

A SPACE-TIME SPECTRAL PETROV-GALERKIN SPECTRAL METHOD FOR TIME FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. We develop in this paper a space-time Petrov-Galerkin spectral method for linear and nonlinear time fractional diffusion equations (TFDEs) involving either a Caputo or Riemann-Liouville derivative. Our space-time spectral method are based on generalized Jacobi functions (GJFs) in time and Fourier-like basis functions in space. A complete error analysis is carried out for both linear and nonlinear TFDEs. Numerical experiments are presented to demonstrate the effectiveness of the proposed method.

1. INTRODUCTION

We consider in this paper the following time fractional diffusion equation

$${}_0^C D_t^\alpha u(x, t) - \gamma \Delta u(x, t) + \mathcal{N}(u(x, t), t) = 0, \quad \forall (x, t) \in \Omega = (-1, 1) \times (0, T], \quad (1.1)$$

with the initial and boundary conditions

$$\begin{aligned} u(x, 0) &= u_0(x), & \forall x \in (-1, 1), \\ u(\pm 1, t) &= 0, & \forall t \in [0, T]. \end{aligned} \quad (1.2)$$

In the above, $\alpha \in (0, 1)$, γ is a diffusion constant, \mathcal{N} is a linear or nonlinear operator, and ${}_0^C D_t^\alpha$ refers to left-sided Caputo fractional derivative of order α (see the definition in (2.4)). For the sake of simplicity, we choose to concentrate on the case of one spatial dimension. However, the proposed method and analysis can be directly extended to the multi-dimensional case with rectangular domains thanks to the Fourier-like basis functions that we employ in this paper.

The TFDEs (1.1) are frequently used to model anomalous diffusion in heterogeneous media, and its numerical solution has attracted much attention recently, including [22, 15, 29, 28] which use finite difference methods, and [26, 12, 11, 13] which use finite element methods.

Two main difficulties in solving fractional PDEs such as (1.1) are (i) fractional derivatives are non-local operators and generally lead to full matrices; and (ii) their solutions are often singular so polynomial based approximations are not efficient.

Since spectral methods are capable of providing exceedingly accurate numerical results with less degrees of freedoms, they have been widely used for numerical approximations [1, 9, 19, 20]. In particular, well designed spectral methods appear to be particularly attractive to deal with the difficulties associated with fractional PDEs mentioned above. Polynomial based spectral methods

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have been developed for TFDEs, e.g., [14, 23, 3]. However, these methods use polynomial basis functions which are not particularly suitable for TFDEs whose solutions are generally non-smooth at $t = 0$. Zayernouri and Karniadakis [25] first proposed to approximate the singular solutions by Jacobi poly-fractionomials, which were defined as eigenfunctions of a fractional Sturm-Liouville problem. Chen, Shen and Wang [4] constructed efficient Petrov-Galerkin methods for fractional PDEs by using the generalized Jacobi functions (GJFs) which include Jacobi poly-fractionomials as special cases. Subsequently, some authors developed spectral methods by using nodal GJFs for fractional PDEs [10, 27, 30, 24]. Most of these work are concerned with linear equations only.

We recall that the method presented in [4] is very efficient and accurate for fractional differential equations in time of the kind:

$${}_0^C D_t^\alpha u(t) = f(t), \quad u(0) = u_0.$$

On the other hand, an efficient and accurate space-time spectral method based on Fourier-like basis functions is developed in [21] for solving linear or nonlinear parabolic equations. In this paper, we combine the approaches in [4] and [21] to construct an efficient space-time spectral method for solving TFDEs. We highlight below the main contributions of this paper.

- For the time variable, the choice of GJFs can be tuned to match the singularities of the underlying solutions; and for the spatial variables, we use the Fourier-like basis functions. This combination greatly improves the accuracy and simplifies the implementation.
- We carry out error analysis of the proposed method for linear and nonlinear cases.

The organization of this paper is as follows. In the next section, we introduce the basis functions in both time and spatial directions, present some useful properties of fractional calculus and define some functional spaces. In Section 3, we develop efficient Petrov-Galerkin methods for linear/and nonlinear TFDEs. In Section 4, we derive error estimates for linear/nonlinear problems. We present in Section 5 some illustrative numerical results. Some concluding remarks are given in the last section.

2. PRELIMINARIES

In this section, we first review basics of fractional integrals/derivatives, define some functional spaces endowed with norms and inner products, and introduce some properties of the shifted generalized Jacobi functions and the shifted Legendre polynomial.

2.1. Fractional derivatives. We start with some preliminary definitions of fractional derivatives (see, e.g. [5, 17]). To fix the idea, we restrict our attentions to the interval I .

For $\rho \in \mathbb{R}^+$, the left-sided and right-sided Riemann-Liouville integrals are respectively defined as

$$\begin{aligned} {}_0 I_t^\rho u(t) &= \frac{1}{\Gamma(\rho)} \int_0^t \frac{u(s)}{(t-s)^{1-\rho}} ds, \quad t \in I, \\ {}_t I_T^\rho u(t) &= \frac{1}{\Gamma(\rho)} \int_t^T \frac{u(s)}{(s-t)^{1-\rho}} ds, \quad t \in I, \end{aligned} \tag{2.1}$$

where $\Gamma(\cdot)$ is the usual Gamma function.

For $\nu \in [m-1, m)$ with $m \in \mathbb{N}$, the left-sided Riemann-Liouville fractional derivative of order ν is defined by

$${}_0 D_t^\nu u(t) = \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dt^m} \int_0^t \frac{u(s)}{(t-s)^{\nu-m+1}} ds, \quad t \in I, \tag{2.2}$$

and the right-sided Riemann-Liouville fractional derivative of order ν is defined by

$${}_t D_T^\nu u(t) = \frac{(-1)^m}{\Gamma(m-\nu)} \frac{d^m}{dt^m} \int_t^T \frac{u(s)}{(s-t)^{\nu-m+1}} ds, \quad t \in I. \quad (2.3)$$

For $\nu \in [m-1, m)$ with $m \in \mathbb{N}$, the left-sided Caputo fractional derivative of order ν is defined by

$${}_0^C D_t^\nu u(t) = \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{u^{(m)}(s)}{(t-s)^{\nu-m+1}} ds, \quad t \in I, \quad (2.4)$$

and the right-sided Caputo fractional derivative of order ν is defined by

$${}_t^C D_T^\nu u(t) = \frac{(-1)^m}{\Gamma(m-\nu)} \int_t^T \frac{u^{(m)}(s)}{(s-t)^{\nu-m+1}} ds, \quad t \in I, \quad (2.5)$$

where $u^{(m)}$ denotes n -th derivative of u .

The following lemma shows the relationship between the Riemann-Liouville and Caputo fractional derivatives (see, e.g., [5, 17]).

Lemma 2.1. *For $\nu \in [k-1, k)$ with $k \in \mathbb{N}$, we have*

$${}_0 D_t^\nu u(t) = {}_0^C D_t^\nu u(t) + \sum_{j=0}^{k-1} \frac{u^{(j)}(0)}{\Gamma(1+j-\nu)} t^{j-\nu}. \quad (2.6)$$

2.2. Some functional spaces and their properties. We now introduce some functional spaces endowed with norms and inner products that are used hereafter.

Let $C_0^\infty(I)$ be the space of smooth functions with compact support in I , and $H_0^s(I)$ denote the closure of $C_0^\infty(I)$ with respect to norm $\|\cdot\|_{s,I}$. We also define the space

$$\begin{aligned} {}_0 C^\infty(I) &= \{v \mid v \in C^\infty(I) \text{ with compact support in } (0, 1]\}, \\ {}_r C^\infty(I) &= \{v \mid v \in C^\infty(I) \text{ with compact support in } [0, 1)\}. \end{aligned}$$

The space ${}_0 H^s(I)$ and ${}_r H^s(I)$ denotes the closure of ${}_0 C^\infty(I)$ and ${}_r C^\infty(I)$ respectively, with respect to norm $\|\cdot\|_{s,I}$. For the Sobolev space X with norm $\|\cdot\|_X$, let

$$\begin{aligned} H^s(I; X) &:= \{v \mid \|v(\cdot, t)\|_X \in H^s(I)\}, \quad s \geq 0, \\ {}_0 H^s(I; X) &:= \{v \mid \|v(\cdot, t)\|_X \in {}_0 H^s(I)\}, \quad s \geq 0, \\ {}_r H^s(I; X) &:= \{v \mid \|v(\cdot, t)\|_X \in {}_r H^s(I)\}, \quad s \geq 0, \end{aligned}$$

endowed with the norm:

$$\|v\|_{H^s(I; X)} := \|\|v(\cdot, t)\|_X\|_{s,I}.$$

We define space

$$B^s(\Omega) := L^2(I, L^2(\Lambda)) \cap H^s(I, L^2(\Lambda)) \cap L^2(I, H_0^1(\Lambda)),$$

equipped with the norm:

$$\|v\|_{B^s(\Omega)} := \left(\|v\|_{L^2(I, L^2(\Lambda))}^2 + \|v\|_{H^s(I, L^2(\Lambda))}^2 + \|v\|_{L^2(I, H_0^1(\Lambda))}^2 \right)^{1/2}.$$

It can be verified that $B^s(\Omega)$ is a Banach space. Next, we introduce following definitions (cf. [14])

Definition 2.1. Given $s > 0$. Let $H_t^s(I)$ be the closure of ${}_0 C^\infty(I)$ with respect to the norm

$$\|v\|_{H_t^s(I)} := \left(\|v\|_{L^2(I)} + |v|_{H_t^s(I)} \right)^{\frac{1}{2}}. \quad (2.7)$$

We also define the semi-norm $|v|_{H_t^s(I)} := \|{}_0 D_t^s v\|_{L^2(I)}$.

Definition 2.2. Given $s > 0$. Let let $H_r^s(I)$ be the closure of ${}^r_0C^\infty(I)$ with respect to the norm

$$\|v\|_{H_r^s(I)} := \left(\|v\|_{L^2(I)} + |v|_{H_r^s(I)} \right)^{\frac{1}{2}}. \quad (2.8)$$

We also define the semi-norm $|v|_{H_r^s(I)} := \|{}_tD_T^s v\|_{L^2(I)}$.

Definition 2.3. Given $s > 0$ and $s \neq n + 1/2$. Let let $H_c^s(I)$ be the closure of $C_0^\infty(I)$ with respect to the norm

$$\|v\|_{H_c^s(I)} := \left(\|v\|_{L^2(I)} + |v|_{H_c^s(I)} \right)^{\frac{1}{2}}. \quad (2.9)$$

We also define the semi-norm $|v|_{H_c^s(I)} := |({}_0D_t^s v, {}_tD_T^s v)_I|^{\frac{1}{2}}$

Lemma 2.2. For $s > 0$, $s \neq n + 1/2$, the spaces $H_l^s(I)$, $H_c^s(I)$ and $H_0^s(I)$ are equal in the sense that their seminorms as well as norms are equivalent.

Lemma 2.3. (see, e.g., [14]) For all $0 < \alpha < 1$, if $w \in {}_0H^1(I)$, $v \in {}_0H^{\frac{\alpha}{2}}(I)$, then

$$({}_0D_t^\alpha v, w)_I = ({}_0D_t^{\frac{\alpha}{2}} v, {}_tD_T^{\frac{\alpha}{2}} w)_I. \quad (2.10)$$

Using an argument similar to the proof of Theorem 2.10 in [7], we can prove the following:

Lemma 2.4. (Fractional Poincaré-Friedrichs) Give $s > 0$. Then, for $v \in H_l^s(I)$, we have

$$\|v\|_{L^2(I)} \lesssim |v|_{H_l^s(I)}, \quad (2.11)$$

and for $v \in H_r^s(I)$, we have

$$\|v\|_{L^2(I)} \lesssim |v|_{H_r^s(I)}. \quad (2.12)$$

2.3. Trial & test functions in time. We introduce below the Trial & test functions that we will use for the time variable. For $\alpha, \beta > -1$, let $P_n^{(\alpha, \beta)}(x)$, $x \in \Lambda$ be the standard Jacobi polynomial of degree n , and denote the weight function $\chi^{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta$. The set of Jacobi polynomials is a complete $L_{\chi^{(\alpha, \beta)}}^2(\Lambda)$ -orthogonal system, i.e.,

$$\int_{-1}^1 P_l^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) \chi^{(\alpha, \beta)}(x) dx = \gamma_l^{(\alpha, \beta)} \delta_{l,m}, \quad (2.13)$$

where $\delta_{l,m}$ is the Kronecker function, and

$$\gamma_l^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}}{(2l+\alpha+\beta+1)} \frac{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{l!\Gamma(l+\alpha+\beta+1)}.$$

In particular, $P_0^{(\alpha, \beta)}(x) = 1$.

The shifted Jacobi polynomial of degree n is defined by

$$\tilde{P}_n^{(\alpha, \beta)}(t) = P_n^{(\alpha, \beta)}\left(\frac{2t-T}{T}\right), \quad t \in I, \quad n \geq 0. \quad (2.14)$$

Clearly, the set of $\{\tilde{P}_n^{(\alpha, \beta)}(t)\}_{n \geq 0}$ is a complete $L_{\omega^{(\alpha, \beta)}}^2(I)$ -orthogonal system with the weight function $\omega^{(\alpha, \beta)}(t) = (T-t)^\alpha t^\beta$, by (2.13) and (2.14) we get that

$$\int_I \tilde{P}_l^{(\alpha, \beta)}(t) \tilde{P}_m^{(\alpha, \beta)}(t) \omega^{(\alpha, \beta)}(t) dt = \left(\frac{T}{2}\right)^{\alpha+\beta+1} \gamma_l^{(\alpha, \beta)} \delta_{l,m}. \quad (2.15)$$

For any $\alpha, \beta > -1$, the shifted generalized Jacobi functions on I is defined by (cf. [18])

$$J_n^{(\alpha, \beta)}(t) = t^\beta \tilde{P}_n^{(\alpha, \beta)}(t), \quad t \in I, \quad n \geq 0. \quad (2.16)$$

and our approximation space on I is defined by

$$\mathcal{F}_N^{(\alpha)}(I) := \{t^\alpha \psi(t) : \psi(t) \in \mathcal{P}_N(I)\} = \text{span}\{J_n^{(-\alpha, \alpha)}(t) = t^\alpha \tilde{P}_n^{(-\alpha, \alpha)}(t) : 0 \leq n \leq N\}, \quad (2.17)$$

which incorporates the homogeneous boundary conditions at $t = 0$.

A particular case is the shifted Legendre polynomial $L_n(t)$, $t \in I$ is defined by

$$L_n(t) = P_n^{0,0}\left(\frac{2t}{T} - 1\right), \quad n = 0, 1, 2, \dots \quad (2.18)$$

The set of $L_n(t)$ is a complete $L^2(I)$ -orthogonal system, namely,

$$\int_I L_l(t) L_m(t) dt = \frac{T}{2l+1} \delta_{l,m}. \quad (2.19)$$

Clearly, we derive from (2.16) - (2.18) and a direct calculation that (cf. [4])

$${}_0D_t^\alpha J_n^{(-\alpha, \alpha)}(t) = \frac{\Gamma(n + \alpha + 1)}{n!} L_n(t). \quad (2.20)$$

3. PETROV-GALERKIN SPECTRAL METHOD

In this section, we will propose a Petrov-Galerkin spectral method for (1.1). We shall start by considering a special case of (1.1) with $\mathcal{N}u = \beta u$:

$${}_0D_t^\alpha u(x, t) - \gamma \Delta u(x, t) + \beta u(x, t) = f(x, t), \quad \forall (x, t) \in \Omega, \quad (3.1)$$

with the initial and boundary conditions (1.2). The algorithm we develop for (3.1) will play a key role in solving (1.1).

3.1. Petrov-Galerkin spectral method for (3.1). For the case of non-homogeneous initial conditions $u(x, 0) = u_0(x)$, we first decompose the solution $u(x, t)$ into two parts as

$$u(x, t) = u^h(x, t) + u_0(x), \quad (3.2)$$

with $u^h(x, 0) = 0$. Hence, by (3.2) and (2.6), the equation (3.1) is equivalent to the following equation with Riemann-Liouville fractional derivative:

$${}_0D_t^\alpha u^h(x, t) - \gamma \Delta u^h(x, t) + \beta u(x, t) = g(x, t), \quad \forall (x, t) \in \Omega, \quad (3.3)$$

where

$$g(x, t) = f(x, t) + \gamma \partial_x^2 u_0(x) - \beta u_0(x),$$

with homogeneous initial and boundary conditions

$$\begin{aligned} u^h(x, 0) &= 0, & \forall x \in \Lambda, \\ u^h(\pm 1, t) &= 0, & \forall t \in I. \end{aligned} \quad (3.4)$$

For the space variable, we shall use the standard polynomial space

$$V_M = \{u \in \mathcal{P}_M(\Lambda) : u(\pm 1) = 0\}. \quad (3.5)$$

For the time variable, we shall use the fractional polynomial space $\mathcal{F}_N^{(\alpha)}(I)$ defined in (2.17). Then, the space-time Petrov-Galerkin spectral method for (3.3) is to seek $u_L^h(x, t) := u_{MN}^h \in V_M \otimes \mathcal{F}_N^{(\alpha)}$, such that

$$\mathcal{A}(u_L^h, v) := ({}_0D_t^\alpha u_L^h, v)_\Omega + \gamma (\partial_x u_L^h, \partial_x v)_\Omega + \beta (u_L^h, v)_\Omega = (g, v)_\Omega, \quad \forall v \in V_M \otimes \mathcal{P}_N. \quad (3.6)$$

Next, we shall construct suitable basis functions of V_M and $\mathcal{F}_N^{(\alpha)}(I)$ so that the above system can be solved efficiently.

We consider first basis functions for the space variable. Setting $h_k(x) = \frac{1}{\sqrt{2(2k+3)}}(P_k(x) - P_{k+2}(x))$, with $0 \leq k \leq M-2$. Denote by \mathbf{M}^x and \mathbf{S}^x be the spatial mass and stiffness matrix respectively. One verifies readily that \mathbf{S}^x (with entries $s_{pq}^x = -(h_p'', h_q)$) is an identity matrix, and \mathbf{M}^x (with entries $m_{pq}^x = (h_p, h_q)$) is a symmetric positive definite penta-diagonal matrix. Next, we construct a set of basis functions which lead to diagonal stiffness and mass matrices. Let $\mathbf{E}^x := (\bar{y}_0, \dots, \bar{y}_{M-2}) = (e_{pq})_{p,q=0,\dots,M-2}$ be the matrix formed by the orthonormal eigenvectors of the generalized eigenvalue problem $\mathbf{S}^x \bar{y}_j = \lambda_j^x \mathbf{M}^x \bar{y}_j$ and $\Lambda^x = \text{diag}(\lambda_0^x, \dots, \lambda_{M-2}^x)$, i.e.,

$$\mathbf{S}^x \mathbf{E}^x = \mathbf{M}^x \mathbf{E}^x \Lambda, \quad (\mathbf{E}^x)^T \mathbf{S}^x \mathbf{E}^x = \Lambda^x, \quad (\mathbf{E}^x)^T \mathbf{M}^x \mathbf{E}^x = \mathbf{I}_{M-1}. \quad (3.7)$$

Following [21], we set

$$\phi_m(x) = \sum_{j=0}^{M-2} e_{jm} h_j(x), \quad 0 \leq m \leq M-2, \quad (3.8)$$

which satisfying

$$\begin{aligned} (\phi_p, \phi_q) &= \sum_{k,j=0}^{M-2} e_{kp} e_{jq} (h_k, h_j) = \sum_{k,j=0}^{M-2} e_{jq} m_{jk}^x e_{kp} = ((\mathbf{E}^x)^T \mathbf{M}^x \mathbf{E}^x)_{pq} = \delta_{pq}, \\ -(\phi_p'', \phi_q) &= -\sum_{k,j=0}^{M-2} e_{kp} e_{jq} (h_k'', h_j) = \sum_{k,j=0}^{M-2} e_{jq} s_{jk}^x e_{kp} = ((\mathbf{E}^x)^T \mathbf{S}^x \mathbf{E}^x)_{pq} = \lambda_q^x \delta_{pq}. \end{aligned} \quad (3.9)$$

Accordingly, we have

$$V_M = \text{span}\{\phi_m : 0 \leq m \leq M-2\}. \quad (3.10)$$

For the time variable, we use the generalized Jacobi functions $J_n^{(-\alpha, \alpha)}(t)$ defined in the last section:

$$\mathcal{F}_N^{(\alpha)}(I) = \text{span}\{J_n^{(-\alpha, \alpha)}(t) : 0 \leq n \leq N\}, \quad (3.11)$$

and for $\mathcal{P}_N(I)$ in the test space, we simple use the scaled Legendre polynomials, namely:

$$\mathcal{P}_N(I) = \text{span}\left\{L_n^{(\alpha)}(t) := \kappa_{n,\alpha} L_n(t) : 0 \leq n \leq N\right\}, \quad \text{with } \kappa_{n,\alpha} = \frac{n!(2n+1)}{T \cdot \Gamma(n+\alpha+1)}. \quad (3.12)$$

We now describe the numerical implementations for (3.6) under this set of basis functions. We write

$$u_L^h(x, t) = \sum_{m=0}^{M-2} \sum_{n=0}^N \tilde{u}_{nm}^h \phi_m(x) J_n^{(-\alpha, \alpha)}(t). \quad (3.13)$$

Substituting (3.13) into (3.6), and taking $v_L = \phi_p(x) L_q^{(\alpha)}(t)$ we obtain that

$$\begin{aligned} &\sum_{m=0}^{M-2} \sum_{n=0}^N \tilde{u}_{nm}^h \left\{ (\phi_m, \phi_p) ({}_0D_t^\alpha J_n^{(-\alpha, \alpha)}, L_q^{(\alpha)}) - \gamma (\phi_m'', \phi_p) (J_n^{(-\alpha, \alpha)}, L_q^{(\alpha)}) + \beta (\phi_m, \phi_p) (J_n^{(-\alpha, \alpha)}, L_q^{(\alpha)}) \right\} \\ &= (f, \phi_p L_q^{(\alpha)})_\Omega. \end{aligned} \quad (3.14)$$

Denote

$$\begin{aligned} f_{nm} &= (f, \phi_m(x) L_n^{(\alpha)}(t))_\Omega, \quad F = (f_{nm})_{0 \leq n \leq N, 0 \leq m \leq M-2}, \\ s_{pq}^t &= \int_I {}_0D_t^\alpha J_q^{(-\alpha, \alpha)}(t) L_p^{(\alpha)}(t) dt, \quad m_{pq}^t = \int_I J_q^{(-\alpha, \alpha)}(t) L_p^{(\alpha)}(t) dt, \\ \mathbf{S}^t &= (s_{pq}^t)_{0 \leq p, q \leq N}, \quad \mathbf{M}^t = (m_{pq}^t)_{0 \leq p, q \leq N}, \quad U = (\tilde{u}_{n,m}^h)_{0 \leq n \leq N, 0 \leq m \leq M-2}. \end{aligned} \quad (3.15)$$

It can be verified easily from (2.19), (2.20) and (3.12) that $\mathbf{S}^t = \mathbf{I}$ with \mathbf{I} being the identity matrix. On the other hand,

$$m_{pq}^t = \int_I J_q^{(-\alpha, \alpha)}(t) L_p^{(\alpha)}(t) dt = \int_I \tilde{P}_q^{(-\alpha, \alpha)}(t) L_p^{(\alpha)}(t) t^\alpha dt. \quad (3.16)$$

So \mathbf{M}^t is not sparse but can be accurately computed by Jacobi-Gauss quadrature with index $(0, \alpha)$.

Then, from (3.9) and (3.15), we find that (3.6) is equivalent to the following linear system:

$$U + \gamma \mathbf{M}^t U \Lambda^x + \beta \mathbf{M}^t U = F. \quad (3.17)$$

Hence, the m -th column of the above matrix equation becomes:

$$(\mathbf{I} + (\gamma \lambda_m^x + \beta) \mathbf{M}^t) \mathbf{u}_m^h = \mathbf{f}_m, \quad 0 \leq m \leq M-2, \quad (3.18)$$

with

$$\mathbf{u}_m^h = (\tilde{u}_{0,m}^h, \tilde{u}_{1,m}^h, \dots, \tilde{u}_{N,m}^h)^T, \quad \mathbf{f}_m = (f_{0m}, f_{1m}, \dots, f_{Nm})^T.$$

Finally, we obtain the numerical solutions of (3.1) by $u_L = u_L^h + u_0$.

3.2. Petrov-Galerkin spectral method for nonlinear problems (1.1). As above, we first rewrite the equation (1.1) using Riemann-Liouville fractional derivative as follows:

$${}_0D_t^\alpha u^h(x, t) - \gamma \Delta u^h(x, t) + \tilde{\mathcal{N}}(u^h(x, t), t) = g(x, t), \quad \forall (x, t) \in \Omega, \quad (3.19)$$

where

$$g(x, t) = \gamma \Delta u_0(x), \quad \tilde{\mathcal{N}}(u^h(x, t), t) = \mathcal{N}(u^h(x, t) + u_0(x), t),$$

with the initial and boundary conditions (3.4).

The corresponding space-time Petrov-Galerkin spectral method for (3.19) is to find $u_L^h(x, t) \in V_M \otimes \mathcal{F}_N^{(\alpha)}$, such that

$$({}_0D_t^\alpha u_L^h, v)_\Omega + \gamma (\partial_x u_L^h, \partial_x v)_\Omega + (\tilde{\mathcal{N}}(u_L^h, t), v)_\Omega = (g, v)_\Omega, \quad \forall v \in V_M \otimes S_N^{(\alpha)}. \quad (3.20)$$

We set

$$\begin{aligned} u_L^h(x, t) &= \sum_{m=0}^{M-2} \sum_{n=0}^N \tilde{u}_{nm}^h \phi_m(x) J_n^{(-\alpha, \alpha)}(t), \quad \tilde{u}_{nm} = (\tilde{\mathcal{N}}(u_L^h, t), \phi_m(x) L_n^{(\alpha)}(t)). \\ \mathbf{u}_m^h &= (\tilde{u}_{0,m}^h, \tilde{u}_{1,m}^h, \dots, \tilde{u}_{N,m}^h)^T, \quad \mathbf{n}_m = (\tilde{n}_{0,m}, \tilde{n}_{1,m}, \dots, \tilde{n}_{N,m})^T, \\ g_{nm} &= (g, \phi_m(x) L_n^{(\alpha)}(t))_\Omega, \quad \mathbf{g}_m = (g_{0m}, \dots, g_{Nm})^T. \end{aligned} \quad (3.21)$$

As in the linear case, we find that the unknown vector \mathbf{u}_m^h satisfies the following system of nonlinear algebraic equations:

$$(\mathbf{I} + \gamma \lambda_m^x \mathbf{M}^t) \mathbf{u}_m^h + \mathbf{n}_m(\mathbf{u}^h) = \mathbf{g}_m, \quad 0 \leq m \leq M-2, \quad (3.22)$$

which can be solved, e.g., by a Newton-type iterative procedure that we describe below.

We rewrite the above system (3.22) as

$$\mathbf{N}_m(\mathbf{u}^h) := (\mathbf{I} + \gamma \lambda_m^x \mathbf{M}^t) \mathbf{u}_m^h + \mathbf{n}_m(\mathbf{u}^h) - \mathbf{g}_m = 0, \quad 0 \leq m \leq M-2. \quad (3.23)$$

Then, the Newton iterative algorithm applied to (3.23) is:

$$\mathbf{N}'_m(\mathbf{u}^{h,(n)}) (\mathbf{u}^{h,(n+1)} - \mathbf{u}^{h,(n)}) = -\mathbf{N}_m(\mathbf{u}^{h,(n)}), \quad 0 \leq m \leq M-2, \quad (3.24)$$

where

$$\mathbf{N}'(\mathbf{u}^h) = \begin{pmatrix} \frac{\partial \mathbf{N}_{0,m}}{\partial \bar{u}_{0,m}^h} & \cdots & \frac{\partial \mathbf{N}_{0,m}}{\partial \bar{u}_{N,m}^h} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{N}_{N,m}}{\partial \bar{u}_{0,m}^h} & \cdots & \frac{\partial \mathbf{N}_{N,m}}{\partial \bar{u}_{N,m}^h} \end{pmatrix}.$$

We observe from (3.23) that

$$\begin{aligned} & \mathbf{N}'_m(\mathbf{u}^{h,(n)})(\mathbf{u}^{h,(n+1)} - \mathbf{u}^{h,(n)}) \\ &= (\mathbf{I}_{N+1} + \gamma \lambda_m^x \mathbf{M}^t)(\mathbf{u}_m^{h,(n+1)} - \mathbf{u}_m^{h,(n)}) + \mathbf{n}'_m(\mathbf{u}^{h,(n)})(\mathbf{u}^{h,(n+1)} - \mathbf{u}^{h,(n)}) \end{aligned}$$

where

$$\mathbf{n}'_m(\mathbf{u}^h) = \begin{pmatrix} \frac{\partial \mathbf{n}_{0,m}}{\partial \bar{u}_{0,m}^h} & \cdots & \frac{\partial \mathbf{n}_{0,m}}{\partial \bar{u}_{N,m}^h} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{n}_{N,m}}{\partial \bar{u}_{0,m}^h} & \cdots & \frac{\partial \mathbf{n}_{N,m}}{\partial \bar{u}_{N,m}^h} \end{pmatrix}.$$

Hence, we can rewrite (3.24) as

$$\begin{aligned} & (\mathbf{I} + \gamma \lambda_m^x \mathbf{M}^t) \mathbf{u}_m^{h,(n+1)} + \mathbf{n}'_m(\mathbf{u}^{h,(n)}) \mathbf{u}_m^{h,(n+1)} \\ &= (\mathbf{I} + \gamma \lambda_m^x \mathbf{M}^t) \mathbf{u}_m^{h,(n)} + \mathbf{n}'_m(\mathbf{u}^{h,(n)}) \mathbf{u}_m^{h,(n)} - \mathbf{N}_m(\mathbf{u}^{h,(n)}), \quad 0 \leq m \leq M-2, \end{aligned} \quad (3.25)$$

A combination of (3.23) and (3.25) yields

$$(\mathbf{I} + \gamma \lambda_m^x \mathbf{M}^t) \mathbf{u}_m^{h,(n+1)} + \mathbf{n}'_m(\mathbf{u}^{h,(n)}) \mathbf{u}_m^{h,(n+1)} = \mathbf{n}'_m(\mathbf{u}^{h,(n)}) \mathbf{u}_m^{h,(n)} - \mathbf{n}_m(\mathbf{u}^{h,(n)}) + \mathbf{g}_m. \quad (3.26)$$

The above system (with a variable coefficient $\mathbf{n}'_m(\mathbf{u}^{h,(n)})$) can be efficiently solved by using a preconditioned CG iteration with a preconditioner, based on replacing the variable coefficient $\mathbf{n}'_m(\mathbf{u}^{h,(n)})$ by a suitable constant β , which can be applied very fast as we described in the last subsection.

4. ERROR ESTIMATES FOR PETROV-GALERKIN SPECTRAL METHOD (1.1)

In this section, we conduct an error analysis for the Petrov-Galerkin method described in the previous section. Since the full error analysis for the nonlinear system in its most general form is not feasible, we shall only consider the analysis for following two special cases of (3.19).

Case I: a linear case

$$\tilde{\mathcal{N}}(u^h) = \beta(u^h + u_0).$$

Case II: with a Burgers-like nonlinearity

$$\tilde{\mathcal{N}}(u^h) = \frac{1}{2} \partial_x (u^h + u_0)^2.$$

We start with two projection operators (in space and time direction) and the corresponding approximation results which will be used later. The first one $\Pi_M^{1,0} : H_0^1(\Lambda) \rightarrow V_M$ is defined by

$$((\Pi_M^{1,0} v - v)', \phi')_\Lambda = 0, \quad \forall \phi \in V_M. \quad (4.1)$$

According to [2, 8], we have

Lemma 4.5. *If $v \in H_0^1(\Lambda)$ and $\partial_x^k v \in L_{\chi^{k-1}}^2(\Lambda)$ with $1 \leq k \leq r$, then we have*

$$\|\partial_x^\mu (\Pi_M^{1,0} v - v)\|_\Lambda \lesssim M^{\mu-r} \|\partial_x^r v\|_{\Lambda, \chi^{r-1}}, \quad \mu \leq r, \quad \mu = 0, 1. \quad (4.2)$$

In order to characterize the regularity of u in t , we define the following non-uniformly weighted space involving fractional derivatives

$$\mathcal{B}_{\alpha,\beta}^s(I) := \{v \in L_{\omega(\alpha,-\beta)}^2(I) : {}_0D_t^{\beta+r}v \in L_{\omega(\alpha+\beta+r,r)}^2(I) \text{ for } 0 \leq r \leq s\}, \quad s \in \mathbb{N}_0.$$

The second projection operator is $\pi_N^{(-\alpha,-\alpha)} : L_{\omega(-\alpha,-\alpha)}^2(I) \rightarrow \mathcal{F}_N^{(\alpha)}(I)$ defined by

$$(\pi_N^{(-\alpha,-\alpha)}v - v, \psi)_{\omega(-\alpha,-\alpha)} = 0, \quad \forall \psi \in \mathcal{F}_N^{(\alpha)}(I). \quad (4.3)$$

It is shown in [4] that

$$({}_0D_t^\alpha(\pi_N^{(-\alpha,-\alpha)}v - v), p) = 0, \quad \forall p \in \mathcal{P}_N(I). \quad (4.4)$$

We recall from [18] that

Lemma 4.6. *Let $\alpha \in (0, 1)$, for any $v \in \mathcal{B}_{-\alpha,\alpha}^s(I)$, with integer $0 \leq s \leq N$. Then*

$$\|\pi_N^{(-\alpha,-\alpha)}v - v\|_{\omega(-\alpha,-\alpha)} \lesssim N^{-(\alpha+s)} \|{}_0D_t^{\alpha+s}v\|_{\omega(s,s)}. \quad (4.5)$$

and

$$\|{}_0D_t^\alpha(\pi_N^{(-\alpha,-\alpha)}v - v)\|_I \lesssim N^{-s} \|{}_0D_t^{\alpha+s}v\|_{\omega(s,s)}. \quad (4.6)$$

Next, we define suitable functional spaces for our space-time approximation. We denote by $E^r(\Omega)$ (respectively $G^s(\Omega)$ and $K^s(\Omega)$) a function space consisting of measurable functions satisfying $\|u\|_{E^r(\Omega)} < \infty$ (respectively $\|u\|_{G^s(\Omega)} < \infty$ and $\|u\|_{K^s(\Omega)} < \infty$), where, for integers $r \geq 2$ and $s \geq 0$,

$$\begin{aligned} \|u\|_{E^r(\Omega)} &= \left(\|\partial_x^r v\|_{L_{\chi^{r-1}}^2(\Lambda, L^2(I))}^2 + \|\partial_x^{r-1}({}_0D_t^\alpha v)\|_{L_{\chi^{r-2}}^2(\Lambda, L^2(I))}^2 \right)^{\frac{1}{2}}, \\ \|u\|_{G^{r,s}(\Omega)} &= \left(\|\partial_x^{r-1}({}_0D_t^{\alpha+s}v)\|_{L_{\chi^{r-2}}^2(\Lambda, L_{\omega(s,s)}^2(I))}^2 \right)^{\frac{1}{2}}, \\ \|u\|_{K^s(\Omega)} &= \left(\|{}_0D_t^{\alpha+s}v\|_{L^2(\Lambda, L_{\omega(s,s)}^2(I))}^2 + \|\partial_x^2({}_0D_t^{\alpha+s}v)\|_{L^2(\Lambda, L_{\omega(s,s)}^2(I))}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Case I. This corresponds to (3.3) with the space-time Petrov-Galerkin approximation (3.6) considered in Subsection 3.1.

Theorem 4.1. *Let u^h be the solution of (3.3) and u_L^h be the numerical solution (3.6). If $u^h \in H^\alpha(I; H_0^1(\Lambda)) \cap E^r(\Omega) \cap G^s(\Omega) \cap K^s(\Omega)$, there holds*

$$\|u^h - u_L^h\|_{B^\alpha(\Omega)} \lesssim M^{1-r} \|u^h\|_{E^r(\Omega)} + N^{-(\alpha+s)} M^{-r} \|u^h\|_{G^{r,s}(\Omega)} + N^{-s} \|u^h\|_{K^s(\Omega)}. \quad (4.7)$$

Proof. Let us denote $\tilde{u}_L^h := \pi_N^{(-\alpha,-\alpha)} \Pi_M^{1,0} u^h = \Pi_M^{1,0} \pi_N^{(-\alpha,-\alpha)} u^h$ and $e_L := \tilde{u}_L^h - u_L^h$. We derive from (3.3) and (3.6) that

$$\begin{aligned} a(e_L, v) &:= ({}_0D_t^\alpha e_L, v)_\Omega + \gamma(\partial_x e_L, \partial_x v)_\Omega + \beta(e_L, v)_\Omega \\ &= ({}_0D_t^\alpha(\tilde{u}_L^h - u^h), v)_\Omega + \gamma(\partial_x(\tilde{u}_L^h - u^h), \partial_x v)_\Omega + \beta(\tilde{u}_L^h - u^h, v)_\Omega, \end{aligned} \quad (4.8)$$

for all $v \in V_M \otimes S_N$. Due to (4.1) and (4.4), the above equation can be simplified to

$$a(e_L, v) = ({}_0D_t^\alpha(\Pi_M^{1,0} u^h - u^h), v)_\Omega + \gamma(\partial_x(\pi_N^{(-\alpha,-\alpha)} u^h - u^h), \partial_x v)_\Omega + \beta(\tilde{u}_L^h - u^h, v)_\Omega. \quad (4.9)$$

Taking $v = {}_0D_t^\alpha e_L \in V_M \otimes S_N$ in above equation, we obtain that

$$\begin{aligned} &({}_0D_t^\alpha e_L, {}_0D_t^\alpha e_L)_\Omega + \gamma(\partial_x e_L, \partial_x({}_0D_t^\alpha e_L))_\Omega + \beta(e_L, {}_0D_t^\alpha e_L)_\Omega \\ &= ({}_0D_t^\alpha(\Pi_M^{1,0} u^h - u^h), {}_0D_t^\alpha e_L)_\Omega + \gamma(\partial_x(\pi_N^{(-\alpha,-\alpha)} u^h - u^h), \partial_x({}_0D_t^\alpha e_L))_\Omega + \beta(e_L, {}_0D_t^\alpha e_L)_\Omega. \end{aligned} \quad (4.10)$$

We recall the following result [16]:

$$H_0^{\frac{\alpha}{2}}(I) = H^{\frac{\alpha}{2}}(I) = {}_0H^{\frac{\alpha}{2}}(I), \quad \text{for any } 0 < \alpha < 1. \quad (4.11)$$

Since $e_L(x, 0) = 0$, we derive from Lemma 2.2 - Lemma 2.4, (2.9) and (4.11) that

$$\begin{aligned} \|\partial_x e_L\|_{\Omega}^2 &\lesssim \|\partial_x {}_0D_t^{\frac{\alpha}{2}} e_L\|_{\Omega}^2 \cong \|\partial_x e_L\|_{L^2(\Lambda; H_c^{\frac{\alpha}{2}}(I))}^2 \\ &= (\partial_x ({}_0D_t^{\frac{\alpha}{2}} e_L), \partial_x ({}_tD_T^{\frac{\alpha}{2}} e_L))_{\Omega} = (\partial_x ({}_0D_t^{\alpha} e_L), \partial_x e_L)_{\Omega}, \end{aligned} \quad (4.12)$$

and

$$\|e_L\|_{\Omega}^2 \lesssim \|{}_0D_t^{\frac{\alpha}{2}} e_L\|_{\Omega}^2 \cong \|e_L\|_{L^2(\Lambda; H_c^{\frac{\alpha}{2}}(I))}^2 = ({}_0D_t^{\frac{\alpha}{2}} e_L, {}_tD_T^{\frac{\alpha}{2}} e_L)_{\Omega} = ({}_0D_t^{\alpha} e_L, e_L)_{\Omega}. \quad (4.13)$$

This, along with equation (4.10), yields

$$\begin{aligned} \|{}_0D_t^{\alpha} e_L\|_{\Omega}^2 + \|\partial_x e_L\|_{\Omega}^2 + \|e_L\|_{\Omega}^2 &\lesssim \|{}_0D_t^{\alpha} (\Pi_M^{(1,0)} u^h - u^h)\|_{\Omega} \|{}_0D_t^{\alpha} e_L\|_{\Omega} \\ &\quad + \|\partial_x^2 (\pi_N^{(-\alpha, \alpha)} u^h - u^h)\|_{\Omega} \|{}_0D_t^{\alpha} e_L\|_{\Omega} + \|\tilde{u}_L^h - u^h\|_{\Omega} \|{}_0D_t^{\alpha} e_L\|_{\Omega}. \end{aligned}$$

Hence, by the definition of the norm $\|\cdot\|_{B^s(\Omega)}$ leads to

$$\|e_L\|_{B^{\alpha}(\Omega)} \lesssim \|{}_0D_t^{\alpha} (\Pi_M^{(1,0)} u^h - u^h)\|_{\Omega} + \|\partial_x^2 (\pi_N^{(-\alpha, \alpha)} u^h - u^h)\|_{\Omega} + \|\tilde{u}_L^h - u^h\|_{\Omega}. \quad (4.14)$$

The two terms at the right-hand side can be bounded by using Lemma 4.5 and Lemma 4.6 as follows:

$$\begin{aligned} \|{}_0D_t^{\alpha} (\Pi_M^{(1,0)} u^h - u^h)\|_{\Omega} &\lesssim M^{1-r} \|\partial_x^{r-1} ({}_0D_t^{\alpha} u^h)\|_{L^2_{\chi^{r-2}}(\Lambda, L^2(I))}, \\ \|\partial_x^2 (\pi_N^{(-\alpha, \alpha)} u^h - u^h)\|_{\Omega} &\lesssim N^{-(\alpha+s)} \|\partial_x^2 ({}_0D_t^{\alpha+s} u^h)\|_{L^2(\Lambda, L^2_{\omega^{(s,s)}}(I))}, \end{aligned}$$

and

$$\begin{aligned} \|\tilde{u}_L^h - u^h\|_{\Omega} &\lesssim \|\Pi_M^{1,0} (u^h - \pi_N^{(-\alpha, \alpha)} u^h)\|_{\Omega} + \|u^h - \Pi_M^{1,0} u^h\|_{\Omega} \\ &\lesssim \|(\Pi_M^{1,0} - I)(u^h - \pi_N^{(-\alpha, \alpha)} u^h)\|_{\Omega} + \|(u^h - \pi_N^{(-\alpha, \alpha)} u^h)\|_{\Omega} + \|u^h - \Pi_M^{1,0} u^h\|_{\Omega} \\ &\lesssim N^{-(\alpha+s)} M^{1-r} \|\partial_x^{r-1} ({}_0D_t^{\alpha+s} u^h)\|_{L^2_{\chi^{r-2}}(\Lambda, L^2_{\omega^{(s,s)}}(I))} \\ &\quad + N^{-(\alpha+s)} \|{}_0D_t^{\alpha+s} u^h\|_{L^2(\Lambda, L^2_{\omega^{(s,s)}}(I))} + M^{1-r} \|\partial_x^{r-1} u^h\|_{L^2_{\chi^{r-2}}(\Lambda, L^2(I))}. \end{aligned} \quad (4.15)$$

A combination of the above and (4.14) leads to

$$\begin{aligned} \|e_L\|_{B^{\alpha}(\Omega)} &\lesssim M^{1-r} \|\partial_x^{r-1} ({}_0D_t^{\alpha} u^h)\|_{L^2_{\chi^{r-2}}(\Lambda, L^2(I))} \\ &\quad + N^{-(\alpha+s)} \|\partial_x^2 ({}_0D_t^{\alpha+s} u^h)\|_{L^2(\Lambda, L^2_{\omega^{(s,s)}}(I))} \\ &\quad + N^{-(\alpha+s)} M^{-r} \|\partial_x^{r-1} ({}_0D_t^{\alpha+s} u^h)\|_{L^2_{\chi^{r-2}}(\Lambda, L^2_{\omega^{(s,s)}}(I))} \\ &\quad + N^{-(\alpha+s)} \|{}_0D_t^{\alpha+s} u^h\|_{L^2(\Lambda, L^2_{\omega^{(s,s)}}(I))} + M^{1-r} \|\partial_x^{r-1} u^h\|_{L^2_{\chi^{r-2}}(\Lambda, L^2(I))}. \end{aligned} \quad (4.16)$$

On the other hand, we have $u^h - u_L^h = u - \tilde{u}_L^h + e_L$. Then, using Lemma 4.5 and Lemma 4.6 again yields

$$\begin{aligned} \|\partial_x (u^h - \tilde{u}_L^h)\|_{\Omega} &\lesssim \|\partial_x \Pi_M^{1,0} (u^h - \pi_N^{(-\alpha, \alpha)} u^h)\|_{\Omega} + \|\partial_x (u^h - \Pi_M^{1,0} u^h)\|_{\Omega} \\ &\lesssim \|\partial_x (u^h - \pi_N^{(-\alpha, \alpha)} u^h)\|_{\Omega} + \|\partial_x (u^h - \Pi_M^{1,0} u^h)\|_{\Omega} \\ &\lesssim N^{-(\alpha+s)} \|\partial_x ({}_0D_t^{\alpha+s} u^h)\|_{L^2(\Lambda, L^2_{\omega^{(s,s)}}(I))} + M^{1-r} \|\partial_x^r u^h\|_{L^2_{\chi^{r-1}}(\Lambda, L^2(I))}, \\ \|{}_0D_t^{\alpha} (u^h - \tilde{u}_L^h)\|_{\Omega} &\lesssim \|{}_0D_t^{\alpha} \pi_N^{(-\alpha, \alpha)} (u^h - \Pi_M^{1,0} u^h)\|_{\Omega} + \|{}_0D_t^{\alpha} (u^h - \pi_N^{(-\alpha, \alpha)} u^h)\|_{\Omega} \\ &\lesssim \|{}_0D_t^{\alpha} (u^h - \Pi_M^{1,0} u^h)\|_{\Omega} + \|{}_0D_t^{\alpha} (u^h - \pi_N^{(-\alpha, \alpha)} u^h)\|_{\Omega} \\ &\lesssim M^{1-r} \|\partial_x^{r-1} ({}_0D_t^{\alpha} u^h)\|_{L^2_{\chi^{r-2}}(\Lambda, L^2(I))} + N^{-s} \|{}_0D_t^{\alpha+s} u^h\|_{L^2(\Lambda, L^2_{\omega^{(s,s)}}(I))}. \end{aligned}$$

Consequently, the desired result follows from the above estimates and the triangle inequality. \square

Case II. We consider (3.19) with $\tilde{\mathcal{N}}(u^h, t) = \frac{1}{2} \partial_x (u^h + u_0)^2$. We first establish a stability result.

Lemma 4.7. Let $u_0 \in H_0^1(\Lambda)$ and u_L^h be the solution of (3.20) with $\tilde{\mathcal{N}}(u^h, t) = \frac{1}{2}\partial_x(u^h + u_0)^2$. There holds

$$\|u_L^h\|_\Omega \leq \|{}_0D_t^{\frac{\alpha}{2}} u_L^h\|_\Omega^2 \leq K, \quad \|\partial_x u_L^h\|_\Omega \leq K, \quad (4.17)$$

where K is a positive constant independent u_L^h .

Proof. Taking $v = u_L^h + u_0$ in (3.20) with $\tilde{\mathcal{N}}(u^h, t) = \frac{1}{2}\partial_x(u^h + u_0)^2$, using the fact that $(\frac{1}{2}\partial_x(u_L^h + u_0))^2, u_L^h + u_0)_\Omega = 0$, as well as (2.10) and (4.11), we get that

$$\begin{aligned} & ({}_0D_t^\alpha u_L^h, u_L^h)_\Omega + \gamma(\partial_x u_L^h, \partial_x u_L^h)_\Omega \\ &= -({}_0D_t^\alpha u_L^h, u_0)_\Omega - \gamma(\partial_x u_L^h, \partial_x u_0)_\Omega + \gamma(\Delta u_0, u_L^h + u_0)_\Omega \\ &\leq -({}_0D_t^{\frac{\alpha}{2}} u_L^h, {}_tD_T^{\frac{\alpha}{2}} u_0)_\Omega + \gamma\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda + \gamma\|\partial_x u_0\|_\Lambda(\|\partial_x u_L^h\|_\Omega + T\|\partial_x u_0\|_\Lambda) \\ &\leq \|{}_0D_t^{\frac{\alpha}{2}} u_L^h\|_\Omega\|{}_tD_T^{\frac{\alpha}{2}} u_0\|_\Omega + 2\gamma\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda + \gamma T\|\partial_x u_0\|_\Lambda^2 \\ &= \|{}_0D_t^{\frac{\alpha}{2}} u_L^h\|_\Omega \int_I t^{-\frac{\alpha}{2}} dt \|u_0\|_\Lambda + 2\gamma\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda + \gamma T\|\partial_x u_0\|_\Lambda^2 \\ &= \frac{T^{\frac{\alpha}{2}}}{1 - \frac{\alpha}{2}} \|{}_0D_t^{\frac{\alpha}{2}} u_L^h\|_\Omega \|u_0\|_\Lambda + 2\gamma\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda + \gamma T\|\partial_x u_0\|_\Lambda^2 \\ &\leq \frac{1}{2}\|{}_0D_t^{\frac{\alpha}{2}} u_L^h\|_\Omega^2 + \frac{T^\alpha}{2(1 - \frac{\alpha}{2})^2} \|u_0\|_\Lambda^2 + 2\gamma\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda + \gamma T\|\partial_x u_0\|_\Lambda^2. \end{aligned} \quad (4.18)$$

Then, we derive from (2.10) and (4.11) that

$$\begin{aligned} & \|{}_0D_t^{\frac{\alpha}{2}} u_L^h\|_\Omega^2 + \gamma\|\partial_x u_L^h\|_\Omega^2 = ({}_0D_t^{\frac{\alpha}{2}} u_L^h, {}_tD_T^{\frac{\alpha}{2}} u_L^h)_\Omega + \gamma\|\partial_x u_L^h\|_\Omega^2 \\ &\leq \frac{1}{2}\|{}_0D_t^{\frac{\alpha}{2}} u_L^h\|_\Omega^2 + \frac{T^\alpha}{2(1 - \frac{\alpha}{2})^2} \|u_0\|_\Lambda^2 + \frac{\gamma}{2}\|\partial_x u_L^h\|_\Omega^2 + \frac{1}{4}\|\partial_x u_0\|_\Lambda^2 + \gamma T\|\partial_x u_0\|_\Lambda^2. \end{aligned} \quad (4.19)$$

or equivalently,

$$\|{}_0D_t^{\frac{\alpha}{2}} u_L^h\|_\Omega^2 + \|\partial_x u_L^h\|_\Omega^2 \lesssim \|u_0\|_\Lambda^2 + \|\partial_x u_0\|_\Lambda^2. \quad (4.20)$$

Thus, by using Lemma 2.4, we can derive the desired result. \square

Theorem 4.2. Let u^h be the solution of (3.19) and u_L^h be the numerical solution (3.20) with $\tilde{\mathcal{N}}(u^h, t) = \frac{1}{2}\partial_x(u^h + u_0)^2$. If $u^h \in H^\alpha(I; H_0^1(\Lambda)) \cap E^r(\Omega) \cap K^s(\Omega)$, there holds

$$\|{}_0D_t^\alpha(u^h - u_L^h)\|_\Omega + \|\partial_x(u^h - u_L^h)\|_\Omega \lesssim M^{1-r}\|u^h\|_{E^r(\Omega)} + N^{-s}\|u^h\|_{K^s(\Omega)}. \quad (4.21)$$

Proof. We derive from (3.19) and (3.20) that

$$\begin{aligned} \tilde{a}(e_L, v) &:= ({}_0D_t^\alpha e_L, v)_\Omega + \gamma(\partial_x e_L, \partial_x v)_\Omega \\ &= ({}_0D_t^\alpha(\tilde{u}_L^h - u^h), v)_\Omega + \gamma(\partial_x(\tilde{u}_L^h - u^h), \partial_x v)_\Omega + (\tilde{\mathcal{N}}(u_L^h, t) - \tilde{\mathcal{N}}(\tilde{u}_L^h, t), v)_\Omega \\ &\quad + (\tilde{\mathcal{N}}(\tilde{u}_L^h, t) - \tilde{\mathcal{N}}(u^h, t), v)_\Omega, \quad \forall v \in V_M \otimes S_N. \end{aligned} \quad (4.22)$$

Thanks to (4.1) and (4.4), the above equation can be simplified to

$$\begin{aligned} \tilde{a}(e_L, v) &= ({}_0D_t^\alpha(\Pi_M^{(1,0)} u^h - u^h), v)_\Omega + \gamma(\partial_x(\pi_N^{(-\alpha, \alpha)} u^h - u^h), \partial_x v)_\Omega \\ &\quad + (\tilde{\mathcal{N}}(u_L^h, t) - \tilde{\mathcal{N}}(\tilde{u}_L^h, t), v)_\Omega + (\tilde{\mathcal{N}}(\tilde{u}_L^h, t) - \tilde{\mathcal{N}}(u^h, t), v)_\Omega. \end{aligned} \quad (4.23)$$

Taking $v = {}_0D_t^\alpha e_L (\in V_M \otimes S_N)$ in the above equation, we obtain

$$\begin{aligned} & ({}_0D_t^\alpha e_L, {}_0D_t^\alpha e_L)_\Omega + \gamma(\partial_x e_L, \partial_x ({}_0D_t^\alpha e_L))_\Omega \\ &= ({}_0D_t^\alpha (\Pi_M^{(1,0)} u^h - u^h), {}_0D_t^\alpha e_L)_\Omega + \gamma(\partial_x (\pi_N^{(-\alpha, \alpha)} u^h - u^h), \partial_x ({}_0D_t^\alpha e_L))_\Omega \\ & \quad + (\tilde{\mathcal{N}}(u_L^h, t) - \tilde{\mathcal{N}}(\tilde{u}_L^h, t), {}_0D_t^\alpha e_L)_\Omega + (\tilde{\mathcal{N}}(\tilde{u}_L^h, t) - \tilde{\mathcal{N}}(u^h, t), {}_0D_t^\alpha e_L)_\Omega. \end{aligned} \quad (4.24)$$

Since $\tilde{\mathcal{N}}(u^h, t) = \frac{1}{2} \partial_x (u^h + u_0)^2$, we derive from (4.12) and integration by parts that

$$\begin{aligned} \|{}_0D_t^\alpha e_L\|_\Omega^2 + \|\partial_x e_L\|_\Omega^2 &\lesssim \|{}_0D_t^\alpha (\Pi_M^{(1,0)} u^h - u^h)\|_\Omega \|{}_0D_t^\alpha e_L\|_\Omega \\ & \quad + \|\partial_x^2 (\pi_N^{(-\alpha, \alpha)} u^h - u^h)\|_\Omega \|{}_0D_t^\alpha e_L\|_\Omega \\ & \quad + \|\partial_x (u_L^h + u_0)^2 - \partial_x (\tilde{u}_L^h + u_0)^2\|_\Omega \|{}_0D_t^\alpha e_L\|_\Omega \\ & \quad + \|\partial_x (\tilde{u}_L^h + u_0)^2 - \partial_x (u^h + u_0)^2\|_\Omega \|{}_0D_t^\alpha e_L\|_\Omega, \end{aligned}$$

which leads to

$$\begin{aligned} \|{}_0D_t^\alpha e_L\|_\Omega + \|\partial_x e_L\|_\Omega &\lesssim \|{}_0D_t^\alpha (\Pi_M^{(1,0)} u^h - u^h)\|_\Omega + \|\partial_x^2 (\pi_N^{(-\alpha, \alpha)} u^h - u^h)\|_\Omega \\ & \quad + \|\partial_x (u_L^h + u_0)^2 - \partial_x (\tilde{u}_L^h + u_0)^2\|_\Omega + \|\partial_x (\tilde{u}_L^h + u_0)^2 - \partial_x (u^h + u_0)^2\|_\Omega. \end{aligned} \quad (4.25)$$

The two terms at the right-hand side can be bounded by using Lemma 4.5 and Lemma 4.6 as follows:

$$\begin{aligned} \|{}_0D_t^\alpha (\Pi_M^{(1,0)} u^h - u^h)\|_\Omega &\lesssim M^{1-r} \|\partial_x^{r-1} ({}_0D_t^\alpha u^h)\|_{L^2_{\chi^{r-2}}(\Lambda, L^2(I))}, \\ \|\partial_x^2 (\pi_N^{(-\alpha, \alpha)} u^h - u^h)\|_\Omega &\lesssim N^{-(\alpha+s)} \|\partial_x^2 ({}_0D_t^{\alpha+s} u^h)\|_{L^2(\Lambda, L^2_{\omega(s,s)}(I))}, \end{aligned}$$

and

$$\begin{aligned} \|\partial_x (u_L^h + u_0)^2 - \partial_x (\tilde{u}_L^h + u_0)^2\|_\Omega &\lesssim \|u_L^h + \tilde{u}_L^h\|_\Omega \|\partial_x e_L\|_\Omega \lesssim \|\partial_x e_L\|_\Omega, \\ \|\partial_x (\tilde{u}_L^h + u_0)^2 - \partial_x (u^h + u_0)^2\|_\Omega &\lesssim \|\tilde{u}_L^h + u^h\|_\Omega \|\partial_x (\tilde{u}_L^h - u^h)\|_\Omega. \end{aligned}$$

A combination of the above and (4.14) leads to

$$\begin{aligned} \|{}_0D_t^\alpha e_L\|_\Omega + \|\partial_x e_L\|_\Omega &\lesssim M^{1-r} \|\partial_x^{r-1} ({}_0D_t^\alpha u^h)\|_{L^2_{\chi^{r-2}}(\Lambda, L^2(I))} \\ & \quad + N^{-(\alpha+s)} \|\partial_x^2 ({}_0D_t^{\alpha+s} u^h)\|_{L^2(\Lambda, L^2_{\omega(s,s)}(I))} \\ & \quad + N^{-(\alpha+s)} \|\partial_x ({}_0D_t^{\alpha+s} u^h)\|_{L^2(\Lambda, L^2_{\omega(s,s)}(I))} + M^{1-r} \|\partial_x^r u^h\|_{L^2_{\chi^{r-1}}(\Lambda, L^2(I))}. \end{aligned} \quad (4.26)$$

On the other hand, we have $u^h - u_L^h = u - \tilde{u}_L^h + e_L$. Consequently, the desired result follows from the above estimates and the triangle inequality. \square

5. NUMERICAL RESULTS FOR PETROV-GALERKIN METHOD

We now present some numerical results obtained by the proposed space-time spectral methods introduced in Section 3 for both linear and nonlinear cases.

5.1. Numerical Results for time fractional diffusion equation (3.1). we present below three numerical experiments to validate our algorithms and error estimates.

Example 1. We consider (3.1) with the exact solution $u(x, t) = \sin(\pi x) \cdot t^\alpha \sin(\pi t)$, and non-homogeneous initial conditions $u_0(x) = \sin(\pi x)$. The main purpose is to check the convergence behavior of numerical solutions with respect to M and N for several α . In Figure 5.1, we list the $B^\alpha(\Omega)$ -errors in semi-log scale. Clearly, the exact solution has a singularity at $t = 0$, but for the weighted Sobolev spaces involving fractional derivatives $\mathcal{B}_{-\alpha, \alpha}^s(I)$, we have $u|_t \in \mathcal{B}_{-\alpha, \alpha}^s(I)$ for any large s so spectral accuracy is still expected despite the singularity at $t = 0$. We observe from Figure 5.1 that the numerical errors decay exponentially as M or N increases.

Example 2. We consider time fractional subdiffusion equation (3.1) with the exact solution $u(x, t) = \sin(\pi x) \cdot \sin(\pi t^\alpha)$, and the homogeneous initial conditions $u_0(x) = 0$. The singularity at $t = 0$ is more complicated than Example 1, and its approximation in time can not achieve spectral accuracy. In Figure 5.2 (a), we list the $B^\alpha(\Omega)$ -errors in semi-log scale against various M and α with $N = 40$. We also list the $B^\alpha(\Omega)$ -errors in log-log scale against various N and α with $M = 20$ in Figure 5.2 (b). We observe that the numerical errors decay exponentially as M increases since the solution is smooth in x , but only algebraically, as N increases since the GLF basis can only resolve the leading singular term of the solution.

Example 3. We consider the time fractional subdiffusion equation (3.1) with a given source function $f(x, t) = \sin(\pi x) \cdot \sin(\pi t)$, and homogeneous initial conditions. The exact solution is unknown but expected to have singularity at $t = 0$ due to the time fractional derivative. We compute a numerical solution with $M = 40$ and $N = 80$ as the reference solution. In Figure 5.3 (a), we list the $B^\alpha(\Omega)$ -errors in log-log scale against various N and α with $M = 20$. As expected, we observe that the numerical errors in time variable decay algebraically. In Figure 5.3 (b), we list the $B^\alpha(\Omega)$ -errors in semi-log scale against various M and α with $N = 80$. We observe that the errors decay exponentially.

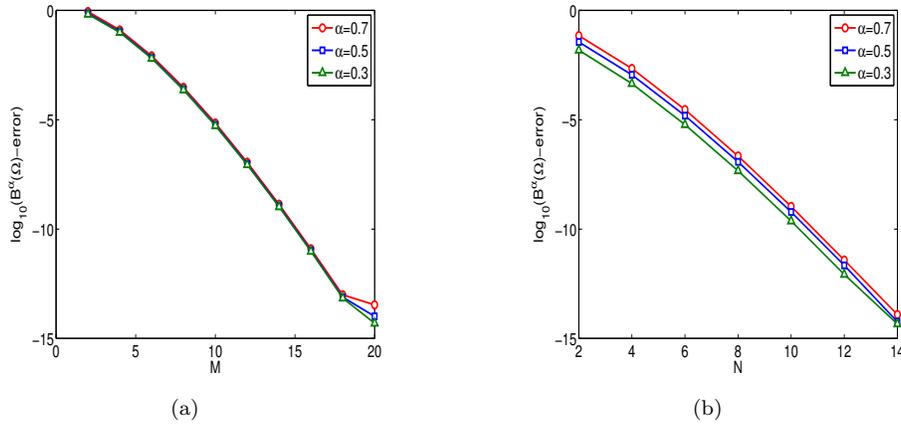


FIGURE 5.1. (a). The $B^\alpha(\Omega)$ -error in semi-log scale against various M and α for Example 1 with $N = 20$; (b). The $B^\alpha(\Omega)$ -error in semi-log scale against various N and α for Example 1 with $M = 20$.

5.2. Numerical Results for nonlinear problem (1.1). Consider the following time fractional Burgers equation:

$${}_0^C D_t^\alpha u(x, t) - \gamma \Delta u(x, t) + \partial_x \left(\frac{1}{2} u^2(x, t) \right) = 0, \quad \forall (x, t) \in \Omega, \quad (5.1)$$

with the initial data $u_0(x) = -\sin(\pi x)$.

We initially solved this problem by using the preconditioned Jacobian-free Newton-Krylov method. For the case with $N = 40$, $M = 60$, $\alpha = 0.7$, and $\gamma = 0.025$, 90 iterations are needed to achieve 10^{-7} -digit accuracy. To accelerate the convergence, we employed a two-grid approach. Namely, we use the coarse approximation solution with $\tilde{N} \approx N/2$ and $\tilde{M} \approx M/2$ as the initial data for the iteration for the fine approximation solution with N and M . This two-grid approach significantly improved the convergence, only less than 10 iterations are needed to achieve 10^{-7} -digit accuracy.

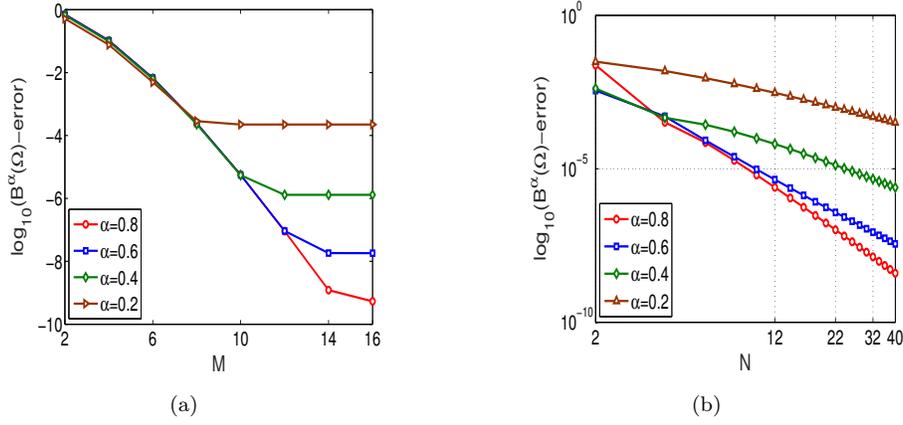


FIGURE 5.2. (a). The $B^\alpha(\Omega)$ -error in semi-log scale against various M and α for Example 2 with $N = 40$; (b). The $B^\alpha(\Omega)$ -error in log-log scale against various N and α for Example 2 with $M = 20$.

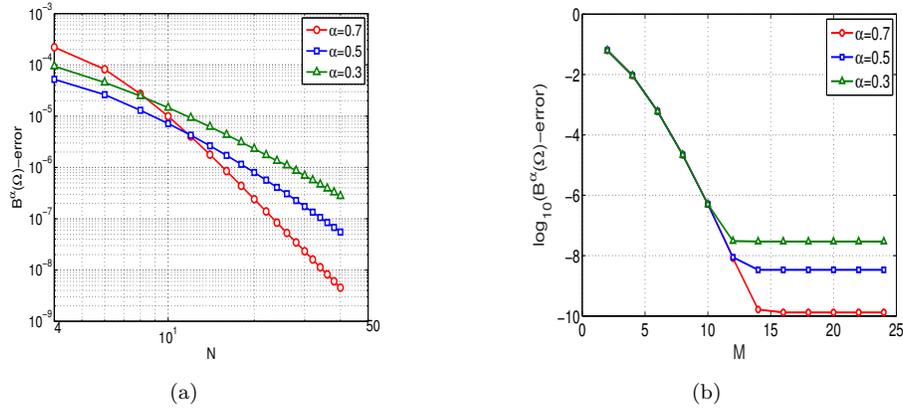


FIGURE 5.3. (a). The $B^\alpha(\Omega)$ -error in log-log scale against various N and α for Example 3 with $M = 20$; (b). The $B^\alpha(\Omega)$ -error in semi-log scale against various M and α for Example 3 with $N = 80$.

The numerical solutions obtained using Petrov-Galerkin spectral method with $M = 40$, $N = 60$, $\alpha = 0.7$, $\gamma = 0.025$ for various time are shown in Figure 5.4 (a). We compute a numerical solution with $M = 40$ and $N = 100$ as a reference solution. In Figure 5.4 (b), we plot the $B^\alpha(\Omega)$ -errors in log-log scale against various N and α of (5.1).

We plots in Figure 5.5 (a) the numerical solution with $M = 40$, $N = 60$, $\alpha = 0.7$, $\gamma = 0.025$ at $t = 0, 0.1, 0.25, 0.5, 1, 1.5$. We observe that, similar to the usual Burgers equation, the profile of solution become steeper as t increases to about $t = 0.25$, and the solution start to relax towards zero for $t > 0.25$. To investigate the influence of α on the solution, we plot the numerical solution with $\gamma = 0.025$, $N = 60$, $M = 100$ at $t = 0.971$ for various α in Figure 5.5 (b). As a comparison, we also plot the solution with $\alpha = 1$, i.e., the usual Burgers equation. We observe that, as α increases, the profile of the solution becomes steeper and dissipates faster.

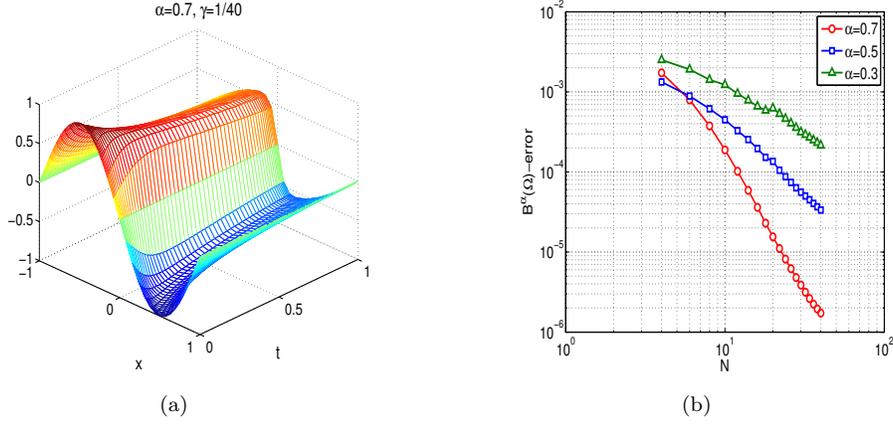


FIGURE 5.4. (a). The numerical solution with $\alpha = 0.7$, $\gamma = 0.025$, $N = 60$, $M = 40$; (b). The $B^\alpha(\Omega)$ -error in log-log scale against various N and α for (5.1).

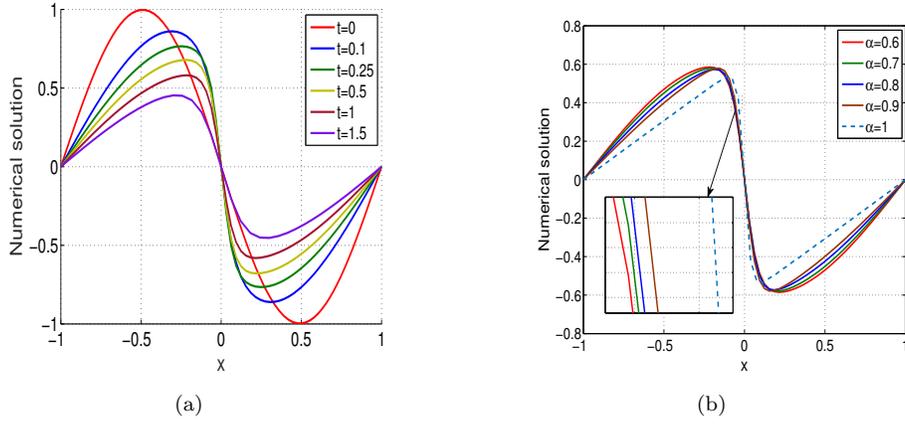


FIGURE 5.5. (a). The numerical solution with $\alpha = 0.7$, $\gamma = 0.025$, $N = 60$, $M = 40$; (b). The numerical solution with $\gamma = 0.025$, $N = 60$, $M = 100$ at $t = 0.971$.

5.3. Numerical Results for nonlinear system. In this subsection, we shall consider the following time fractional Volterra-Lotka systems with diffusion (cf. [6]):

$$\begin{cases} {}_0^C D_t^\alpha u - d_1 \Delta u - u(\alpha_1 - \beta_1 u - \gamma_1 v - \delta_1 w) = 0, & x \in \Lambda, \\ {}_0^C D_t^\alpha v - d_2 \Delta v - v(\alpha_2 - \beta_2 u - \gamma_2 v - \delta_2 w) = 0, & x \in \Lambda, \\ {}_0^C D_t^\alpha w - d_3 \Delta w - w(s + \varepsilon_1 u - \varepsilon_2 v) = 0, & x \in \Lambda, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \end{cases} \quad (5.2)$$

with homogeneous boundary conditions (1.2). Clearly, the above system can be derived in a similar fashion as in section 3.1, we obtain that

$$\begin{cases} {}_0D_t^\alpha u^h - d_1 \Delta u^h - (u^h + u_0)(\alpha_1 - \beta_1(u^h + u_0) - \gamma_1(v^h + v_0) - \delta_1(w^h + w_0)) = g_1, & x \in \Lambda, \\ {}_0D_t^\alpha v^h - d_2 \Delta v^h - (v^h + v_0)(\alpha_2 - \beta_2(u^h + u_0) - \gamma_2(v^h + v_0) - \delta_2(w^h + w_0)) = g_2, & x \in \Lambda, \\ {}_0D_t^\alpha w^h - d_3 \Delta w^h - (w^h + w_0)(s + \varepsilon_1(u^h + u_0) - \varepsilon_2(v^h + v_0)) = g_3, & x \in \Lambda, \\ u^h(x, 0) = 0, \quad v^h(x, 0) = 0, \quad w^h(x, 0) = 0, \end{cases} \quad (5.3)$$

where $g_1 = d_1 u_0''$, $g_2 = d_2 v_0''$, $g_3 = d_3 w_0''$.

The corresponding space-time Petrov-Galerkin spectral method for (5.3) are to find $u_L^h(x, t)$, $v_L^h(x, t)$, $w_L^h(x, t) \in V_M \otimes \mathcal{F}_N^{(\alpha)}$, such that

$$\begin{cases} ({}_0D_t^\alpha u_L^h, v)_\Omega + \gamma(\partial_x u_L^h, \partial_x v)_\Omega + (\mathcal{N}_1(u_L^h, v_L^h, w_L^h, t), v)_\Omega = (g_1, v)_\Omega, & v \in V_M \otimes S_N^{(\alpha)}, \\ ({}_0D_t^\alpha v_L^h, v)_\Omega + \gamma(\partial_x v_L^h, \partial_x v)_\Omega + (\mathcal{N}_2(u_L^h, v_L^h, w_L^h, t), v)_\Omega = (g_2, v)_\Omega, & v \in V_M \otimes S_N^{(\alpha)}, \\ ({}_0D_t^\alpha w_L^h, v)_\Omega + \gamma(\partial_x w_L^h, \partial_x v)_\Omega + (\mathcal{N}_3(u_L^h, v_L^h, w_L^h, t), v)_\Omega = (g_3, v)_\Omega, & v \in V_M \otimes S_N^{(\alpha)}, \end{cases} \quad (5.4)$$

where

$$\begin{aligned} \mathcal{N}_1(u_L^h, v_L^h, w_L^h, t) &= -(u^h + u_0)(\alpha_1 - \beta_1(u^h + u_0) - \gamma_1(v^h + v_0) - \delta_1(w^h + w_0)), \\ \mathcal{N}_2(u_L^h, v_L^h, w_L^h, t) &= -(v^h + v_0)(\alpha_2 - \beta_2(u^h + u_0) - \gamma_2(v^h + v_0) - \delta_2(w^h + w_0)), \\ \mathcal{N}_3(u_L^h, v_L^h, w_L^h, t) &= -(w^h + w_0)(s + \varepsilon_1(u^h + u_0) - \varepsilon_2(v^h + v_0)). \end{aligned}$$

Set

$$\begin{aligned} u_L^h(x, t) &= \sum_{m=0}^{M-2} \sum_{n=0}^N \tilde{u}_{nm}^h \phi_m(x) J_n^{(-\alpha, \alpha)}(t), & v_L^h(x, t) &= \sum_{m=0}^{M-2} \sum_{n=0}^N \tilde{v}_{nm}^h \phi_m(x) J_n^{(-\alpha, \alpha)}(t), \\ w_L^h(x, t) &= \sum_{m=0}^{M-2} \sum_{n=0}^N \tilde{w}_{nm}^h \phi_m(x) J_n^{(-\alpha, \alpha)}(t), & {}^1\tilde{n}_{nm} &= (\mathcal{N}_1(u_L^h, v_L^h, w_L^h, t), \phi_m(x) L_n^{(\alpha)}(t)), \\ {}^2\tilde{n}_{nm} &= (\mathcal{N}_2(u_L^h, v_L^h, w_L^h, t), \phi_m(x) L_n^{(\alpha)}(t)), & {}^3\tilde{n}_{nm} &= (\mathcal{N}_3(u_L^h, v_L^h, w_L^h, t), \phi_m(x) L_n^{(\alpha)}(t)), \\ \mathbf{u}_m^h &= (\tilde{u}_{0,m}^h, \tilde{u}_{1,m}^h, \dots, \tilde{u}_{N,m}^h)^T, & \mathbf{v}_m^h &= (\tilde{v}_{0,m}^h, \tilde{v}_{1,m}^h, \dots, \tilde{v}_{N,m}^h)^T, \\ \mathbf{w}_m^h &= (\tilde{w}_{0,m}^h, \tilde{w}_{1,m}^h, \dots, \tilde{w}_{N,m}^h)^T, & {}^1\mathbf{n}_m &= (\tilde{n}_{0,m}^1, \tilde{n}_{1,m}^1, \dots, \tilde{n}_{N,m}^1)^T, \\ {}^2\mathbf{n}_m &= (\tilde{n}_{0,m}^2, \tilde{n}_{1,m}^2, \dots, \tilde{n}_{N,m}^2)^T, & {}^3\mathbf{n}_m &= (\tilde{n}_{0,m}^3, \tilde{n}_{1,m}^3, \dots, \tilde{n}_{N,m}^3)^T. \end{aligned} \quad (5.5)$$

Similarly, we obtain the following linear system:

$$\begin{cases} (\mathbf{I}_{N+1} + d_1 \lambda_m^x \mathbf{M}^t) \mathbf{u}_m^h + {}^1\mathbf{n}_m(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h) = \mathbf{g}_m^1, & 0 \leq m \leq M-2, \\ (\mathbf{I}_{N+1} + d_2 \lambda_m^x \mathbf{M}^t) \mathbf{v}_m^h + {}^2\mathbf{n}_m(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h) = \mathbf{g}_m^2, & 0 \leq m \leq M-2, \\ (\mathbf{I}_{N+1} + d_3 \lambda_m^x \mathbf{M}^t) \mathbf{w}_m^h + {}^3\mathbf{n}_m(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h) = \mathbf{g}_m^3, & 0 \leq m \leq M-2, \end{cases} \quad (5.6)$$

which can be solved by using the preconditioned Jacobian-free Newton-Krylov method as in the previous case.

Nest, we present numerical results for the following two cases.

Example 4. We let f and the initial data be such that the equation (5.2) has the solution $u(x, t) = \sin(\pi x)(t^\alpha \sin(\pi t) + 1)$, $v(x, t) = \sin(\pi x)(t^\alpha \cos(\pi t) + 1)$, $w(x, t) = \sin(\pi x)(t^\alpha e^{-t} + 1)$. In Figure 5.6, we list the max-errors of example 4 with $\alpha_1 = \frac{1}{2}$, $\beta_1 = 1$, $\gamma_1 = 0.4$, $\delta_1 = 0.3$, $\alpha_2 = 0.34$, $\beta_2 = 1$, $\gamma_2 = 0.3$, $\delta_2 = 0.1$, $s = 1$, $\varepsilon_1 = 2/3$, $\varepsilon_2 = 0.1$, $d_1 = \frac{1}{2}$, $d_2 = 0.7$, $d_3 = 1$. We observe that

the numerical errors decay exponentially as M and N increases. They indicate that our algorithm is effective for this special problem.

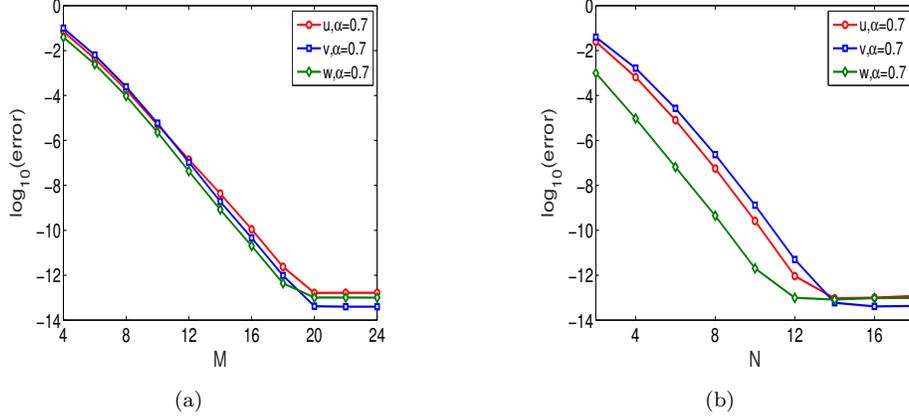


FIGURE 5.6. (a). The spatial max-errors in semi-log scale against M with $\alpha = 0.7$ and $N = 20$ for example 4; (b). The time max-errors in semi-log scale against N with $\alpha = 0.7$ and $M = 30$ for example 4.

Example 5. We consider (5.2) with the initial data $u_0(x) = v_0(x) = w_0(x) = \sin(\pi x)$. To examine the error of spatial and time direction, we fix $M = 50$ and $N = 80$ as the exact solution. Figure 5.7 (a) plot the max-errors in semi-log scale against various M . The figure clearly indicate that the error decays like $O(e^{-cM})$. In Figure 5.7 (b), we plot the max-errors in log-log scale against various N . They indicate that the numerical errors decay algebraically as N increases. We also present plots of the numerical solution u_L, v_L, w_L with $\alpha = 0.7$ $M = 40, M = 60$ for various time in Figure 5.8.

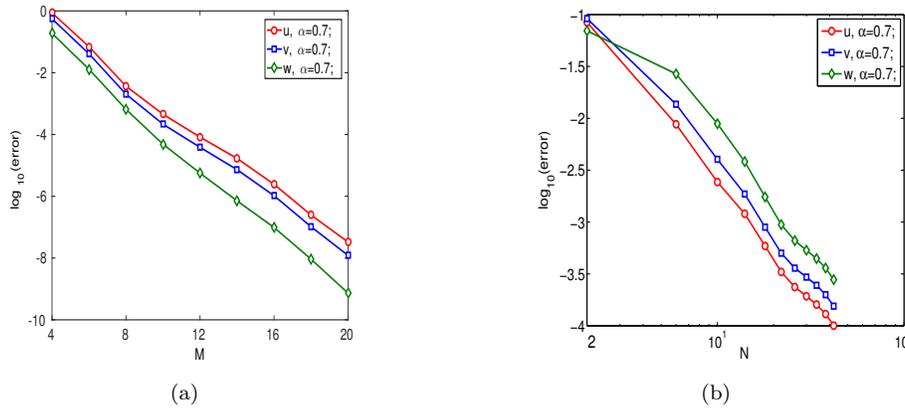


FIGURE 5.7. (a). The spatial max-errors in semi-log scale against M with $\alpha = 0.7$ for example 5; (b). The time max-errors in log-log scale against N with $\alpha = 0.7$ for example 5.

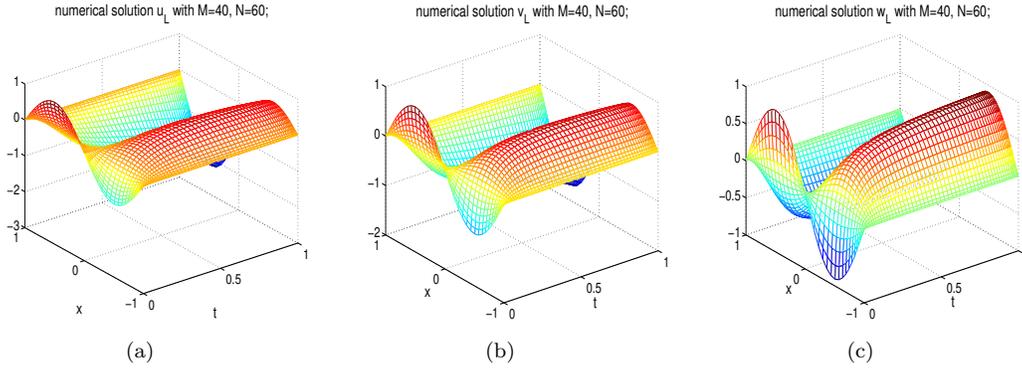


FIGURE 5.8. (a) (b) (c): the evolution of numerical solution u_L , v_L , w_L with $\alpha = 0.7$ $M = 40$, $M = 60$ for example 5;

6. CONCLUDING REMARKS

We developed a new space-time spectral method for nonlinear fractional subdiffusion equation. Our new scheme is based on a set of Fourier-like basis functions in the spatial variable and GJFs in time variable. The Fourier-like basis functions are discrete eigenfunctions of the Laplace operator, and lead to diagonal stiffness and mass matrices; The GJFs are chosen to match the leading singularity of the underlying problem so that they provide better performance than polynomial basis. We also presented error analysis for typical linear and nonlinear problems, and numerical results to validate our algorithms and error estimates.

Although we only dealt with one-dimensional spatial direction in this paper, the algorithm and the error analysis can be directly extended to multiple spatial dimensions of tensor product domains.

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