3.34), we can define the following norm:

$$\|v\|_{X^{\alpha,\beta}(Q)} := \left(\|{}^R D_t^{\alpha} v\|_Q^2 - (D^{2\beta} v, v)_Q \right)^{1/2}.$$

Theorem 3.12. *Let u^h and u_L^h be the solutions of* (3.50) *and* (3.52), *respectively. Then we have the following error estimates:*

$$
\begin{aligned}
&\|u-u_L\|_{X^{\alpha,\beta}(Q)}\\
&\lesssim M^{1-(2\beta+m)}\|D^{2\beta-1+m}({}^R D_t^{\alpha}u)\|_{L^2_{\omega^{(\beta-1+m,\beta-1+m)}}(\Lambda;L^2(I))}\\
&\quad+N^{-(\alpha+n)}\big\|{}^R D_t^{\alpha+n}(D^{2\beta}u)\big\|_{L^2(\Lambda;L^2_{\omega^{(n,n)}}(I))}\\
&\quad+M^{1-m}\|D^{2\beta-1+m}u\|_{L^2_{\omega^{(\beta-1+m,\beta-1+m)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\
&\quad+N^{-n}\big\|{}^R D_t^{\alpha+n}u\big\|_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2_{\omega^{(n,n)}}(I))}.
\end{aligned}
$$

Proof. Let us denote $\tilde u_L^h := {}^-\pi_N^{(-\alpha,-\alpha)}\,\pi_M^{(-\beta,-\beta)}u^h = \pi_M^{(-\beta,-\beta)}\,{}^-\pi_N^{(-\alpha,-\alpha)}u^h$ and $e_L := \tilde u_L^h - u_L^h$. We derive from (3.49) and (3.52) that for all $\forall v_L \in \mathbb{F}_M^{-\beta,-\beta}\otimes\mathcal{P}_N$, we have

$$
\begin{aligned}
b(e_L,v_L) :&= ({}^R D_t^{\alpha}e_L,v)_Q-(D^{2\beta}e_L,v_L)_Q\\
&=({}^R D_t^{\alpha}(\tilde u_L^h-u^h),v_L)_Q-(D^{2\beta}(\tilde u_L^h-u^h),v_L)_Q\\
&=\big({}^R D_t^{\alpha}(\pi_M^{(-\beta,-\beta)}u^h-u^h),v_L\big)_Q-(D^{2\beta}({}^-\pi_N^{(-\alpha,-\alpha)}u^h-u^h),v_L)_Q.
\end{aligned}
$$

Taking $v_L = {}^R D_t^{\alpha}e_L$ $(\in \mathbb{F}_M^{-\beta,-\beta}\otimes\mathcal{P}_N)$ in the above equation, we obtain that

$$
\begin{aligned}
&({}^R D_t^{\alpha}e_L,{}^R D_t^{\alpha}e_L)_Q-(D^{2\beta}e_L,{}^R D_t^{\alpha}e_L)_Q\\
&=({}^R D_t^{\alpha}(\pi_M^{(-\beta,-\beta)}u^h-u^h),{}^R D_t^{\alpha}e_L)_Q-(D^{2\beta}({}^-\pi_N^{(-\alpha,-\alpha)}u^h-u^h),{}^R D_t^{\alpha}e_L)_Q.
\end{aligned}
\tag{3.59}
$$

Thanks to the generalized Poincaŕe inequality [8]:

$$\|u\|_{L^2(I)}\le c\|{}^R D_t^{\alpha}u\|_{L^2(I)},$$

we derive from Lemmas 2.2-2.3, (3.59) that

$$
\begin{aligned}
(D^{2\beta}e_L,e_L)_Q &\lesssim (D^{2\beta}({}^R D_t^{\frac{\alpha}{2}}e_L),{}^R D_t^{\frac{\alpha}{2}}e_L\big)_Q\\
&\cong (D^{2\beta}({}_t^R D^{\frac{\alpha}{2}}e_L),{}^R D_t^{\frac{\alpha}{2}}e_L\big)_Q=(D^{2\beta}e_L,{}^R D_t^{\alpha}e_L)_Q,
\end{aligned}
\tag{3.60}
$$

This, along with equation (3.59), yields

$$
\begin{aligned}
\|{}^R D_t^{\alpha}e_L\|_Q^2+(D^{2\beta}e_L,e_L)_Q &\le \|{}^R D_t^{\alpha}(\pi_M^{(-\beta,-\beta)}u^h-u^h)\|_Q\,\|{}^R D_t^{\alpha}e_L\big\|_Q\\
&\quad+\|D^{2\beta}({}^-\pi_N^{(-\alpha,-\alpha)}u^h-u^h)\|_Q\,\|{}^R D_t^{\alpha}e_L\|_Q,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\|e_L\|^2_{X^{\alpha,\beta}(Q)}:=\|{}^R D_t^{\alpha}e_L\|_Q^2+(D^{2\beta}e_L,e_L)_Q\\
&\lesssim \|{}^R D_t^{\alpha}(\pi_M^{(-\beta,-\beta)}u^h-u^h)\|_Q^2+\|D^{2\beta}({}^-\pi_N^{(-\alpha,-\alpha)}u^h-u^h)\|_Q^2.
\end{aligned}
$$

The two terms at the right-hand side can be bounded by using Lemma 2.2 and Lemma 2.4 as follows:

$$\begin{aligned}&\|{}^R D_t^{\alpha}(\pi_M^{(-\beta,-\beta)}u^h - u^h)\|_Q \lesssim \|{}^R D_t^{\alpha}(\pi_M^{(-\beta,-\beta)}u^h - u^h)\|_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2(I))}\\ &\lesssim M^{1-(2\beta+m)}\|D^{2\beta-1+m}({}^R D_t^{\alpha}u)\|_{L^2_{\omega^{(\beta-1+m,\beta-1+m)}}(\Lambda;L^2(I))},\end{aligned}$$

and

$$\begin{aligned}&\|D^{2\beta}({}^-\pi_N^{(-\alpha,-\alpha)}u^h - u^h)\|_Q \lesssim \|D^{2\beta}({}^-\pi_N^{(-\alpha,-\alpha)}u^h - u^h)\|_{L^2(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &\lesssim N^{-(\alpha+n)}\left\|{}^R D_t^{\alpha+n}(D^{2\beta}u)\right\|_{L^2(\Lambda;L^2_{\omega^{(n,n)}}(I))}.\end{aligned}$$

Combining the above estimates, we arrive at

$$\begin{aligned}\|e_L\|_{X^{\alpha,\beta}(Q)} \lesssim & M^{1-(2\beta+m)}\|D^{2\beta-1+m}({}^R D_t^{\alpha}u)\|_{L^2_{\omega^{(\beta-1+m,\beta-1+m)}}(\Lambda;L^2(I))}\\ &+ N^{-(\alpha+n)}\left\|{}^R D_t^{\alpha+n}(D^{2\beta}u)\right\|_{L^2(\Lambda;L^2_{\omega^{(n,n)}}(I))}.\end{aligned} \tag{3.61}$$

On the other hand, we have $u^h - u_L^h = u - \tilde{u}_L^h + e_L$. Then, using Lemma 2.2 and Lemma 2.4 again yields

$$\begin{aligned}&(D^{2\beta}(\tilde{u}^h - u^h), \tilde{u}^h - u^h)_Q\\ &\le \|D^{2\beta}(\tilde{u}^h - u^h)\|_{L^2_{\omega^{(\beta,\beta)}}(\Lambda;L^2_{\omega^{(\alpha,\alpha)}}(I))}\|\tilde{u}^h - u^h\|_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &\le \|D^{2\beta}(\tilde{u}^h - u^h)\|^2_{L^2_{\omega^{(\beta,\beta)}}(\Lambda;L^2_{\omega^{(\alpha,\alpha)}}(I))} + \|\tilde{u}^h - u^h\|^2_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &:= I_1 + I_2.\end{aligned}$$

Similarly, these two terms can be estimated as follows:

$$\begin{aligned}I_1 &\le \|D^{2\beta}\pi_M^{(-\beta,-\beta)}({}^-\pi_N^{(-\alpha,-\alpha)}u^h - u^h)\|^2_{L^2_{\omega^{(\beta,\beta)}}(\Lambda;L^2_{\omega^{(\alpha,\alpha)}}(I))}\\ &\quad + \|D^{2\beta}(\pi_M^{(-\beta,-\beta)}u^h - u^h)\|^2_{L^2_{\omega^{(\beta,\beta)}}(\Lambda;L^2_{\omega^{(\alpha,\alpha)}}(I))}\\ &\le \|D^{2\beta}({}^-\pi_N^{(-\alpha,-\alpha)}u^h - u^h)\|^2_{L^2_{\omega^{(\beta,\beta)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &\quad + \|D^{2\beta}(\pi_M^{(-\beta,-\beta)}u^h - u^h)\|^2_{L^2_{\omega^{(\beta,\beta)}}(\Lambda;L^2_{\omega^{(\alpha,\alpha)}}(I))}\\ &\le cN^{-2(\alpha+n)}\left\|{}^R D_t^{\alpha+n}(D^{2\beta}u)\right\|^2_{L^2(\Lambda;L^2_{\omega^{(n,n)}}(I))}\\ &\quad + M^{2-2m}\|D^{2\beta-1+m}u\|^2_{L^2_{\omega^{(\beta-1+m,\beta-1+m)}}(\Lambda;L^2(I))}.\end{aligned}$$

and

$$\begin{aligned}I_2 &\le \|{}^-\pi_N^{(-\alpha,-\alpha)}(\pi_M^{(-\beta,-\beta)}u^h - u^h)\|^2_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &\quad + \|{}^-\pi_N^{(-\alpha,-\alpha)}u^h - u^h\|^2_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &\le \|\pi_M^{(-\beta,-\beta)}u^h - u^h\|^2_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &\quad + \|{}^-\pi_N^{(-\alpha,-\alpha)}u^h - u^h\|^2_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &\le cM^{2-2(2\beta+m)}\|D^{2\beta-1+m}u\|^2_{L^2_{\omega^{(\beta-1+m,\beta-1+m)}}(\Lambda;L^2_{\omega^{(-\alpha,-\alpha)}}(I))}\\ &\quad + cN^{-2(\alpha+n)}\left\|{}^R D_t^{\alpha+n}u\right\|^2_{L^2_{\omega^{(-\beta,-\beta)}}(\Lambda;L^2_{\omega^{(n,n)}}(I))}.\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
&\|{}^RD_t^{\alpha}(\pi_M^{(-\beta,-\beta)}\,{}^{-}\pi_N^{(-\alpha,-\alpha)}u^h-u^h)\|_Q\\
&\le \|{}^RD_t^{\alpha}\,{}^{-}\pi_N^{(-\alpha,-\alpha)}(\pi_M^{(-\beta,-\beta)}u^h-u^h)\|_Q+\|{}^RD_t^{\alpha}({}^{-}\pi_N^{(-\alpha,-\alpha)}u^h-u^h)\|_Q\\
&\le \|{}^RD_t^{\alpha}(\pi_M^{(-\beta,-\beta)}u^h-u^h)\|_Q+\|{}^RD_t^{\alpha}({}^{-}\pi_N^{(-\alpha,-\alpha)}u^h-u^h)\|_Q\\
&\lesssim cM^{1-(2\beta+m)}\|D^{2\beta-1+m}({}^RD_t^{\alpha}u)\|_{L^2_{\omega^{(\beta-1+m,\beta-1+m)}}(\Lambda;L^2(I))}\\
&\quad +cN^{-n}\left\|{}^RD_t^{\alpha+n}u\right\|_{L^2(\Lambda;L^2_{\omega^{(n,n)}}(I))}.
\end{aligned}
$$

Consequently, the desired result follows from the above estimates and the triangle inequality. □

Remark 3.2. Note that the error estimate in the above theorem can not be easily expressed in terms of the data (f, u_0). In particular, the space-time Petrov-Galerkin method (3.52) will not lead to high-order convergence, even if f and u_0 are sufficiently smooth, due to the singularities of the solution at $t = 0$ and $x = \pm 1$. However, the leading singular term in time and in space is included in our approximation space so our method will lead to better convergence rate than those based on the polynomial approximations.

4 Concluding remarks

We presented in this article essential properties of the GJFs and their application to a class of fractional differential equations. In particular, we showed that (i) by using suitable GJFs, the non-local fractional operators become local operators in the space spanned by GJFs; (ii) for simple FDEs, the spectral methods using GJFs can lead to exponential convergence rate despite the non-smoothness of the solution in usual Sobolev spaces; and (iii) for more general FDEs, a suitable spectral method using GJFs is still very efficient as the non-local fractional stiffness matrices can be easily computed, and furthermore, it is also more accurate than using a usual polynomial based method as the GJFs include the leading singular term of the underlying FDEs.

We only consider one dimensional FDEs in this paper. For multi-dimensional fractional PDEs with only fractional derivative in time, one can couple the GJF spectral method in time with a usual spatial approximation to construct a space-time Petrov-Galerkin method. It can still be efficiently solved by using the matrix diagonalization method as in the last subsection, we refer to [27] for more detail. As for FDEs with multi-dimensional fractional operators in space, one has to construct appropriate numerical methods with respect to the specific definitions of fractional operator. In particular, the GJFs for Riesz equation in one-dimension can be extended to deal with fractional Laplacian by the integral definition on the multi-dimensional balls [20]. On the other hand, for fractional Laplacian defined through the spectral decomposition of the Laplacian operator, one can use the Caffarelli-Silvestre extension to cast the fractional Laplacian equation in d-dimension into an extended problem in $d+1$-dimension with regular derivatives and a weakly singular weight in the extended direction. Then, one can construct efficient and accurate spectral method in the extended direction to couple with any consistent approximation in space, for more detail, we refer to [6]. For other types of multi-dimensional fractional PDEs, we refer to a recent review paper [19] for a nice presentation on different definitions of fractional Laplacian and their numerical treatments.

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] George E Andrews, Richard Askey, Ranjan Roy, et al. Special functions, encyclopedia of mathematics and its applications, 1999.

[3] Harry Bateman. The solution of linear differential equations by means of definite integrals. *Trans. Camb. Phil. Soc*, 21:171–196, 1909.

[4] Lizhen Chen, Zhiping Mao, and Huiyuan Li. Jacobi-galerkin spectral method for eigenvalue problems of riesz fractional differential equations. arXiv preprint arXiv:1803.03556.

[5] S. Chen, J. Shen, and L. L. Wang. Generalized Jacobi functions and their applications to fractional differential equations. *Math. Comput.*, 85(300):1603–1638, 2016.

[6] Sheng Chen and Jie Shen. An efficient and accurate method for the fractional Laplacian equation using the Caffarelli-Silvestre extension. Preprint.

[7] K. Diethelm. *The Analysis of Fractional Differential Equations.* Springer-Verlag, Berlin, 2010.

[8] V. J. Ervin and J. P. Roop. Variational formulation for the stationary fractional advection dispersion equation. *Numer. Meth. Part. D. E.*, 22(3):558–576, 2006.

[9] V.J. Ervin, N. Heuer, and J.P. Roop. Regularity of the solution to 1-d fractional order diffusion equations. Preprint, 2016.

[10] Ben-Yu Guo, Jie Shen, and Li-Lian Wang. Generalized jacobi polynomials/functions and their applications. *Applied Numerical Mathematics*, 59(5):1011–1028, 2009.

[11] Dianming Hou and Chuanju Xu. A fractional spectral method with applications to some singular problems. *Adv. Comput. Math.*, 43(5):911–944, 2017.

[12] Can Huang, Yujian Jiao, Li-Lian Wang, and Zhimin Zhang. Optimal fractional integration preconditioning and error analysis of fractional collocation method using nodal generalized Jacobi functions. *SIAM J. Numer. Anal.*, 54(6):3357–3387, 2016.

[13] Can Huang and Martin Stynes. A spectral collocation method for a weakly singular Volterra integral equation of the second kind. *Adv. Comput. Math.*, 42(5):1015–1030, 2016.

[14] Yujian Jiao, Li-Lian Wang, and Can Huang. Well-conditioned fractional collocation methods using fractional Birkhoff interpolation basis. *J. Comput. Phys.*, 305:1–28, 2016.

[15] Ehsan Kharazmi, Mohsen Zayernouri, and George Em Karniadakis. Petrov-Galerkin and spectral collocation methods for distributed order differential equations. *SIAM J. Sci. Comput.*, 39(3):A1003–A1037, 2017.

[16] Ehsan Kharazmi, Mohsen Zayernouri, and George Em Karniadakis. A Petrov-Galerkin spectral element method for fractional elliptic problems. *Comput. Methods Appl. Mech. Engrg.*, 324:512–536, 2017.

[17] X. Li and C. Xu. A space-time spectral method for the time fractional diffusion equation. *SIAM Journal on Numerical Analysis*, 47(3):2108–2131, 2009.

[18] X. Li and C. Xu. Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation. *Communications in Computational Physics*, 8(5):1016, 2010.

[19] Anna Lischke, Guofei Pang, Mamikon Gulian, Fangying Song, Christian Glusa, Xiaoning Zheng, Zhiping Mao, Mark M. Meerschaert Wei Cai, Mark Ainsworth, and George Em Karniadakis. What is the fractional Laplacian? arXiv:1801.09767.

[20] Suna Ma, Huiyuan Li, and Zhiming Zhang. Efficient spectral methods for the fractional laplacian. Preprint.

[21] Zhiping Mao, Sheng Chen, and Jie Shen. Efficient and accurate spectral method using generalized jacobi functions for solving riesz fractional differential equations. *Applied Numerical Mathematics*, 106:165–181, 2016.

[22] Zhiping Mao and George Em Karniadakis. A spectral method (of exponential convergence) for singular solutions of the diffusion equation with general two-sided fractional derivative. *SIAM J. Numer. Anal.*, 56(1):24–49, 2018.

[23] Zhiping Mao and Jie Shen. Efficient spectral-Galerkin methods for fractional partial differential equations with variable coefficients. *J. Comput. Phys.*, 307:243–261, 2016.

[24] I. Podlubny. *Fractional Differential Equations.* Academic Press, New York, 1999.

[25] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and*

Derivatives:Theory and Applications. Gordon and Breach Science, Yverdon, Switzerland, 1993.

[26] Jie Shen. Efficient spectral-galerkin method i. direct solvers of second-and fourth-order equations using legendre polynomials. *SIAM Journal on Scientific Computing*, 15(6):1489–1505, 1994.

[27] Jie Shen and Changtao Sheng. An efficient space-time method for time fractional diffusion equation. Preprint.

[28] Jie Shen, Changtao Sheng, and Zhongqing Wang. Generalized Jacobi spectral-Galerkin method for nonlinear Volterra integral equations with weakly singular kernels. *J. Math. Study*, 48(4):315–329, 2015.

[29] Jie Shen, Tao Tang, and Li-Lian Wang. *Spectral methods: algorithms, analysis and applications*, volume 41. Springer Science & Business Media, 2011.

[30] G. Szegö. *Orthogonal Polynomials (fourth edition)*, volume 23. AMS Coll. Publ., 1975.

[31] Mohsen Zayernouri, Mark Ainsworth, and George Em Karniadakis. A unified Petrov-Galerkin spectral method for fractional PDEs. *Comput. Methods Appl. Mech. Engrg.*, 283:1545–1569, 2015.

[32] Mohsen Zayernouri and George Em Karniadakis. Fractional Sturm-Liouville eigen-problems: theory and numerical approximation. *J. Comput. Phys.*, 252:495–517, 2013.

[33] Mohsen Zayernouri and George Em Karniadakis. Exponentially accurate spectral and spectral element methods for fractional ODEs. *J. Comput. Phys.*, 257(part A):460–480, 2014.

[34] Mohsen Zayernouri and George Em Karniadakis. Fractional spectral collocation methods for linear and nonlinear variable order FPDEs. *J. Comput. Phys.*, 293:312–338, 2015.

[35] Fanhai Zeng, Zhiping Mao, and George Em Karniadakis. A generalized spectral collocation method with tunable accuracy for fractional differential equations with end-point singularities. *SIAM J. Sci. Comput.*, 39(1):A360–A383, 2017.

[36] Zhongqiang Zhang, Fanhai Zeng, and George Em Karniadakis. Optimal error estimates of spectral Petrov-Galerkin and collocation methods for initial value problems of fractional differential equations. *SIAM J. Numer. Anal.*, 53(4):2074–2096, 2015.