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Long Time Stability and Convergence for Fully Discrete Nonlinear Galerkin Methods

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Abstract The aim of this paper is to analyze the fully discrete nonlinear Galerkin methods, which are well suited to the long time integration of dissipative partial differential equations.
With the help of several time discrete Gronwall lemmas, we are able to prove the $L^\infty(\mathbb{R}^+, H^\alpha)$ $(\alpha = 0, 1)$ stabilities of the fully discrete nonlinear Galerkin methods under a less restrictive time step constraint than that of the classical Galerkin methods.

KEY WORDS: Nonlinear Galerkin methods, long time stability.
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1 Introduction and description of the method

The long time integration of the Navier-Stokes equations (N.S.E) is of great importance for numerical approximations of the permanent regime of flows. It is well known that the permanent regime of the flows can be represented by a finite number of determining modes, e.g. by the universal (global) attractors or the inertial manifolds (if they exist) whose dimensions are finite (see for instance [9]). One of the difficulties in numerical simulations of the permanent regime of the flows is to construct an appropriate finite dimensional system which can capture the long time behavior of these flows.

The inertial manifold (see [2], [9]), whenever it exists, is a positively invariant finite dimensional Lipschitz manifold which attracts exponentially all the trajectories, whereas the convergence of the trajectories towards the attractor can be very slow. Although the existence of inertial manifolds for some dynamical systems, for instance the 2-D N.S.E., is still unknown, it has been proven that the approximate inertial manifolds (see [1], [11], [12]) provide better approximations to the solution than the flat manifold $P_m H$ (see the definition below). Therefore, it is of interest to construct numerical schemes corresponding to these approximate inertial manifolds. This observation motivated the construction of the nonlinear Galerkin methods (see [6]) and the numerical tests presented in [3] and [10].

We consider in this paper time discretizations of the nonlinear Galerkin methods. In order to make the implementation of the schemes simpler, we restrict ourselves

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to schemes of semi-implicit type: here the dissipative term is treated implicitly to avoid severe time step constraints while keeping the nonlinear terms explicit so that the corresponding discrete systems are easily invertible. It is well known that this type of schemes is only stable under a restriction on the time step size, which has an important impact on the efficiency of the schemes since we are interested in the long time integrations.

The upper bound and the error estimate for approximate solutions to evolutionary partial differential equations, given by a large number of existing stability analyses, often increases indefinitely when the time interval [0,T] goes to infinity. Such a stability result is certainly irrelevant for the long time integrations. By using several discrete analogs of Gronwall lemmas, which are essential for proving stabilities in arbitrary large time intervals, we are able to show that solutions of the fully discrete nonlinear Galerkin schemes are uniformly (independent of time and space mesh sizes) bounded in $L^\infty(\mathbb{R}^+; H^\alpha)$ $(\alpha = 0, 1)$ under a less restrictive constraint on the time step size than what should be verified by the classical Galerkin methods. The convergence of the schemes in corresponding functional spaces are also established, and the appropriate choice for the parameter d in nonlinear Galerkin methods (see below) is suggested as well. Let us mention that a local stability analysis for a discrete nonlinear Galerkin method was also carried out in [3].

The technique used here for proving the long time stability is quite general. It can be used to obtain uniform upper bounds and error estimates in large time intervals for a fairly large class of numerical schemes to some evolutionary partial differential equations (see already [7]).

To be more specific, we restrict ourselves to the 2-D N.S.E. Similar schemes and analyses are applicable to other dissipative dynamical systems.

1.1 Functional setting of the N.S.E.

The 2-D unsteady Navier-Stokes equations in the primitive variable formulation are written as:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = F \tag{1}$$

$$div u = 0 \tag{2}$$

$$u(0) = u_0 \tag{3}$$

where Ω is an open bounded set in $\mathbb{R}^2$ with sufficient smooth boundary, $\nu > 0$ is the kinematic viscosity and F=F(x,t) represents the external body force. The unknowns are the vector function u (velocity) and the scalar function p (pressure).

We will consider either the homogeneous Dirichlet boundary conditions, for which we denote:

$$V = \{v \in (H_0^1(\Omega))^2 : div v = 0\}$$

or the periodic boundary conditions for which

$$V = \{v \in (H_p^1(\Omega))^2 : divv = 0, \int_\Omega v(x)dx = 0\}.$$

In both cases, we set

$$H = \text{closure of V in } (L^2(\Omega))^2.$$

Let P be the orthonormal projection of $(L^2(\Omega))^2$ onto H, we define the Stokes operator

$$Au = -P\Delta u, \ \forall u \in D(A) = V \cap (H^2(\Omega))^2,$$

and the bilinear operator

$$B(u,v) = P[(u \cdot \nabla)v], \ \forall u, v \in V.$$

The Stokes operator A is an unbounded positive self-adjoint closed operator in H with domain $D(A)$ and its inverse A^{-1} is compact in H. Consequently, there exists an orthonormal basis of H consisting of the eigenvectors w_j of A:

$$Aw_j = \lambda_j w_j, \ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j \to +\infty.$$

We denote the norms in H and V respectively by

$$|u| = (\int_\Omega |u(x)|^2 dx)^{\frac{1}{2}} \text{ and } ||u|| = (\int_\Omega |\nabla u(x)|^2 dx)^{\frac{1}{2}}.$$

The corresponding scalar products are denoted by $(\cdot,\cdot)$ and $((\cdot,\cdot))$ respectively.

We define a trilinear form on $V \times V \times V$ by

$$b(u,v,w) = < B(u,v), w >_{V',V}, \ \forall u, v, w \in V.$$

It is easy to verify that b satisfies the following important property

$$b(u,v,w) = -b(u,w,v), \ \forall u, v, w \in V. \tag{4}$$

We recall some of the continuity properties satisfied by B and b:

$$|b(u,v,w)| \le c_1 |u|^{\frac{1}{2}} ||u||^{\frac{1}{2}} ||v|| |w|^{\frac{1}{2}} ||w||^{\frac{1}{2}}, \tag{5}$$

$$|B(u,v)| \le c_2 |u|^{\frac{1}{2}} ||u||^{\frac{1}{2}} ||v||^{\frac{1}{2}} |Av|^{\frac{1}{2}}, \tag{6}$$

$$|B(u,v)| \le c_3 |u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} ||v||. \tag{7}$$

Under the above notations, the system (1)-(3) is equivalent to the following abstract equation:

$$\frac{du}{dt} + \nu Au + B(u,u) = f \tag{8}$$

$$u(0) = u_0 \tag{9}$$

where f=PF. The following results are well known (see for instance [9]).

Theorem. *We assume $f \in L^\infty(\mathbb{R}^+; H)$. Then for $u_0 \in H$, the system (8)-(9) admits a unique solution $u \in C(\mathbb{R}^+; H) \cap L^2(0,T;V)$, $\forall T > 0$.*

Moreover, if $u_0 \in V$, then $u \in C(\mathbb{R}^+; V) \cap L^2(0,T;D(A))$, $\forall T > 0$.

We denote hereafter $M_1 = \sup_{t\geq 0} ||u(t)||$, $M_f = \sup_{t\geq 0} |f(t)|$.

1.2 Description of the nonlinear Galerkin methods

Let us first explain briefly the idea of Foias-Manley-Temam [1] for constructing an approximate inertial manifold of (8)-(9).

We select a cut-off value m and define

P_m : the projection operator onto $H_m = span\{w_1, \ldots, w_m\}$;

$Q_m = I - P_m$.

Therefore, we can write

$$u = P_m u + Q_m u = y_m + z_m.$$

y_m corresponding to the small eigenvalues represents the large eddies of the flow, while z_m corresponding to the large eigenvalues represents the small eddies. Now we apply respectively P_m and Q_m to (8):

$$\frac{\partial y_m}{\partial t} + \nu A y_m + P_m B(y_m + z_m, y_m + z_m) = P_m f, \tag{10}$$

$$\frac{\partial z_m}{\partial t} + \nu A z_m + Q_m B(y_m + z_m, y_m + z_m) = Q_m f. \tag{11}$$

It can be proven (see [1] and [11]) that z_m only carries a small part of the kinematic energy after a transient time, namely

$$\begin{cases} |z_m(t)| \leq k_0 L_m^{\frac{1}{2}} \lambda_{m+1}^{-1} \\ |z_m'(t)| \leq k_0 L_m^{\frac{1}{2}} \lambda_{m+1}^{-1} \end{cases}, \quad \text{for t large,} \tag{12}$$

where $L_m = 1 + \log \frac{\lambda_{m+1}}{\lambda_1}$. It is then reasonable to neglect z_m in some circumstances. This leads them [1] (see also [11], [12] for other type of approximations) to approximate (11) by

$$\nu A \tilde{z}_m + Q_m B(y_m, y_m) = Q_m f \tag{13}$$

since $|Q_m B(y_m + z_m, y_m + z_m)| \sim |Q_m B(y_m, y_m)| >> |z'_m(t)|$.

If we define

$$\Phi_1(p) = (\nu A)^{-1} Q_m [f - B(p,p)] \ , \ \forall \ p \in H_m,$$

then $\tilde{z}_m = \Phi_1(y^m)$. It has been proven that the finite dimensional manifold $\mathcal{M}_1 = \mathcal{M}_{1,m}$ defined as the graph of Φ_1 is a better approximate manifold to the universal attractor than the flat manifold $\mathcal{M}_0 = P_m H$, namely, we have (see [1])

$$dist(u(t), \mathcal{M}_1) \leq k_1 L_m \lambda_{m+1}^{-\frac{3}{2}}. \tag{14}$$

This result was the motivation of the paper by M. Marion & R. Temam [6] where they introduced a finite dimensional version of (13) called *nonlinear Galerkin methods.* The stability and convergence of the methods *without time discretizations* were established in [6]. Our aim in this paper is to analyze the time discretization of the nonlinear Galerkin methods.

Let $P = P_m$, $Q = P_{dm} - P_m$ ($d = d(m) > 0$ is of our choice), and $f^n = \frac{1}{k}\int_{nk}^{(n+1)k} f(t)dt$. Our first scheme (corresponding to the approximate inertial manifold $\mathcal{M}_1$) is the following:

Given $y_0 = Pu_0, z^0 = Qu_0$, find $y^{n+1} = y_m^{n+1} \in H_m$ and $z^{n+1} = z_m^{n+1} \in H_{dm} - H_m$ such that

$$\nu A z^{n+1} = Q[f^n - B(y^n, y^n)], \tag{15}$$

$$\frac{y^{n+1} - y^n}{k} + \nu A y^{n+1} = P[f^n - B(y^n, y^n) - B(z^{n+1}, y^n) - B(y^n, z^{n+1})] \tag{16}$$

where k is the time step.

The advantage of this scheme over the classical Galerkin scheme (which corresponds to (16) with $z^{n+1} = 0$) were clarified in [10], [6] and [3].

The efficiency of the scheme (comparing to the classical Galerkin scheme with dm modes) depends clearly on the choice of d. We suggest d to be chosen according to the following arguments.

We note that (15) defines a m-dimensional manifold $\mathcal{M}_{1,d}$ as the graph of

$$\Phi_{1,d}(p) = (\nu A)^{-1}(P_{dm} - P_m)[f - B(p,p)] \ , \ \forall \ p \in H_m.$$

Therefore, let $p = P_m u$, by using (14)

$$\begin{aligned} dist(u(t), \mathcal{M}_{1,d}) &\leq dist(u(t), \mathcal{M}_1) + dist(\mathcal{M}_1, \mathcal{M}_{1,d}) \\ &\leq k_1 L_m \lambda_{m+1}^{-\frac{3}{2}} + |\Phi_1(p) - \Phi_{1,d}(p)|. \end{aligned} \tag{17}$$

On the other hand, we derive from the definition of $\Phi_1, \Phi_{1,d}$ that

$$\nu A(\Phi_1(p) - \Phi_{1,d}(p)) = (I - P_{dm})[f - B(p,p)]. \tag{18}$$

We recall (see for instance [1]) that

$$B(u,v) \leq c_4||u||||v||(1 + log\frac{|Au|^2}{\lambda_1||u||^2})^{\frac{1}{2}}. \tag{19}$$

Hence

$$|B(p,p)| \leq c_4||p||^2(1 + log\frac{|Ap|^2}{\lambda_1||p||^2})^{\frac{1}{2}} \leq c_4 M_1^2 L_m^{\frac{1}{2}}.$$

Since $\Phi_1(p) - \Phi_{1,d}(p) \in (I - P_{dm})H$ and

$$|Aq| \geq \lambda_{dm+1}|q| \;,\; \forall\, q \in (I - P_{dm})H$$

we derive

$$\begin{aligned} \nu\lambda_{dm+1}|\Phi_1(p) - \Phi_{1,d}(p)| &\leq |\nu A(\Phi_1(p) - \Phi_{1,d}(p))| \\ &\leq |f| + |B(p,p)| \leq |f| + c_4 M_1^2 L_m^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$|\Phi_1(p) - \Phi_{1,d}(p)| \leq (\nu\lambda_{dm+1})^{-1}[|f| + c_4 M_1^2 L_m^{\frac{1}{2}}] \leq k_2 L_m^{\frac{1}{2}} \lambda_{dm+1}^{-1}.$$

Hence

$$dist(u(t), \mathcal{M}_{1,d}) \leq k_1 L_m \lambda_{m+1}^{-\frac{3}{2}} + k_2 L_m^{\frac{1}{2}} \lambda_{dm+1}^{-1}. \tag{20}$$

For fixed m, we should then choose d such that

$$k_1 L_m \lambda_{m+1}^{-\frac{3}{2}} \sim k_2 L_m^{\frac{1}{2}} \lambda_{dm+1}^{-1}. \tag{21}$$

We recall that for the 2-D N.S.E. (see [5]) $\lambda_m \sim m$. We then derive that (21) is equivalent to

$$d \sim \left(\frac{m}{logm}\right)^{\frac{1}{2}}. \tag{22}$$

We note that with this choice of d, the error for (15)-(16) is of the same order as that of the classical Galerkin scheme with dm modes (see (12)).

2 Uniform stability

From now on, we will use c_i to denote some absolute constants, R_i, b_i, B_i and G_i to denote constants depending on some data: $R_i = R_i(\nu, \lambda_1, f)$, $b_i = b_i(\nu, \lambda_1, f, u_0)$, $B_i = B_i(\nu, \lambda_1, f, u_0, n)$, $G_i = G_i(\nu, \lambda_1, f, u_0, T)$. We will assume hereafter $k \leq K_0$ (for some $K_0 > 0$ fixed).

2.1 Uniform stability in $L^\infty(\mathbb{R}^+; H)$

Theorem 1 *We assume that k and m are such that*

$$\begin{cases} 16c_1^2k^2\lambda_m^2 b_0 \leq \frac{1}{2} - \delta \\ 16c_1^2 k\lambda_m \nu^{-1} b_0 \leq 1 - \delta \end{cases} \tag{23}$$

where $\delta \in (0, \frac{1}{2})$ and

$$b_0 = |u_0|^2 + \frac{2(1 + K_0\nu\delta\lambda_1)}{\nu^2\lambda_1^2\delta} M_f^2. \tag{24}$$

Then, we have

$$|y^n|^2 \leq \frac{1}{(1 + k\nu\delta\lambda_1)^{n+1}} |u^0|^2 + \frac{2(1 + K_0\nu\delta\lambda_1)}{\nu^2\lambda_1^2\delta} M_f^2 = B_0(n) \leq b_0,$$

$$\forall T > 0,\ \delta \sum_{n=0}^{\frac{T}{k}-1} |y^{n+1} - y^n|^2 + k\nu\delta \sum_{n=0}^{\frac{T}{k}-1} [||y^{n+1}||^2 + ||z^{n+1}||^2] \leq b_0 + \frac{2T}{\nu\lambda_1} M_f^2 = G_0(T).$$

Remark 1 We emphysize that the stabiblity condition (23) only involves λ_m. In other word, no matter how large the d is, the time step constraints for (15)-(16) are always the same. It can be proven that the stability condition for the classical Galerkin scheme with dm modes is $k \sim \lambda_{dm}^{-1}$. This suggest that we can use larger time step size for the nonlinear Galerkin scheme (15)-(16) than for the classical Galerkin scheme. This may lead to substantial savings in cpu when doing long time integrations of the N.S.E.

Before proving Theorem 1, let us first recall a simple inequality which is the time discrete counterpart of the Gronwall lemma.

Lemma 1 *Let a^n, b^n be two positive series satisfying*

$$\frac{a^{n+1} - a^n}{k} + \lambda a^{n+1} \leq b^n \text{ and } \mathrm{b}^n \leq \mathrm{b},\ \forall \mathrm{n} \geq 0.$$

Then

$$a^n \leq \frac{1}{(1+k\lambda)^n}a^0 + \frac{1+k\lambda}{\lambda}(1-\frac{1}{(1+k\lambda)^{n+1}})b \ , \ \forall \ n \geq 0$$

provided $k, 1+k\lambda > 0$.

PROOF OF THE THEOREM 1: Taking the scalar product of (15) with $2kz^{n+1}$, (16) with $2ky^{n+1}$, by using the relation

$$(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2, \tag{25}$$

we obtain

$$2k\nu||z^{n+1}||^2 = 2k(f^n, z^{n+1}) - 2kb(y^n, y^n, z^{n+1}), \tag{26}$$

and

$$\begin{aligned} |y^{n+1}|^2 \quad &- \quad |y^n|^2 + |y^{n+1} - y^n|^2 + 2k\nu||y^{n+1}||^2 = 2k(f^n, y^{n+1}) \\ &- \quad 2k[b(y^n, y^n, y^{n+1}) + b(y^n, z^{n+1}, y^{n+1}) + b(z^{n+1}, y^n, y^{n+1})]. \end{aligned} \tag{27}$$

The following inequalities will be used repeatly in the rest of the paper.

$$\begin{cases} ||u|| \geq \lambda_1^{\frac{1}{2}}|u| \ , \ \forall \ u \in V \\ ||y|| \leq \lambda_m^{\frac{1}{2}}|y| \ , \ \forall \ y \in P_mV \\ ||z|| \geq \lambda_{m+1}^{\frac{1}{2}}|z| \geq \lambda_m^{\frac{1}{2}}|z| \ , \ \forall \ z \in (I-P_m)V \end{cases} \tag{28}$$

By using sucessively Schwarz inequality and (28), we derive

$$\begin{aligned} 2k(f^n, y^{n+1} + z^{n+1}) \quad &\leq \quad 2k|f^n||y^{n+1} + z^{n+1}| \\ &\leq \quad 2k\lambda_1^{-\frac{1}{2}}|f^n|(||y^{n+1}|| + ||z^{n+1}||) \\ &\leq \quad k\nu(||y^{n+1}||^2 + ||z^{n+1}||^2) + \frac{2k}{\nu\lambda_1}M_f^2. \end{aligned} \tag{29}$$

Then the summation of (26), (27) and (29) leads to

$$\begin{aligned} |y^{n+1}|^2 \quad &- \quad |y^n|^2 + |y^{n+1} - y^n|^2 + k\nu(||y^{n+1}||^2 + ||z^{n+1}||^2) \\ &\leq \quad \frac{2k}{\nu\lambda_1}M_f^2 - \{2kb(y^n, y^n, y^{n+1}) + 2kb(z^{n+1}, y^n, y^{n+1}) \\ &+ \quad 2k[b(y^n, z^{n+1}, y^{n+1}) + b(y^n, y^n, z^{n+1})]\}. \end{aligned} \tag{30}$$

The nonlinear terms in the above inequality can be majorized as follows:

By using repeatly (4), (5), (28) and Schwarz inequality, we obtain

$$\begin{aligned} 2kb(y^n, y^n, y^{n+1}) &= 2kb(y^n, y^n, y^{n+1} - y^n) \\ &\leq 2kc_1|y^n|^{\frac{1}{2}}||y^n||^{\frac{3}{2}}||y^{n+1} - y^n||^{\frac{1}{2}}|y^{n+1} - y^n|^{\frac{1}{2}} \\ &\leq 2kc_1\lambda_m^{\frac{1}{2}}|y^n| \cdot ||y^n|| \cdot |y^{n+1} - y^n| \\ &\leq \frac{1}{8}|y^{n+1} - y^n|^2 + 8c_1^2k^2\lambda_m|y^n|^2||y^n||^2, \end{aligned} \tag{31}$$

$$\begin{aligned} 2kb(z^{n+1}, y^n, y^{n+1}) &= 2kb(z^{n+1}, y^n, y^{n+1} - y^n) \\ &\leq 2kc_1|z^{n+1}|^{\frac{1}{2}}||z^{n+1}||^{\frac{1}{2}}||y^n||||y^{n+1} - y^n||^{\frac{1}{2}}|y^{n+1} - y^n|^{\frac{1}{2}} \\ &\leq 2kc_1||z^{n+1}|| \cdot ||y^n|| \cdot |y^{n+1} - y^n| \\ &\leq \frac{1}{8}|y^{n+1} - y^n|^2 + 8c_1^2k^2||y^n||^2||z^{n+1}||^2 \\ &\leq \frac{1}{8}|y^{n+1} - y^n|^2 + 8c_1^2k^2\lambda_m|y^n|^2||z^{n+1}||^2, \end{aligned} \tag{32}$$

and

$$\begin{aligned} 2k[b(y^n, z^{n+1}, y^{n+1}) &+ b(y^n, y^n, z^{n+1})] = 2kb(y^n, z^{n+1}, y^{n+1} - y^n) \\ &\leq 2kc_1|y^n|^{\frac{1}{2}}||y^n||^{\frac{1}{2}}||z^{n+1}||||y^{n+1} - y^n||^{\frac{1}{2}}|y^{n+1} - y^n|^{\frac{1}{2}} \\ &\leq 2kc_1\lambda_m^{\frac{1}{2}}||z^{n+1}|| \cdot |y^n| \cdot |y^{n+1} - y^n| \\ &\leq \frac{1}{4}|y^{n+1} - y^n|^2 + 4c_1^2k^2\lambda_m|y^n|^2||z^{n+1}||^2. \end{aligned} \tag{33}$$

Combining (31) to (33) into (30), we arrive to

$$\begin{aligned} |y^{n+1}|^2 &- |y^n|^2 + \frac{1}{2}|y^{n+1} - y^n|^2 + k\nu(||y^{n+1}||^2 + ||z^{n+1}||^2) \\ &\leq \frac{2k}{\nu\lambda_1}M_f^2 + 12k^2c_1^2\lambda_m|y^n|^2||z^{n+1}||^2 + 8c_1^2k^2\lambda_m|y^n|^2||y^n||^2. \end{aligned} \tag{34}$$

We derive from (28) that

$$\begin{aligned} ||y^n||^2 &\leq 2||y^{n+1} - y^n||^2 + 2||y^{n+1}||^2 \\ &\leq 2\lambda_m|y^{n+1} - y^n|^2 + 2||y^n||^2. \end{aligned} \tag{35}$$

Using (35), we can rewrite (34) as

$$\begin{aligned} |y^{n+1}|^2 &- |y^n|^2 + (\frac{1}{2} - 16c_1^2k^2\lambda_m^2|y^n|^2)|y^{n+1} - y^n|^2 \\ &+ k\nu(1 - 16c_1^2k\lambda_m\nu^{-1}|y^n|^2)(||y^{n+1}||^2 + ||z^{n+1}||^2) \\ &\leq \frac{2k}{\nu\lambda_1}M_f^2. \end{aligned} \tag{36}$$

Assuming that k and m verify the hypothesis (23), we are going to prove by induction that

$$|y^q|^2 \leq B_0(q) \leq b_0\ ,\ \forall\ q. \tag{37}$$

- (37) at $q = 0$ is obvious;
- assuming (37) is true up to $q = n$, then by using (23), the inequality (36) becomes

$$|y^{n+1}|^2 - |y^n|^2 + \delta|y^{n+1} - y^n|^2 + k\nu\delta(||y^{n+1}||^2 + ||z^{n+1}||^2) \leq \frac{2k}{\nu\lambda_1}M_f^2. \tag{38}$$

Therefore, by using (28) and dropping some unnecessary terms, we arrive to

$$\frac{|y^{n+1}|^2 - |y^n|^2}{k} + \nu\delta\lambda_1|y^{n+1}|^2 \leq \frac{2}{\nu\lambda_1}M_f^2.$$

We can now apply Lemma 1 to this last inequality with $a^n = |y^n|^2$, $b^n = \frac{2}{\nu\lambda_1}M_f^2$ and $\lambda = \nu\delta\lambda_1$. From which we derive

$$\begin{aligned} |y^{n+1}|^2 &\leq \frac{1}{(1 + k\nu\delta\lambda_1)^{n+1}}|y^0|^2 + \frac{2(1 + K_0\nu\delta\lambda_1)}{\nu^2\lambda_1^2\delta}(1 - \frac{1}{(1 + k\nu\delta\lambda_1)^{n+2}})M_f^2 \\ &\leq B_0(n+1) \leq |u^0|^2 + \frac{2(1 + K_0\nu\delta\lambda_1)}{\nu^2\lambda_1^2\delta}M_f^2 = b_0. \end{aligned} \tag{39}$$

The proof of (37) is complete.

In order to prove the last inequality of Theorem 1, $\forall T > 0$ given, we take the sum of (38) for n from 0 to $\frac{T}{k} - 1$, which lead to

$$|y^{\frac{T}{k}}|^2 + \delta\sum_{n=0}^{\frac{T}{k}-1}|y^{n+1} - y^n|^2 + k\nu\delta\sum_{n=0}^{\frac{T}{k}-1}[||y^{n+1}||^2 + ||z^n||^2] \leq b_0 + \frac{2T}{\nu\lambda_1}M_f^2 = G_0(T). \tag{40}$$

This completes the proof of Theorem 1. ¶

Corollary 1 *Under the hypotheses of Theorem 1, we have*

$$|z^n| \leq \frac{c_2}{\nu}B_0(n) + \frac{1}{\nu\lambda_1}M_f = B_1(n) \leq \frac{c_2}{\nu}b_0 + \frac{1}{\nu\lambda_1}M_f = b_1 \ , \ \forall \ n,$$

$$k\sum_{n=0}^{\frac{T}{k}-1}|z^n|^2 \leq \lambda_m^{-1}(\frac{2T}{\nu^2}M_f^2 + \frac{2c_2b_0}{\nu\delta}G_0(T)) = \lambda_m^{-1}G_1(T) \to 0 \ \ (\text{as m}^{-1} \to 0) \ , \ \forall \ \text{T} > 0.$$

Moreover, there exists $R_0 = R_0(\nu, \lambda_1, f)$ *such that* $\forall u_0 \in H$, *we can find* $N(u_0) > 0$ *such that for all* $n > N(u_0)$, *we have*

$$|y^n|^2 + |z^n|^2 \leq [\frac{6(1+K_0\nu\delta\lambda_1)}{\nu^2\lambda_1^2\delta} + \frac{4}{\nu^2\lambda_1^2}]M_f^2 = R_0. \tag{41}$$

PROOF: To majorize z^n, we take the scalar product of (15) with z^{n+1}

$$\nu||z^{n+1}||^2 = (f^n, z^{n+1}) - b(y^n, y^n, z^{n+1}).$$

By using (28), (6) and Schwarz inequality, we find

$$\nu\lambda_m|z^{n+1}|^2 \leq \nu||z^{n+1}||^2 \leq M_f|z^{n+1}| + c_2|y^n|^{\frac{1}{2}}||y^n|||Ay^n|^{\frac{1}{2}}|z^{n+1}|$$

which implies

$$\begin{aligned} \nu\lambda_m|z^{n+1}| &\leq M_f + c_2|y^n|^{\frac{1}{2}}||y^n|||Ay^n|^{\frac{1}{2}} \\ &\leq M_f + c_2\lambda_m^{\frac{1}{2}}|y^n|||y^n|| \\ &\leq M_f + c_2\lambda_m|y^n|^2 \ , \ \forall \ n. \end{aligned} \tag{42}$$

Therefore

$$|z^{n+1}|^2 \leq \frac{2}{\nu^2\lambda_m^2}M_f^2 + \frac{2c_2}{\lambda_m}|y^n|^2||y^n||^2.$$

By using the results of Theorem 1, we derive

$$\begin{aligned} k\sum_{n=0}^{\frac{T}{k}-1}|z^n|^2 &\leq \frac{2T}{\nu^2\lambda_m^2}M_f^2 + \frac{2c_2b_0}{\lambda_m}k\sum_{n=0}^{\frac{T}{k}-1}||y^n||^2 \\ &\leq \lambda_m^{-1}(\frac{2T}{\nu^2}M_f^2 + \frac{2c_2b_0}{\nu\delta}G_0(T)) \to 0 \ \ (\text{as m}^{-1} \to 0) \ , \ \forall \ \text{T} > 0, \end{aligned}$$

also from (42) and (39)

$$|z^{n+1}| \leq \frac{1}{\nu\lambda_1}M_f + \frac{c_2}{\nu}B_0(n) = B_1(n).$$

(41) is then a direct consequence of (39) and the last inequality. ¶

2.2 Uniform stability in $L^\infty(\mathbb{R}^+; V)$

We can also prove that the scheme (15)-(16) is stable in stronger topologies. To this end, we need the following time discrete Gronwall lemmas.

Let us recall first the time discrete counterpart of the usual Gronwall lemma.

Lemma 2 *Let d^n, g^n, h^n be three series satisfying*

$$\frac{d^{n+1} - d^n}{k} \leq g^n d^n + h^n \ , \ \forall \ n.$$

Then, $\forall N > 0$,

$$d^n \leq d^0 \exp(k\sum_{i=0}^{N} g^i) + k\sum_{i=0}^{N} h^i \exp(k\sum_{j=i}^{N} g^j) \ , \ \forall \ n \leq N+1.$$

PROOF: For the reader's convenience, we give below the proof of the lemma. Using recursively the following relation

$$d^{n+1} \leq (1 + kg^n)d^n + kh^n, \tag{43}$$

we derive

$$d^n \leq d^0 \prod_{i=0}^{n-1}(1 + kg^i) + k\sum_{i=0}^{n-1} h^i \prod_{j=i}^{n-1}(1 + kg^j).$$

On the other hand, since $(1 + x) \leq e^x \ , \ \forall \ x \in \mathbb{R}$, we derive

$$\prod_{i=q}^{n-1}(1 + kg^i) \leq \prod_{i=q}^{n-1}\exp(kg^i) = \exp(k\sum_{i=q}^{n-1} g^i) \ , \ \forall \ q \leq n-1.$$

Therefore

$$d^n \leq d^0 \exp(k\sum_{i=0}^{N} g^i) + k\sum_{i=0}^{N} h^i \exp(k\sum_{j=i}^{N} g^j) \ , \ \forall \ n \leq N+1. ¶$$

Now let us establish the time discrete counterpart of the uniform Gronwall lemma.

Lemma 3 *Let d^n, g^n, h^n be three series satisfying*

$$\frac{d^{n+1}-d^n}{k} \leq g^n d^n + h^n \ , \ \forall\ n \geq n_0,$$

and

$$\begin{cases} k\sum_{n=k_0}^{N+k_0} g^n \leq a_1 & \\ k\sum_{n=k_0}^{N+k_0} h^n \leq a_2 & , \ \forall\ k_0 \geq n_0 \\ k\sum_{n=k_0}^{N+k_0} d^n \leq a_3 & \end{cases}$$

with $kN = r$. Then

$$d^n \leq (a_2 + \frac{a_3}{r})\exp(a_1) \ , \ \forall\ n \geq n_0 + N.$$

PROOF: For any m_1, m_2 such that $n_0 \leq m_1 \leq m_2 \leq m_1 + N$, we use recursively (43) to get

$$\begin{aligned} d^{m_1+N} &= d^{m_2} \prod_{n=m_2}^{m_1+N-1} (1+kg^n) \\ &+ k \sum_{n=m_2}^{m_1+N-1} h^n \prod_{j=n}^{m_1+N-1} (1+kg^j). \end{aligned} \tag{44}$$

As in the proof of the previous lemma, we have

$$\begin{aligned} \prod_{n=m_1}^{m_1+N} (1+kg^n) &\leq \sum_{n=m_1}^{m_1+N} \exp(kg^i) \\ &= \exp(k \sum_{n=m_1}^{m_1+N} g^i) \leq \exp(a_1). \end{aligned}$$

Applying the above inequality to (44), we arrive to

$$\begin{aligned} d^{m_1+N} &\leq d^{m_2}\exp(a_1) + k \sum_{n=m_2}^{m_1+N-1} h^n \exp(a_1) \\ &\leq \exp(a_1)(d^{m_2} + a_2). \end{aligned}$$

We rewrite this inequality as

$$kd^{m_1+N} \leq \exp(a_1)(kd^{m_2} + ka_2).$$

Finally, by adding the above inequality for m_2 from m_1 to $m_1 + N - 1$, keeping in mind $kN = r$, we obtain

$$\begin{aligned} rd^{m_1+N} &\leq \exp(a_1)(k \sum_{m_2=m_1}^{m_1+N-1} d^{m_2} + ra_2) \\ &\leq \exp(a_1)(a_3 + ra_2).¶ \end{aligned}$$

Theorem 2 *We assume $u_0 \in V$, k,n satisfying (23) and in addition*

$$\frac{k\nu\lambda_m}{2} \leq 1 - \delta, \tag{45}$$

where δ is the same as in Theorem 1. Then, we have

$$||y^n||^2 \leq b_2 \ , \ \forall \ n,$$

$$\forall T > 0,\ \delta \sum_{n=0}^{\frac{T}{k}-1} ||y^{n+1} - y^n||^2 + \frac{k\nu}{2} \sum_{n=0}^{\frac{T}{k}-1} [|Ay^{n+1}|^2 + |Az^{n+1}|^2] \leq b_2 + \frac{2T}{\nu\lambda_1} M_f^2 = G_2(T),$$

where b_2 is to be given explicitly in the process of the proof.

PROOF: As in the proof of Theorem 1, we take the scalar product of (15) with $2kAz^{n+1}$, (16) with $2kAy^{n+1}$ respectively

$$2k\nu|Az^{n+1}|^2 = 2k(f^n, Az^{n+1}) - 2kb(y^n, y^n, Az^{n+1}),$$

and

$$\begin{aligned} ||y^{n+1}||^2 &- ||y^n||^2 + ||y^{n+1} - y^n||^2 + +2k\nu|Ay^{n+1}|^2 \leq 2k(f^n, Ay^{n+1}) \\ &- 2k[b(y^n, y^n, Ay^{n+1}) + b(y^n, z^{n+1}, Ay^{n+1}) + b(z^{n+1}, y^n, Ay^{n+1})]. \end{aligned}$$

We then add them up to get

$$\begin{aligned} ||y^{n+1}||^2 &- ||y^n||^2 + ||y^{n+1} - y^n||^2 + 2k\nu(A|y^{n+1}|^2 + |Az^{n+1}|^2) \\ &\leq 2k(f^n, Ay^{n+1} + Az^{n+1}) \\ &- \{2kb(y^n, y^n, Ay^{n+1}) + 2kb(z^{n+1}, y^n, Ay^{n+1}) \\ &+ 2kb(y^n, z^{n+1}, Ay^{n+1}) - 2kb(y^n, y^n, Az^{n+1})]\}. \end{aligned} \tag{46}$$

We majorize the right hand side terms as follows:

$$2k(f^n, Ay^{n+1} + Az^{n+1}) \leq 2k|f^n||Ay^{n+1} + Az^{n+1}| \leq \frac{k\nu}{4}(|Ay^{n+1}|^2 + |Az^{n+1}|^2) + \frac{8k}{\nu}M_f^2. \tag{47}$$

By using sucessively Schwarz inequality and (6) and Hölder inequality, we derive

$$\begin{aligned} 2kb(y^n, y^n, Ay^{n+1}) &\leq 2k|B(y^n, y^n)||Ay^{n+1}| \\ &\leq 2kc_2|y^n|^{\frac{1}{2}}||y^n|||Ay^n|^{\frac{1}{2}}|Ay^{n+1}|^{\frac{1}{2}} \\ &\leq \frac{k\nu}{4}|Ay^{n+1}|^2 + \frac{k\nu}{16}|Ay^n|^2 + c_5\nu^{-3}k|y^n|^2||y^n||^4. \end{aligned} \tag{48}$$

Similarly

$$\begin{aligned} 2kb(y^n, z^{n+1}, Ay^{n+1}) &\leq 2kc_2|y^n|^{\frac{1}{2}}||y^n||^{\frac{1}{2}}||z^{n+1}||^{\frac{1}{2}}|Az^{n+1}|^{\frac{1}{2}}|Ay^{n+1}| \\ &\leq \frac{k\nu}{4}(|Ay^{n+1}|^2 + |Az^{n+1}|^2) + c_6\nu^{-3}k|y^n|^2||y^n||^2||z^{n+1}||^2, \end{aligned} \tag{49}$$

$$\begin{aligned} 2kb(z^{n+1}, y^n, Ay^{n+1}) &\leq 2kc_2|z^{n+1}|^{\frac{1}{2}}||z^{n+1}||^{\frac{1}{2}}||y^n||^{\frac{1}{2}}|Ay^n|^{\frac{1}{2}}|Ay^{n+1}| \\ &\leq \frac{k\nu}{4}|Ay^{n+1}|^2 + \frac{k\nu}{8}|Ay^n|^2 + c_7\nu^{-3}k|z^{n+1}|^2||z^{n+1}||^2||y^n||^2, \end{aligned} \tag{50}$$

and

$$\begin{aligned} 2kb(y^n, y^n, Az^{n+1}) &\leq 2kc_2|y^n|^{\frac{1}{2}}||y^n|||Ay^n|^{\frac{1}{2}}|Az^{n+1}| \\ &\leq \frac{k\nu}{16}|Ay^n|^2 + \frac{k\nu}{4}|Az^{n+1}|^2 + c_5\nu^{-3}k|y^n|^2||y^n||^4. \end{aligned} \tag{51}$$

Combining these inequalities to (46), we obtain

$$\begin{aligned} ||y^{n+1}||^2 &- ||y^n||^2 + ||y^{n+1} - y^n||^2 + k\nu(|Ay^{n+1}|^2 + |Az^{n+1}|^2) \\ &\leq \frac{8k}{\nu}M_f^2 + \frac{k\nu}{4}|Ay^n|^2 + g^n||y^n||^2 \end{aligned} \tag{52}$$

with

$$g^n = c_8\nu^{-3}k(|y^n|^2||y^n||^2 + |y^n|^2||z^{n+1}||^2 + |z^{n+1}|^2||z^{n+1}||^2).$$

By using the results of Theorem 1, we derive

$$g^n \le c_8 \nu^{-3} k[b_0(||y^n||^2 + ||z^{n+1}||^2) + b_1 ||z^{n+1}||^2].$$

Hence

$$k \sum_{n=k_0}^{N+k_0} g^n \le \frac{c_8 \nu^{-4}}{\delta}(2b_0 + b_1) G_0(1) = b_5, \quad \text{for } \mathrm{N} = \frac{1}{\mathrm{k}}\ , \ \forall\ \mathrm{k}_0 > 0. \tag{53}$$

Using the relation

$$|Ay^n|^2 \le 2\lambda_m ||y^{n+1} - y^n||^2 + 2|Ay^{n+1}|^2, \tag{54}$$

we can rewrite (52) as

$$\begin{aligned} ||y^{n+1}||^2 \quad &- \quad ||y^n||^2 + (1 - \frac{k\nu\lambda_m}{2})||y^{n+1} - y^n||^2 \\ &+ \quad \frac{k\nu}{2}(|Ay^{n+1}|^2 + |Az^{n+1}|^2) \le \frac{8k}{\nu} M_f^2 + g^n ||y^n||^2. \end{aligned} \tag{55}$$

We assume k, m satisfying in addition (45). By dropping some unnecessary terms in (55), we get

$$\frac{||y^{n+1}||^2 - ||y^n||^2}{k} \le g^n ||y^n||^2 + \frac{8}{\nu} M_f^2. \tag{56}$$

The idea for deriving a uniform upper bound of $||y^n||^2$ is the following:

- applying Lemma 2 to get an upper bound of $||y^n||^2$ for $n \le N = \frac{1}{k}$;
- applying the time discrete uniform Gronwall lemma (Lemma 3) to get an upper bound of $||y^n||^2$ for $n \ge N = \frac{1}{k}$.

We apply first Lemma 2 with $d^n = ||y^n||^2$, $h_m = \frac{8}{\nu} M_f^2$ and $N = \frac{1}{k}$. Since

$$k \sum_{n=k_0}^{N+k_0} h^n = \frac{8}{\nu} M_f^2\ , \ \forall\ k_0 \ge 0,$$

we derive from Lemma 2 and (53) that

$$\begin{aligned} ||y^n||^2 &\le ||y^0||^2 exp(k \sum_{i=0}^{N} g^i) + k \sum_{i=0}^{N} h^i \exp(k \sum_{j=i}^{N} g^j) \\ &\le ||y^0||^2 exp(b_5) + \frac{8}{\nu} M_f^2 exp(b_5) = b_3\ , \ \forall\ n \le N. \end{aligned} \tag{57}$$

In order to derive a uniform bound for $n \geq N$, we apply Lemma 3. Using $|y^n|^2 \leq b_0$, we derive from (40) that

$$k \sum_{n=k_0}^{N+k_0} d^n \leq (b_0 + \frac{2}{\nu\lambda_1} M_f^2)(\nu\delta)^{-1} = b_4 \quad \text{for } \mathrm{N} = \frac{1}{\mathrm{k}} \ , \ \forall \ \mathrm{k}_0 > 0.$$

Therefore, the hypotheses of Lemma 3 are all verified. We derive from Lemma 3 that

$$d^n = ||y^n||^2 \leq (\frac{8}{\nu} M_f^2 + b_4) exp(b_5) = b_6 \ , \ \forall \ n > N.$$

Therefore

$$||y^n||^2 \leq b_2 = \max(b_3, b_6) \ , \ \forall \ n \geq 0.$$

Finally, $\forall T > 0$ given, taking the sum of (55) for n from 0 to $\frac{T}{k} - 1$, we recover the last inequality of Theorem 2. ¶

Corollary 2 *Under the hypotheses of Theorem 2, we have*

$$||z^n|| \leq \nu^{-1}\lambda_1^{-\frac{1}{2}} M_f + c_2 \lambda_1^{-\frac{1}{2}} b_2 \ , \ \forall \ n,$$

$$k \sum_{n=0}^{\frac{T}{k}-1} ||z^n||^2 \leq 2\lambda_m^{-1}(M_f^2 + c_2\lambda_1^{-1} b_2 G_2(T)) \to 0 \ \ (\text{as m}^{-1} \to 0) \ , \ \forall \ \mathrm{T} > 0.$$

Moreover, There exists $R_1 = R_1(\nu, \lambda_1, f)$ such that $\forall u_0 \in H$, we can find $N(u_0) > 0$ such that for all $n > N(u_0)$, we have

$$||y^n||^2 + ||z^n||^2 \leq R_1. \tag{58}$$

PROOF: The proof is similar to that of Corollary 1.
By taking the scalar product of (15) with Az^{n+1}, we derive

$$\nu|Az^{n+1}|^2 \leq M_f|Az^{n+1}| + c_2|y^n|^{\frac{1}{2}}||y^n|||Ay^n|^{\frac{1}{2}}|Az^{n+1}|.$$

By using (28)

$$\nu\lambda_m^{\frac{1}{2}}||z^{n+1}|| \leq \nu|Az^{n+1}| \leq M_f + c_2\lambda_1^{-\frac{1}{2}}||y^n|| \cdot |Ay^n| \tag{59}$$

$$\leq M_f + c_2(\frac{\lambda_m}{\lambda_1})^{\frac{1}{2}}||y^n||^2. \tag{60}$$

Therefore, from (59) and Theorem 2, we derive

$$k\nu \sum_{n=0}^{\frac{T}{k}-1} ||z^{n+1}||^2 \le 2\lambda_m^{-1}(M_f^2 + c_2\lambda_1^{-1}b_2G_2(T)) \to 0 \text{ (as m}^{-1} \to 0),$$

also from (60)

$$||z^{n+1}|| \le \nu^{-1}\lambda_m^{-\frac{1}{2}}M_f + c_2\lambda_1^{-\frac{1}{2}}b_2.$$

(58) is then a consequence of Theorem 2 and this inequality. ¶

Remark 2 A direct consequence of Corollaries 1 & 2 is that there exist uniform (independent of k, m) absorbing sets $\mathcal{B}_H = \mathcal{B}_V(0, R_0)$ in H and $\mathcal{B}_V = \mathcal{B}_V(0, R_1)$ in V for solutions of the approximate system (15)-(16). In virtue of a general theorem (Thm 1.1 of [9]), there exist universal attractors $\mathcal{A}_{k,m}$ in H_m for the system (15)-(16) which are uniformly (independent of k, m) compact in H. The analysis for the convergence and the error estimate of $\mathcal{A}_{k,m}$ to $\mathcal{A}$ (as $k, m^{-1} \to 0$) is beyond the scope of this paper. The related problems in a different context will be reported in a subsequent paper (see [7]).

3 Convergence

With the stability results we established in the previous section, the procedure to prove the convergence of the scheme is rather standard. We will only sketch it rapidly.

Let us first introduce some approximate functions of $u(t)$.

DEFINITION:

- $u_1(t) = u_1^{(k,m)}(t) : \mathbb{R}^+ \to H$ is the piecewise constant function which equals to y^n on $[nk, (n+1)k)$;

- $u_2(t) = u_2^{(k,m)}(t) : \mathbb{R}^+ \to H$ is the piecewise constant function which equals to y^n on $[nk, (n+1)k)$;

- $u_3(t) = u_3^{(k,m)}(t) : \mathbb{R}^+ \to H$ is the continuous function which is linear on $[nk, (n+1)k)$ and $u_3(nk) = y^n$, $u_3((n+1)k) = y^{n+1}$; and

- $z(t) = z^{(k,m)}(t) : \mathbb{R}^+ \to H$ is the piecewise constant function which equals to z^{n+1} on $[nk, (n+1)k)$.

The main results in this section are

Theorem 3 *Under the hypothesis (23), we have*

- $u_i^{(k,m)} \to u$ *(as* $k, m^{-1} \to 0)$ $i = 1,2,3$, *in* $L^2(0,T;V) \cap L^p(0,T;H)$ *strongly,* $\forall T > 0$, $1 \leq p < +\infty$, *provided* $k\lambda_m \to 0$ *in case of i=1,3.*

- *Moreover if* $u_0 \in V$ *and k,m satisfying in addition (45), then*

  $u_i^{(k,m)} \to u$ *(as* $k, m^{-1} \to 0)$ $i = 1,2,3$ *in* $L^2(0,T;D(A)) \cap L^p(0,T;V)$ *strongly* , $\forall\ T > 0$, $1 \leq p < +\infty$, *provided* $k\lambda_m \to 0$ *in case of i=1,3.*

PROOF: By using these definitions, we can reformulate the system (15)-(16) as

$$\nu Az(t) = Q[f^n - B(u_1(t), u_1(t))], \tag{61}$$

$$\begin{aligned}\frac{\partial u_3(t)}{\partial t} + \nu Au_2(t) &= P[f^n - B(u_1(t), u_1(t)) \\ &\quad - B(u_1(t), z(t)) - B(z(t), u_1(t))].\end{aligned} \tag{62}$$

Let us first derive an estimate on u_3. From the definition of u_3, we have

$$\begin{aligned}
\int_0^T \quad ||u_3(t)||^2 dt &= \sum_{n=0}^{\frac{T}{k}-1} \int_{nk}^{(n+1)k} \quad ||u_3(t)||^2 dt \\
&= \sum_{n=0}^{\frac{T}{k}-1} \int_{nk}^{(n+1)k} \left\{ \frac{||y^{n+1}||^2 - ||y^n||^2}{k} t + [||y^n||^2 - n(||y^{n+1}||^2 - ||y^n||^2)] \right\} dt \\
&= \sum_{n=0}^{\frac{T}{k}-1} \left\{ (n + \frac{1}{2})k(||y^{n+1}||^2 - ||y^n||^2) + k||y^n||^2 - kn(||y^{n+1}||^2 - ||y^n||^2) \right\} \\
&= \frac{k}{2} \sum_{n=0}^{\frac{T}{k}-1} (||y^{n+1}||^2 - ||y^n||^2) + k \sum_{n=0}^{\frac{T}{k}-1} ||y^n||^2 \\
&= \frac{k}{2} \sum_{n=0}^{\frac{T}{k}-1} (||y^{n+1}||^2 + ||y^n||^2) \leq (b_0 + \frac{2T}{\nu\lambda_1})\nu^{-1}\delta^{-1}.
\end{aligned}$$

The last inequality comes from the results of Theorem 1.

This inequality and Theorem 1 can be reinterpreted as

$$\left\{ \begin{array}{l} u_i(k,m)(t), i = 1,2,3 \text{ and } z(t) \text{ are bounded uniformly in } L^\infty(\mathbb{R}^+; H) \\ u_i^{(k,m)}(t), i = 1,2,3 \text{ and } z(t) \text{ are bounded uniformly in } L^2(0,T;V)\ ,\ \forall\ T > 0. \\ \frac{\partial}{\partial t} u_3^{(k,m)}(t) \in L^2(0,T;V') \end{array} \right. \tag{63}$$

A direct consequence of (63) is that there exists $U_i \in L^\infty(\mathbb{R}^+;H) \cap L^2(0,T;V)$, $\forall T > 0$, $i = 1,2,3$ and a subsequence (k', m') such that

$$\begin{cases} u_i = u_i^{(k',m')} \to U_i \text{ (as k, m}^{-1} \to 0) \text{ in } L^\infty(\mathbb{R}^+; H) \text{ weak} - \text{star} \\ u_i = u_i^{(k',m')} \to U_i \text{ (as k, m}^{-1} \to 0) \text{ in } L^2(0,T;V) \text{ weakly} \\ \frac{\partial u_3(t)}{\partial t} = \frac{\partial u_3^{(k',m')}(t)}{\partial t} \to \frac{\partial U_3(t)}{\partial t} \text{ (as k, m}^{-1} \to 0) \text{ in } L^2(0,T;V') \text{ weakly} \end{cases} \tag{64}$$

From the definitions of u_i, we have

$$\begin{aligned} \int_0^T |u_1(t) - u_3(t)|^2 dt &= \sum_{n=0}^{\frac{T}{k}-1} \int_{nk}^{(n+1)k} |y^n - [\frac{y^{n+1} - y^n}{k} t + y^n - n(y^{n+1} - y^n)]|^2 dt \\ &\le \sum_{n=0}^{\frac{T}{k}-1} \int_{nk}^{(n+1)k} |y^{n+1} - y^n|^2 (\frac{t}{k} - n)^2 dt \\ &= \sum_{n=0}^{\frac{T}{k}-1} |y^{n+1} - y^n|^2 \int_{nk}^{(n+1)k} (\frac{t}{k} - n)^2 dt \\ &= \frac{k}{3} \sum_{n=0}^{\frac{T}{k}-1} |y^{n+1} - y^n|^2 \le \frac{k}{3\delta} G_0(T). \end{aligned} \tag{65}$$

Similarly,

$$\int_0^T |u_2(t) - u_3(t)|^2 dt \le \frac{k}{3\delta} G_0(T). \tag{66}$$

It is then obvious that $U_1 = U_2 = U_3 = u^*$. In virtue of a classical compactness theorem (see for example [4]), we derive from (64) that

$$u_3^{(k,m)} \to u^* \text{ in } L^2(0,T;H) \text{ strongly.}$$

Then from (65)-(66), we have also

$$u_i^{(k,m)} \to u^* \text{ in } L^2(0,T;H) \text{ strongly, } i = 1,2.$$

Finally, we derive from Corollary 1 that

$$z^{(k,m)} \to 0 \text{ in } L^2(0,T;H) \text{ strongly.}$$

With these strong convergence results, the passage to the limit in (61)-(62) is standard (see [8] for more details) and we find out that u^* is indeed the solution of the N.S.E.

It remains to prove the strong convergence in $L^2(0,T;V)$. Let us define

$$\begin{aligned}
X &= X^{(k,m)} = |u_3(T) - u(T)|^2 \\
&+ \sum_{n=0}^{\frac{T}{k}-1} |y^{n+1} - y^n|^2 + 2\nu \int_0^T (||u_2(t) - u(t)||^2 + ||z(t)||^2)dt \\
&= \left\{ |u_3(T)|^2 + \sum_{n=0}^{\frac{T}{k}-1} |y^{n+1} - y^n|^2 + 2\nu \int_0^T (||u_2(t)||^2 + ||z(t)||^2)dt \right\} \\
&+ \left\{ |u(T)|^2 + 2\nu \int_0^T ||u(t)||^2 dt \right\} \\
&+ \left\{ -2(u_3(T), u(T)) - 4\nu \int_0^T ((u_2(t), u(t)))dt \right\} \\
&= X_1^{(k,m)} + X_2^{(k,m)} + X_3^{(k,m)}.
\end{aligned}$$

From (64), we derive that

$$X_3^{(k,m)} \to -2|u(T)|^2 - 4\nu \int_0^T ||u(t)||^2 dt.$$

Taking the sum of (27) for n from 0 to $\frac{T}{k} - 1$, we obtain

$$\begin{aligned}
|u_3(T)|^2 &+ \sum_{n=0}^{\frac{T}{k}-1} |y^{n+1} - y^n|^2 + 2\nu \int_0^T (||u_2(t)||^2 + ||z(t)||^2)dt \\
&= |u_3(0)|^2 + 2 \int_0^T (f^n, u_2(t) + z(t))dt \\
&- \int_0^T [b(u_1(t), u_1(t), u_2(t)) + b(u_1(t), z(t), u_2(t)) + b(z(t), u_1(t), u_2(t))]dt.
\end{aligned}$$

Let $k, m^{-1} \to 0$ in the last relation, by using the strong convergences of $u_i^{(k,m)}$, $z^{(k,m)}$ and (4), we derive

$$X_1^{(k,m)} \to |u(0)|^2 + 2 \int_0^T (f, u(t))dt.$$

Finally, by taking the scalar product of (8) with u, we find

$$\frac{\partial |u(t)|^2}{\partial t} + 2\nu ||u(t)||^2 = 2(f, u(t)).$$

The integration of which over [0,T] implies

$$|u(T)|^2 + 2\nu \int_0^T ||u(t)||^2 dt = |u(0)|^2 + 2\int_0^T (f, u(t))dt.$$

Combining all these relations, we derive

$$X^{(k,m)} = X_1^{(k,m)} + X_2^{(k,m)} + X_3^{(k,m)} \to 0 \ \ (\text{as } k, m^{-1} \to 0).$$

This implies

$$u_2^{(k,m)} \to u \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{V}) \text{ strongly}.$$

By the definition of $u_1(t), u_2(t)$

$$\int_0^T ||u_1(t) - u_2(t)||^2 dt = k \sum_{n=0}^{\frac{T}{k}-1} ||y^{n+1} - y^n||^2 \le k\lambda_m \sum_{n=0}^{\frac{T}{k}-1} |y^{n+1} - y^n|^2 \le \frac{k\lambda_m}{\delta} G_0(T).$$

Similarly as in (65)

$$\int_0^T ||u_3(t) - u_2(t)||^2 dt \le \frac{k}{3} \sum_{n=0}^{\frac{T}{k}-1} ||y^{n+1} - y^n||^2 \le \frac{k}{3}\lambda_m \sum_{n=0}^{\frac{T}{k}-1} |y^{n+1} - y^n|^2 \le \frac{k\lambda_m}{3\delta} G_0(T).$$

Therefore

$$u_1^{(k,m)}, u_3^{(k,m)} \to u \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{V}) \text{ strongly provided } \mathrm{k}\lambda_\mathrm{m} \to 0.$$

This completes the proof of the first part of Theorem 3. We omit the proof of the second part since the procedure is exactly the same. ¶

4 Another nonlinear Galerkin scheme

We consider in this section a second scheme corresponding to a better approximate inertial manifold of N.S.E.

Find $y^{n+1} = y_m^{n+1} \in H_m$ and $z^{n+1} = z_m^{n+1} \in H_{dm} - H_m$ such that

$$\frac{z^{n+1} - z^n}{k} + \nu A z^{n+1} = Q[f^n - B(y^n, y^n) - B(y^n, z^n) - B(z^n, y^n)], \tag{67}$$

$$\begin{aligned}\frac{y^{n+1} - y^n}{k} + \nu A y^{n+1} &= P[f^n - B(y^n,^n) - B(z^{n+1}, y^n) \\ &\quad -B(y^n, z^{n+1}) - B(z^n, z^{n+1})].\end{aligned} \tag{68}$$

Theorem 4 *We assume k and m satisfying*

$$\begin{cases} 16c_1^2 k^2 \lambda_m \lambda_{dm} b_0 \leq \frac{1}{2} - \delta \\ 16c_1^2 k \lambda_m^{\frac{1}{2}} \lambda_{dm}^{\frac{1}{2}} \nu^{-1} b_0 \leq \frac{1}{3}(1 - \delta) \end{cases} \tag{69}$$

Then, we have

$$|y^n|^2 + |z^n|^2 \leq B_0(n) \ , \ \forall \ n,$$

$$\begin{aligned}\forall T > 0, \quad \delta &\sum_{n=0}^{\frac{T}{k}-1} [|y^{n+1} - y^n|^2 + |z^{n+1} - z^n|^2] \\ + \quad k\nu\delta &\sum_{n=0}^{\frac{T}{k}-1} [||y^{n+1}||^2 + ||z^{n+1}||^2] \leq b_0 + \frac{2T}{\nu\lambda_1} M_f^2 = G_0(T),\end{aligned}$$

and

$$k\nu \sum_{n=0}^{\frac{T}{k}-1} |z^n|^2 \leq \lambda_m^{-1} \left\{ |z^0|^2 + \frac{4T}{\nu\lambda_m} M_f^2 + [1 + 2(\frac{\lambda_{dm}}{\lambda_m})^{\frac{1}{2}}] \frac{4c_2^2}{\nu} b_0 G_0(T) \right\} \to 0, \tag{70}$$

where $\delta, b_0, B_0(n)$ are the same as in the Theorem 1.

Proof: We take respectively the scalar product of (67) with $2kz^{n+1}$, (68) with $2ky^{n+1}$ and add the corresponding equalities, we obtain

$$\begin{aligned}|z^{n+1}|^2 &- |z^n|^2 + |z^{n+1} - z^n|^2 + |y^{n+1}|^2 - |y^n|^2 \\ &+ |y^{n+1} - y^n|^2 + 2k\nu(||y^{n+1}||^2 + ||z^{n+1}||^2) \\ &= 2k(f^n, y^{n+1} + z^{n+1}) \\ &\quad -\{2kb(y^n, y^n, y^{n+1}) + 2kb(z^{n+1}, y^n, y^{n+1}) \\ &\quad +2k[b(y^n, z^{n+1}, y^{n+1}) + b(y^n, y^n, z^{n+1})] \\ &\quad +2k[b(z^n, z^{n+1}, y^{n+1}) + b(z^n, y^n, z^{n+1})] \\ &\quad +2kb(y^n, z^n, z^{n+1})\} \\ &= 2k(f^n, y^{n+1} + z^{n+1}) - B_1 - B_2 - B_3 - B_4 - B_5.\end{aligned} \tag{71}$$

As in the proof of Theorem 1, B_1, B_2, B_3 can be majorized respectively by (31-33). Other nonlinear terms can be majorized by using (6), (28) and Schwarz inequality, namely

$$\begin{aligned} B_4 &= 2kb(z^n, z^{n+1}, y^{n+1} - y^n) \\ &\leq 2kc_1|z^n|^{\frac{1}{2}}||z^n||^{\frac{1}{2}}||z^{n+1}|||y^{n+1} - y^n|^{\frac{1}{2}}||y^{n+1} - y^n||^{\frac{1}{2}} \\ &\leq 2kc_1\lambda_m^{\frac{1}{4}}\lambda_{dm}^{\frac{1}{4}}||z^{n+1}|| \cdot |z^n| \cdot |y^{n+1} - y^n| \\ &\leq \frac{1}{8}|y^{n+1} - y^n|^2 + 8c_1^2k^2\lambda_m^{\frac{1}{2}}\lambda_{dm}^{\frac{1}{2}}|z^n|^2||z^{n+1}||^2. \end{aligned}$$

Similarly

$$\begin{aligned} B_5 &= 2kb(y^n, z^n, z^{n+1} - z^n) \\ &\leq 2kc_1|\lambda_m^{\frac{1}{4}}\lambda_{dm}^{\frac{1}{4}}||z^n|| \cdot |y^n| \cdot |z^{n+1} - z^n| \\ &\leq \frac{1}{8}|z^{n+1} - z^n|^2 + 8c_1^2k^2\lambda_m^{\frac{1}{2}}\lambda_{dm}^{\frac{1}{2}}|y^n|^2||z^n||^2. \end{aligned}$$

By using these inequalities and (29), (82) becomes

$$\begin{aligned} |z^{n+1}|^2 &- |z^n|^2 + \frac{1}{2}|z^{n+1} - z^n|^2 + |y^{n+1}|^2 - |y^n|^2 + \frac{1}{2}|y^{n+1} - y^n|^2 \\ &+ k\nu(||y^{n+1}||^2 + ||z^{n+1}||^2) \\ &\leq \frac{2k}{\nu\lambda_1}M_f^2 + 16k^2c_1^2\{\lambda_m|y^n|^2||z^{n+1}||^2 + \lambda_m^{\frac{1}{2}}\lambda_{dm}^{\frac{1}{2}}|z^n|^2||z^{n+1}||^2\} \\ &+ 8c_1^2k^2\{\lambda_m|y^n|^2||y^n||^2 + \lambda_m^{\frac{1}{2}}\lambda_{dm}^{\frac{1}{2}}|y^n|^2||z^n||^2\}. \end{aligned} \tag{72}$$

By using (35), we can rewrite (72) as

$$\begin{aligned} |y^{n+1}|^2 &- |y^n|^2 + (\frac{1}{2} - 16c_1^2k^2\lambda_m^2|y^n|^2)|y^{n+1} - y^n|^2 \\ &+ |z^{n+1}|^2 - |z^n|^2 + (\frac{1}{2} - 16c_1^2k^2\lambda_m\lambda_{dm}|y^n|^2)|z^{n+1} - z^n|^2 \\ &+ k\nu(1 - 16c_1^2k\lambda_m\nu^{-1}|y^n|^2)||y^{n+1}||^2 \\ &+ k\nu[1 - 16c_1^2k\nu^{-1}\lambda_m^{\frac{1}{2}}\lambda_{dm}^{\frac{1}{2}}(2|y^n|^2 + |z^n|^2)]||z^{n+1}||^2 \\ &\leq \frac{2k}{\nu\lambda_1}M_f^2. \end{aligned} \tag{73}$$

Assuming that k and m verifies the hypothesis (69), using exactly the same technique as in the proof of Theorem 1, we can prove by induction that

$$|y^q|^2 + |z^q|^2 \leq B_0(q) \ , \ \forall \ q. \tag{74}$$

Therefore, $\forall T > 0$ given, taking the sum of (72) for n from 0 to $\frac{T}{k} - 1$, we obtain

$$\delta \sum_{n=0}^{\frac{T}{k}-1} [|y^{n+1} - y^n|^2 + |z^{n+1} - z^n|^2] \quad + \quad k\nu\delta \sum_{n=0}^{\frac{T}{k}-1} [||y^{n+1}||^2 + ||z^{n+1}||^2]$$

$$\leq \quad b_0 + \frac{2T}{\nu\lambda_1} M_f^2 = G_0(T). \tag{75}$$

To prove (70), we take the scalar product of (67) with $2kz^{n+1}$, using repeatly (28) and (6), after some lengthy but easy computations, we arrive to

$$|z^{n+1}|^2 - |z^n|^2 + k\nu\lambda_m |z^{n+1}|^2 \quad \leq \quad \frac{4k}{\nu\lambda_m} M_f^2 + \frac{4c_2^2}{\nu} k |y^n|^2 ||y^n||^2$$

$$\leq \quad \frac{4c_2^2}{\nu} (\frac{\lambda_{dm}}{\lambda_m})^{\frac{1}{2}} k (|z^n|^2 ||y^n||^2 + |y^n|^2 ||z^n||^2).$$

Therefore, by taking the sum of this last inequality for n from 0 to $\frac{T}{k} - 1$, using (74) and (75), we derive

$$k\nu \sum_{n=0}^{\frac{T}{k}-1} |z^{n+1}|^2 \leq \lambda_m^{-1} \left\{ |z^0|^2 + \frac{4T}{\nu\lambda_m} M_f^2 + [1 + 2(\frac{\lambda_{dm}}{\lambda_m})^{\frac{1}{2}}] \frac{4c_2^2}{\nu} b_0 G_0(T) \right\} . ¶$$

Theorem 5 *If* $u_0 \in V$, *k,n satisfying (69) and in addition*

$$\frac{k\nu\lambda_{dm}}{4} \leq 1 - \delta. \tag{76}$$

Then, we have

$$||y^n||^2 + ||z^n||^2 \leq b_7 \ , \ \forall \ n,$$

$$\forall T > 0, \quad \delta \quad \sum_{n=0}^{\frac{T}{k}-1} [||y^{n+1} - y^n||^2 + ||z^{n+1} - z^n||^2]$$

$$+ \quad \frac{k\nu}{2} \sum_{n=0}^{\frac{T}{k}-1} [|Ay^{n+1}|^2 + |Az^{n+1}|^2] \leq b_7 + \frac{2T}{\nu\lambda_1} M_f^2 = G_3(T), \tag{77}$$

$$k\nu \sum_{n=0}^{\frac{T}{k}-1} ||z^{n+1}||^2 \leq \lambda_m^{-1} \left\{ ||z^0||^2 + \frac{4T}{\nu} M_f^2 + \frac{8c_2^2}{\nu} (\lambda_m \lambda_1)^{-\frac{1}{2}} b_7 G_3(T) \right\} \to 0, \tag{78}$$

where δ is the same as in the Theorem 1 and b_7 is to be given explicitly in the process of the proof.

PROOF: Taking the scalar product of (67) with $2kAz^{n+1}$, (68) with $2kAy^{n+1}$ respectively, after some lengthy computations, we arrive to

$$\begin{aligned} ||y^{n+1}||^2 \ &+ \ ||z^{n+1}||^2 - ||y^n||^2 - ||z^n||^2 + ||y^{n+1} - y^n||^2 \\ &+ \ ||z^{n+1} - z^n||^2 + k\nu(|Ay^{n+1}|^2 + |Az^{n+1}|^2) \\ &\leq \ \frac{2k}{\nu} M_f^2 + \frac{k\nu}{4}(|Ay^n|^2 + |Az^n|^2) \\ &+ \ g^n(||y^n||^2 + ||z^n||^2) \end{aligned} \tag{79}$$

with

$$g^n = c_9 \nu^{-3} k (|y^n|^2 ||y^n||^2 + |y^n|^2 ||z^{n+1}||^2 + |z^{n+1}|^2 ||y^n||^2 + |z^{n+1}|^2 ||z^{n+1}||^2).$$

We then derive from (74) and (75) that

$$k \sum_{n=k_0}^{N+k_0} g^n \leq 2c_9 \nu^{-4} \delta^{-1} b_0 G_0(1) = b_9 \ , \ \text{for N} = \frac{1}{\text{k}}. \tag{80}$$

Using the relation (54), we can rewrite (79) as

$$\begin{aligned} ||y^{n+1}||^2 \ &+ \ ||z^{n+1}||^2 - ||y^n||^2 - ||z^n||^2 + (1 - \frac{k\nu\lambda_m}{4})||y^{n+1} - y^n||^2 \\ &+ \ (1 - \frac{k\nu\lambda_{dm}}{4})||z^{n+1} - z^n||^2 + \frac{k\nu}{2}(|Ay^{n+1}|^2 + |Az^{n+1}|^2) \\ &\leq \ \frac{2k}{\nu} M_f^2 + g^n(||y^n||^2 + ||z^n||^2). \end{aligned} \tag{81}$$

After dropping some unnecessary terms in (81), we get

$$\frac{(||y^{n+1}||^2 + ||z^{n+1}||^2) - (||y^n||^2 + ||z^n||^2)}{k} \leq \frac{2k}{\nu} M_f^2 + g^n(||y^n||^2 + ||z^n||^2). \tag{82}$$

We apply first Lemma 2 to this last inequality.

Let $d^n = ||y^n||^2 + ||z^n||^2$, $h_m = \frac{8}{\nu}M_f^2, N = \frac{1}{k}$. We derive from Lemma 2 and (80) that

$$||y^n||^2 + ||z^n||^2 \leq (||y^0||^2 + ||z^0||^2)exp(k\sum_{i=0}^{N} g^i) + k\sum_{i=0}^{N} h^i \exp(k\sum_{j=i}^{N} g^j)$$

$$\leq ||u^0||^2 exp(b_9) + \frac{8}{\nu}M_f^2 exp(b_9) = b_8 \ , \ \forall \ n \leq N. \qquad (83)$$

We can now apply Lemma 3 to derive a upper bound of d^n for n large. We derive from the definitions of y^n, h^n and (75) that

$$\begin{cases} k\sum_{n=k_0}^{N+k_0} y^n \leq (b_0 + \frac{2}{\nu\lambda_1}M_f^2)\nu^{-1}\delta^{-1} = b_4 \\ k\sum_{n=k_0}^{N+k_0} h^n \leq \frac{8}{\nu}M_f^2 \end{cases} , \text{ for } N = \frac{1}{k} \ , \ \forall \ k_0 > 0.$$

The hypertheses of Lemma 3 are then all verified. We derive from Lemma 3 that

$$||y^n||^2 + ||z^n||^2 \leq (\frac{8}{\nu}M_f^2 + b_4)exp(b_9) = b_{10}. \qquad (84)$$

Let $b_7 = \max(b_8, b_{10})$, (81) can be obtained by taking the sum of (79) for n from 0 to $\frac{T}{k} - 1$.

To prove (78), we take the scalar product of (67) with $2kAz^{n+1}$, using repeatly (28) and (6), we can derive

$$\begin{aligned} ||z^{n+1}||^2 - ||z^n||^2 + k\nu\lambda_m||z^{n+1}||^2 &\leq \frac{4k}{\nu}M_f^2 + \frac{4c_2^2}{\nu\lambda_1}k||y^n||^2|Ay^n|^2 \\ &\leq \frac{4c_2^2}{\nu}(\lambda_m\lambda_1)^{-\frac{1}{2}}k(||z^n||^2|Ay^n|^2 + ||y^n||^2|Az^n|^2). \end{aligned}$$

Hence, by taking the sum of this inequality for n from 0 to $\frac{T}{k} - 1$, using (77) and (84), we derive

$$k\nu\sum_{n=0}^{\frac{T}{k}-1} ||z^{n+1}||^2 \leq \lambda_m^{-1}\left\{||z^0||^2 + \frac{4T}{\nu}M_f^2 + \frac{8c_2^2}{\nu}(\lambda_m\lambda_1)^{-\frac{1}{2}}b_7G_3(T)\right\}.$$

The proof is complete. ¶

Remark 3 The stability condition in (69) is not independent of d but is still better than that of the classical Galerkin scheme with dm modes which needs $k \sim \lambda_{dm}^{-1}$.

Let $u_i(t), i = 1,2,3$ and $z(t)$ be defined as in the section 3. By using the same procedures as in the section 3, We can prove the following convergence theorem.

Theorem 6 *Under the hypothesis (69), we have*

- $u_i \to u$ *(as* $k, m^{-1} \to 0)$ $i = 1,2,3$, *in* $L^2(0,T;V) \cap L^p(0,T;H)$ *strongly,* $\forall T > 0,\ 1 \le p < +\infty$, *provided* $k\lambda_m \to 0$ *in case of i=1,3.*
- *Moreover if* $u_0 \in V$ *and k,m satisfying in addition (76), then* $u_i \to u$ *(as* $k, m^{-1} \to 0)$ $i = 1,2,3$ *in* $L^2(0,T;D(A)) \cap L^p(0,T;V)$ *strongly ,* $\forall\ T > 0,\ 1 \le p < +\infty$, *provided* $k\lambda_m \to 0$ *in case of i=1,3.*

Remark 4 For the sake of simplicity, we have only analyzed two first order (in time) semi-implicit schemes. Higher order schemes of semi-implicit type such as Crank-Nicolson & Adams Bashforth-scheme, etc. are suggested in practice to increase the efficiency.

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