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# Error Analysis for Mapped Jacobi Spectral Methods

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Approximation properties of mapped Jacobi polynomials and of interpolations based on mapped Jacobi–Gauss–Lobatto points are established. These results play an important role in numerical analysis of mapped Jacobi spectral methods. As examples of applications, optimal error estimates for several popular regular and singular mappings are derived.

**KEY WORDS:** Spectral method; mapped Jacobi polynomials; orthogonal system.

**Mathematics Subject Classification (1991):** 65N35; 65N15; 65N50.

## 1. INTRODUCTION

Standard spectral methods are capable of providing very accurate approximations to well-behaved smooth functions with significantly less degrees of freedom when compared with finite difference or finite element methods (cf. [6, 7, 11]). However, if a function exhibits localized behaviors such as sharp interfaces, very thin internal or boundary layers, using a standard Gauss-type grid usually fails to produce an accurate approximation with a reasonable number of degrees of freedom. Thus, it is advisable to use a grid which is adapted to the local behaviors of the underlying function. Since spectral methods can not gracefully handle an arbitrarily locally refined grid, a popular strategy is to use a suitable mapping which transforms a function having sharp interfaces in the physical domain to a well behaved function on the computational domain. Thus, to better understand what are the impacts of the mapping on the approximation, it is necessary to study the properties of the mapped polynomials.

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In a recent work [25], we studied the mapped Legendre polynomials and derived optimal error estimates featuring explicit expressions on the mapping parameters for several popular mappings. However, the analysis in [25] limited to the Legendre case with regular mappings and is not applicable in some important situations such as: (i) singular mappings (see, for instance [21,28]) which are very efficient for resolving thin boundary layers; and (ii) Chebyshev mappings which are often used in practice (see, for instance [3,4,20]). The purpose of this paper is to investigate the approximate properties of the mapped Jacobi polynomials and apply them to study the error behaviors of the mapped Jacobi spectral method with several popular regular and singular mappings.

The paper is organized as follows. In Secs. 2 and 3, we consider the mapped Jacobi spectral method with regular and singular mappings, respectively. In Sec. 4, we present some numerical results illustrating our theoretical estimates. Some useful properties of Jacobi polynomials are gathered in Appendix A.

## 2. JACOBI APPROXIMATIONS USING REGULAR MAPPINGS

This section is devoted to the study of the approximation properties of mapped Jacobi polynomials using general regular mappings, and explore the dependence of the approximate errors on the parameters of some given mappings.

We now introduce some notations. Let $\omega(x)$ be a given weight function in $I := (-1, 1)$, which is not necessary in $L^1(I)$. We denote by $H^r_\omega(I)$ $(r = 0, 1, \dots)$ the weighted Sobolev spaces whose inner products, norms and semi-norms are $(\cdot,\cdot)_{r,\omega}$, $\|\cdot\|_{r,\omega}$ and $|\cdot|_{r,\omega}$, respectively. For real $r > 0$, we define the space $H^r_\omega(I)$ by space interpolation. In particular, the norm and inner product of $L^2_\omega(I) = H^0_\omega(I)$ are denoted by $\|\cdot\|_\omega$ and $(\cdot,\cdot)_\omega$, respectively. The subscript $\omega$ will be omitted from the notations in case of $\omega(x) \equiv 1$.

Let $\mathbb{N}$ be the set of all non-negative integers. For any $N \in \mathbb{N}$, we denote by $\mathcal{P}_N$ the set of all algebraic polynomials of degree $\leqslant N$. We shall use $c$ to denote a generic positive constant independent of any function and $N$, and we use the expression $A \lesssim B$ to mean that there exists a generic positive constant $c$ such that $A \leqslant cB$.

### 2.1. General Setup

We consider the following general one-to-one coordinate transformation:

$$x = s(y;\mu),\ \frac{dx}{dy} = s'(y;\mu) > 0, \quad s(\pm 1, \mu) = \pm 1, \quad x, y \in \bar{I},\ \mu \in D_\mu, \quad (2.1)$$

where $D_\mu$ is the feasible domain of the parameter vector $\mu$. Without loss of generality, we assume that the inverse of (2.1) exists, and for certain $r \geqslant 1$,

$$s,\, s^{-1} \in C(\bar{I}) \cap C^r(I), \quad \mu \in D_\mu. \tag{2.2}$$

Indeed, some interesting mappings proposed in [3,4,20,26] are of this general type.

For a given mapping, instead of considering the mapped differential equation which could be cumbersome and case dependent, we consider its approximation using a new family of orthogonal functions $\{p_k(s^{-1}(x;\mu))\}$ obtained by applying the mapping to a classical orthogonal polynomial (see, for instance [5,14,17]). The analysis of this approach will require approximation results by using the new family of orthogonal functions. A particular advantage of this approach is that once these approximation results are established, it can be directly (i.e. without using a transform) applied to a large class of problems.

Let $J_k^{\alpha,\beta}$ be the $k$-th degree classical Jacobi polynomials whose properties are summarized in the appendix, we define the mapped Jacobi polynomials as

$$j_{\mu,n}^{\alpha,\beta}(x) := J_n^{\alpha,\beta}(y), \quad x = s(y;\mu), \quad x, y \in I, \quad \mu \in D_\mu, \quad \alpha, \beta > -1. \tag{2.3}$$

Throughout this section, the variables $x$ and $y$ are always connected by the mapping $x = s(y;\mu)$.

We infer from (A.3) that (2.3) defines a new family of orthogonal functions $\{j_{\mu,n}^{\alpha,\beta}\}$, i.e.,

$$\int_I j_{\mu,n}^{\alpha,\beta}(x) j_{\mu,m}^{\alpha,\beta}(x) \omega_\mu^{\alpha,\beta}(x) dx = \gamma_n^{\alpha,\beta} \delta_{m,n}, \tag{2.4}$$

where the weight function

$$\omega_\mu^{\alpha,\beta}(x) = \omega^{\alpha,\beta}(y) \frac{dy}{dx} = \omega^{\alpha,\beta}(y)(s'(y;\mu))^{-1} > 0 \tag{2.5}$$

with $\omega^{\alpha,\beta}(y) = (1-y)^\alpha (1+y)^\beta$.

We now consider error estimates of approximations using the orthogonal system $\{j_{\mu,n}^{\alpha,\beta}\}$. Since $\{j_{\mu,n}^{\alpha,\beta}\}$ forms a complete orthogonal system in $L^2_{\omega_\mu^{\alpha,\beta}}(I)$, for any $u \in L^2_{\omega_\mu^{\alpha,\beta}}(I)$, we write

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_{\mu,n}^{\alpha,\beta} j_{\mu,n}^{\alpha,\beta}(x) \quad \text{with } \hat{u}_{\mu,n}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} (u, j_{\mu,n}^{\alpha,\beta})_{\omega_\mu^{\alpha,\beta}}. \tag{2.6}$$

We define the approximation space as

$$V_{\mu,N}^{\alpha,\beta} := \text{span}\{j_{\mu,0}^{\alpha,\beta}, j_{\mu,1}^{\alpha,\beta}, \cdots j_{\mu,N}^{\alpha,\beta}\}. \tag{2.7}$$

The orthogonal projection $\pi_{\mu,N}^{\alpha,\beta}$: $L^2_{\omega_\mu^{\alpha,\beta}}(I) \to V_{\mu,N}^{\alpha,\beta}$ is defined by

$$(\pi_{\mu,N}^{\alpha,\beta} v - v, \phi)_{\omega_\mu^{\alpha,\beta}} = 0, \qquad \forall \phi \in V_{\mu,N}^{\alpha,\beta}. \tag{2.8}$$

For clarity, the following notations will be used in the sequel:

$$U_\mu(y) = u(x),\ \widetilde{\omega}_\mu^{\alpha,\beta}(x) = s'(y;\mu)\omega^{\alpha+1,\beta+1}(y), \quad x = s(y;\mu), \quad x, y \in I, \mu \in D_\mu, \tag{2.9}$$

$$A^m_{\omega_\mu^{\alpha,\beta}}(I) = \{u \in L^2_{\omega_\mu^{\alpha,\beta}}(I) : U_\mu(y) = u(x) \quad \text{and} \quad U_\mu \in A^m_{\omega^{\alpha,\beta}}(I)\}, \quad m \in \mathbb{N} \tag{2.10}$$

with the semi-norm

$$|u|_{A^m_{\omega_\mu^{\alpha,\beta}}} = \|(1-y^2)^{\frac{m}{2}} \partial_y^m U_\mu\|_{\omega^{\alpha,\beta}}, \tag{2.11}$$

where $A^m_{\omega^{\alpha,\beta}}(I)$ is a function space defined in (A.17).

We have the following fundamental approximation results.

**Theorem 2.1.** For any $u \in A^m_{\omega_\mu^{\alpha,\beta}}(I)$, we have

$$\|\partial_x(\pi_{\mu,N}^{\alpha,\beta} u - u)\|_{\widetilde{\omega}_\mu^{\alpha,\beta}} + N\|\pi_{\mu,N}^{\alpha,\beta} u - u\|_{\omega_\mu^{\alpha,\beta}} \lesssim N^{1-m} |u|_{A^m_{\omega_\mu^{\alpha,\beta}}}, \quad m \geqslant 1, \tag{2.12}$$

$$|(\pi_{\mu,N}^{\alpha,\beta} u - u)(1)| \lesssim N^{1+\alpha-m} |u|_{A^m_{\omega_\mu^{\alpha,\beta}}}, \quad m > \alpha + 1, \tag{2.13}$$

$$|(\pi_{\mu,N}^{\alpha,\beta} u - u)(-1)| \lesssim N^{1+\beta-m} |u|_{A^m_{\omega_\mu^{\alpha,\beta}}}, \quad m > \beta + 1. \tag{2.14}$$

*Proof.* By (A.15), (2.1), and (2.6), we have

$$\hat{u}_{\mu,n}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}}(u, j_{\mu,n}^{\alpha,\beta})_{\omega_\mu^{\alpha,\beta}} = \frac{1}{\gamma_n^{\alpha,\beta}}(U_\mu, J_n^{\alpha,\beta})_{\omega^{\alpha,\beta}} = \widehat{U}_{\mu,n}^{\alpha,\beta}. \tag{2.15}$$

Now, let $\hat{\pi}_N^{\alpha,\beta}$ be the $L^2_{\omega^{\alpha,\beta}}$-orthogonal projector as in (A.18). Then, by (A.3), (2.4), and Lemma A.1,

$$\|\pi_{\mu,N}^{\alpha,\beta}u-u\|^2_{\omega_\mu^{\alpha,\beta}}=\sum_{n=N+1}^{\infty}(\hat{u}_{\mu,n}^{\alpha,\beta})^2\gamma_n^{\alpha,\beta}=\sum_{n=N+1}^{\infty}(\widehat{U}_{\mu,n}^{\alpha,\beta})^2\gamma_n^{\alpha,\beta}$$
$$=\|\hat{\pi}_N^{\alpha,\beta}U_\mu-U_\mu\|^2_{\omega^{\alpha,\beta}}\lesssim N^{-2m}\|\partial_y^mU_\mu\|^2_{\omega^{\alpha+m,\beta+m}}\lesssim N^{-2m}|u|^2_{A^m_{\omega_\mu^{\alpha,\beta}}}. \tag{2.16}$$

Next, we deduce from (2.3) and the orthogonality of $\{\partial_yJ_n^{\alpha,\beta}\}$ that $\{\partial_x j_{\mu,n}^{\alpha,\beta}\}$ is $L^2_{\widetilde{\omega}_\mu^{\alpha,\beta}}-$ orthogonal, and $\|\partial_x j_{\mu,n}^{\alpha,\beta}\|^2_{\widetilde{\omega}_\mu^{\alpha,\beta}}=\|\partial_yJ_n^{\alpha,\beta}\|^2_{\omega^{\alpha,\beta}}=\lambda_n^{\alpha,\beta}\gamma_n^{\alpha,\beta}$. Therefore, by (2.15) and Lemma A.1,

$$\|\partial_x(\pi_{\mu,N}^{\alpha,\beta}u-u)\|^2_{\widetilde{\omega}_\mu^{\alpha,\beta}}=\sum_{n=N+1}^{\infty}\lambda_n^{\alpha,\beta}\gamma_n^{\alpha,\beta}(\hat{u}_{\mu,n}^{\alpha,\beta})^2=\sum_{n=N+1}^{\infty}\lambda_n^{\alpha,\beta}\gamma_n^{\alpha,\beta}(\widehat{U}_{\mu,n}^{\alpha,\beta})^2$$
$$=\|\partial_y(\pi_N^{\alpha,\beta}U_\mu-U_\mu)\|^2_{\omega^{\alpha+1,\beta+1}}$$
$$\lesssim N^{2(1-m)}\|(1-y^2)^{\frac{m}{2}}\partial_y^mU_\mu\|^2_{\omega^{\alpha,\beta}}$$
$$\lesssim N^{2(1-m)}|u|^2_{A^m_{\omega_\mu^{\alpha,\beta}}}. \tag{2.17}$$

A combination of (2.16) and (2.17) leads to (2.12).

Now, we prove (2.13). By using the Stirling formula (cf. [8]),

$$\Gamma(s+1)=\sqrt{2\pi s}s^se^{-s}(1+O(s^{-(1/5)})),\quad s\gg1, \tag{2.18}$$

we derive from (A.7), (A.4), and (2.1) that

$$\gamma_n^{\alpha,\beta}\sim n^{-1},\quad |j_{\mu,n}^{\alpha,\beta}(1)|\sim n^\alpha,\quad |j_{\mu,n}^{\alpha,\beta}(-1)|\sim n^\beta,\quad n\gg1. \tag{2.19}$$

The above facts, together with (A.16) and (2.15), lead to

$$|(\pi_{\mu,N}^{\alpha,\beta}u-u)(1)|\leqslant\sum_{n=N+1}^{\infty}|\hat{u}_{\mu,n}^{\alpha,\beta}||j_{\mu,n}^{\alpha,\beta}(1)|\lesssim\sum_{n=N+1}^{\infty}n^\alpha|\widehat{U}_{\mu,n}^{\alpha,\beta}|$$
$$\le\left(\sum_{n=N+1}^{\infty}n^{2\alpha}(\lambda_n^{\alpha,\beta})^{-m}(\gamma_n^{\alpha,\beta})^{-1}\right)^{1/2}\left(\sum_{n=N+1}^{\infty}(\lambda_n^{\alpha,\beta})^m\gamma_n^{\alpha,\beta}(\widehat{U}_{\mu,n}^{\alpha,\beta})^2\right)^{1/2}$$
$$\lesssim\left(\sum_{n=N+1}^{\infty}n^{2\alpha}(\lambda_n^{\alpha,\beta})^{-m}(\gamma_n^{\alpha,\beta})^{-1}\right)^{1/2}\|\partial_y^mU_\mu\|_{\omega^{\alpha+m,\beta+m}}.$$

By (A.2) and (2.19), we have that for $m > \alpha+1$,

$$\sum_{n=N+1}^{\infty} n^{2\alpha}(\lambda_n^{\alpha,\beta})^{-m}(\gamma_n^{\alpha,\beta})^{-1} \lesssim \sum_{n=N+1}^{\infty} n^{1+2\alpha-2m} \leqslant \int_N^{\infty} x^{1+2\alpha-2m}dx \lesssim N^{2(1+\alpha-m)}.$$

This leads to (2.13).

Finally, (2.14) can be proved in the same fashion as above. □

We now consider Gauss–Lobatto interpolations based on mapped Jacobi polynomials. We note that the Gauss or Gauss–Radau interpolations are also very useful, especially for problems in unbounded domains (cf. [13, 15, 16]), but their treatments are very similar to the Gauss–Lobatto case and it is straightforward to extend what follows to the Gauss or Gauss–Radau interpolations.

Applying the mapping (2.1) to the standard Jacobi–Gauss–Lobatto (JGL) interpolation, we come up with the mapped JGL points and weights:

$$\zeta_{\mu,N,j}^{\alpha,\beta} := s(\xi_{N,j}^{\alpha,\beta};\mu), \quad \rho_{\mu,N,j}^{\alpha,\beta} := \omega_{N,j}^{\alpha,\beta}, \quad 0 \leqslant j \leqslant N, \ \ \mu \in D_\mu. \tag{2.20}$$

Due to (A.20), we have the following exactness on the mapped JGL quadrature:

$$\int_I u(x)\omega_\mu^{\alpha,\beta}(x)dx = \sum_{j=0}^{N} u(\zeta_{\mu,N,j}^{\alpha,\beta})\rho_{\mu,N,j}^{\alpha,\beta}, \quad \forall u \in V_{\mu,2N-1}^{\alpha,\beta}. \tag{2.21}$$

Accordingly, we can define the discrete inner product and discrete norm:

$$\begin{aligned}(u,v)_{\omega_\mu^{\alpha,\beta},N} &= \sum_{j=0}^{N} u(\zeta_{\mu,N,j}^{\alpha,\beta})v(\zeta_{\mu,N,j}^{\alpha,\beta})\rho_{\mu,N,j}^{\alpha,\beta},\\ \|u\|_{\omega_\mu^{\alpha,\beta},N} &= (u,u)_{\omega_\mu^{\alpha,\beta},N}^{1/2}, \quad \forall u,v \in C(\bar{I}).\end{aligned}$$

As a direct result of (A.26),

$$\|u_N\|_{\omega_\mu^{\alpha,\beta}} \leqslant \|u_N\|_{\omega_\mu^{\alpha,\beta},N} \leqslant \sqrt{2+\frac{\alpha+\beta+1}{N}}\,\|u_N\|_{\omega_\mu^{\alpha,\beta}}, \quad \forall u_N \in V_{\mu,N}^{\alpha,\beta}. \tag{2.22}$$

Let $\mathcal{I}_{\mu,N}^{\alpha,\beta}$ be the Lagrange interpolation operator associated with the mapped JGL points. We have the following result on error estimate of the mapped JGL interpolation.

**Theorem 2.2.** For any $u \in A^m_{\omega_\mu^{\alpha,\beta}}(I)$ and $m > \max(\alpha+1, \beta+1)$, we have

$$\|\partial_x(\mathcal{I}^{\alpha,\beta}_{\mu,N}u - u)\|_{\widetilde{\omega}_\mu^{\alpha,\beta}} + N\|\mathcal{I}^{\alpha,\beta}_{\mu,N}u - u\|_{\omega_\mu^{\alpha,\beta}} \lesssim N^{1-m}|u|_{A^m_{\omega_\mu^{\alpha,\beta}}}. \tag{2.23}$$

*Proof.* By (A.27) and (2.20),

$$\begin{aligned}
\|\mathcal{I}^{\alpha,\beta}_{\mu,N}u\|^2_{\omega_\mu^{\alpha,\beta},N} &= |u(-1)|^2\rho^{\alpha,\beta}_{\mu,N,0} + |u(1)|^2\rho^{\alpha,\beta}_{\mu,N,N} + \sum_{j=1}^{N-1}|u(\zeta^{\alpha,\beta}_{\mu,N})|^2\rho^{\alpha,\beta}_{\mu,N,j} \\
&= |u(-1)|^2\omega^{\alpha,\beta}_{N,0} + |u(1)|^2\omega^{\alpha,\beta}_{N,N} + \sum_{j=1}^{N-1}|U_\mu(\xi^{\alpha,\beta}_{\mu,N})|^2\omega^{\alpha,\beta}_{N,j} \\
&\lesssim |u(-1)|^2\omega^{\alpha,\beta}_{N,0} + |u(1)|^2\omega^{\alpha,\beta}_{N,N} + \|U_\mu\|^2_{\omega^{\alpha,\beta}} + N^{-2}\|\partial_y U_\mu\|^2_{\omega^{\alpha+1,\beta+1}} \\
&\lesssim |u(-1)|^2\omega^{\alpha,\beta}_{N,0} + |u(1)|^2\omega^{\alpha,\beta}_{N,N} + \|u\|^2_{\omega_\mu^{\alpha,\beta}} + N^{-2}\|\partial_x u\|^2_{\widetilde{\omega}_\mu^{\alpha,\beta}}.
\end{aligned} \tag{2.24}$$

Using the Stirling formula (2.18), we have the asymptotic behaviors of the weights given in (A.22) and (A.23):

$$\omega^{\alpha,\beta}_{N,0} \sim N^{-2-2\beta}, \quad \omega^{\alpha,\beta}_{N,N} \sim N^{-2-2\alpha}, \quad N \gg 1. \tag{2.25}$$

Thus, a combination of (2.22), (2.24), and (2.25) leads to

$$\begin{aligned}
\|\mathcal{I}^{\alpha,\beta}_{\mu,N}u\|_{\omega_\mu^{\alpha,\beta}} \leqslant \|\mathcal{I}^{\alpha,\beta}_{\mu,N}u\|_{\omega_\mu^{\alpha,\beta},N} &\lesssim N^{-1-\beta}|u(-1)| + N^{-1-\alpha}|u(1)| \\
&+ \|u\|_{\omega_\mu^{\alpha,\beta}} + N^{-1}\|\partial_x u\|_{\widetilde{\omega}_\mu^{\alpha,\beta}}.
\end{aligned} \tag{2.26}$$

Let $\pi^{\alpha,\beta}_{\mu,N}$ be the orthogonal projector defined in (2.8). Since $\pi^{\alpha,\beta}_{\mu,N}u \in V^{\alpha,\beta}_{\mu,N}$, we have from (2.26) and Theorem 2.1 that

$$\begin{aligned}
\|\mathcal{I}^{\alpha,\beta}_{\mu,N}u - \pi^{\alpha,\beta}_{\mu,N}u\|_{\omega_\mu^{\alpha,\beta}} &= \|\mathcal{I}^{\alpha,\beta}_{\mu,N}(\pi^{\alpha,\beta}_{\mu,N}u - u)\|_{\omega_\mu^{\alpha,\beta}} \\
&\lesssim N^{-1-\beta}|(\pi^{\alpha,\beta}_{\mu,N}u - u)(-1)| + N^{-1-\alpha}|(\pi^{\alpha,\beta}_{\mu,N}u - u)(1)| \\
&+ \|\pi^{\alpha,\beta}_{\mu,N}u - u\|_{\omega_\mu^{\alpha,\beta}} + N^{-1}\|\partial_x(\pi^{\alpha,\beta}_{\mu,N}u - u)\|_{\widetilde{\omega}_\mu^{\alpha,\beta}} \\
&\lesssim N^{-m}|u|_{A^m_{\omega_\mu^{\alpha,\beta}}}.
\end{aligned} \tag{2.27}$$

Next, we use the inverse inequality (3.4) in [13] to obtain that

$$\|\partial_x u\|_{\widetilde{\omega}_\mu^{\alpha,\beta}} = \|\partial_y U_\mu\|_{\omega^{\alpha+1,\beta+1}} \lesssim N\|U_\mu\|_{\omega^{\alpha,\beta}} \lesssim N\|u\|_{\omega_\mu^{\alpha,\beta}}, \quad \forall u \in V^{\alpha,\beta}_{\mu,N}. \tag{2.28}$$

Therefore, by (2.27) and (2.28),

$$\|\partial_x(\mathcal{I}^{\alpha,\beta}_{\mu,N}u-\pi^{\alpha,\beta}_{\mu,N}u)\|_{\widetilde{\omega}^{\alpha,\beta}_\mu}\lesssim N\|\mathcal{I}^{\alpha,\beta}_{\mu,N}u-\pi^{\alpha,\beta}_{\mu,N}u\|_{\omega^{\alpha,\beta}_\mu}\lesssim N^{1-m}|u|_{A^m_{\omega^{\alpha,\beta}_\mu}}. \quad (2.29)$$

Finally, (2.23) follows from (2.27), (2.29) and Theorem 2.1. □

Theorems 2.1 and 2.2 are fundamental results on the mapped Jacobi approximations. However, the error estimates are bounded by the semi-norm $|u|_{A^m_{\omega^{\alpha,\beta}_\mu}}$, which is defined in terms of the derivatives of $U_\mu(y)$. Next, we apply these results to a specific mapping, estimate the upper bounds for $|u|_{A^m_{\omega^{\alpha,\beta}_\mu}}$ and explore the dependence of the error estimates on the mapping parameters.

### 2.2. Application to a Specific Mapping

We consider the following mapping introduced in [4]:

$$x=s(y;\mu):=\mu_2+\frac{1}{\mu_1}\tan(a_1(y-a_0)),\quad \mu\in D_\mu, \quad (2.30)$$

where $D_\mu=\{(\mu_1,\mu_2):\ \mu_1>0,\ -1\leqslant\mu_2\leqslant 1\}$. For large values of $\mu_1$, this function is nearly discontinuous with a region of rapid variations near $x=\mu_2$. Consequently, as $\mu_1$ increases, more and more points are clustered near $x=\mu_2$. The constants $a_0$ and $a_1$ are chosen to satisfy (2.1), and the values are

$$\begin{aligned}&a_0=a_0(\mu):=\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2},\quad a_1=a_1(\mu):=\frac{\kappa_1+\kappa_2}{2},\\&\kappa_1=\arctan(\mu_1(1+\mu_2)),\quad \kappa_2=\arctan(\mu_1(1-\mu_2)).\end{aligned} \quad (2.31)$$

With the above choice, we find that

$$-1\leqslant a_0\leqslant 1,\qquad 0<a_1<\frac{\pi}{2}. \quad (2.32)$$

This mapping is explicitly invertible:

$$y=s^{-1}(x;\mu)=a_0+\frac{1}{a_1}\text{arc}\quad\tan(\mu_1(x-\mu_2)) \quad (2.33)$$

and sufficiently regular:

$$\frac{dx}{dy}=\frac{a_1(\mu_1+1)^2}{\mu_1}\Big(\frac{1}{(\mu_1+1)^2}+\frac{\mu_1^2}{(\mu_1+1)^2}(x-\mu_2)^2\Big):=C_\mu(D_1+D_2(x-\mu_2)^2), \quad (2.34)$$

where

$$C_\mu \lesssim \mu_1 + 1, \qquad 0 < C_\mu^{-1}, D_1, D_2 \leqslant 1. \tag{2.35}$$

Next, let $U_\mu(y)$ be the same as in (2.9). Direct calculations yield

$$\partial_y U_\mu(y) = \partial_x u(x) \frac{dx}{dy} = C_\mu Q_2(x-\mu_2)\partial_x u(x),$$
$$\partial_y^2 U_\mu(y) = \frac{d(\partial_y U_\mu(y))}{dx}\frac{dx}{dy} = C_\mu^2 (Q_4(x-\mu_2)\partial_x^2 u(x) + Q_3(x-\mu_2)\partial_x u(x)),$$

where $Q_l(x-\mu_2)$ is some polynomial of degree $l$ with respect to $x-\mu_2$ with coefficients in terms of $D_1, D_2$ and $C_\mu^{-1}$. Hence, by an induction argument, we find that for $k \geqslant 1$,

$$\partial_y^k U_\mu(y) = C_\mu^k \sum_{j=1}^{k} Q_{k+j}(x-\mu_2)\partial_x^j u(x), \tag{2.36}$$

where $Q_{k+j}(x-\mu_2)$ $(1 \leqslant j \leqslant k)$ are uniformly bounded for all $x \in \bar{I}$ and $\mu \in D_\mu$. Consequently, we derive from the definition of $|u|_{A^m_{\omega_\mu^{\alpha,\beta}}}$ and (2.36) that

$$|u|_{A^m_{\omega_\mu^{\alpha,\beta}}} \lesssim (1+\mu_1)^m \|u\|_{m,\tilde{\omega}_\mu^{\alpha+m,\beta+m}}, \quad \forall u \in H^m_{\tilde{\omega}_\mu^{\alpha+m,\beta+m}}(I), \tag{2.37}$$

where

$$\tilde{\omega}_\mu^{\alpha+m,\beta+m}(x) = (1-y^2)^m \omega^{\alpha,\beta}(y), \quad y = a_0 + \frac{1}{a_1}\arctan(\mu_1(x-\mu_2)).$$

As a direct result of the above facts and Theorems 2.1 and 2.2, we have the following Corollary.

**Corollary 2.1.** For any $u \in H^m_{\tilde{\omega}_\mu^{\alpha+m,\beta+m}}(I)$, we have

$$\|\partial_x(\pi^{\alpha,\beta}_{\mu,N} u - u)\|_{\tilde{\omega}_\mu^{\alpha,\beta}} + N\|\pi^{\alpha,\beta}_{\mu,N} v - v\|_{\omega_\mu^{\alpha,\beta}}$$
$$\lesssim (1+\mu_1)^m N^{1-m} \|u\|_{m,\tilde{\omega}_\mu^{\alpha+m,\beta+m}}, \quad m \geqslant 1, \tag{2.38}$$

$$\|\partial_x(\mathcal{I}^{\alpha,\beta}_{\mu,N} u - u)\|_{\tilde{\omega}_\mu^{\alpha,\beta}} + N\|\mathcal{I}^{\alpha,\beta}_{\mu,N} u - u\|_{\omega_\mu^{\alpha,\beta}}$$
$$\lesssim (1+\mu_1)^m N^{1-m} \|u\|_{m,\tilde{\omega}_\mu^{\alpha+m,\beta+m}}, \quad m > \max(\alpha+1, \beta+1). \tag{2.39}$$

**Remark 2.1.** The general results in Theorems 2.1 and 2.2 are also applicable to several other interesting mappings in [3,20,26]. In particular, these results with $\alpha=\beta=-(1/2)$ can be used to analyze the mapped Chebyshev methods.

## 3. JACOBI APPROXIMATIONS USING SINGULAR MAPPINGS

The mapped Jacobi spectral methods using regular mappings are very useful in the approximation of PDEs' solutions with localized rapid variations in the interior of the domain. But for problems with very thin boundary layers, it is more efficient to use a mapping which is singular at the boundary (cf. [21–24,28]).

In this section, we analyze a mapped Jacobi spectral method with the following *singular mapping of index-*$(k,l)$*:*

$$x=s(y;k,l),\ \ s(\pm 1;k,l)=\pm 1,\ \ \frac{dx}{dy}=s'(y;k,l)=g(y;k,l)(1-y)^k(1+y)^l,$$
$$g\in C^\infty([-1,1]),\quad 0<g_0(k,l)\leqslant g(y;k,l)\leqslant g_1(k,l),\ \ x,y\in\bar{I},\ \ k,l\in\mathbb{N}. \tag{3.1}$$

Note that as $k$ (resp. $l$) increases, the collocation points are increasingly clustered to the boundary $x=1$ (resp. $x=-1$). In particular, the singular mappings used in [21] and [28] are of index-$(k,k)$ and index-$(2^k-1,2^k-1)$, respectively. However, singular mappings can not be analyzed using the general framework developed in the last section so they need to be treated separately.

### 3.1. Generalized Jacobi Polynomials

We study here approximation properties of a family of generalized Jacobi polynomials, which are essential for the mapped Jacobi spectral method with singular mappings given in (3.1).

For any $k,l\in\mathbb{N}$, we define

$$\psi_n^{k,l}(y):=(1-y)^{k+1}(1+y)^{l+1}J_n^{k+1,l+1}(y),\quad y\in I. \tag{3.2}$$

Thanks to the orthogonality relation (A.3), we find that

$$\int_I \psi_n^{k,l}(y)\psi_m^{k,l}(y)\omega^{-k-1,-l-1}(y)dy=0\quad \text{for } n\neq m. \tag{3.3}$$

Hence, $\psi_n^{k,l}$ can be considered as generalized Jacobi polynomials with negative parameters $\alpha=-k-1$ and $\beta=-l-1$. Although some properties of these generalized Jacobi polynomials were studied in [12], additional results are needed for the analysis of singular mappings.

We note that $\{\psi_n^{k,l}\}$ have the following attractive properties:

- By (A.3), $\{\psi_n^{k,l}\}$ forms an orthogonal system in $L^2_{\omega^{-k-1,-l-1}}(I)$.
- We have from (A.12) (with $\alpha=k, \beta=l$ and $m=1$) that

$$\partial_y \psi_n^{k,l}(y) = -2(n+1)(1-y)^k(1+y)^l J_{n+1}^{k,l}(y). \tag{3.4}$$

  Therefore, $\{\partial_y \psi_n^{k,l}\}$ forms an orthogonal system in $L^2_{\omega^{-k,-l}}(I)$.
- It is shown in Appendix A.2 that there exists a unique set of constants $\{a_{n,j}^{k,l}\}$ such that

$$\psi_n^{k,l}(y) = \sum_{j=-k-l}^{k+l+2} a_{n,j}^{k,l} J_{n+j}^{k,l}(y) \quad \text{with } a_{n,j}^{k,l} \equiv 0 \quad \text{if } n+j<0. \tag{3.5}$$

- It is also shown in Appendix A.2 that for $k \geqslant l$,

$$\partial_y^m \psi_n^{k,l}(y) = \begin{cases} A_{m,n}^{k,l}\omega^{k-m+1,l-m+1}(y) J_{n+m}^{k-m+1,l-m+1}(y), & m \leqslant l+1 \leqslant k+1, \\ B_{m,n}^{k,l}\omega^{k-m+1,0}(y) J_{n+l+1}^{k-m+1,m-l-1}(y), & l+1 \leqslant m \leqslant k+1, \\ C_{m,n}^{k,l} J_{n-m+k+l+2}^{m-k-1,m-l-1}(y), & l+1 \leqslant k+1 \leqslant m, \end{cases} \tag{3.6}$$

  where

$$\begin{aligned} A_{m,n}^{k,l} &= (-2)^m \frac{(n+m)!}{n!}, \\ B_{m,n}^{k,l} &= \left( (-1)^{m-l-1} \prod_{j=1}^{m-l-1} (n+k+2-j) \right) A_{l+1,n}^{k,l}, \\ C_{m,n}^{k,l} &= \frac{(n+m)!}{2^{m-k-1}(n+k+1)} B_{k+1,n}^{k,l}. \end{aligned} \tag{3.7}$$

- The derivative relation (3.6), together with (A.3), implies that for $k,l,m \in \mathbb{N}$ and $m \leqslant n+k+l$, $\{\partial_y^m \psi_n^{k,l}\}$ forms a mutually orthogonal system in $L^2_{\omega^{m-k-1,m-l-1}}(I)$, and for $k \geqslant l$,

$$\int_{-1}^{1} \partial_y^m \psi_n^{k,l}(y) \partial_y^m \psi_{n'}^{k,l}(y) \omega^{m-k-1,m-l-1}(y) dy = D_{m,n}^{k,l} \delta_{n,n'} \tag{3.8}$$

where

$$D^{k,l}_{m,n}=\begin{cases}(A^{k,l}_{m,n})^2\gamma^{k-m+1,l-m+1}_{n+m}, & m\leqslant l+1\leqslant k+1,\\ (B^{k,l}_{m,n})^2\gamma^{k-m+1,m-l-1}_{n+l+1}, & l+1\leqslant m\leqslant k+1,\\ (C^{k,l}_{m,n})^2\gamma^{m-k-1,m-l-1}_{n-m+k+l+2}, & l+1\leqslant k+1\leqslant m.\end{cases} \tag{3.9}$$

For $k<l$, the constant $D^{k,l}_{m,n}$ in (3.8) is replaced by $D^{l,k}_{m,n}$.

Now, we consider the approximation properties of the orthogonal system $\{\psi^{k,l}_n\}$, and define the approximation space as

$$V_N:=V_N(y;k,l):=\mathrm{span}\{\psi^{k,l}_0,\psi^{k,l}_1,\dots,\psi^{k,l}_N\}\subseteq\mathcal{P}_{N+k+l+2}. \tag{3.10}$$

The orthogonal projection $P^{k,l}_N:L^2_{\omega^{-k-1,-l-1}}(I)\to V_N$ is defined by

$$(P^{k,l}_N v-v,v_N)_{\omega^{-k-1,-l-1}}=0,\quad \forall v_N\in V_N. \tag{3.11}$$

**Theorem 3.1.** Let $k,l,m\in\mathbb{N}$. We have

$$\|P^{k,l}_N v-v\|_{\omega^{k,l}}\lesssim N^{-m}\|\partial^m_y v\|_{\omega^{k+m,l+m}},\quad \forall v\in L^2_{\omega^{-k-1,-l-1}}(I)\cap A^m_{\omega^{k,l}}(I), \tag{3.12}$$

$$\begin{aligned}&\|\partial_x(P^{k,l}_N v-v)\|_{\omega^{-k,-l}}\lesssim N^{1-m}\|\partial^m_y v\|_{\omega^{m-k-1,m-l-1}},\\ &\forall v\in L^2_{\omega^{-k-1,-l-1}}(I)\cap A^m_{\omega^{-k-1,-l-1}}(I),\ m\geqslant 1.\end{aligned} \tag{3.13}$$

*Proof.* For any $v\in L^2_{\omega^{-k-1,-l-1}}(I)$, we write

$$v(y)=\sum_{n=0}^{\infty}\widehat{v}^{k,l}_n\psi^{k,l}_n(y)\quad\text{with }\ \widehat{v}^{k,l}_n=\frac{1}{\|\psi^{k,l}_n\|^2_{\omega^{-k-1,-l-1}}}(v,\psi^{k,l}_n)_{\omega^{-k-1,-l-1}}. \tag{3.14}$$

By (3.5), we have

$$v(y)=\sum_{n=0}^{\infty}\widehat{v}^{k,l}_n\left(\sum_{j=-k-l}^{k+l+2}a^{k,l}_{n,j}J^{k,l}_{n+j}(y)\right)=\sum_{n=0}^{\infty}\widetilde{v}^{k,l}_n J^{k,l}_n(y), \tag{3.15}$$

where

$$\widetilde{v}^{k,l}_n=\sum_{j=-k-l}^{k+l+2}\widehat{v}^{k,l}_{n-j}a^{k,l}_{n-j,j},\quad\text{with }\widehat{v}^{k,l}_{n-j}\equiv 0,\quad\text{if } n-j<0.$$

Using (A.11) inductively yields

$$\partial_y^m J_n^{k,l}(y) = \eta_{m,n}^{k,l} J_{n-m}^{k+m,l+m}(y), \quad \eta_{m,n}^{k,l} = \frac{(m+n+k+l)!}{2^m(n+k+l)!}, \quad n \geqslant m, \tag{3.16}$$

which implies $\{\partial_y^m J_n^{k,l}\}$ is $L^2_{\omega^{k+m,l+m}}$-orthogonal and $\|\partial_y^m J_n^{k,l}\|^2_{\omega^{k+m,l+m}} = (\eta_{m,n}^{k,l})^2 \gamma_{n-m}^{k+m,l+m}$. In view of this fact, we have from (3.15) that

$$\begin{aligned}\|P_N^{k,l} v - v\|^2_{\omega^{k,l}} &= \sum_{n=N'}^{\infty} (\widetilde{v}_n^{k,l})^2 \gamma_n^{k,l} \leqslant C_{m,N} \sum_{n=N'}^{\infty} (\widetilde{v}_n^{k,l})^2 (\eta_{m,n}^{k,l})^2 \gamma_{n-m}^{k+m,l+m} \\ &\leqslant C_{m,N} \|\partial_y^m v\|^2_{\omega^{k+m,l+m}},\end{aligned} \tag{3.17}$$

where $N' = N + k + l + 3$ and

$$C_{m,N} = \max_{n \geqslant N'} \{\gamma_n^{k,l} (\eta_{m,n}^{k,l})^{-2} (\gamma_{n-m}^{k+m,l+m})^{-1}\}.$$

By the Stirling formula (2.18),

$$\gamma_n^{k,l} \sim \gamma_{n-m}^{k+m,l+m} \sim n^{-1}, \quad \eta_{m,n}^{k,l} \sim n^m, \quad n \gg 1, \ m, k, l \in \mathbb{N}. \tag{3.18}$$

Thus, $C_{m,N} \lesssim N^{-2}$, and (3.12) follows directly from this fact and (3.17).

We now prove (3.13). By (A.3), (3.4), and (3.14),

$$\begin{aligned}\|\partial_y (P_N^{k,l} v - v)\|^2_{\omega^{-k,-l}} &= \sum_{n=N+1}^{\infty} 4(n+2)^2 \widehat{v}_n^2 \gamma_{n+1}^{k,l} \\ &\leqslant \max_{n \geqslant N+1} \{(n+2)^2 (D_{m,n}^{k,l})^{-1} \gamma_{n+1}^{k,l}\} \sum_{n=N}^{\infty} (\widehat{v}_n^{k,l})^2 D_{m,n}^{k,l} \\ &\leqslant \max_{n \geqslant N+1} \{(n+2)^2 (D_{m,n}^{k,l})^{-1} \gamma_{n+1}^{k,l}\} \|\partial_y^m v\|^2_{\omega^{m-k-1,m-l-1}},\end{aligned} \tag{3.19}$$

where we used the orthogonality (3.8) to derive the last step. Next, by the Stirling formula (2.18), (3.7), and (3.9),

$$\gamma_{n+1}^{k,l} \sim n^{-1}, \quad (D_{m,n}^{k,l})^{-1} \sim n^{1-2m}, \quad n \gg 1, \ k, l, m \in \mathbb{N}. \tag{3.20}$$

Therefore, a combination of (3.19) and (3.20) leads to (3.13). □

### 3.2. Mapped Jacobi Spectral Method with Singular Mappings

Since the singular mappings are commonly used for problems with thin boundary layers, we now consider, as an example, the singularly perturbed differential equation

$$-\varepsilon u'' + pu' + qu = f, \qquad u(\pm 1) = 0, \qquad x \in I, \tag{3.21}$$

where $\varepsilon > 0$ is a small parameter, and $p, q, f$ are given functions with $\| f(\cdot, \varepsilon)\|_{L^\infty(I)} < c$ (independent of $\varepsilon$). This equation in general possesses a thin boundary layer of width $O(\varepsilon^\gamma)(\gamma > 0)$.

#### *3.2.1. Spectral-Galerkin Approximation*

As suggested in [21] and [22], we define the approximation space as

$$X_N := \{u \in H_0^1(I) : u(x) = v(y), \quad v(y) \in Y_N(y; k, l),\ x = s(y; k, l)\}, \tag{3.22}$$

where

$$Y_N := Y_N(y; k, l) = \{v : v'(y) = \omega^{k,l}(y)\psi(y),\ \psi \in \mathcal{P}_N\}. \tag{3.23}$$

The spectral-Galerkin approximation to (3.21) is to find $u_N \in X_N$ such that

$$a_\varepsilon(u_N, \phi) := \varepsilon(u_N', \phi') + (pu_N', \phi) + (qu_N, \phi) = (f, \phi), \quad \forall \phi \in X_N. \tag{3.24}$$

Note that the above scheme using transformed basis functions is equivalent to the weighted spectral-Galerkin scheme for the transformed equation: find $v_N \in Y_N$ such that

$$\begin{aligned} \hat{a}_\varepsilon(v_N, \psi) &:= \varepsilon(g^{-1} v_N', \psi')_{\omega^{-k,-l}} + (\hat{p} v_N', \psi) + (g\hat{q} v_N, \psi)_{\omega^{k,l}} \\ &= (g\hat{f}, \psi)_{\omega^{k,l}}, \quad \forall \psi \in Y_N, \end{aligned} \tag{3.25}$$

where $\hat{p}(y) = p(x)$, $\hat{q}(y) = q(x)$, $\hat{f}(y) = f(x)$ with $x = s(y; k, l)$ and $g$ is defined in (3.1).

Without loss of generality, we assume that $p', p, q \in L^\infty(I)$, and satisfy one of the following coercive conditions

$$\begin{aligned} &\text{(i)} \quad \widehat{\omega}(x) := -\frac{p'(x)}{2} + q(x) \geqslant 0, \quad x \in \bar{I}; \\ &\text{(ii)} \quad \widehat{\omega}(x) := -\frac{p'(x)}{2} + q(x) \geqslant \gamma > 0, \quad x \in \bar{I}, \end{aligned} \tag{3.26}$$

which ensures the well-posedness of the scheme (3.24).

**Lemma 3.1.** Let $u_N$ be the solution of (3.24). If (i) of (3.26) holds, then

$$\varepsilon|u_N|_1^2 + \|u_N\|_{\widehat{\omega}}^2 \lesssim \varepsilon^{-1}\|f\|^2. \tag{3.27}$$

If (ii) of (3.26) holds, then

$$\varepsilon|u_N|_1^2 + \|u_N\|_{\widehat{\omega}}^2 \lesssim \|f\|^2. \tag{3.28}$$

*Proof.* We first prove (3.27). Taking $\phi = u_N$ in (3.24), we have from (3.26), integration by parts and the Poincaré inequality that

$$\begin{aligned} a_\varepsilon(u_N, u_N) &= \varepsilon|u_N|_1^2 + \|u_N\|_{\widehat{\omega}}^2 \leqslant |(f, u_N)| \\ &\leqslant \|u_N\|\|f\| \leqslant |u_N|_1\|f\| \leqslant \frac{\varepsilon}{2}|u_N|_1^2 + \frac{c}{\varepsilon}\|f\|^2. \end{aligned} \tag{3.29}$$

This implies (3.27). Similarly, we can prove (3.28) by using the fact

$$|(f, u_N)| \leqslant \frac{1}{2}\|u_N\|_{\widehat{\omega}}^2 + \frac{1}{2}\|f\|_{\widehat{\omega}^{-1}}^2 \leqslant \frac{1}{2}\|u_N\|_{\widehat{\omega}}^2 + \frac{c}{2}\|f\|^2. \tag{3.30}$$

This ends the proof. □

We now turn to the error estimates. Let $u$ and $u_N$ be respectively the solutions of (3.21) and (3.24), and denote $e = u - u_N$.

**Theorem 3.2.** Let $u(x) = v(y)$ with $x = s(y; k, l)$. Suppose that $u \in H_0^1(I)$ and $v \in A^m_{\omega^{-k-1,-l-1}}(I)$ with $m, k, l \in \mathbb{N}$ and $m \geqslant 1$.
If (i) or (ii) of (3.26) holds, then

$$\varepsilon\|\partial_x e\|^2 + \|e\|_{\widehat{\omega}}^2 \lesssim \varepsilon^{-1}N^{-2m}\|\partial_y^m v\|_{\omega^{k+m,l+m}} + \varepsilon N^{2-2m}\|\partial_y^m v\|^2_{\omega^{m-k-1,m-l-1}}. \tag{3.31}$$

If, in addition, $p \equiv 0$ and (ii) of (3.26) holds, then

$$\varepsilon\|\partial_x e\|^2 + \|e\|_{\widehat{\omega}}^2 \lesssim N^{-2m}\|\partial_y^m v\|_{\omega^{k+m,l+m}} + \varepsilon N^{2-2m}\|\partial_y^m v\|^2_{\omega^{m-k-1,m-l-1}}. \tag{3.32}$$

*Proof.* By (3.21) and (3.24), we have

$$a_\varepsilon(e, v_N) = 0, \quad \forall v_N \in X_N. \tag{3.33}$$

For any $\Phi_N \in X_N$, we denote $e_N = \Phi_N - u_N$. Then (3.33) implies

$$a_\varepsilon(e_N, v_N) = a_\varepsilon(\Phi_N - u, v_N), \quad \forall v_N \in X_N. \tag{3.34}$$

Taking $v_N = e_N$ in (3.34) and using integration by parts, we have from (3.26) that

$$\varepsilon|e_N|_1^2 + \|e_N\|_{\hat{\omega}}^2 \leqslant \varepsilon|\Phi_N u - u|_1|e_N|_1 + \|p\|_\infty\|\Phi_N u - u\||e_N|_1 + (\|p'\|_\infty + \|q\|_\infty)\|\Phi_N u - u\|\|e_N\|. \quad (3.35)$$

By the Poincaré inequality and Cauchy–Schwartz inequality, we have

$$\frac{\varepsilon}{2}|e_N|_1^2 + \|e_N\|_{\hat{\omega}}^2 \lesssim \varepsilon|\Phi_N - u|_1^2 + \varepsilon^{-1}\|\Phi_N - u\|^2. \quad (3.36)$$

Thus, using the triangle inequality gives

$$\frac{\varepsilon}{2}|e|_1^2 + \|e\|_{\hat{\omega}}^2 \leqslant \frac{\varepsilon}{2}|e_N|_1^2 + \|e_N\|_{\hat{\omega}}^2 + \frac{\varepsilon}{2}|\Phi_N - u|_1^2 + \|\Phi_N - u\|_{\hat{\omega}}^2 \lesssim \varepsilon\|\Phi_N - u\|_1^2 + \varepsilon^{-1}\|\Phi_N - u\|^2. \quad (3.37)$$

Then it remains to choose $\Phi_N \in X_N$ such that the upper bound for the right-hand side of (3.37) is as sharp as possible.

For $u \in H_0^1(I)$, we have $v(\pm 1) = 0$ and $\partial_y v \in L^2_{\omega^{-k,-l}}(I)$. Now, we take $\Phi_N(x) = (P_{N-1}^{k,l} v)(y) \in Y_N$, and obtain from (3.1) and Lemma 3.1 that

$$\|\Phi_N - u\|^2 \leqslant g_1\|P_{N-1}^{k,l} v - v\|_{\omega^{k,l}}^2 \lesssim N^{-2m}\|\partial_y^m v\|_{\omega^{k+m,l+m}}, \quad (3.38)$$

and

$$\|\partial_x(\Phi_N - u)\|^2 \leqslant g_0^{-1}\|\partial_y(P_{N-1}^{k,l} v - v)\|_{\omega^{-k,-l}}^2 \lesssim N^{2-2m}\|\partial_y^m v\|_{\omega^{m-k-1,m-l-1}}, \quad (3.39)$$

A combination of (3.37)–(3.39) leads to (3.31).

We now prove (3.32). Since $p \equiv 0$ and (ii) of (3.26) holds, we have from (3.34) with $v_N = e_N$ that

$$\varepsilon|e_N|_1^2 + \|e_N\|_{\hat{\omega}}^2 \leqslant \varepsilon|\Phi_N - u|_1|e_N|_1 + \|q\|_\infty\|\Phi_N - u\|_{\hat{\omega}^{-1}}\|e_N\|_{\hat{\omega}} \leqslant \frac{\varepsilon}{2}|e_N|_1^2 + \frac{\varepsilon}{2}|\Phi_N - u|_1^2 + \frac{1}{2}\|e_N\|_{\hat{\omega}}^2 + c\|\Phi_N - u\|^2. \quad (3.40)$$

The rest of the proof is straightforward. □

For problems with very thin boundary layers, the dominated terms in error estimates are usually associated with higher-order of derivatives of the solutions (cf. [21,22]). To see the error bounds in Theorem 3.2 more clearly, we assume that the solution of (3.21) satisfies (cf. [19]):

$$|\partial_x^n u(x)| \leqslant C_1 + C_2\varepsilon^{-n/2}\left(e^{-\mu_1(1-x)/\sqrt{\varepsilon}} + e^{-\mu_2(1+x)/\sqrt{\varepsilon}}\right), \quad n = 1, 2, 3, \ldots, \quad (3.41)$$

where $C_1, C_2, \mu_1$ and $\mu_2$ are generic positive constants independent of $\varepsilon$.

By (3.1), we have

$$\frac{d^n x}{dy^n} = g_n(y;k,l)(1-y)^{k+1-n}(1+y)^{l+1-n}, \quad k,l \geqslant n-1, \tag{3.42}$$

where $\{g_n\}$ are some sufficiently smooth and uniformly bounded functions for all $y \in \bar{I}$ and $k,l,n \in \mathbb{N}$. Let $v(y) = u(x)$ with $x = s(y;k,l)$ be the same as before. Then by (3.42),

$$\begin{aligned}
\partial_y v &= \partial_x u \frac{dx}{dy} = g_1 \omega^{k,l} \partial_x u, \\
\partial_y^2 v &= \partial_x^2 u \Big(\frac{dx}{dy}\Big)^2 + \partial_x u \frac{d^2 x}{dy^2} = g_1^2 \omega^{2k,2l} \partial_x^2 u + g_2 \omega^{k-1,l-1} \partial_x u, \\
\partial_y^3 v &= \partial_x^3 u \Big(\frac{dx}{dy}\Big)^3 + 3\partial_x^2 u \frac{dx}{dy}\frac{d^2 x}{dy^2} + \partial_x u \frac{d^3 x}{dy^3} \\
&= g_1^3 \omega^{3k,3l} \partial_x^3 u + g_1 g_2 \omega^{2k-1,2l-1} \partial_x^2 u + g_3 \omega^{k-2,l-2} \partial_x u.
\end{aligned} \tag{3.43}$$

Consequently,

$$\begin{aligned}
\|\partial_y^3 v\|^2_{\omega^{k+3,l+3}} &\lesssim \|\partial_x u\|^2_{\widetilde{\omega}^{2k-1,2l-1}} + \|\partial_x^2 u\|^2_{\widetilde{\omega}^{4k+1,4l+1}} + \|\partial_x^3 u\|_{\widetilde{\omega}^{6k+3,6l+3}}, \\
\|\partial_y^3 v\|^2_{\omega^{2-k,2-l}} &\lesssim \|\partial_x u\|^2_{\widetilde{\omega}^{-2,-2}} + \|\partial_x^2 u\|^2_{\widetilde{\omega}^{2k,2l}} + \|\partial_x^3 u\|^2_{\widetilde{\omega}^{4k+2,4l+2}},
\end{aligned} \tag{3.44}$$

where the weight function $\widetilde{\omega}^{\alpha,\beta}(x) = \omega^{\alpha,\beta}(y)$ with $x = s(y;k,l)$. We observe from (3.1) the following relation:

$$\begin{aligned}
\widetilde{\omega}^{\alpha,\beta}(x) = (1-y)^{\alpha}(1+y)^{\beta} &\lesssim (1-x)^{\frac{\alpha}{k+1}}(1+x)^{\frac{\beta}{l+1}}, \\
&\alpha > -k-1,\ \beta > -l-1,\ k,l \geqslant 1,\ k,l \in \mathbb{N}.
\end{aligned} \tag{3.45}$$

If the solution satisfies (3.41), then we have from (3.45) that

$$\begin{aligned}
\int_I (\partial_x^n u)^2 \widetilde{\omega}^{\alpha,\beta}(x)dx &\lesssim \int_I (\partial_x^n u)^2 \omega^{\frac{\alpha}{k+1},\frac{\beta}{l+1}}(x)dx \\
&\lesssim C_1^2 + C_2^2 \varepsilon^{-n} \int_I \Big(e^{-\mu_1(1-x)/\sqrt{\varepsilon}} + e^{-\mu_2(1+x)/\sqrt{\varepsilon}}\Big)^2 \omega^{\frac{\alpha}{k+1},\frac{\beta}{l+1}}(x)dx \\
&\lesssim 1 + \varepsilon^{-n} \int_I \Big(e^{-2\mu_1(1-x)/\sqrt{\varepsilon}} + e^{-2\mu_2(1+x)/\sqrt{\varepsilon}}\Big) \omega^{\frac{\alpha}{k+1},\frac{\beta}{l+1}}(x)dx \\
&\lesssim 1 + \varepsilon^{-n+\frac{1}{2}+\frac{\alpha}{2(k+1)}} \int_0^{\frac{2}{\sqrt{\varepsilon}}} e^{-2\mu_1 t} t^{\frac{\alpha}{k+1}} dt + \varepsilon^{-n+\frac{1}{2}+\frac{\beta}{2(l+1)}} \int_0^{\frac{2}{\sqrt{\varepsilon}}} e^{-2\mu_2 t} t^{\frac{\beta}{l+1}} dt \\
&\lesssim 1 + \varepsilon^{-n+\frac{1}{2}} \Big(\varepsilon^{\frac{\alpha}{2(k+1)}} + \varepsilon^{\frac{\beta}{2(l+1)}}\Big).
\end{aligned} \tag{3.46}$$

A combination of (3.44), (3.46) and Theorem 3.2 with $m=3$ (other cases can be considered accordingly) leads to the following convergence result.

**Corollary 3.1.** Let $k,l\geqslant 2$ and $k,l\in\mathbb{N}$, and assume that the solution of (3.21) satisfies (3.41). If (i) or (ii) of (3.26) holds, then

$$\varepsilon\|\partial_x e\|^2+\|e\|^2_{\hat{\omega}}\lesssim\varepsilon^{-1/2}\Big(\varepsilon^{-\frac{3}{2(k+1)}}+\varepsilon^{-\frac{3}{2(l+1)}}\Big)N^{-6}+\varepsilon^{1/2}\Big(\varepsilon^{-\frac{1}{k+1}}+\varepsilon^{-\frac{1}{l+1}}\Big)N^{-4}. \tag{3.47}$$

If, in addition, $p\equiv 0$ and (ii) of (3.26) holds, then

$$\varepsilon\|\partial_x e\|^2+\|e\|^2_{\hat{\omega}}\lesssim\varepsilon^{1/2}\Big(\varepsilon^{-\frac{3}{2(k+1)}}+\varepsilon^{-\frac{3}{2(l+1)}}\Big)N^{-6}+\varepsilon^{1/2}\Big(\varepsilon^{-\frac{1}{k+1}}+\varepsilon^{-\frac{1}{l+1}}\Big)N^{-4}. \tag{3.48}$$

### *3.2.2. Applications to Two Specific Mappings*

We first consider the following two-parameter mapping:

$$x=s(y;k,l)=-1+\sigma_{k,l}\int_{-1}^{y}(1-t)^k(1+t)^l dt,\quad x,y\in\bar{I},\ k,l\in\mathbb{N} \tag{3.49}$$

with

$$\sigma_{k,l}=2\Big/\int_{-1}^{1}(1-y)^k(1+y)^l dy. \tag{3.50}$$

Clearly, we have

$$\frac{dx}{dy}=s'(y;k,l)=\sigma_{k,l}(1-y)^k(1+y)^l=\sigma_{k,l}\omega^{k,l}(y),\quad y\in I. \tag{3.51}$$

Note that the symmetric cases (i.e., $k=l$) was used in [21,22]. As indicated by the numerical results in Sec. 5, the non-symmetric cases with $k$ or $l=0$ are very effective in resolving one-side boundary layers.

Since the mapping (3.49) is a singular mapping of index-$(k,l)$ so the general results in Theorem 3.2 and Corollary 3.1 can be applied directly. Moreover, repeating a procedure as in (3.42)–(3.46), we can also derive a convergence result for $k\geqslant 2$ and $l=0$. Indeed, assuming that the solution of (3.21) satisfies (3.41), if (i) or (ii) of (3.26) holds, then

$$\varepsilon\|\partial_x e\|^2+\|e\|^2_{\hat{\omega}}\lesssim\varepsilon^{-\frac{k+4}{2(k+1)}}N^{-6}+\varepsilon^{\frac{k-1}{2(k+1)}}N^{-4}. \tag{3.52}$$

If, in addition, $p \equiv 0$ and (ii) of (3.26) holds, then

$$\varepsilon \|\partial_x e\|^2 + \|e\|^2_{\hat{\omega}} \lesssim \varepsilon^{\frac{k-2}{2(k+1)}} N^{-6} + \varepsilon^{\frac{k-1}{2(k+1)}} N^{-4}. \tag{3.53}$$

We also note that under the mapping (3.49), the system of (3.24) with $p(x) \equiv \bar{p}$ and $q(x) \equiv \bar{q}$ is sparse. We choose the basis for $X_N$ as

$$\phi_n^{k,l}(x) := \psi_n^{k,l}(y), \quad x = s(y; k, l).$$

Then

$$X_N = \text{span}\{\phi_0^{k,l}, \phi_1^{k,l}, \dots, \phi_N^{k,l}\}$$

and we deduce from (3.5) and (3.8) that

$$\begin{aligned}
a_{ij} &:= a_{ij}(k,l) = (\partial_x \phi_j^{k,l}, \partial_x \phi_i^{k,l}) = \sigma_{k,l}^{-1} (\partial_y \psi_j^{k,l}, \partial_y \psi_i^{k,l})_{\omega^{-k,-l}} = 0, \quad \forall i \neq j,\\
b_{ij} &:= b_{ij}(k,l) = (\phi_j^{k,l}, \phi_i^{k,l}) = \sigma_{k,l} (\psi_j^{k,l}, \psi_i^{k,l})_{\omega^{k,l}} = 0, \quad \forall |i-j| > 2k + 2l + 2,\\
c_{ij} &:= c_{ij}(k,l) = (\partial_x \phi_j^{k,l}, \phi_i^{k,l}) = (\partial_y \psi_j^{k,l}, \psi_i^{k,l}) = 0, \quad \forall |i-j| > k + l + 1.
\end{aligned} \tag{3.54}$$

Setting

$$\begin{aligned}
&A = (a_{ij})_{i,j=0,\dots,N-1}, \quad B = (b_{ij})_{i,j=0,\dots,N-1}, \quad C = (c_{ij})_{i,j=0,\dots,N-1},\\
&u_N(x) = \sum_{j=0}^{N-1} \hat{u}_j \phi_j^{k,l}(x), \quad \mathbf{u} = (\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{N-1})^t,\\
&\hat{f}_i = (f, \phi_i^{k,l}), \quad \mathbf{f} = (\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{N-1})^t,
\end{aligned}$$

the linear system of (3.24) with constant coefficients $p(x) \equiv \bar{p}$ and $q(x) \equiv \bar{q}$ becomes

$$(\varepsilon A + \bar{p} C + \bar{q} B)\mathbf{u} = \mathbf{f}. \tag{3.55}$$

We see that the band-widths of the matrices $A, B$, and $C$ are independent of $N$, $A, B$ are symmetric positive, and $C$ is skew-symmetric. The entries of them can be explicitly determined by using properties of the Jacobi polynomials in Appendix.

As a second example, we consider the iterated mappings introduced by Tang and Trummer [28]:

$$x_0 = y, \ \ x_j = \sin\left(\frac{\pi}{2} x_{j-1}\right), \ \ y \in \bar{I}, \ \ j \geqslant 1, \ j \in \mathbb{N}, \tag{3.56}$$

which are very effective mappings for problems with thin boundary layers. This mapping with $j = 1$ was discussed in Kosloff and Tal-Ezer [20]. This mapping has the following property.

**Lemma 3.2.** We have

$$1 \pm x_j = \left(\frac{\pi^2}{2^3}\right)^j (1 \pm y)^{2^j} (1 + o(1)), \quad y \to \pm 1, \ j \in \mathbb{N}, \tag{3.57}$$

$$\frac{dx_j}{dy} = \left(\frac{\pi}{2}\right)^{2j} \left(\frac{\pi^2}{2^3}\right)^{\frac{j(j-1)}{2}} (1 \pm y)^{2^j - 1} (1 + o(1)), \quad y \to \pm 1, \ j \in \mathbb{N}. \tag{3.58}$$

*Proof.* Since $x_j(\pm 1) = \pm 1$, we have from the Taylor expansion theorem that

$$1 \pm x_j = 1 \pm \sin\left(\frac{\pi}{2} x_{j-1}\right) = \frac{\pi^2}{2^3} (1 \pm x_{j-1})^2 (1 + o(1)), \quad y \to \pm 1. \tag{3.59}$$

In particular,

$$1 \pm x_1 = \frac{\pi^2}{2^3} (1 \pm y)^2 (1 + o(1)), \quad y \to \pm 1. \tag{3.60}$$

Thus, (3.57) follows from (3.59) to (3.60) and an induction argument.

We now prove (3.58). By (3.56),

$$\frac{dx_1}{dy} = \frac{\pi}{2} \cos\left(\frac{\pi}{2} y\right) = \left(\frac{\pi}{2}\right)^2 (1 \pm y)(1 + o(1)), \quad y \to \pm 1. \tag{3.61}$$

and

$$\frac{dx_j}{dy} = \frac{\pi}{2} \cos\left(\frac{\pi}{2} x_{j-1}\right) \frac{dx_{j-1}}{dy}, \quad j \geqslant 1. \tag{3.62}$$

Inductively, we have from (3.57) and (3.62) that

$$\begin{aligned} \frac{dx_j}{dy} &= \left(\frac{\pi}{2}\right)^j \prod_{k=0}^{j-1} \cos\left(\frac{\pi}{2} x_k\right) = \left(\frac{\pi}{2}\right)^j \prod_{k=0}^{j-1} \left(\frac{\pi}{2} (1 \pm x_k)(1 + o(1))\right) \\ &= \left(\frac{\pi}{2}\right)^{2j} \prod_{k=0}^{j-1} \left[\left(\frac{\pi^2}{2^3}\right)^k (1 \pm y)^{2^k} (1 + o(1))\right] \\ &= \left(\frac{\pi}{2}\right)^{2j} \left(\frac{\pi^2}{2^3}\right)^{\frac{j(j-1)}{2}} (1 \pm y)^{2^j - 1} (1 + o(1)). \end{aligned} \tag{3.63}$$

This ends the proof. □

The above lemma indicates that (3.56) is a singular mapping of index–$(2^j-1, 2^j-1)$. Therefore, the results in Theorem 3.2 and Corollary 3.1 with $k=l=2^j-1$ and $j>1$ are applicable for this mapping.

We note that Liu and Tang [22] carried out an error analysis of (3.25) with these two mappings. Our results in this section are more general, and in many cases, improve the results in [22].

## 4. NUMERICAL RESULTS AND DISCUSSION

We next present some numerical results with emphasis on how the accuracy depends on the mapping and its parameters.

### 4.1. Example 1

We consider the approximation of the function

$$u(x)=\exp(-\sigma(x-x_0)^2), \quad \sigma>0,\ x_0\in I. \tag{4.1}$$

This function and its derivatives exhibit rapid variations near the region of $x=x_0$, when $\sigma$ is large.

We now apply a Jacobi approximation to (4.1) using the mapping (2.30), and set

$$v(y)=u(x), \quad v_N(y)=u_N(x)=(\pi^{\alpha,\beta}_{\mu,N}u)(x), \quad x=s(y;\mu), \tag{4.2}$$

where $\pi^{\alpha,\beta}_{\mu,N}u$ is the orthogonal projection defined in (2.8). In the following computations, we take $\sigma=2000$, $x_0=0$ in (4.1), and $\mu_1=30, \mu_2=0$, $\alpha=\beta=1$ in (4.2).

In Figs. 1 and 2, we plot the test function $u$ (resp. $v$) and its derivatives $u'$ (resp. $v'$) vs. the approximations $u_N$ (resp. $v_N$) and $u'_N$ (resp. $v'_N$) with $N=100$ in physical domain (resp. computational domain). By using the mapping (2.30), the mapped JGL points $\{\xi^{\alpha,\beta}_{\mu,N}\}$ are clustered to the region with rapid variations, and consequently, the test function with strong local behaviors in $x$ becomes very smooth in the computational domain (in $y$).

To illustrate the theoretical results presented in Sec. 2, we plot in Fig. 3 the discrete weighted $L^2-$ and $H^1-$ norms of $u-u_N$ and $\partial_x(u-u_N)$ with mapping (the below two lines) and without mapping (the above two lines), respectively. It indicates an exponential convergence of the mapped Jacobi approximation as predicted by Theorem 2.1 and Corollary 2.1.

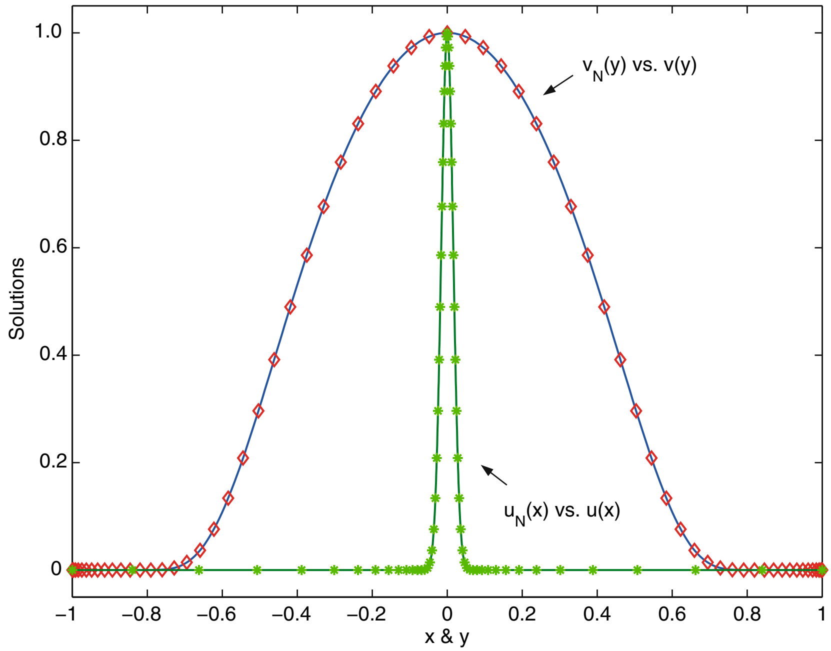


**Fig. 1.** Test function vs. its approximations in $x$ and $y$.

We see that the use of mapping leads to significant reduction in the number of collocation points required in order to obtain a given level of accuracy.

Next, we examine how the accuracy depends on the parameters of the mapping (2.30). In Fig. 4, we plot the errors $\|\partial_x(\mathcal{I}^{\alpha,\beta}_{\mu,N}u-u)\|_{\tilde{\omega}^{\alpha,\beta},N}$ vs. various $\mu_1$ and $\mu_2$ with $N=100$ and $\alpha=\beta=1$. As predicted in Theorem 2.1 and Corollary 2.1, the errors increase as the intensity $\mu_1$ increases. We also note that the accuracy is sensitive to the values of $\mu_2$ (the location of the large variation), but less sensitive to the choices of $\mu_1$ (the intensity of clustering the points).

### 4.2. Example 2

We consider the initial-value Fisher equation (cf. [9]):

$$\partial_t u = \partial_x^2 u + u(1-u), \quad u(x,0) = u_0(x) \tag{4.3}$$

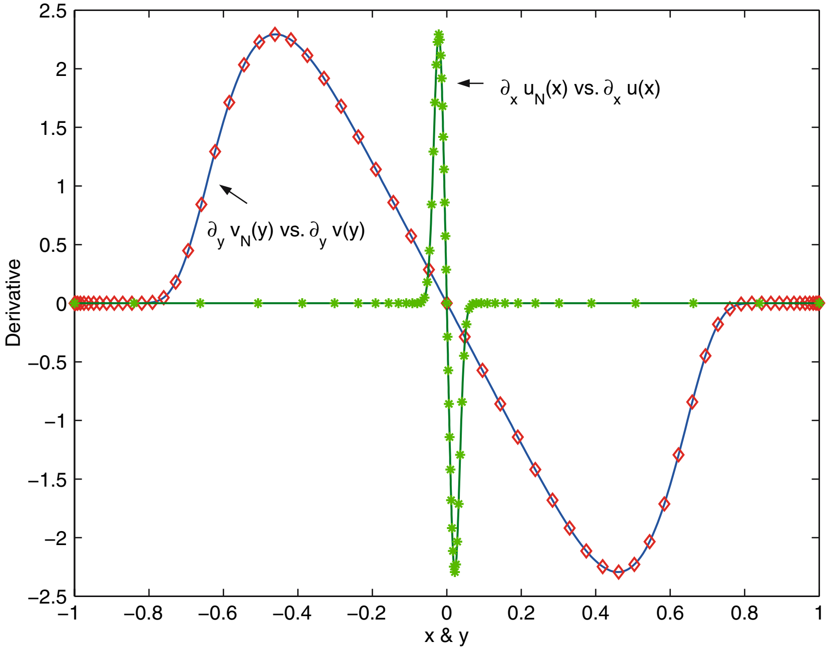


**Fig. 2.** Derivative of test function vs. its approximation in $x$ and $y$.

with the travelling solution

$$u(x,t)=\left(1+\exp\left(\frac{x}{\sqrt{6}}-\frac{5}{6}t\right)\right)^{-2}, \tag{4.4}$$

and the wave speed $c=5/\sqrt{6}$.

Since $u(x,t)$ tends to 0 (resp. 1) exponentially as $x\to+\infty$ (resp. $x\to-\infty$), we can approximate the initial-value problem (4.3) by an initial-boundary-value problem in $(-L,L)$ as long as the wave front doesn't reach the boundaries.

In actual computation, we rescale the problem (4.3): $x\to Lx$, $t\to L^2t$, and consider the scaled equation:

$$\begin{aligned} &\partial_t u=\partial_x^2 u+L^2u(1-u), \quad x\in(-1,1),\\ &u(-1,t)=1, \quad u(1,t)=0,\\ &u(x,0)=u_0(x) \end{aligned} \tag{4.5}$$

with the wave speed $c=5L/\sqrt{6}$.

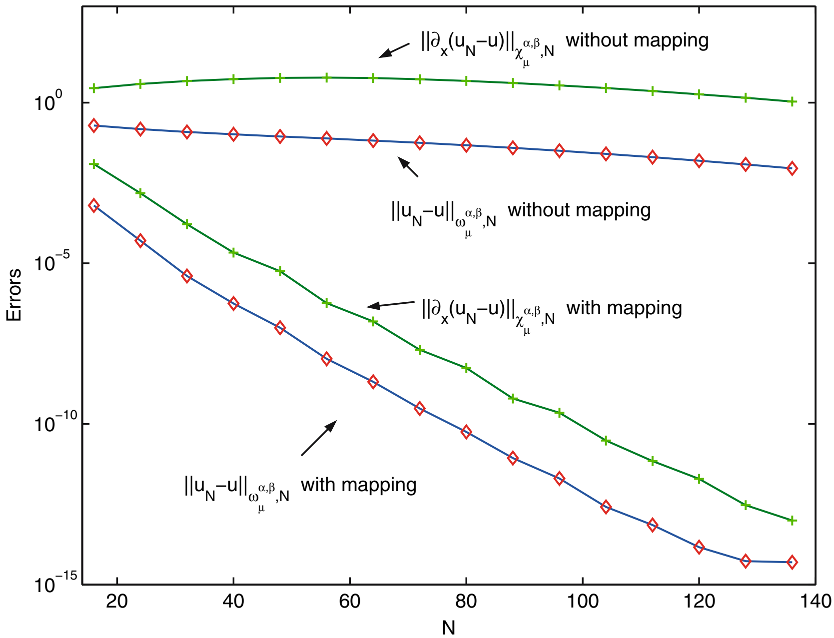


**Fig. 3.** Errors vs. $N$.

By setting $w=u-(1-x)/2$, we convert (4.5) to the following homogeneous problem:

$$\begin{aligned} &\partial_t w = \partial_x^2 w + L^2 F(w), \quad x\in(-1,1) \\ &w(\pm 1,t)=0, \quad w(x,0)=w_0(x):=u_0(x)-(1-x)/2, \end{aligned} \tag{4.6}$$

where the nonlinear term $F(w):=((1-x)/2+w)((1+x)/2-w)$.

For a given time step $\tau$, we set $t_k=k\tau$ and $v^k=v(x,t_k)$. We still use the mapping (2.30), and take $\mu_1=30$ and $\mu_2=ct_k=\frac{5L}{\sqrt{6}}t_k$ to track the wave front. We define the approximation space:

$$V_N^{\mu}:=\text{span}\{j_{\mu,n+2}^{0,0}(x)-j_{\mu,n}^{0,0}(x): n=0,1,\dots,N-2\}. \tag{4.7}$$

The fully-discrete Crank–Nicolson leap-frog mapped spectral-Galerkin approximation to (4.6) is to find $w_N^{k+1}\in V_N^{\mu}$ such that

$$\frac{1}{2\tau}(w_N^{k+1}-w_N^{k-1},v_N)-\frac{1}{2}(\partial_x(w_N^{k+1}+w_N^{k-1}),\partial_x v_N) = L^2(F(w_N^k),v_N), \quad \forall v_N\in V_N^{\mu} \tag{4.8}$$

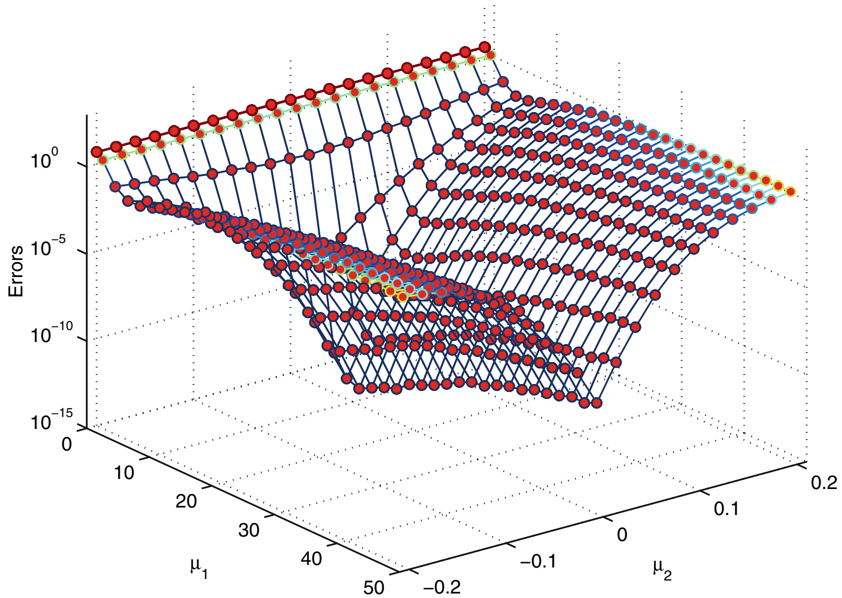


**Fig. 4.** Errors vs. parameters $\mu_1$ & $\mu_2$.

with $w_N^0 = \pi_{\mu,N}^{0,0} w_0$, and $w_N^1$ being a suitable approximation to $w(\cdot, t_1)$, which for instance, can be computed by using a one-step semi-implicit scheme.

Then the numerical solution of (4.3) can be evaluated by

$$u_N(x, L^2 t_{k+1}) = w_N(L^{-1}x, t_{k+1}) + \frac{1 - L^{-1}x}{2}, \quad x \in (-L, L). \tag{4.9}$$

We plot the exact solution (4.4) vs. the numerical solution $u_N$ ($N = 100$ and $L = 100$) without mapping (see Fig. 5) and with mapping (see Fig. 6). It is obvious that the use of an appropriate mapping can provide not only a very good approximation, but also a high resolution.

### 4.3. Example 3

We now present some numerical results on the mapped Jacobi spectral methods using singular mapping (3.49) and (3.50). As an example, we consider the following diffusion equation:

$$-\varepsilon \partial_x^2 u + u = -\frac{x+1}{2}, \quad x \in I, \quad u(\pm 1) = 0 \tag{4.10}$$

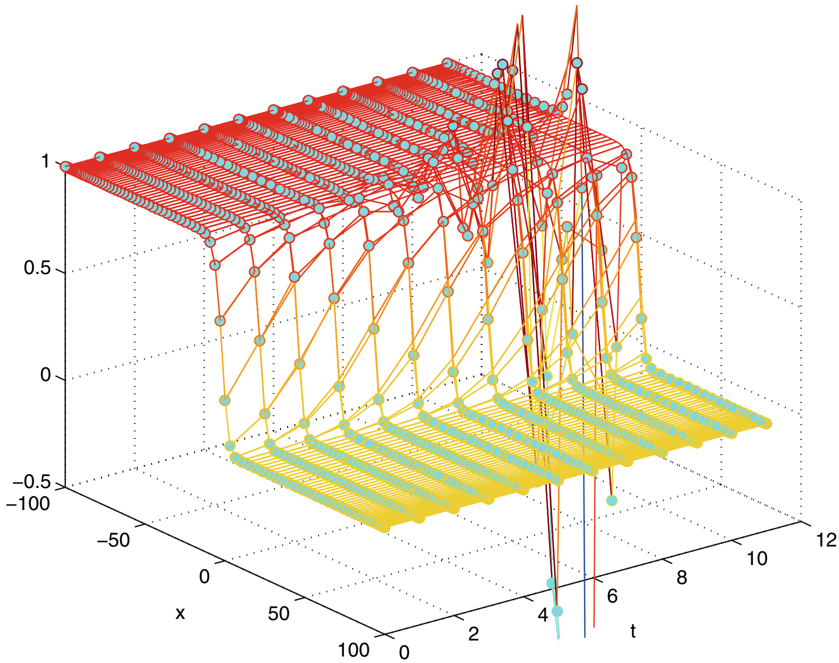


**Fig. 5.** Exact solution vs. numerical solutions without mapping.

with the exact solution

$$u(x) = \frac{\exp\left(\frac{1+x}{\sqrt{\varepsilon}}\right) - \exp\left(-\frac{1+x}{\sqrt{\varepsilon}}\right)}{\exp\left(\frac{2}{\sqrt{\varepsilon}}\right) - \exp\left(-\frac{2}{\sqrt{\varepsilon}}\right)} - \frac{1+x}{2}. \tag{4.11}$$

This solution has a boundary layer of width $O(\sqrt{\varepsilon})$ at $x=1$.

Let $v(y)=u(x)$ and $x=s(y;k,l)$ be the mapping given in (3.49) and (3.50). We use the scheme (3.24) to solve (4.10) numerically, and obtain the numerical solutions: $u_N(x)$ in physical domain and $v_N(y)$ in computational domain. In Fig. 7, we plot the exact solution vs. the numerical solution in $x$ and $y$ with $N=128, \varepsilon=10^{-12}, k=4$ and $l=0$. We see that the use of the singular mapping (3.49) and (3.50) stretches the boundary layer and the corresponding mapped Jacobi spectral method provides a high accuracy in resolving very thin boundary layer.

In Fig. 8, we plot the maximum point-wise errors (on the mapped Gauss–Legendre–Lobatto points) vs. $N$ with $k=4, l=0$ and various $\varepsilon$. We observe a fast convergence is achieved even for very small $\varepsilon$, which is in agreement with the theoretical results in Sec. 4.2.

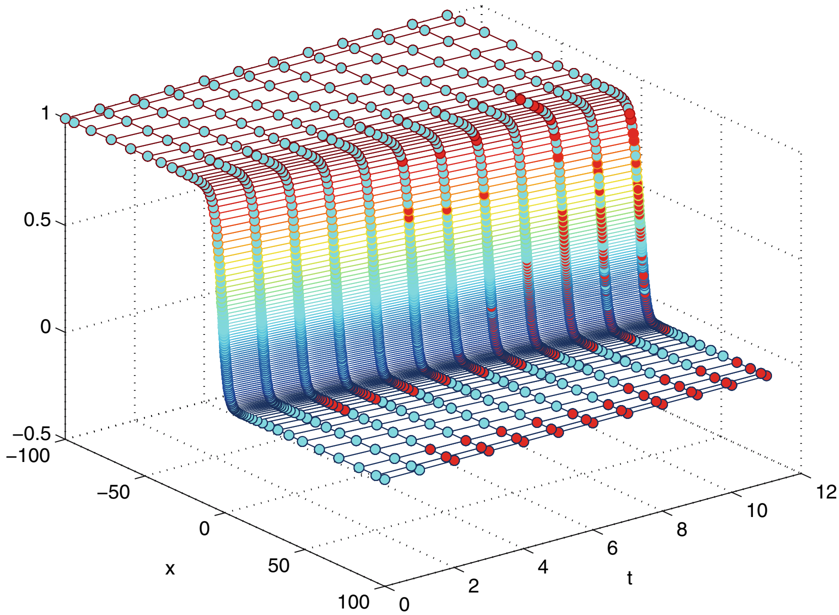


**Fig. 6.** Exact solution vs. numerical solutions with mapping.

In Figs. 9 and 10, we plot the maximum point-wise errors of mapped Jacobi spectral method using a symmetric mapping $k = l = 4$, and of the standard spectral method (without mapping: $k = l = 0$). Compared with the errors in Fig. 8, we find that a much better accuracy can be obtained by using a non-symmetric singular mapping for problems with one-side boundary layers, and that the standard spectral method does not converge when the boundary layers are very thin.

## APPENDIX A. SOME PROPERTIES OF JACOBI POLYNOMIALS

The classical Jacobi polynomials $\{J_n^{\alpha,\beta}\}$ are the eigenfunctions of the Sturm–Liouville problem:

$$\partial_y((1-y)^{\alpha+1}(1+y)^{\beta+1}\partial_y J_n^{\alpha,\beta}(y)) + \lambda_n^{\alpha,\beta}(1-y)^{\alpha}(1+y)^{\beta} J_n^{\alpha,\beta}(y) = 0 \quad \text{(A.1)}$$

with the eigenvalues:

$$\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1), \quad n \geqslant 0, \quad \alpha, \beta > -1. \quad \text{(A.2)}$$

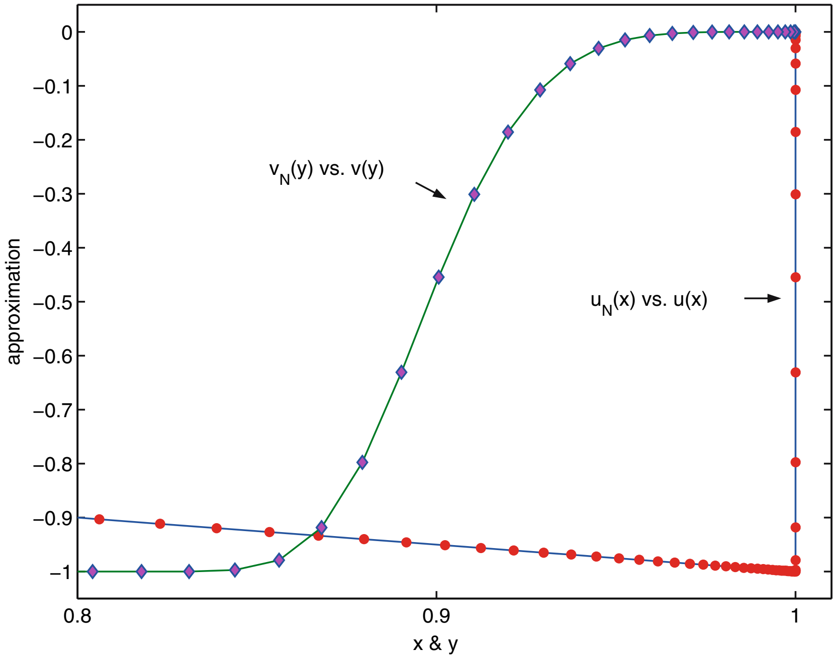


**Fig. 7.** Exact solution vs. numerical solution.

Let $\omega^{\alpha,\beta}(y)=(1-y)^{\alpha}(1+y)^{\beta}$ be the Jacobi weight function. For $\alpha,\beta>-1$, the Jacobi polynomials are mutually orthogonal in $L^2_{\omega^{\alpha,\beta}}(I)$, i.e.,

$$\int_I J_n^{\alpha,\beta}(y)J_m^{\alpha,\beta}(y)\omega^{\alpha,\beta}(y)dy=\gamma_n^{\alpha,\beta}\delta_{n,m}, \tag{A.3}$$

where $\delta_{n,m}$ is the Kronecker function, and

$$\gamma_n^{\alpha,\beta}=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{A.4}$$

They satisfy the following recurrence relations (see Szegö [27] and Askey [1]):

$$\begin{aligned} yJ_n^{\alpha,\beta}(y)&=a_n^{(1)}J_{n-1}^{\alpha,\beta}(y)+b_n^{(1)}J_n^{\alpha,\beta}(y)+c_n^{(1)}J_{n+1}^{\alpha,\beta}(y),\\ J_0^{\alpha,\beta}(y)&=1,\quad J_1^{\alpha,\beta}(y)=\tfrac{1}{2}(\alpha+\beta+2)y+\tfrac{1}{2}(\alpha-\beta), \end{aligned} \tag{A.5}$$

$$(1-y^2)\partial_y J_n^{\alpha,\beta}(y)=a_n^{(2)}J_{n-1}^{\alpha,\beta}(y)+b_n^{(2)}J_n^{(\alpha,\beta)}(y)+c_n^{(2)}J_{n+1}^{\alpha,\beta}(y), \tag{A.6}$$

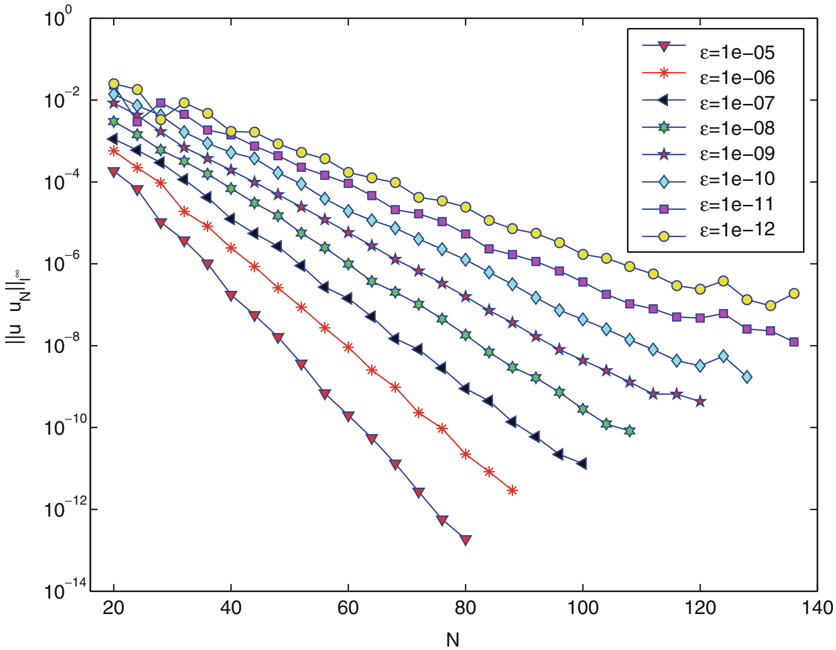


**Fig. 8.** Errors vs. $N$.

$$J_n^{\alpha,\beta}(-y) = (-1)^n J_n^{\beta,\alpha}(y), \quad J_n^{\alpha,\beta}(1) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}, \tag{A.7}$$

$$J_{n-1}^{\alpha,\beta}(y) = J_n^{\alpha,\beta-1}(y) - J_n^{\alpha-1,\beta}(y), \quad \alpha, \beta > 0 \;\; n \geqslant 1, \tag{A.8}$$

$$J_n^{\alpha,\beta}(y) = \frac{1}{n+\alpha+\beta}[(n+\beta) J_n^{\alpha,\beta-1}(y) + (n+\alpha) J_n^{\alpha-1,\beta}(y)], \quad \alpha, \beta > 0, \tag{A.9}$$

$$(1-y) J_n^{\alpha+1,\beta}(y) = \frac{2}{2n+\alpha+\beta+2}[(n+\alpha+1) J_n^{\alpha,\beta}(y) - (n+1) J_{n+1}^{\alpha,\beta}(y)], \tag{A.10}$$

$$\partial_y J_n^{\alpha,\beta}(y) = \tfrac{1}{2}(n+\alpha+\beta+1) J_{n-1}^{\alpha+1,\beta+1}(y), \quad n \geqslant 1, \tag{A.11}$$

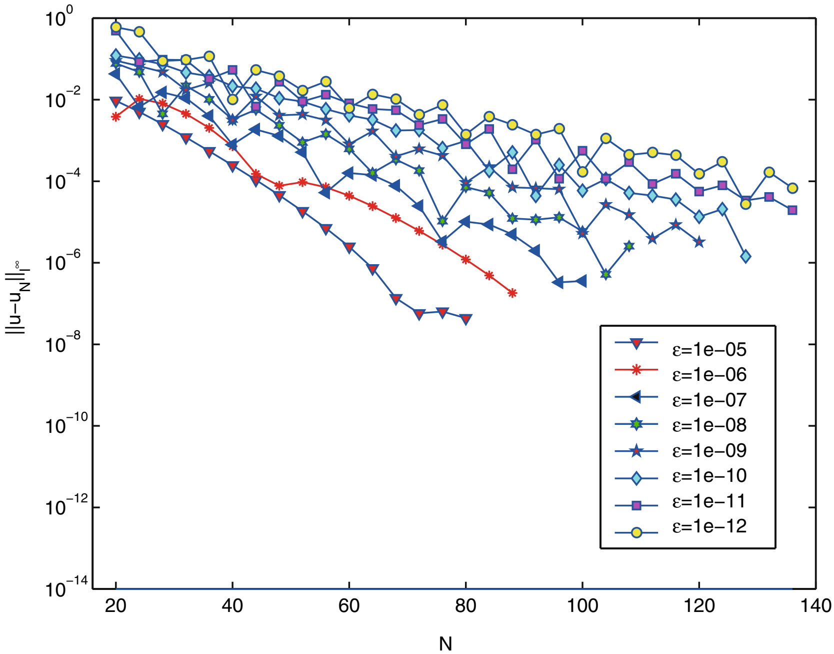


**Fig. 9.** Errors vs. $N$ with a symmetric mapping.

$$(1-y)^{\alpha}(1+y)^{\beta}J_n^{\alpha,\beta}(y)=\frac{(-1)^m(n-m)!}{2^m n!}\frac{d^m}{dy^m}\{(1-y)^{\alpha+m}(1+y)^{\beta+m}J_{n-m}^{\alpha+m,\beta+m}(y)\},\ \ n\geqslant m\geqslant 0. \tag{A.12}$$

Here, $a_n^{(i)}, b_n^{(i)}$ and $c_n^{(i)}$ $(i=1,2)$ in (A.5) and (A.6) are constants in terms of $\alpha, \beta$ and $n$, whose explicit expressions are given in [27].

Using the above properties, we can show

$$\partial_y((1-y)^{\alpha}J_n^{\alpha,\beta}(y))=-(n+\alpha)(1-y)^{\alpha-1}J_n^{\alpha-1,\beta+1}(y),\quad \alpha>0,\ \beta>-1, \tag{A.13}$$

$$\partial_y((1+y)^{\beta}J_n^{\alpha,\beta}(y))=(n-\beta)(1+y)^{\beta-1}J_n^{\alpha+1,\beta-1}(y),\quad \alpha>-1,\ \beta>0. \tag{A.14}$$

Indeed, by (A.8)–(A.11),

$$\partial_y((1-y)^{\alpha}J_n^{\alpha,\beta}(y))$$

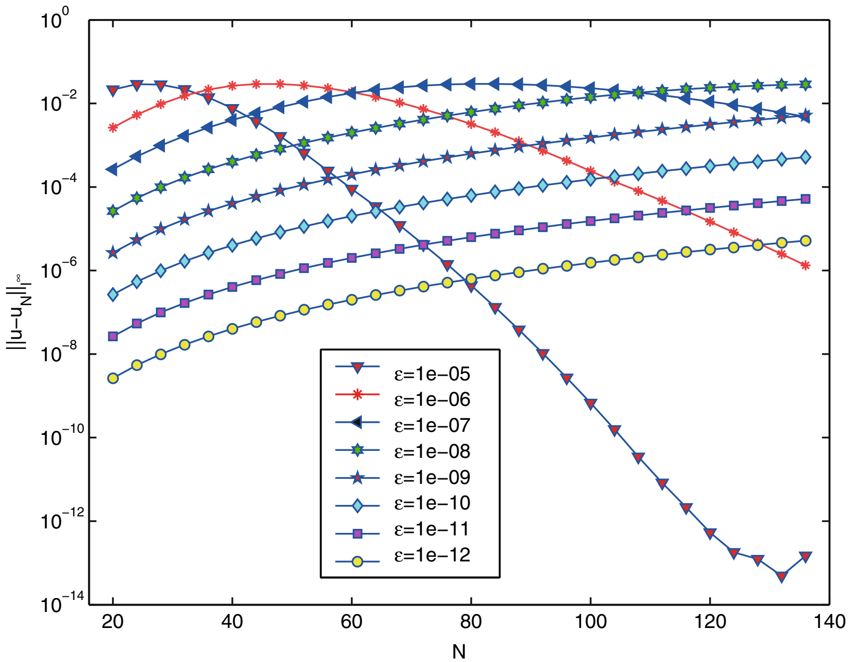


**Fig. 10.** Errors vs. $N$ without mapping.

$$
\begin{aligned}
&\overset{(A.11)}{=} (1-y)^{\alpha-1}\Big(-\alpha J_n^{\alpha,\beta}(y)+\frac{1}{2}(n+\alpha+\beta+1)(1-y)J_{n-1}^{\alpha+1,\beta+1}(y)\Big)\\
&\overset{(A.10)}{=} (1-y)^{\alpha-1}\Big(-\alpha J_n^{\alpha,\beta}(y)+\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\Big((n+\alpha)J_{n-1}^{\alpha,\beta+1}(y)-nJ_n^{\alpha,\beta+1}(y)\Big)\Big)\\
&\overset{(A.8)}{=} (1-y)^{\alpha-1}\Big(-\alpha J_n^{\alpha,\beta}(y)+\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\Big((n+\alpha)J_n^{\alpha,\beta}(y)\\
&\qquad -(n+\alpha)J_n^{\alpha-1,\beta+1}(y)-nJ_n^{\alpha,\beta+1}(y)\Big)\Big)\\
&= (1-y)^{\alpha-1}\Big(\frac{n}{2n+\alpha+\beta+1}\Big((n+\beta+1)J_n^{\alpha,\beta}(y)\\
&\qquad -(n+\alpha+\beta+1)J_n^{\alpha,\beta+1}(y)\Big)\\
&\qquad -\frac{(n+\alpha)(n+\alpha+\beta+1)}{2n+\alpha+\beta+1}J_n^{\alpha-1,\beta+1}(y)\Big)\\
&\overset{(A.9)}{=} (1-y)^{\alpha-1}\Big(\frac{-n(n+\alpha)}{2n+\alpha+\beta+1}J_n^{\alpha-1,\beta+1}(y)\\
&\qquad -\frac{(n+\alpha)(n+\alpha+\beta+1)}{2n+\alpha+\beta+1}J_n^{\alpha-1,\beta+1}(y)\Big)\\
&= -(n+\alpha)(1-y)^{\alpha-1}J_n^{\alpha-1,\beta+1}(y).
\end{aligned}
$$

This leads to (A.13). Similarly, we can prove (A.14).

### A.1. Some Results on Jacobi Approximations

For any $v \in L^2_{\omega^{\alpha,\beta}}(I)$, we write

$$v(y) = \sum_{n=0}^{\infty} \hat{v}_n^{\alpha,\beta} J_n^{\alpha,\beta}(y), \quad \text{with } \hat{v}_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} (v, J_n^{\alpha,\beta})_{\omega^{\alpha,\beta}}. \tag{A.15}$$

As pointed out in [15], we have the following equivalence:

$$\|\partial_y^m v\|_{\omega^{\alpha+m,\beta+m}} \sim \Big( \sum_{n=m}^{\infty} (\lambda_n^{\alpha,\beta})^m \gamma_n^{\alpha,\beta} (\hat{v}_n^{\alpha,\beta})^2 \Big)^{1/2}, \quad \forall v \in A^m_{\omega^{\alpha,\beta}}(I), \ m \in \mathbb{N}, \tag{A.16}$$

where

$$A^m_{\omega^{\alpha,\beta}}(I) := \{ v \in L^2_{\omega^{\alpha,\beta}}(I) : \ \partial_y^k v \in L^2_{\omega^{\alpha+k,\beta+k}}(I), \ 0 \leqslant k \leqslant m \}. \tag{A.17}$$

Now, we define the $L^2_{\omega^{\alpha,\beta}}(I)$-orthogonal projection: $\hat{\pi}_N^{\alpha,\beta}: \ L^2_{\omega^{\alpha,\beta}}(I) \to \mathcal{P}_N$, such that

$$(\hat{\pi}_N^{\alpha,\beta} v - v, v_N)_{\omega^{\alpha,\beta}} = 0, \quad \forall v_N \in \mathcal{P}_N. \tag{A.18}$$

The following result was proved in [10] (also see [2,18]):

**Lemma A.1.**

$$\|\partial_y^l (\hat{\pi}_N^{\alpha,\beta} v - v)\|_{\omega^{\alpha+l,\beta+l}} \lesssim N^{l-m} \|\partial_y^m v\|_{\omega^{\alpha+m,\beta+m}}, \quad 0 \leqslant l \leqslant m, \forall v \in A^m_{\omega^{\alpha,\beta}}(I). \tag{A.19}$$

Next, let $\{\xi_{N,j}^{\alpha,\beta}\}_{j=0}^N$ be the set of JGL points, which are the zeros of the Jacobi polynomials $(1-y^2)\partial_y J_N^{\alpha,\beta}(y)$. We assume that $\{\xi_{N,j}^{\alpha,\beta}\}_{j=0}^N$ are arranged in ascending order. Then, there exists a unique set of quadrature weights $\{\omega_{N,j}^{\alpha,\beta}\}_{j=0}^N$ (cf. [27]) such that

$$\int_I v(y) \omega^{\alpha,\beta}(y) dy = \sum_{j=0}^{N} v(\xi_{N,j}^{\alpha,\beta}) \omega_{N,j}^{\alpha,\beta}, \quad \forall v \in \mathcal{P}_{2N-1}. \tag{A.20}$$

We have the following explicit expressions for the weights:

$$\omega_{N,j}^{\alpha,\beta} = -\frac{2^{\alpha+\beta}(2N+\alpha+\beta)\Gamma(N+\alpha)\Gamma(N+\beta)}{(N+\alpha+\beta+1)N!\Gamma(N+\alpha+\beta)}\Big((J_N^{\alpha,\beta}\partial_y J_{N-1}^{\alpha,\beta})(\xi_{N,j}^{\alpha,\beta})\Big)^{-1}, \quad 1 \leqslant j \leqslant N-1, \tag{A.21}$$

$$\omega_{N,0}^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}(\beta+1)\Gamma^2(\beta+1)\Gamma(N)\Gamma(N+\alpha+1)}{\Gamma(N+\beta+1)\Gamma(N+\alpha+\beta+2)}, \tag{A.22}$$

$$\omega_{N,N}^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}(\alpha+1)\Gamma^2(\alpha+1)\Gamma(N)\Gamma(N+\beta+1)}{\Gamma(N+\alpha+1)\Gamma(N+\alpha+\beta+2)}. \tag{A.23}$$

Indeed, the formula (A.21) comes from (3.5.2) of [10], we now prove (A.22) and (A.23) below.

Let

$$\phi(y) := \frac{(1-y)\partial_y J_N^{\alpha,\beta}(y)}{2\partial_y J_N^{\alpha,\beta}(-1)}.$$

Clearly, $\phi \in \mathcal{P}_N$ and $\phi(\xi_{N,0}^{\alpha,\beta}) = 1$. Since $\{\xi_{N,j}^{\alpha,\beta}\}_{j=1}^N$ are the zeros of $(1-y)\partial_y J_N^{\alpha,\beta}(y)$, we have $\phi(\xi_{N,j}^{\alpha,\beta}) = 0$, $1 \leqslant j \leqslant N$. Actually, $\phi$ is the Lagrangian base function corresponding to the node $y = \xi_{N,0}^{\alpha,\beta}$. Then the weight

$$\begin{aligned}
\omega_{N,0}^{\alpha,\beta} &= \int_I \phi(y)\omega^{\alpha,\beta}(y)dy = \frac{1}{2\partial_y J_N^{\alpha,\beta}(-1)}\int_I \partial_y J_N^{\alpha,\beta}(y)\omega^{\alpha+1,\beta}(y)dy \\
&\overset{(A.11)}{=} \frac{1}{2J_{N-1}^{\alpha+1,\beta+1}(-1)}\int_I J_{N-1}^{\alpha+1,\beta+1}(y)\omega^{\alpha+1,\beta}(y)dy \\
&= \frac{1}{2J_{N-1}^{\alpha+1,\beta+1}(-1)}(J_{N-1}^{\alpha+1,\beta+1}, 1)_{\omega^{\alpha+1,\beta}}.
\end{aligned} \tag{A.24}$$

As pointed out in [1], $J_{N-1}^{\alpha+1,\beta+1}$ is a linear combination of $\{J_n^{\alpha+1,\beta}\}_{n=0}^{N-1}$:

$$J_{N-1}^{\alpha+1,\beta+1}(y) = a_0 + \sum_{n=1}^{N-1} a_n J_n^{\alpha+1,\beta}(y) \tag{A.25}$$

with

$$a_0 = \frac{(-1)^{N-1}(\alpha+\beta+2)\Gamma(\alpha+\beta+2)\Gamma(N+\alpha+1)}{\Gamma(\alpha+2)\Gamma(N+\alpha+\beta+2)}.$$

Plugging (A.25) into (A.24), we obtain from (A.7) and (A.3) that

$$\omega_{N,0}^{\alpha,\beta}=\frac{a_0\gamma_0^{\alpha+1,\beta}}{2J_{N-1}^{\alpha+1,\beta+1}(-1)}=\frac{2^{\alpha+\beta+1}(\beta+1)\Gamma^2(\beta+1)\Gamma(N)\Gamma(N+\alpha+1)}{\Gamma(N+\beta+1)\Gamma(N+\alpha+\beta+2)}.$$

To prove (A.23), we take

$$\phi(y)=\frac{(1+y)\partial_y J_N^{\alpha,\beta}(y)}{2\partial_y J_N^{\alpha,\beta}(1)},$$

and then

$$\omega_{N,N}^{\alpha,\beta}=\int_I \phi(y)\omega^{\alpha,\beta}(y)dy.$$

A similar procedure as for (A.22) leads to (A.23).

For notational convenience, we introduce the discrete inner product and norm associated with the quadrature rule (A.20):

$$(u,v)_{\omega^{\alpha,\beta},N}=\sum_{j=0}^{N}u(\xi_{N,j}^{\alpha,\beta})v(\xi_{N,j}^{\alpha,\beta})\omega_{N,j}^{\alpha,\beta},\quad \|v\|_{\omega^{\alpha,\beta},N}=(v,v)_{\omega^{\alpha,\beta},N}^{1/2},\quad \forall u,v\in C(\bar{I}).$$

As shown in (2.26) of [13], we have

$$\|v_N\|_{\omega^{\alpha,\beta}}\leqslant\|v_N\|_{\omega^{\alpha,\beta},N}\leqslant\sqrt{2+\frac{\alpha+\beta+1}{N}}\|v_N\|_{\omega^{\alpha,\beta}},\quad \forall v_N\in\mathcal{P}_N. \tag{A.26}$$

Besides, by Theorem 4.9 of [13],

$$\left(\sum_{j=1}^{N-1}|v(\xi_{N,j}^{\alpha,\beta})|^2\omega_{N,j}^{\alpha,\beta}\right)^{1/2}\lesssim\|v\|_{\omega^{\alpha,\beta}}+N^{-1}\|\partial_y v\|_{\omega^{\alpha+1,\beta+1}},\quad \forall v\in A^1_{\omega^{\alpha,\beta}}(I). \tag{A.27}$$

### A.2. Proofs of (3.5) and (3.6)

We first prove (3.5). By (A.6) and (A.11),

$$\psi_n^{k,l}(y)\overset{(A.11)}{=}\frac{2}{n+k+l+2}(1-y)^{k+1}(1+y)^{l+1}\partial_y J_{n+1}^{k,l}(y)$$
$$\overset{(A.6)}{=}\frac{2}{n+k+l+2}(1-y)^k(1+y)^l(a_{n+1}^{(2)}J_n^{k,l}(y)+b_{n+1}^{(2)}J_{n+1}^{k,l}(y)+c_{n+1}^{(2)}J_{n+2}^{k,l}(y)).$$

Since $(1-y)^k(1+y)^l$ is a polynomial of $y$ with degree $\leqslant k+l$, we can use(A.5) repeatedly to obtain (3.5).

Now, we consider (3.6). If (i) $m \leqslant l+1 \leqslant k+1$, then we take $\alpha = k - m+1$, $\beta = l-m+1$, replace $n$ by $n+m$ in (A.12), and derive (3.6) directly.

In case of (ii) $l+1 \leqslant m \leqslant k+1$, we obtain from the above proved case (i) (with $m = l+1$) that

$$\partial_y^m \psi_n^{k,l}(y) = \partial_y^{m-l-1}\{\partial_y^{l+1}\psi_n^{k,l}(y)\} = A_{l+1,n}^{k,l}\partial_y^{m-l-1}\{(1-y)^{k-l}J_{n+l+1}^{k-l,0}(y)\}.$$

Thus, by using (A.13) inductively, we can obtain (3.6) with $l+1 \leqslant m \leqslant k+1$.

Similarly, if (iii) $l+1 \leqslant k+1 \leqslant m$, then (3.6) follows from the proved case (ii) and using (A.11) inductively.

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