Applied Numerical Mathematics 33 (2000) 183–190

APPLIED NUMERICAL MATHEMATICS

www.elsevier.nl/locate/apnum

A new fast Chebyshev–Fourier algorithm for Poisson-type equations in polar geometries ☆

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Abstract

A fast Chebyshev–Fourier algorithm for Poisson-type equations in polar geometries is presented in this paper. The new algorithm improves upon the algorithm of Jie Shen (1997), by taking advantage of the odd–even parity of the Fourier expansion in the azimuthal direction, and it is shown to be more efficient in terms of CPU and memory.

Keywords: Chebyshev–Fourier expansion; Polar coordinates; Spectral–Galerkin

1. Introduction

In a recent paper [5], the author introduced an efficient spectral–Galerkin algorithm, for solving elliptic equations in polar geometries, which has quasi-optimal computational complexity while being spectrally accurate. However, it does not take into account the odd–even parity of the Fourier expansion in the azimuthal direction, and consequently, it uses collocation points which are unnecessarily clustered near the origin. The aim of this paper is to develop a more efficient Chebyshev–Fourier algorithm for the Poisson-type equations in polar geometries by taking advantage of the odd–even parity in the Fourier expansion. Note that there are several very good algorithms (cf. [2–4]) which preserve the odd–even parity, but the new algorithm in this paper appears to be more efficient.

2. A Chebyshev–Fourier interpolation operator on a disk

A key aspect of spectral methods is to find an appropriate transform between the values of a function at a set of collocation points to the spectral representation of its interpolating function. Thus, we describe

☆ This work was supported in part by NSF grants DMS-9623020, DMS-9706951 and by a research fellowship at the Texas Institute of Computational and Applied Mathematics.

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0168-9274/00/$20.00
PII: S0168-9274(99)00082-3

first a Chebyshev–Fourier interpolation operator on the disk to be used in our Chebyshev–Fourier algorithm.

Let $V(x, y)$ be a continuous function on the unit disk $\Omega = \{(x, y)\colon x^2 + y^2 < 1\}$. We denote $v(r,\theta) := V(r\cos\theta, r\sin\theta)$ to be the function in $D = (0, 1) \times (0, 2\pi)$ after the polar transform $x = r\cos\theta$, $y = r\sin\theta$. Since v is periodic in θ, we may write

$$v(r,\theta) = \sum_{|m|=0}^{\infty} v_m(r)\,\mathrm{e}^{\mathrm{i}m\theta} \quad \text{with} \quad v_{-m}(r) = \bar{v}_m(r) \quad \text{for all } m, \tag{2.1}$$

where $\bar{v}_m$ is the complex conjugate of v_m. We emphasize that the expansion coefficients in (2.1) cannot be arbitrary. In fact, it is well known (see, for instance, [1,3,4]) that $v_m(r)$ has the same parity as m and can be expanded *smoothly* to the interval $(-1, 0)$, i.e., if v_m is in $H^k(0, 1)$ then the expanded function is in $H^k(-1, 1)$. In particular, for $v \in C(\Omega)$ we have

$$v_m \in Y^{(m)} := \bigl\{v \in C(-1, 1)\colon\ v(-r) = (-1)^m v(r),\ r \in (0, 1)\bigr\},$$

and consequently,

$$v \in Y := \left\{ v \in \sum_{|m|=0}^{\infty} v_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}\colon\ v_{-m}(r) = \bar{v}_m(r),\ v_m \in Y^{(m)} \right\}.$$

Note that $Y^{(m)}$ (and $X^{(m)}$ defined later) are complex spaces while Y (and X defined below) are real spaces. We define below approximations of $Y^{(m)}$ and Y which preserve the odd–even parity.

Given a pair of even integers (N, M), we denote P_N to be the space of polynomials of degree less than or equal to N, and let

$$\psi_j^{(m)}(r) = \begin{cases} T_{2j}(r), & \text{if } m \text{ is even}, \\ T_{2j+1}(r), & \text{if } m \text{ is odd}, \end{cases} \tag{2.2}$$

where $T_k(r)$ is the kth-degree Chebyshev polynomial. We define respectively approximation of $Y^{(m)}$ and Y by

$$Y_N^{(m)} := \left\{ v = \sum_{j=0}^{N/2-\mathrm{mod}(m,2)} v_j \psi_j^{(m)}(r)\colon\ v_j \text{ are complex numbers} \right\}, \tag{2.3}$$

and

$$Y_{NM} := \left\{ v = \sum_{|m|=0}^{M} v_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}\colon\ v_m \in Y_N^{(m)},\ v \text{ is real} \right\}. \tag{2.4}$$

We define a set of collocation points in $\overline{D}$ associated to Y_{NM} by

$$\Sigma_{NM} := \left\{ (r_k, \theta_j)\colon\ \begin{array}{l} k = 0, 1, \ldots, N/2 - 1,\ j = 0, 1, \ldots, 2M - 1 \\ k = N/2,\ j = 0, 1, \ldots, M - 1 \end{array} \right\}, \tag{2.5}$$

where $r_k = \cos(k\pi/N)$ and $\theta_j = j\pi/M$. Note that for $v \in Y$ or Y_{NM}, $v(0,\theta) = v(0, \pi + \theta)$ for all θ. Hence, the points (r_k, θ_j) with $k = N/2$ and $j = M, M + 1, \ldots, 2M - 1$ are excluded from Σ_{NM}.

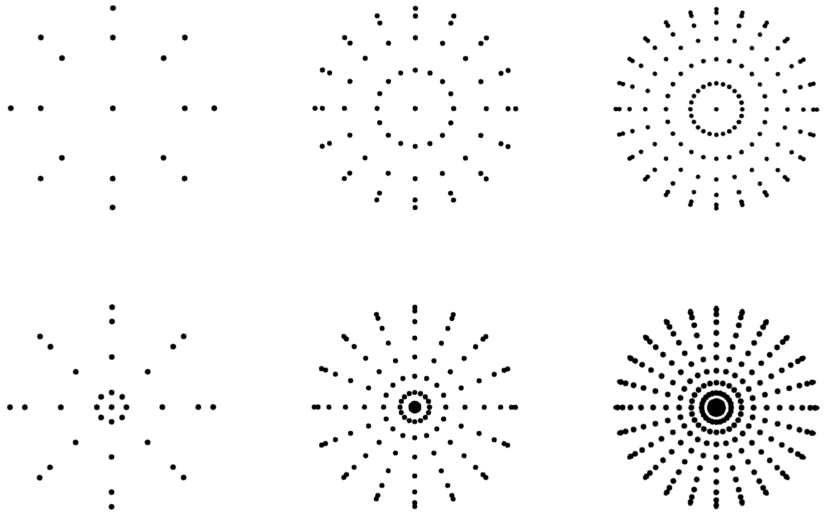

Fig. 1. Distribution of collocation points: first row—Σ_{NM}; second row—$\widetilde{\Sigma}_{NM}$.

One can now readily check that a unique interpolation operator I_{NM} from $Y \cap C(\overline{D})$ to Y_{NM} is defined by

$$\begin{aligned} &I_{NM}g(r,\theta) = \sum_{|m|=0}^{M} \sum_{n=0}^{N/2-\mathrm{mod}(m,2)} g_{nm}\psi_n^{(m)}(r)\,\mathrm{e}^{\mathrm{i}m\theta} \in Y_{NM}, \\ &I_{NM}g(r_k,\theta_j) = g(r_k,\theta_j), \quad (r_k,\theta_j) \in \Sigma_{NM}. \end{aligned} \tag{2.6}$$

Remark 1. A different interpolation operator was used in [5]. More precisely, the change of variable $r = (t+1)/2$ is applied so that the functions in the transformed spaces no longer satisfy the odd–even parity condition and that the corresponding set of collocation points in $\overline{D}$ is

$$\widetilde{\Sigma}_{NM} := \big\{\big((t_k+1)/2, \theta_j\big):\ k = 0, 1, \ldots, N,\ j = 0, 1, \ldots, 2M-1\big\}$$

with $t_k = \cos(k\pi/N)$ and $\theta_j = j\pi/M$. Not only $\widetilde{\Sigma}_{NM}$ has twice as many points as Σ_{NM}, but also its points are unnecessarily clustered in the radial direction near the pole ($r = 0$), see Fig. 1. Indeed, the smallest distance in the Cartesian coordinates between two adjacent points in $\widetilde{\Sigma}_{NM}$ (respectively in Σ_{NM}) near the pole is of order $\mathrm{O}(N^{-2}M^{-1})$ (respectively $\mathrm{O}(N^{-1}M^{-1})$).

3. Description of the algorithm

Consider the Poisson type equation on the unit disk

$$\begin{aligned} &-\Delta U + \alpha U = F \quad \text{in } \Omega = \big\{(x,y):\ x^2+y^2<1\big\}, \\ &U = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Applying the polar transformation $x = r\cos\theta$, $y = r\sin\theta$ to (3.1), and setting $u(r,\theta) = U(r\cos\theta, r\sin\theta)$, $f(r,\theta) = F(r\cos\theta, r\sin\theta)$, we obtain

$$\begin{aligned} &-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}u\right) - \frac{1}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}u + \alpha u = f, \quad (r,\theta)\in D, \\ &u(1,\theta) = 0, \quad \theta\in[0,2\pi), \qquad u \text{ periodic in } \theta. \end{aligned} \tag{3.2}$$

Writing

$$u(r,\theta) = \sum_{|m|=0}^{\infty} u_m(r)\,\mathrm{e}^{\mathrm{i}m\theta} \quad \text{with} \quad u_{-m}(r) = \bar{u}_m(r) \quad \text{for all } m, \tag{3.3}$$

(likewise for f), and substituting the expansions in (3.2), we find

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}u_m\right) + \left(\frac{m^2}{r^2} + \alpha\right)u_m = f_m, \quad r\in(0,1), \qquad u_m(1) = 0. \tag{3.4}$$

Below we develop an algorithm which takes advantage of the odd–even parity by seeking approximation of u_m in the space

$$X_N^{(m)} := \{v \in Y_N^{(m)}\colon\ v(1) = 0\}. \tag{3.5}$$

Given a pair of even integers (N, M), we consider the following weighted (the weight function is $r^2\omega(r)$) spectral–Galerkin approximation to (3.4) for $m = 0, 1, \ldots, M$: find $u_N^{(m)} \in X_N^{(m)}$ such that

$$\begin{aligned} &-\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}u_N^{(m)}\right) r v\omega\,\mathrm{d}r + m^2\int_0^1 u_N^{(m)} v\omega\,\mathrm{d}r + \alpha\int_0^1 r^2 u_N^{(m)} v\omega\,\mathrm{d}r = \int_0^1 r^2 f_N^{(m)} v\omega\,\mathrm{d}r \\ &\forall v\in X_N^{(m)}, \end{aligned} \tag{3.6}$$

where $\omega(r) = (1-r^2)^{-1/2}$ is the Chebyshev weight, and $f_N^{(m)}$ is the mth component of

$$I_{NM} f = \sum_{|m|=0}^{M} \sum_{n=0}^{N/2-\mathrm{mod}(m,2)} f_{nm}\psi_n^{(m)}(r)\,\mathrm{e}^{\mathrm{i}m\theta},$$

namely,

$$f_N^{(m)} = \sum_{n=0}^{N/2-\mathrm{mod}(m,2)} f_{nm}\psi_n^{(m)}(r).$$

Then, the approximation to u is given by

$$u_{NM}(r,\theta) = \sum_{|m|=0}^{M} u_N^{(m)}(r)\,\mathrm{e}^{\mathrm{i}m\theta}, \quad u_N^{(-m)} = \bar{u}_N^{(m)}.$$

Note that the approximate solution u_{NM} does not satisfy the pole conditions "$u_N^{(m)}(0) = 0$ for $m \neq 0$", hence, u_{NM} is neither necessarily single valued nor differentiable at the pole in the Cartesian coordinates. However, u_{NM} still converges to u exponentially provided that f is smooth in the Cartesian coordinates (see [6] for a related discussion and on how to extract from u_{NM} an approximate function which is single valued and differentiable at the pole).

It is clear that $X_N^{(m)}$ is a $N/2$ (respectively $N/2-1$) dimensional space if m is even (respectively odd) and that

$$\phi_j^{(m)}(r) := (1-r^2)\psi_j^{(m)}(r) \in X_N^{(m)}. \tag{3.7}$$

Therefore,

$$X_N^{(m)} = \operatorname{span}\{\phi_j^{(m)}:\ j=0,1,\ldots,N/2-1-\operatorname{mod}(m,2)\}. \tag{3.8}$$

Thus, letting

$$\begin{aligned}
u_N^{(m)} &= \sum_{k=0}^{N/2-1-\operatorname{mod}(m,2)} x_k^{(m)}\phi_k^{(m)}, && \mathbf{x}^{(m)} = (x_k^{(m)}),\\
a_{kj}^{(m)} &:= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\phi_j^{(m)}\right) r\phi_k^{(m)}\omega\,\mathrm{d}r, && A^{(m)} = (a_{kj}^{(m)}),\\
b_{kj}^{(m)} &:= \int_0^1 \phi_j^{(m)}\phi_k^{(m)}\omega\,\mathrm{d}r, && B^{(m)} = (b_{kj}^{(m)}),\\
c_{kj}^{(m)} &:= \int_0^1 r^2\phi_j^{(m)}\phi_k^{(m)}\omega\,\mathrm{d}r, && C^{(m)} = (c_{kj}^{(m)}),\\
f_k^{(m)} &:= \int_0^1 r^2 f_N^{(m)}\phi_k^{(m)}\omega\,\mathrm{d}r, && \mathbf{f}^{(m)} = (f_k^{(m)}),
\end{aligned} \tag{3.9}$$

the formulation (3.6) is reduced to the linear system

$$\left(A^{(m)} + m^2B^{(m)} + \alpha C^{(m)}\right)\mathbf{x}^{(m)} = \mathbf{f}^{(m)}. \tag{3.10}$$

Note that although an index m is used in $A^{(m)}$, $B^{(m)}$, $C^{(m)}$ and $X_N^{(m)}$, these matrices only depend on the parity of m, rather than the actual value of m.

Proposition 2. *For m even or odd, $A^{(m)}$, $B^{(m)}$ are penta-diagonal matrices, and $C^{(m)}$ are seven-diagonal matrices.*

Proof. Notice that all the integrands in $a_{kj}^{(m)}$, $b_{kj}^{(m)}$ and $c_{kj}^{(m)}$ are even functions. Therefore, we can replace $\int_0^1$ by $\frac{1}{2}\int_{-1}^1$. Then, thanks to the orthogonality relation of the Chebyshev polynomials and the special form of the basis functions (3.7), one derives immediately that $B^{(m)}$ and $C^{(m)}$ are respectively penta- and seven-diagonal symmetric matrix. By the same argument, we have

$$\begin{aligned}
a_{kj}^{(m)} &= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\phi_j^{(m)}\right) r\phi_k^{(m)}\omega\,\mathrm{d}r\\
&= -\frac{1}{2}\int_{-1}^1 \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\phi_j^{(m)}\right) r\phi_k^{(m)}\omega\,\mathrm{d}r = 0 \quad \text{for } j<k-2.
\end{aligned}$$

On the other hand, integrating by parts twice, using (3.7) and the identity $\omega'(r) = (r/(1-r^2))\omega(r)$, we have

$$\begin{aligned}
a_{kj}^{(m)} &= \frac{1}{2}\int_{-1}^{1}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\phi_j^{(m)}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(r\phi_k^{(m)}\omega\right)\mathrm{d}r \\
&= \frac{1}{2}\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}r}\phi_j^{(m)}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\phi_k^{(m)} + \frac{r}{1-r^2}\phi_k^{(m)}\right)\omega\,\mathrm{d}r \\
&= -\frac{1}{2}\int_{-1}^{1}\phi_j^{(m)}\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\phi_k^{(m)} + \frac{r}{1-r^2}\phi_k^{(m)}\right) + \left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\phi_k^{(m)} + \frac{r}{1-r^2}\phi_k^{(m)}\right)\frac{r}{1-r^2}\right\}\omega\,\mathrm{d}r \\
&= -\frac{1}{2}\int_{-1}^{1}\psi_j^{(m)}\left\{(1-r^2)\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\phi_k^{(m)} + r\psi_k^{(m)}\right) + \left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\phi_k^{(m)} + r\psi_k^{(m)}\right)r\right\}\omega\,\mathrm{d}r.
\end{aligned}$$

Then, thanks to the special form of the basis function (3.7), we find that the function between the pair of brackets is a polynomial of degree $2k+4$ (respectively $2k+5$) for m even (respectively odd). Therefore, we have $a_{kj}^{(m)} = 0$ if $k < j-2$. □

The entries of $A^{(m)}$, $B^{(m)}$ and $C^{(m)}$ can be evaluated analytically, but this process can be quite tedious. Alternatively, one can compute these entries automatically by using the Chebyshev–Gauss–Lobatto quadrature of degree $N+2$.

In summary, the new Chebyshev–Fourier algorithm for solving (3.2) consists of three steps:

(1) Compute $I_{NM} f$ from the values of f on Σ_{NM}—$\mathrm{O}(NM\log(NM))$ operations;
(2) Solve $u_N^{(m)}$ from (3.10) for $m = 0, 1, \ldots, M$—$\mathrm{O}(NM)$ operations;
(3) Compute $u_{NM}(r,\theta) = \sum_{|m|=0}^{M} u_N^{(m)}(r)\,\mathrm{e}^{\mathrm{i}m\theta}$ on Σ_{NM}—$\mathrm{O}(NM\log(NM))$ operations.

Hence, the overall computational complexity is $\mathrm{O}(NM\log(NM))$ which is quasi-optimal.

Remark 3. Although we have only described the algorithm for Poisson-like equations with homogeneous Dirichlet boundary conditions on a disk, it can be readily extended, as in [5], to problems with more general boundary conditions, to three-dimensional cylindrical domains, to problems with variable coefficients and to higher-order elliptic equations, we refer to [5] for more details.

4. Numerical results

We now present some numerical results using the new algorithm. Since the Chebyshev–Fourier–Galerkin (CFG) algorithm in [5] is already shown to be more efficient and more accurate than those in [2,7] (see [5]), we shall only compare the new CFG algorithm with that in [5].

We consider the Poisson equation on a unit disk with the exact solution

$$U(x,y) = \left(x^2+y^2-1\right)\left(\cos\left(\beta(x+y)\right) + \sin\left(\beta(x+y)\right)\right). \tag{4.1}$$

The maximum errors of the two algorithms for the exact solution (4.1) with $\beta = 16$ are listed in Table 1, where new CFG stands for the new Chebyshev–Fourier–Galerkin algorithm described above. This exact

Table 1
Maximum errors: exact solution being (4.1) with $\beta = 16$

$N = M$	32	36	40	44	48	52
New CFG	1.98E−2	3.98E−4	4.63E−6	3.34E−8	1.59E−10	4.58E−13
$N = M$	22	26	30	34	38	42
CFG in [5]	8.68E−3	3.55E−4	8.52E−6	1.17E−7	9.99E−10	5.52E−12

Table 2
Resolution needed for six-digit accuracy: exact solution being (4.1) with different β

β	20	30	40	50	60
New CFG, $N = M$	48	64	80	96	112
CFG in [5], $N = M$	38	52	66	82	98

solution is smooth in both the Cartesian and polar coordinates, so both algorithms converge exponentially fast. Note that to achieve the same accuracy as the CFG in [5] with the pair (N, N), the new CFG should be used, roughly speaking, with the pair $(N + 10, N + 10)$. We recall that *for a fixed pair of* (N, M)*, the number of unknowns and the CPU time of the new algorithm is about half of the algorithm in* [5]. Thus, the new algorithm is still significantly more efficient, in terms of CPU and memory, than the algorithm in [5].

In Table 2, we list the resolution, in terms of (N, M), required to have six-digit accuracy for different values of β in the exact solution (4.1). The results in Table 2 indicates again that to achieve the same accuracy as the CFG in [5] with the pair (N, N), we need to use about $(N + 14, N + 14)$ in the new CFG. Therefore, *asymptotically* speaking, for a fixed pair of (N, M), the new CFG provides about the same accuracy as the CFG in [5] while costing half of the CPU and memory.

5. Concluding remarks

We have presented a new fast Chebyshev–Fourier–Galerkin method for Poisson-type equations in polar geometries. The new algorithm improves upon the algorithm in [5] by taking advantage of the odd–even parity of the Fourier expansion in the zimuthal direction. It is shown that the new algorithm is significantly more efficient in terms of CPU and memory than the algorithm in [5]. Another advantage of the new algorithm is that the collocation points are not unnecessarily clustered in the radial direction near the pole so that a semi-implicit or explicit time marching scheme based on this set of collocation points would not be affected by a unreasonably restrictive CFL stability condition.

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