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NORMAL MODE ANALYSIS OF SECOND-ORDER PROJECTION METHODS FOR INCOMPRESSIBLE FLOWS

Jae-Hong Pyo and Jie Shen

Department of Mathematics
Purdue University
West Lafayette , IN 47907, USA

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Abstract. A rigorous normal mode error analysis is carried out for two second-order projection type methods. It is shown that although the two schemes provide second-order accuracy for the velocity in $\mathbf{L}^2$-norm, their accuracies for the velocity in $\mathbf{H}^1$-norm and for the pressure in L^2-norm are different, and only the consistent splitting scheme introduced in [6] provides full second-order accuracy for all variable in their natural norms. The advantages and disadvantages of the normal mode analysis vs. the energy method are also elaborated.

1. Introduction.

We consider the movement of an incompressible fluid inside Ω whose velocity $\mathbf{u}$ and pressure p are governed by the Navier-Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f}, & \text{in } \Omega\times[0,T], \\ \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega\times[0,T], \\ \mathbf{u}|_\Gamma = \mathbf{0}, \quad \text{and } \mathbf{u}|_{t=0} = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where, for the sake of simplicity, a homogeneous Dirichlet boundary condition is assumed for $\mathbf{u}$ on $\Gamma = \partial\Omega$. One of the main numerical difficulties in solving (1.1) is that the velocity and the pressure are coupled together through the incompressibility condition $\nabla\cdot\mathbf{u} = 0$. The original projection method was introduced by Chorin [2] and Temam [16] in the late 60's to decouple the computation of velocity from the pressure, it quickly gained popularity in the computational fluid dynamics community, and over the years, an enormous amount of efforts have been devoted to develop more accurate and efficient projection type schemes, we refer to [5] for a comprehensive and up-to-date review on this subject.

The numerical analysis of projection type methods are usually carried out by using an energy method (see, for instance, [14, 12, 5] and the references therein) or a normal mode analysis (e.g., [11, 8, 3, 15, 1]): the energy method is capable of providing rigorous estimates for general settings but often overlooks particular error structures of projection errors; on the other hand, the normal mode analysis is only applicable to very special domains such as a periodic channel or a quarter plane but often reveals more precise information on the error behaviors.

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We consider in this paper two second-order projection type schemes, namely the rotational pressure-correction scheme [7] and the consistent splitting scheme [6]. The error analysis of the first scheme was performed in [1] for a semi-infinite periodic strip using a normal mode analysis (for one normal mode only) and in [7] for general domains using an energy method. It is shown in [1] that the velocity and pressure errors are of second-order, but since only one normal mode was considered, it is not clear the estimates hold in which functional spaces. As it is a well known fact by now that the convergence rates of projection type schemes may differ if measured in different norms. In fact, it is shown in [7] that the pressure approximation of this scheme, although more accurate than the standard pressure-correction scheme, is only 3/2-order accurate in $L^2(0,T;L^2(\Omega))$ for general domains. On the other hand, there is no stability and error analysis available for the second-order consistent splitting method. The purposes of this paper are to carry out a systematic and rigorous normal mode error analysis for the two methods, to clarify their different error behaviors, and to exemplify the advantages and shortcomings of normal mode analysis. In particular, We prove that the consistent splitting scheme provides fully second-order accuracy.

The gauge method proposed by E and Liu [4] is another class of projection type methods. Its first-order version has been analyzed in [10, 13, 19]. The original gauge method involves boundary conditions which are not suitable for finite elements, Nochetto and Pyo constructed the finite element gauge-Uzawa method in [9, 13] and proved the convergence of the backward Euler time discrete gauge method by variational approach under realistic regularity of given data in [10, 13]. However, as far as we know, no rigorous stability/error analysis is available for the second-order version of the gauge or gauge-Uzawa method. It turns out that, in the space continuous case, the second-order gauge or gauge-Uzawa method is equivalent to the consistent splitting scheme. Hence, we have also proved that in the special setting considered in this paper the second-order gauge and gauge-Uzawa methods (with properly chosen boundary conditions) are also fully second-order accurate.

The rest of the paper is organized as follows. In §2, we describe the rotational form of pressure-correction projection method (Algorithm 1) and the consistent splitting method (Algorithm 2), and we compute the normal mode reference solution of a second-order coupled scheme. We compare the reference solution with the normal mode solutions of Algorithm 1 in §3 and Algorithm 2 in §4 to estimate their convergence rates. Several interesting issues are addressed in Section 5.

2. **The two schemes and the reference solution.**

In this section, we describe the rotational pressure-correction scheme, the consistent splitting scheme and a second-order coupled scheme as the reference solution. For the sake of simplicity, the time derivative in these schemes will be approximated by the second-order backward difference formula (BDF2). Since the treatment of the nonlinear term does not contribute in any essential way to the error behaviors, we do not choose any specific treatments for the nonlinear term, instead, we simply assume $\mathbf{g}^{n+1}$ is a certain second-order approximation of $\mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u}$ at $t = (n+1)\tau$ where τ is the time step.

2.1. **Descriptions of the two schemes.**

The standard pressure-correction projection scheme was proposed by Van Kan in [18]. It is well known that it suffers from an artificial pressure Neumann boundary condition. Hence, we consider its rotational form which was first introduced in [17]:

Algorithm 1 (The Rotational Form of Pressure-Correction Projection Method). *Set initial values using a suitable first order projection method and repeat for* $2 \le n \le N = [\frac{T}{\tau} - 1]$.

Step 1: *Find* $\widetilde{\mathbf{u}}^{n+1}$ *as the solution of*

$$\begin{cases} \dfrac{3\widetilde{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} + \nabla p^n - \nu \Delta \widetilde{\mathbf{u}}^{n+1} = \mathbf{g}^{n+1}, \\ \widetilde{\mathbf{u}}^{n+1}|_\Gamma = \mathbf{0}. \end{cases} \tag{2.1}$$

Step 2: *Find* ψ^{n+1} *as the solution of*

$$\begin{cases} \Delta \psi^{n+1} = \dfrac{3}{2\tau} \nabla \cdot \widetilde{\mathbf{u}}^{n+1}, \\ \partial_{\boldsymbol{\nu}} \psi^{n+1}|_\Gamma = 0, \end{cases} \tag{2.2}$$

where $\boldsymbol{\nu}$ *is the unit outward normal vector.*

Step 3: *Update* $\mathbf{u}^{n+1}$ *and* p^{n+1} *by:*

$$\begin{aligned} \mathbf{u}^{n+1} &= \widetilde{\mathbf{u}}^{n+1} - \frac{2\tau}{3} \nabla \psi^{n+1}, \\ p^{n+1} &= \psi^{n+1} + p^n - \nu \nabla \cdot \widetilde{\mathbf{u}}^{n+1}. \end{aligned} \tag{2.3}$$

In this scheme, the homogeneous Neumann boundary condition is imposed on a non-physical variable ψ^{n+1} in (2.2). Moreover, by using the identity $\nabla \times \nabla \times \widetilde{\mathbf{u}}^{n+1} = \nabla \times \nabla \times \mathbf{u}^{n+1} = -\Delta \mathbf{u}^{n+1}$, we find that $\mathbf{u}^{n+1}$ satisfies

$$\begin{cases} \dfrac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} + \nabla p^{n+1} - \nu \Delta \mathbf{u}^{n+1} = \mathbf{g}^{n+1}, \\ \nabla \cdot \mathbf{u}^{n+1} = 0, \\ \mathbf{u}^{n+1} \cdot \boldsymbol{\nu}|_\Gamma = 0. \end{cases}$$

Thus, the pressure p^{n+1} does not suffer from the artificial boundary condition as in the standard pressure-correction scheme.

As stated in the introduction, the above scheme is still not fully second-order accurate (cf. [7]). Recently, Guermond and Shen proposed the consistent splitting method in [6] and showed, numerically, that its second-order version is fully accurate. However, no stability and error analysis is available. Below is a second-order version of the consistent splitting scheme based on BDF2:

Algorithm 2 (The Consistent Splitting Method). *Set initial values using a suitable first order projection method and repeat for* $2 \le n \le N = [\frac{T}{\tau} - 1]$.

Step 1: *Find* $\mathbf{u}^{n+1}$ *as the solution of*

$$\begin{cases} \dfrac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} + \nabla(2p^n - p^{n-1}) - \nu \Delta \mathbf{u}^{n+1} = \mathbf{g}^{n+1}, \\ \mathbf{u}^{n+1}|_\Gamma = \mathbf{0}. \end{cases} \tag{2.4}$$

Step 2: *Find* ψ^{n+1} *as the solution of*

$$\begin{cases} \Delta \psi^{n+1} = \nabla \cdot \dfrac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau}, \\ \partial_{\boldsymbol{\nu}} \psi^{n+1}|_\Gamma = 0. \end{cases} \tag{2.5}$$

Step 3: *Update* p^{n+1} *by*

$$p^{n+1} = \psi^{n+1} + 2p^n - p^{n-1} - \nu \nabla \cdot \mathbf{u}^{n+1}. \tag{2.6}$$

2.2. **Normal Mode Analysis of the reference solution.** Since we are concerned with normal mode analysis, we shall restrict our attention to the following linearized Navier-Stokes equations on $\Omega = [-1,1] \times [0, 2\pi]$:

$$\begin{cases} \partial_t \mathbf{u} + \nabla p = \nu \Delta \mathbf{u}, & \text{in } \Omega \times (0,T], \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \times [0,T], \\ \mathbf{u}(\pm 1, y, t) = \mathbf{0}, & \mathbf{u}(x,0,t) = \mathbf{u}(x, 2\pi, t), \quad (x,y,t) \in [-1,1] \times [0,2\pi) \times [0,T], \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, & \text{in } \Omega. \end{cases} \tag{2.7}$$

Although it is customary to choose the solution of (2.7) as the reference solution, since we are only interested in second-order schemes, we shall however use the solution of the following second-order coupled scheme as the reference solution:

$$\begin{cases} \dfrac{3\mathbf{U}^{n+1} - 4\mathbf{U}^n + \mathbf{U}^{n-1}}{2\tau} + \nabla P^{n+1} = \nu \Delta \mathbf{U}^{n+1}, \\ \nabla \cdot \mathbf{U}^{n+1} = 0, \\ \mathbf{U}^{n+1}|_{x=\pm 1} = \mathbf{0}. \end{cases} \tag{2.8}$$

It is obvious that (2.8) is a fully second-order scheme.

We write $(\mathbf{U}^n, P^n)(x,y) = \sum_k (\mathbf{U}^n_k, P^n_k)(x) e^{iky}$ and denote

$$\Delta_k \mathbf{u} := (\partial_x^2 - k^2)\mathbf{u}, \quad \nabla_k p := \begin{pmatrix} \partial_x p \\ ikp \end{pmatrix}, \quad \nabla_k \cdot \mathbf{u} := \partial_x u + ikv,$$

where $\mathbf{u} = (u, v)$. Then, after a Fourier transform in the y variable, the problem (2.8) is reduced to a family (indexed by k) of one-dimensional problems:

$$\begin{cases} \dfrac{3\mathbf{U}^{n+1}_k - 4\mathbf{U}^n_k + \mathbf{U}^{n-1}_k}{2\tau} + \nabla_k P^{n+1}_k = \nu \Delta_k \mathbf{U}^{n+1}_k, \\ \nabla_k \cdot \mathbf{U}^{n+1}_k = 0, \\ \mathbf{U}^{n+1}_k|_{x=\pm 1} = \mathbf{0}. \end{cases} \tag{2.9}$$

To simplify the notation, we shall drop the index k when no confusion would occur, and we assume that the solution of (2.9) takes the following normal mode form

$$(\mathbf{U}^n, P^n)(x) = \rho^n (\overline{\mathbf{u}}, \overline{p})(x). \tag{2.10}$$

Then, (2.9) becomes an ODE system for $(\overline{\mathbf{u}}, \overline{p})$:

$$\begin{cases} \partial_x^2 \overline{\mathbf{u}} - \left(k^2 + \dfrac{3\rho^2 - 4\rho + 1}{2\tau\nu\rho^2}\right) \overline{\mathbf{u}} = \dfrac{1}{\nu} \nabla_k \overline{p}, \\ \nabla_k \cdot \overline{\mathbf{u}} = 0, \\ \Delta_k \overline{p} = 0, \\ \overline{u}(\pm 1) = \overline{v}(\pm 1) = 0. \end{cases} \tag{2.11}$$

Setting

$$\lambda := k^2 + \frac{3\rho^2 - 4\rho + 1}{2\tau\nu\rho^2},$$

and eliminating $\overline{v}$ and $\overline{p}$ from the first equation in (2.11), we obtain

$$\partial_x^4 \overline{u} - (k^2 + \lambda)\partial_x^2 \overline{u} + k^2 \lambda \overline{u} = 0 \quad \text{and} \quad \overline{u}(\pm 1) = 0. \tag{2.12}$$

It can be easily verified that the necessary conditions for (2.12) to have a non-trivial solutions are $k \neq 0$ and $\lambda < 0$. Hence, we set

$$-\mu^2 = \lambda = k^2 + \frac{3\rho^2 - 4\rho + 1}{2\tau\nu\rho^2}, \tag{2.13}$$

so that the solutions of (2.12) are linear combinations of $\cos\mu x$, $\cosh kx$ (symmetric), and $\sin\mu x$, $\sinh kx$ (anti-symmetric). A simple calculation shows that the symmetric solutions of (2.11) are of the form ($k \neq 0$):

$$\begin{cases} \overline{u}(x) = \cos\mu x - \cos\mu \dfrac{\cosh kx}{\cosh k}, \\ \overline{v}(x) = \dfrac{\mu}{ik}\sin\mu x + \dfrac{1}{i}\cos\mu \dfrac{\sinh kx}{\cosh k}, \\ \overline{p}(x) = -\dfrac{k^2+\mu^2}{k}\nu\cos\mu \dfrac{\sinh kx}{\cosh k}. \end{cases} \tag{2.14}$$

The remaining condition $\overline{v}(\pm 1) = 0$ leads to

$$\mu\tan\mu + k\tanh k = 0. \tag{2.15}$$

It can be easily seen that for any given $k \neq 0$, there exists a unique solution μ of (2.15) on each open interval $I_s := \left(\frac{(2s-1)\pi}{2}, \frac{(2s+1)\pi}{2}\right)$, where $s \in \mathbf{Z}$ with $s \neq 0$. We denote it as μ_{k,I_s}. Note however that no solution exists in I_0.

For each $\mu_{k,Is}$, let ρ_{k,I_s} be the solutions of (2.13). We find from (2.13) that

$$\rho_{k,I_s} \in \left(\frac{1}{3}, 1\right) \quad \text{and} \quad \mu_{k,I_s}^2 + k^2 \leq \frac{1}{2\tau\nu}. \tag{2.16}$$

Hence, the solution $\mu_{k,Is}$ of (2.15) may not be compatible with (2.13). In particular, for any given $\tau > 0$, there exists $K_\tau > 0$ such that for $k > K_\tau$, there is no solution for (2.13)-(2.15).

¿From (2.13),

$$\rho_{k,I_s} = \frac{1}{2 \pm \sqrt{1 - 2\tau\nu(\mu_{k,I_s}^2 + k^2)}}.$$

One of them (with the positive sign before the square root) converges to $\frac{1}{3}$ as $\tau \to 0$, and hence corresponds to the trivial solution in view of (2.10). Therefore, we conclude

$$\rho_{k,I_s} = \frac{1}{2 - \sqrt{1 - 2\tau\nu(\mu_{k,I_s}^2 + k^2)}}.$$

We denote the symmetric solutions (2.14) as $\overline{\mathbf{u}}_{k,I_s} = (\overline{u}_{k,I_s}(x), \overline{v}_{k,I_s}(x))$ and $\overline{p}_{k,I_s}(x)$ for each $k \neq 0$ and admissible interval I_s.

Similarly, the antisymmetric solutions are given by

$$\begin{cases} \overline{u}(x) = \sin\mu x - \sin\mu \dfrac{\sinh kx}{\sinh k}, \\ \overline{v}(x) = -\dfrac{\mu}{ik}\cos\mu x + \dfrac{1}{i}\sin\mu \dfrac{\cosh kx}{\sinh k}, \\ \overline{p}(x) = -\dfrac{k^2+\mu^2}{k}\nu\sin\mu \dfrac{\cosh kx}{\sinh k}. \end{cases} \tag{2.17}$$

Since $\overline{v}(\pm 1) = 0$, we find

$$\mu \cot \mu - k \coth k = 0.$$

For any given $k \neq 0$, there exists a unique solution μ on each open interval

$$J_s = \begin{cases} (s\pi, (s+1)\pi), & \forall s \geq 1, \\ ((s-1)\pi, s\pi), & \forall s \leq -1, \end{cases} \tag{2.18}$$

where $s \in \mathbf{Z}$ with $s \neq 0$. We denote it as μ_{k,J_s}. Notes however that no solution exists in $J_0 = (-\pi, \pi)$. In view of (2.13), we have

$$\rho_{k,J_s} \in \left(\frac{1}{3}, 1\right) \quad \text{and} \quad \mu_{k,J_s}^2 + k^2 \leq \frac{1}{2\tau\nu}, \tag{2.19}$$

$$\rho_{k,J_s} = \frac{1}{2 - \sqrt{1 - 2\tau\nu(\mu_{k,J_s}^2 + k^2)}}.$$

We denote the antisymmetric solutions (2.17) as $\overline{\mathbf{u}}_{k,J_s} = (\overline{u}_{k,J_s}(x), \overline{v}_{k,J_s}(x))$ and $\overline{p}_{k,J_s}(x)$ for each $k \neq 0$ and admissible interval J_s.

In summary, the solution of (2.8) can be written as

$$(\mathbf{U}^n, P^n)(x,y) = \sum_{k,s} \big(\alpha_{k,s}\rho_{k,I_s}^n(\overline{\mathbf{u}}_{k,I_s}, \overline{p}_{k,I_s})(x) + \beta_{k,s}\rho_{k,J_s}^n(\overline{\mathbf{u}}_{k,J_s}, \overline{p}_{k,J_s})(x)\big)\exp(iky), \tag{2.20}$$

where $\alpha_{k,s}$ and $\beta_{k,s}$ are determined by the normal mode expansion of the initial velocity, namely:

$$\mathbf{U}^0(x,y) = \sum_{k,s} (\alpha_{k,s}\overline{\mathbf{u}}_{k,I_s}(x) + \beta_{k,s}\overline{\mathbf{u}}_{k,J_s}(x))\exp(iky). \tag{2.21}$$

In view of (2.16) and (2.19), the maximum values of $\mu_{k,I_s}, \mu_{k,J_s}, k$ have to be bounded by $\frac{1}{\tau}$. Hence, (2.20) is a finite series for any given $\tau \neq 0$.

One of the difficulties in the normal mode analysis rigorous is that the normal modes $\{\overline{\mathbf{u}}_{k,I_s}, \overline{\mathbf{u}}_{k,J_s}\}$ are not mutually orthogonal as the Fourier modes are. Hence, we are led to make the following assumption on $\mathbf{U}^0(x,y)$ and its derivatives:

Assumption 1. *We assume that for certain m to be specified later, there exists a positive constant M_m, independent of τ, such that*

$$\begin{aligned} \sum_{k,s} \sum_{\gamma_1+\gamma_2 \leq m} &\big(\big|\alpha_{k,s} D_x^{\gamma_1}\overline{\mathbf{u}}_{k,I_s}(x) D_y^{\gamma_2}\exp(iky)\big| \\ &+ \big|\beta_{k,s} D_x^{\gamma_1}\overline{\mathbf{u}}_{k,J_s}(x) D_y^{\gamma_2}\exp(iky)\big|\big) \leq M_m, \ \forall (x,y) \in \Omega. \end{aligned} \tag{2.22}$$

The above assumptions ensure that the normal mode expansion of $\mathbf{U}^0$ and all its derivatives of order up to m are absolutely convergent. Since (2.22) hold for all $(x,y) \in \Omega$, one can shown that it can be equivalently represented by

$$\sum_{k,s} \sum_{\gamma_1,\gamma_2 \leq m} \left\{ \left(|k|^{\gamma_1} + \mu_{k,I_s}^{\gamma_2}\right)|\alpha_{k,s}| + \left(|k|^{\gamma_1} + \mu_{k,J_s}^{\gamma_2}\right)|\beta_{k,s}| \right\} \leq M_m.$$

3. **Error analysis for the Rotational Pressure-Correction Method.** In this section, we compute normal mode solutions of Algorithm 1 and to estimate the errors by comparing with the reference solution computed in §2.2.

Similarly as for the reference solution, we write the solution of Algorithm 1 as $(\mathbf{u}^n, \widetilde{\mathbf{u}}^n, \psi^n, p^n)(x, y) = \sum_k (\mathbf{u}_k^n, \widetilde{\mathbf{u}}_k^n, \psi_k^n, p_k^n)(x)e^{iky}$. After eliminating $\widetilde{\mathbf{u}}^{n+1}$ from (2.1) by using (2.3), and performing a Fourier transform in y, we obtain a family of one-dimensional problems for $(\mathbf{u}_k^n, \widetilde{\mathbf{u}}_k^n, \psi_k^n, p_k^n)$:

$$\begin{cases} \dfrac{3\mathbf{u}_k^{n+1} - 4\mathbf{u}_k^n + \mathbf{u}_k^{n-1}}{2\tau} - \nu\Delta_k \mathbf{u}_k^{n+1} = -\nabla_k p_k^{n+1}, \\ \Delta_k p_k^{n+1} = 0, \\ \nabla_k \cdot \mathbf{u}_k^{n+1} = 0, \\ \dfrac{3\widetilde{\mathbf{u}}_k^{n+1} - 4\mathbf{u}_k^n + \mathbf{u}_k^{n-1}}{2\tau} - \nu\Delta_k \widetilde{\mathbf{u}}_k^{n+1} = -\nabla_k p_k^n, \\ \Delta_k \psi_k^{n+1} = \dfrac{3}{2\tau}\nabla_k \cdot \widetilde{\mathbf{u}}_k^{n+1}, \\ \widetilde{u}_k^{n+1}(\pm 1, t) = \widetilde{v}_k^{n+1}(\pm 1, t) = u_k^{n+1}(\pm 1, t) = \partial_x \psi_k^{n+1}(\pm 1, t) = 0. \end{cases} \tag{3.1}$$

We shall temporarily drop the index k and assume that solution of the above system takes the following normal mode form:

$$(\mathbf{u}^n, \widetilde{\mathbf{u}}^n, p^n)(x) = \widetilde{\rho}^n(\widehat{\mathbf{u}}, \widehat{\widetilde{\mathbf{u}}}, \widehat{p})(x).$$

Then, (3.1) becomes an ODE system for $(\widehat{\mathbf{u}}, \widehat{\widetilde{\mathbf{u}}}, \widehat{p})$:

$$\begin{cases} \partial_x^2 \widehat{\mathbf{u}} - \left(k^2 + \dfrac{3\widetilde{\rho}^2 - 4\widetilde{\rho} + 1}{2\tau\widetilde{\rho}^2\nu}\right)\widehat{\mathbf{u}} = \dfrac{1}{\nu}\nabla_k \widehat{p}, \\ \Delta_k \widehat{p} = 0, \\ \nabla_k \cdot \widehat{\mathbf{u}} = 0, \\ \partial_x^2 \widehat{\widetilde{\mathbf{u}}} - \left(k^2 + \dfrac{3}{2\tau\nu}\right)\widehat{\widetilde{\mathbf{u}}} = \dfrac{4\widetilde{\rho} - 1}{2\tau\widetilde{\rho}^2\nu}\widehat{\mathbf{u}} + \dfrac{1}{\widetilde{\rho}\nu}\nabla_k \widehat{p}, \\ \Delta_k \widehat{\psi} = \dfrac{3}{2\tau}\nabla_k \cdot \widehat{\widetilde{\mathbf{u}}}, \\ \widehat{\widetilde{u}}(\pm 1, t) = \widehat{\widetilde{v}}(\pm 1, t) = \widehat{u}(\pm 1, t) = \partial_x \widehat{\psi}(\pm 1, t) = 0. \end{cases} \tag{3.2}$$

The procedure for solving (3.2) is essentially the same as for the reference solution. The necessary conditions for (3.2) to have non-trivial solutions are $k \neq 0$ and that there exists a real number $\widetilde{\mu}$ satisfying

$$-\widetilde{\mu}^2 = k^2 + \frac{3\widetilde{\rho}^2 - 4\widetilde{\rho} + 1}{2\tau\widetilde{\rho}^2\nu}. \tag{3.3}$$

Let us denote

$$\lambda^2 := k^2 + \frac{3}{2\tau\nu}. \tag{3.4}$$

Then, the symmetric solutions of the ODE system (3.2) are

$$\begin{cases} \widehat{u}(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k}, \\ \widehat{v}(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}, \\ \widehat{\widehat{u}}(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k} + \dfrac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}\cos\widetilde{\mu}\left(\dfrac{\cosh kx}{\cosh k} - \dfrac{\cosh\lambda x}{\cosh\lambda}\right), \\ \widehat{\widehat{v}}(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k} \\ \qquad -\dfrac{1}{i}\dfrac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}\cos\widetilde{\mu}\left(\dfrac{\sinh kx}{\cosh k} - \dfrac{k}{\lambda}\dfrac{\sinh\lambda x}{\cosh\lambda}\right), \\ \widehat{p}(x) = -\dfrac{\widetilde{\mu}^2+k^2}{k}\nu\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}, \\ \widehat{\psi}(x) = \dfrac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{2\tau\widetilde{\rho}^3}\cos\widetilde{\mu}\left(\dfrac{1}{k}\dfrac{\sinh kx}{\cosh k} - \dfrac{1}{\lambda}\dfrac{\sinh\lambda x}{\cosh\lambda}\right). \end{cases} \tag{3.5}$$

A second relation between μ and ρ can be obtained by the condition $\widehat{\widehat{v}}(\pm 1) = 0$:

$$\widetilde{\mu}\tan\widetilde{\mu} + k\tanh k = \frac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}k\left(\tanh k - \frac{k}{\lambda}\tanh\lambda\right). \tag{3.6}$$

For any given $k \neq 0$, the relations (3.3) and (3.6) determine a unique solution $\widetilde{\mu}_{k,I_s}$ in each open interval I_s ($s \neq 0$). In view of (3.3), we have

$$\widetilde{\rho}_{k,I_s} \in \left(\frac{1}{3}, 1\right) \quad \text{and} \quad \widetilde{\mu}^2_{k,I_s} + k^2 \leq \frac{1}{2\tau\nu}. \tag{3.7}$$

Similarly as for the reference solution, we find that for each $\widetilde{\mu}_{k,I_s}$, (3.3) determines a unique $\widetilde{\rho}_{k,I_s}$ given by

$$\widetilde{\rho}_{k,I_s} = \frac{1}{2 - \sqrt{1 - 2\tau\nu(\widetilde{\mu}^2_{k,I_s} + k^2)}}. \tag{3.8}$$

We will denote the solutions in (3.5) as $\widehat{\mathbf{u}}_{k,I_s} = (\widehat{u}_{k,I_s}\widehat{v}_{k,I_s})$, $\widehat{\widehat{\mathbf{u}}}_{k,I_s} = (\widehat{\widehat{u}}_{k,I_s}, \widehat{\widehat{v}}_{k,I_s})$, $\widehat{p}_{k,I_s}$, and $\widehat{\psi}_{k,I_s}$.

Similarly, the antisymmetric solutions are given by

$$\begin{cases} \widehat{u}(x) = \sin\widetilde{\mu}x - \sin\widetilde{\mu}\dfrac{\sinh kx}{\sinh k}, \\ \widehat{v}(x) = -\dfrac{\widetilde{\mu}}{ik}\cos\widetilde{\mu}x + \dfrac{1}{i}\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k}, \\ \widehat{\widetilde{u}}(x) = \sin\widetilde{\mu}x - \sin\widetilde{\mu}\dfrac{\sinh kx}{\sinh k} + \dfrac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}\sin\widetilde{\mu}\left(\dfrac{\sinh kx}{\sinh k} - \dfrac{\sinh\lambda x}{\sinh\lambda}\right), \\ \widehat{\widetilde{v}}(x) = -\dfrac{\widetilde{\mu}}{ik}\cos\widetilde{\mu}x + \dfrac{1}{i}\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k} \\ \qquad -\dfrac{1}{i}\dfrac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}\sin\widetilde{\mu}\left(\dfrac{\cosh kx}{\sinh k} - \dfrac{k}{\lambda}\dfrac{\cosh\lambda x}{\sinh\lambda}\right), \\ \widehat{p}(x) = -\dfrac{\widetilde{\mu}^2+k^2}{k}\nu\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k}, \\ \widehat{\psi}(x) = \dfrac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{2\tau\widetilde{\rho}^3}\sin\widetilde{\mu}\left(\dfrac{1}{k}\dfrac{\cosh kx}{\sinh k} - \dfrac{1}{\lambda}\dfrac{\cosh\lambda x}{\sinh\lambda}\right). \end{cases} \tag{3.9}$$

We determine from the condition $\widehat{\widetilde{v}}(\pm 1) = 0$ that

$$\widetilde{\mu}\cot\widetilde{\mu} - k\coth k = -\frac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}k\left(\coth k - \frac{k}{\lambda}\coth\lambda\right). \tag{3.10}$$

For any given $k \neq 0$, there is unique solution $\widetilde{\mu}_{k,J_s}$ of (3.10) in each open interval J_s ($s \neq 0$). In view of (3.3), we have

$$\widetilde{\rho}_{k,J_s} \in \left(\frac{1}{3}, 1\right) \quad \text{and} \quad \widetilde{\mu}^2_{k,J_s} + k^2 \leq \frac{1}{2\tau\nu},$$

and

$$\widetilde{\rho}_{k,J_s} = \frac{1}{2 - \sqrt{1 - 2\tau\nu(\widetilde{\mu}^2_{k,J_s} + k^2)}}. \tag{3.11}$$

We will denote the solutions in (3.9) as $\widehat{\mathbf{u}}_{k,J_s} = (\widehat{u}_{k,J_s}, \widehat{v}_{k,J_s})$, $\widehat{\widetilde{\mathbf{u}}}_{k,J_s} = (\widehat{\widetilde{u}}_{k,J_s}, \widehat{\widetilde{v}}_{k,J_s})$, $\widehat{p}_{k,J_s}$, and $\widehat{\psi}_{k,J_s}$.

In summary, the normal mode solution of the Algorithm 1 can be written as

$$\begin{aligned} (\mathbf{u}^n, \widetilde{\mathbf{u}}^n, p^n) = &\sum_{k,s} \alpha_{k,s}\widetilde{\rho}^n_{k,I_s}(\widehat{\mathbf{u}}_{k,I_s}, \widehat{\widetilde{\mathbf{u}}}_{k,J_s}, \widehat{p}_{k,I_s})(x)\exp(iky) \\ &+ \sum_{k,s} \beta_{k,s}\widetilde{\rho}^n_{k,J_s}(\widehat{\mathbf{u}}_{k,J_s}, \widehat{\widetilde{\mathbf{u}}}_{k,J_s}, \widehat{p}_{k,J_s})(x)\exp(iky). \end{aligned} \tag{3.12}$$

Remark 3.1. *We infer from (3.4) that $\lambda \sim O((\nu\tau)^{-\frac{1}{2}})$. Hence, $(\widehat{\widetilde{u}}, \widehat{\widetilde{v}}, \widehat{\psi})$ in (3.5) and (3.9) exhibit a boundary layer of width $O((\nu\tau)^{-\frac{1}{2}})$, and we infer from (3.8) and (3.11) that its magnitude is of order $O((\nu\tau)^2)$. On the other hand, it is clear that $(\widetilde{u}, \widetilde{v}, \widetilde{p})$ are free of any spurious terms, and the only errors are related to $|\widetilde{\mu} - \mu|$ which is shown below to be second-order accurate.*

Remark 3.2 (Initial Value)**.** *Since the spaces spanned by $\{\mathbf{u}_{k,I_s}, \mathbf{u}_{k,J_s}\}$ and $\{\widehat{\mathbf{u}}_{k,I_s}, \widehat{\mathbf{u}}_{k,J_s}\}$ are in general different, it is impossible for Algorithm 1 to have the same initial condition as the reference solution. Instead, we shall assume that $\alpha_{k,s}$ and*

$\beta_{k,s}$ are the same as in the reference initial solution in (2.21) *so the initial velocity for Algorithm 1 becomes*

$$\mathbf{u}^0 = \sum_{k,s} \alpha_{k,s} \widehat{\mathbf{u}}_{k,I_s}(x) \exp(iky) + \sum_{k,s} \beta_{k,s} \widehat{\mathbf{u}}_{k,J_s}(x) \exp(iky).$$

The lemma below indicates that the initial error will be of second-order.

Lemma 3.3. *We have the following error bounds:*

$$\begin{aligned} |\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}| &\le \frac{8\tau^2\nu^2}{3(2|s|-1)\pi} \left(\frac{2|s|+1}{2|s|-1}\right)^4 |k|(\mu_{k,I_s}^2 + k^2)^2, \\ |\mu_{k,J_s} - \widetilde{\mu}_{k,J_s}| &\le \frac{8\tau^2\nu^2}{3|s|\pi} \left(\frac{|s|+1}{|s|}\right)^4 |k|(\mu_{k,J_s}^2 + k^2)^2, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \left|\mu_{k,I_s}^2 - \widetilde{\mu}_{k,I_s}^2\right| &\le \frac{8\tau^2\nu^2}{3} \left(\frac{2|s|+1}{2|s|-1}\right)^5 |k|(\mu_{k,I_s}^2 + k^2)^2, \\ \left|\mu_{k,J_s}^2 - \widetilde{\mu}_{k,J_s}^2\right| &\le \frac{8\tau^2\nu^2}{3} \left(\frac{|s|+1}{|s|}\right)^5 |k|(\mu_{k,J_s}^2 + k^2)^2, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} |\rho_{k,I_s} - \widetilde{\rho}_{k,I_s}| &\le C\tau^3\nu^3(\mu_{k,I_s}^2 + k^2)^3, \\ |\rho_{k,J_s} - \widetilde{\rho}_{k,J_s}| &\le C\tau^3\nu^3(\mu_{k,J_s}^2 + k^2)^3. \end{aligned} \tag{3.15}$$

Proof. We shall only carry out the proof on the interval $I_s = \left(\frac{(2s-1)\pi}{2}, \frac{(2s+1)\pi}{2}\right)$, since the estimate on J_s defined in (2.18) can be derived by the same argument. Subtracting (3.6) from (2.15), we get

$$\begin{aligned} &\mu_{k,I_s} \tan \mu_{k,I_s} - \widetilde{\mu}_{k,I_s} \tan \widetilde{\mu}_{k,I_s} \\ &= -\frac{(3\widetilde{\rho}_{k,I_s} - 1)(\widetilde{\rho}_{k,I_s} - 1)^2}{3\widetilde{\rho}_{k,I_s}^3} k \left(\tanh k - \frac{k}{\lambda_{k,I_s}} \tanh \lambda\right). \end{aligned} \tag{3.16}$$

We will first bound the right hand side of (3.16). From (3.7) and (3.8), we obtain

$$\frac{1-\widetilde{\rho}_{k,I_s}}{\widetilde{\rho}_{k,I_s}} = \frac{2\tau\nu(\widetilde{\mu}_{k,I_s}^2 + k^2)}{1 + \sqrt{1 - 2\tau\nu(\widetilde{\mu}_{k,I_s}^2 + k^2)}} \le 2\tau\nu(\widetilde{\mu}_{k,I_s}^2 + k^2). \tag{3.17}$$

On the other hand, we infer from (3.4) that $k \le \lambda_{k,I_s}$ which yields

$$\left|\tanh k - \tfrac{k}{\lambda_{k,I_s}} \tanh \lambda_{k,I_s}\right| \le 1.$$

Therefore, in light of (3.3) and (3.17), we have

$$\begin{aligned} |\mu_{k,I_s} \tan \mu_{k,I_s} - \widetilde{\mu}_{k,I_s} \tan \widetilde{\mu}_{k,I_s}| &\le \left|\frac{(3\widetilde{\rho}_{k,I_s} - 1)(\widetilde{\rho}_{k,I_s} - 1)^2}{3\widetilde{\rho}_{k,I_s}^3} k\right| \\ &\le \frac{2\nu\tau}{3}|k|(\widetilde{\mu}_{k,I_s}^2 + k^2)\frac{1-\widetilde{\rho}_{k,I_s}}{\widetilde{\rho}_{k,I_s}} \\ &\le \frac{4\tau^2\nu^2}{3}|k|(\widetilde{\mu}_{k,I_s}^2 + k^2)^2. \end{aligned} \tag{3.18}$$

Let $f(x) = x\tan(x)$. It can be easily verified that $|f'(x)| = \left|\frac{x+\sin(x)\cos(x)}{\cos^2(x)}\right| \geq |x|$ on I_s. Since μ_{k,I_s} and $\widetilde{\mu}_{k,I_s} \in I_s$, the mean value theorem and (3.18) yield

$$
\begin{aligned}
|\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}| &\leq \frac{2}{(2|s|-1)\pi}|\mu_{k,I_s}\tan\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}\tan\widetilde{\mu}_{k,I_s}| \\
&\leq \frac{8\tau^2\nu^2}{3(2|s|-1)\pi}|k|(\widetilde{\mu}^2_{k,I_s}+k^2)^2,
\end{aligned} \tag{3.19}
$$

where we note $s \neq 0$. Since $|\mu_{k,I_s} + \widetilde{\mu}_{k,I_s}| \leq (2|s|+1)\pi$, we infer from (3.19) that

$$
\begin{aligned}
\left|\mu^2_{k,I_s} - \widetilde{\mu}^2_{k,I_s}\right| &= |\mu_{k,I_s} + \widetilde{\mu}_{k,I_s}||\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}| \\
&\leq \frac{8\tau^2\nu^2}{3}\frac{2|s|+1}{2|s|-1}|k|(\widetilde{\mu}^2_{k,I_s}+k^2)^2.
\end{aligned} \tag{3.20}
$$

Since $|\widetilde{\mu}_{k,I_s}| \leq \frac{2|s|+1}{2|s|-1}|\mu_{k,I_s}|$, the above two estimates imply (3.13) and (3.14).

Next, we estimate $|\rho_{k,I_s} - \widetilde{\rho}_{k,I_s}|$. Since ρ_{k,I_s} and $\widetilde{\rho}_{k,I_s} \in \left(\frac{1}{3}, 1\right)$, we find by the mean value theorem that

$$
|\rho_{k,I_s} - \widetilde{\rho}_{k,I_s}| \leq |\ln\rho_{k,I_s} - \ln\widetilde{\rho}_{k,I_s}| \leq \sum_{i=1}^{3} A_i, \tag{3.21}
$$

with

$$
\begin{aligned}
A_1 &:= \left|\ln\widetilde{\rho}_{k,I_s} - \frac{3\widetilde{\rho}^2_{k,I_s} - 4\widetilde{\rho}_{k,I_s}+1}{2\widetilde{\rho}^2_{k,I_s}}\right|, \\
A_2 &:= \left|\ln\rho_{k,I_s} - \frac{3\rho^2_{k,I_s} - 4\rho_{k,I_s}+1}{2\rho^2_{k,I_s}}\right|, \\
A_3 &:= \left|\frac{3\rho^2_{k,I_s} - 4\rho_{k,I_s}+1}{2\rho^2_{k,I_s}} - \frac{3\widetilde{\rho}^2_{k,I_s} - 4\widetilde{\rho}_{k,I_s}+1}{2\widetilde{\rho}^2_{k,I_s}}\right|.
\end{aligned}
$$

Using the Taylor expansion $\ln\widetilde{\rho}_{k,I_s} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(\widetilde{\rho}_{k,I_s}-1)^n$, we can write A_1 as

$$
\begin{aligned}
A_1 &= \left|(\widetilde{\rho}_{k,I_s}-1)\left(\frac{3\widetilde{\rho}_{k,I_s}-1}{2\widetilde{\rho}^2_{k,I_s}} - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(\widetilde{\rho}_{k,I_s}-1)^{n-1}\right)\right| \\
&= \left|(\widetilde{\rho}_{k,I_s}-1)\left(\frac{-(2\widetilde{\rho}_{k,I_s}-1)(\widetilde{\rho}_{k,I_s}-1)}{2\widetilde{\rho}^2_{k,I_s}} - \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}(\widetilde{\rho}_{k,I_s}-1)^{n-1}\right)\right| \\
&= \left|(\widetilde{\rho}_{k,I_s}-1)^2\left(\frac{(\widetilde{\rho}_{k,I_s}-1)^2}{2\widetilde{\rho}^2_{k,I_s}} - \sum_{n=3}^{\infty}\frac{(-1)^{n+1}}{n}(\widetilde{\rho}_{k,I_s}-1)^{n-2}\right)\right| \\
&= \left|(\widetilde{\rho}_{k,I_s}-1)^3\left(\frac{-2\widetilde{\rho}^2_{k,I_s}+3\widetilde{\rho}_{k,I_s}-3}{6\widetilde{\rho}^2_{k,I_s}} - \sum_{n=4}^{\infty}\frac{(-1)^{n+1}}{n}(\widetilde{\rho}_{k,I_s}-1)^{n-3}\right)\right|.
\end{aligned} \tag{3.22}
$$

Therefore (3.17) leads to

$$
A_1 \leq C\tau^3\nu^3(\widetilde{\mu}^2_{k,I_s}+k^2)^3.
$$

By the same argument, we obtain $A_2 \le C\tau^3\nu^3(\mu_{k,I_s}^2+k^2)^3$. Subtracting (3.3) from (2.13) yields $A_3 = \nu\tau\left|\mu_{k,I_s}^2 - \widetilde{\mu}_{k,I_s}^2\right|$ which along with (3.20) leads to

$$A_3 \le \frac{8\tau^3\nu^3}{3}\frac{2|s|+1}{2|s|-1}|k|(\widetilde{\mu}_{k,I_s}^2+k^2)^2.$$

In conjunction with $|\widetilde{\mu}|_{k,I_s} \le \frac{2|s|+1}{2|s|-1}|\mu_{k,I_s}|$, inserting above estimates on A_i ($i = 1,2,3$) into (3.21) yields (3.15) and concludes the proof. $\square$

Theorem 3.4. *Let* $(\mathbf{U}^n, P^n)$ *and* $(\mathbf{u}^n, \widetilde{\mathbf{u}}^n, p^n)$ *be respectively the solutions of (2.8) and of Algorithm 1 and Suppose Assumption 1 holds with* $m = 8$. *Then, we have*

$$\begin{aligned} \|\mathbf{U}^n - \mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)} &\le CM_7\nu^3\tau^2, \\ \|P^n - p^n\|_{L^\infty(\Omega)} &\le CM_8\nu^2\tau^2, \qquad \forall 1 \le n \le N, \\ \|\nabla\cdot\widetilde{\mathbf{u}}^n\|_{L^\infty(\Omega)} &\le CM_4\tau^{\frac{3}{2}}. \end{aligned} \tag{3.23}$$

Proof. Let us denote the error functions at time $t = 0$ by:

$$\begin{aligned} \mathbf{E}_{k,I_s} &:= \overline{\mathbf{u}}_{k,I_s} - \widetilde{\mathbf{u}}_{k,I_s}, \quad & \mathbf{E}_{k,J_s} &:= \overline{\mathbf{u}}_{k,J_s} - \widetilde{\mathbf{u}}_{k,J_s}, \\ e_{k,I_s} &:= \overline{p}_{k,I_s} - \widetilde{p}_{k,I_s}, & e_{k,J_s} &:= \overline{p}_{k,J_s} - \widetilde{p}_{k,J_s}. \end{aligned} \tag{3.24}$$

We first compare (2.14) and (3.5) on a fixed interval I_s. Using (3.13)-(3.14), the velocity error can be bounded by

$$\|\mathbf{E}_{k,I_s}\|_{\mathbf{L}^\infty(\Omega)} \le C\left(1+\left|\frac{\mu_{k,I_s}}{k}\right|\right)|\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}| \le C\tau^2\nu^2(\mu_{k,I_s}^2+k^2)^3.$$

Similarly, the pressure error can be bounded by

$$\begin{aligned} \|e_{k,I_s}\|_{L^\infty(\Omega)} \le & C\nu\left(\left|\frac{\mu_{k,I_s}^2+k^2}{k}\right||\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}| + \left|\frac{\mu_{k,I_s}^2 - \widetilde{\mu}_{k,I_s}^2}{k}\right|\right) \\ \le & C\tau^2\nu^3(\mu_{k,I_s}^2+k^2)^3. \end{aligned}$$

Since ρ_{k,J_s} and $\widetilde{\rho}_{k,I_s} \in \left(\frac{1}{3},1\right)$, the mean value theorem leads to $\left|\rho_{k,I_s}^n - \widetilde{\rho}_{k,I_s}^n\right| \le n|\rho_{k,I_s} - \widetilde{\rho}_{k,I_s}|$. So (3.15) leads to (note $n\tau \le T$):

$$\left|\rho_{k,I_s}^n - \widetilde{\rho}_{k,I_s}^n\right| \le n|\rho_{k,I_s} - \widetilde{\rho}_{k,I_s}| \le C\tau^2\nu^3(\mu_{k,I_s}^2+k^2)^3.$$

The errors $|\mathbf{E}_{k,J_s}|$, $\left|\rho_{k,J_s}^n - \widetilde{\rho}_{k,J_s}^n\right|$, and $|e_{k,J_s}|$ can be estimated in a similar fashion.

We now compute the error of Algorithm 1 by comparing (3.12) with (2.20) and using the above three inequalities. Since we have $\widetilde{\rho}_{k,I_s} \in \left(\frac{1}{3},1\right)$ and $\|\overline{\mathbf{u}}_{k,I_s}\|_{\mathbf{L}^\infty(\Omega)} \le C\left(1+\left|\frac{\mu_{k,I_s}}{k}\right|\right)$, we can derive

$$\begin{aligned} &\|\mathbf{U}^n - \mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)} \\ \le & \sum_{k,s}\left\|\alpha_{k,s}\exp(iky)\Big((\rho_{k,I_s}^n - \widetilde{\rho}_{k,I_s}^n)\overline{\mathbf{u}}_{k,I_s} + \widetilde{\rho}_{k,I_s}^n\mathbf{E}_{k,I_s}\Big)\right\|_{\mathbf{L}^\infty(\Omega)} \\ & + \sum_{k,s}\left\|\beta_{k,s}\exp(iky)\Big((\rho_{k,J_s}^n - \widetilde{\rho}_{k,J_s}^n)\overline{\mathbf{u}}_{k,J_s} + \widetilde{\rho}_{k,J_s}^n\mathbf{E}_{k,J_s}\Big)\right\|_{\mathbf{L}^\infty(\Omega)} \\ \le & C\tau^2\nu^3\sum_{k,s}\left(|\alpha_{k,s}|(\mu_{k,I_s}^2+k^2)^3\left|\frac{\mu_{k,I_s}}{k}\right| + |\beta_{k,s}|(\mu_{k,J_s}^2+k^2)^3\left|\frac{\mu_{k,J_s}}{k}\right|\right) \\ \le & CM_7\nu^3\tau^2. \end{aligned}$$

On the other hand, using $\left|\overline{p}_{k,I_s}\right| \le C\frac{\mu_{k,I_s}^2+k^2}{|k|}\nu$, we can obtain

$$\begin{aligned}
\|P^n - p^n\|_{L^\infty(\Omega)} \le& \sum_{k,s} \left\|\alpha_{k,s}\exp(iky)\Big((\rho_{k,I_s}^n - \widetilde{\rho}_{k,I_s}^n)\overline{p}_{k,I_s} + \widetilde{\rho}_{k,I_s}^n e_{k,I_s}\Big)\right\|_{\mathbf{L}^\infty(\Omega)} \\
&+ \sum_{k,s} \left\|\beta_{k,s}\exp(iky)\Big((\rho_{k,J_s}^n - \widetilde{\rho}_{k,J_s}^n)\overline{p}_{k,J_s} + \widetilde{\rho}_{k,J_s}^n e_{k,J_s}\Big)\right\|_{\mathbf{L}^\infty(\Omega)} \\
\le& C\tau^2\nu^4 \sum_{k,s}\left(|\alpha_{k,s}|(\mu_{k,I_s}^2 + k^2)^4 + |\beta_{k,s}|(\mu_{k,J_s}^2 + k^2)^4\right) \\
\le& CM_8\nu^4\tau^2.
\end{aligned}$$

It remains to prove the last inequality in (3.23). We compute $\nabla_k \cdot \widehat{\widetilde{\mathbf{u}}}_{k,I_s}$ from (3.5), and use (3.3)-(3.4), (3.17) to get

$$\begin{aligned}
\left\|\nabla_k \cdot \widehat{\widetilde{\mathbf{u}}}_{k,I_s}\right\|_{L^\infty(\Omega)} \le& \left|\frac{(3\widetilde{\rho}_{k,I_s} - 1)(\widetilde{\rho}_{k,I_s} - 1)^2}{3\widetilde{\rho}_{k,I_s}^3}\frac{\lambda_{k,I_s}^2 - k^2}{\lambda_{k,I_s}}\right| \\
\le& 2\tau\nu\frac{(\widetilde{\mu}_{k,I_s}^2 + k^2)^2}{|\lambda_{k,I_s}|} \\
\le& \frac{2\sqrt{2}}{\sqrt{3}}\tau^{\frac{3}{2}}\nu^{\frac{3}{2}}(\widetilde{\mu}_{k,I_s}^2 + k^2)^2 \le \frac{2\sqrt{2}}{\sqrt{3}}\tau^{\frac{3}{2}}\nu^{\frac{3}{2}}\left(\frac{2|s|+1}{2|s|-1}\right)^4(\mu_{k,I_s}^2 + k^2)^2.
\end{aligned}$$

Similarly, we can also derive

$$\left\|\nabla_k \cdot \widehat{\widetilde{\mathbf{u}}}_{k,J_s}\right\|_{L^\infty(\Omega)} \le \frac{2\sqrt{2}}{\sqrt{3}}\tau^{\frac{3}{2}}\nu^{\frac{3}{2}}\left(\frac{|s|+1}{|s|}\right)^4(\mu_{k,J_s}^2 + k^2)^2.$$

Then, summing up the above two inequalities for k and s yields (3.23). □

We note that (3.23) indicates that the divergence of $\widetilde{\mathbf{u}}^n$ is only $\frac{3}{2}$-order accurate. Consequently, the error of $\widetilde{\mathbf{u}}^n$ in $\mathbf{H}^1(\Omega)$ is at most $\frac{3}{2}$-order accurate.

4. **Error analysis of the Consistent Splitting Method.** Similarly as in the last section, we compute normal mode solutions of Algorithm 2 and estimate the errors by comparing with the reference solution in §2.2.

Writing the solution of **Algorithm 2** as

$$(\mathbf{u}^n, \psi^n, p^n)(x,y) = \textstyle\sum_k (\mathbf{u}_k^n, \psi_k^n, p_k^n)(x)e^{iky},$$

and performing a Fourier transform in y, we obtain a family of one-dimensional problems for $(\mathbf{u}_k^n, \psi_k^n, p_k^n)$:

$$\begin{cases}
\dfrac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} - \nu\Delta_k\mathbf{u}^{n+1} = -\nabla_k(2p^n - p^{n-1}), \\
\Delta_k p^{n+1} = 0, \\
\Delta_k\psi^{n+1} = \nabla_k \cdot \dfrac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau}, \\
p^{n+1} = \psi^{n+1} + 2p^n - p^{n-1} - \nu\nabla_k \cdot \mathbf{u}^{n+1}, \\
u^{n+1}(\pm1, t) = v^{n+1}(\pm1, t) = \partial_x\psi^{n+1}(\pm1, t) = 0.
\end{cases} \tag{4.1}$$

We shall temporarily drop the index k and assume that solution of the above system takes the following normal mode form:

$$(\mathbf{u}^n, p^n, \psi^n)(x) = \widetilde{\rho}^n(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\psi})(x),$$

then (4.1) becomes

$$\begin{cases} \partial_x^2 \widehat{\mathbf{u}} - \left(k^2 + \dfrac{3\widetilde{\rho}^2 - 4\widetilde{\rho} + 1}{2\tau\widetilde{\rho}^2\nu}\right)\widehat{\mathbf{u}} = \dfrac{2\widetilde{\rho} - 1}{\widetilde{\rho}^2\nu}\nabla_k \widehat{p}, \\ \Delta_k \widehat{p} = 0, \\ \partial_x^2 \widehat{\psi} - \left(k^2 + \dfrac{3\widetilde{\rho}^2 - 4\widetilde{\rho} + 1}{2\tau\widetilde{\rho}^2\nu}\right)\widehat{\psi} = -\dfrac{(3\widetilde{\rho} - 1)(\widetilde{\rho} - 1)^3}{2\tau\widetilde{\rho}^4\nu}\widehat{p}, \\ \widehat{u}(\pm 1, t) = \widehat{v}(\pm 1, t) = \partial_x \widehat{\psi}(\pm 1, t) = 0. \end{cases} \tag{4.2}$$

As before, one can show that the necessary conditions for (4.2) to have non-trivial solutions are $k \neq 0$ and that there exists a positive real number $\widetilde{\mu}$ satisfying

$$-\widetilde{\mu}^2 = k^2 + \frac{3\widetilde{\rho}^2 - 4\widetilde{\rho} + 1}{2\tau\widetilde{\rho}^2\nu}. \tag{4.3}$$

The symmetric solutions of ODE system (4.2) are

$$\begin{cases} \widehat{u}(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k} - \dfrac{(\widetilde{\rho} - 1)^2}{\widetilde{\rho}^2}\left(\cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k}\right), \\ \widehat{v}(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k} + \dfrac{(\widetilde{\rho} - 1)^2}{\widetilde{\rho}^2}\left(\dfrac{k}{i\widetilde{\mu}}\sin\widetilde{\mu}x - \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}\right), \\ \widehat{p}(x) = \dfrac{\widetilde{\mu}^2 + k^2}{k}\nu\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}, \\ \widehat{\psi}(x) = \dfrac{(\widetilde{\rho} - 1)^2}{\widetilde{\rho}^2}\dfrac{\widetilde{\mu}^2 + k^2}{\widetilde{\mu}}\nu\left(\sin\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}\right). \end{cases} \tag{4.4}$$

The condition $\widehat{v}(\pm 1) = 0$ leads to

$$\widetilde{\mu}\tan\widetilde{\mu} + k\tanh k = -\frac{(\widetilde{\rho} - 1)^2}{\widetilde{\rho}^2}\left(\frac{k^2}{\widetilde{\mu}}\tan\widetilde{\mu} - k\tanh k\right),$$

which can be rearranged to

$$\frac{\widetilde{\rho}^2\widetilde{\mu}^2 + (\widetilde{\rho} - 1)^2 k^2}{\widetilde{\rho}^2\widetilde{\mu}^2}\widetilde{\mu}\tan\widetilde{\mu} = -\frac{2\widetilde{\rho} - 1}{\widetilde{\rho}^2}k\tanh k.$$

Dividing both sides by $\frac{\widetilde{\rho}^2\widetilde{\mu}^2}{\widetilde{\rho}^2\widetilde{\mu}^2 + (\widetilde{\rho} - 1)^2 k^2}$, we obtain

$$\widetilde{\mu}\tan\widetilde{\mu} + k\tanh k = \frac{(\widetilde{\rho} - 1)^2(\widetilde{\mu}^2 + k^2)}{\widetilde{\rho}^2\widetilde{\mu}^2 + (\widetilde{\rho} - 1)^2 k^2}k\tanh k. \tag{4.5}$$

One can show that for any given $k \neq 0$, there is a unique solution $\widetilde{\mu}_{k,I_s}$ of (4.5) in each open interval I_s $(s \neq 0)$. In view of (4.3), we have

$$\widetilde{\rho}_{k,I_s} \in \left(\frac{1}{3}, 1\right) \quad \text{and} \quad \widetilde{\mu}_{k,I_s}^2 + k^2 \leq \frac{1}{2\tau\nu}. \tag{4.6}$$

We will denote the solution in (4.4) as $\widehat{\mathbf{u}}_{k,I_s} = (\widehat{u}_{k,I_s}, \widehat{v}_{k,I_s})$, $\widehat{p}_{k,I_s}$, and $\widehat{\psi}_{k,I_s}$.

Similarly, the antisymmetric solutions of (4.2) are

$$\begin{cases} \widehat{u}(x) = \sin\widetilde{\mu}x - \sin\widetilde{\mu}\dfrac{\sinh kx}{\sinh k} - \dfrac{(\widetilde{\rho}-1)^2}{\widetilde{\rho}^2}\left(\sin\widetilde{\mu}x - \sin\widetilde{\mu}\dfrac{\sinh kx}{\sinh k}\right), \\ \widehat{v}(x) = -\dfrac{\widetilde{\mu}}{ik}\cos\widetilde{\mu}x + \dfrac{1}{i}\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k} - \dfrac{(\widetilde{\rho}-1)^2}{\widetilde{\rho}^2}\left(\dfrac{k}{i\widetilde{\mu}}\cos\widetilde{\mu}x + \dfrac{1}{i}\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k}\right), \\ \widehat{p}(x) = \dfrac{\widetilde{\mu}^2+k^2}{k}\nu\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k}, \\ \widehat{\psi}(x) = -\dfrac{(\widetilde{\rho}-1)^2}{\widetilde{\rho}^2}\dfrac{\widetilde{\mu}^2+k^2}{\widetilde{\mu}}\nu\left(\cos\widetilde{\mu}x + \sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k}\right). \end{cases} \tag{4.7}$$

We determine from the condition $\widehat{v}(\pm 1) = 0$ that

$$\widetilde{\mu}\cot\widetilde{\mu} - k\coth k = -\frac{(\widetilde{\rho}-1)^2}{\widetilde{\rho}^2}\left(\frac{k^2}{\widetilde{\mu}}\cot\widetilde{\mu} + k\coth k\right),$$

which can be rearranged to

$$\frac{\widetilde{\rho}^2\widetilde{\mu}^2 + (\widetilde{\rho}-1)^2k^2}{\widetilde{\rho}^2\widetilde{\mu}^2}\widetilde{\mu}\cot\widetilde{\mu} = \frac{2\widetilde{\rho}-1}{\widetilde{\rho}^2}k\coth k.$$

Dividing both sides by $\frac{\widetilde{\rho}^2\widetilde{\mu}^2}{\widetilde{\rho}^2\widetilde{\mu}^2+(\widetilde{\rho}-1)^2k^2}$, we obtain

$$\widetilde{\mu}\cot\widetilde{\mu} - k\coth k = -\frac{(\widetilde{\rho}-1)^2(\widetilde{\mu}^2-k^2)}{\widetilde{\rho}^2\widetilde{\mu}^2 + (\widetilde{\rho}-1)^2k^2}k\coth k. \tag{4.8}$$

For any $k \neq 0$, there is a unique solution $\widetilde{\mu}_{k,J_s}$ of (4.8) in each open interval J_s $(s \neq 0)$. In view of (4.3), we have

$$\widetilde{\rho}_{k,J_s} \in \left(\frac{1}{3}, 1\right) \quad \text{and} \quad \widetilde{\mu}^2_{k,J_s} + k^2 \leq \frac{1}{2\tau\nu}.$$

We will denote the solution in (4.7) as $\widehat{\mathbf{u}}_{k,J_s} = (\widehat{u}_{k,J_s}, \widehat{v}_{k,J_s})$, $\widehat{p}_{k,J_s}$, and $\widehat{\psi}_{k,J_s}$.

Hence, the normal mode solution of Algorithm 2 can be written as

$$\begin{aligned} (\mathbf{u}^n, p^n) = &\sum_{k,s} \alpha_{k,s}\widetilde{\rho}^n_{k,I_s}(\widehat{\mathbf{u}}_{k,I_s}, \widehat{p}_{k,I_s})(x)\exp(iky) \\ &+ \sum_{k,s} \beta_{k,s}\widetilde{\rho}^n_{k,J_s}(\widehat{\mathbf{u}}_{k,J_s}, \widehat{p}_{k,J_s})(x)\exp(iky), \end{aligned} \tag{4.9}$$

where $\alpha_{k,s}$ and $\beta_{k,s}$ are given constants in the initial value expansion (2.21) (see Remark 3.2).

Remark 4.1. *We note that, contrary to Algorithm 1 (see Remark 3.1), there is no spurious boundary layer term in the approximation solutions $(\widehat{u}, \widehat{v}, \widehat{p})$ in (4.4) and (4.7).*

Lemma 4.2. *We have the following error bounds:*

$$\begin{aligned} |\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}| \leq &\frac{8\tau^2\nu^2}{(2|s|-1)\pi}\left(\frac{2|s|+1}{2|s|-1}\right)^8 \frac{(\mu^2_{k,I_s}+k^2)^3|k|}{\mu^2_{k,I_s}}, \\ |\mu_{k,J_s} - \widetilde{\mu}_{k,J_s}| \leq &\frac{8\tau^2\nu^2}{|s|\pi}\left(\frac{|s|+1}{|s|}\right)^8 \frac{(\mu^2_{k,J_s}+k^2)^3|k|}{\mu^2_{k,J_s}}, \end{aligned} \tag{4.10}$$

$$\begin{aligned}\left|\mu_{k,I_s}^2-\widetilde{\mu}_{k,I_s}^2\right| &\le 8\tau^2\nu^2\left(\frac{2|s|+1}{2|s|-1}\right)^9\frac{(\mu_{k,I_s}^2+k^2)^3|k|}{\mu_{k,I_s}^2},\\ \left|\mu_{k,J_s}^2-\widetilde{\mu}_{k,J_s}^2\right| &\le 8\tau^2\nu^2\left(\frac{|s|+1}{|s|}\right)^9\frac{(\mu_{k,I_s}^2+k^2)^3|k|}{\mu_{k,I_s}^2},\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}|\rho_{k,I_s}-\widetilde{\rho}_{k,I_s}| &\le C\tau^3\nu^3(\mu_{k,I_s}^2+k^2)^3|k|,\\ |\rho_{k,J_s}-\widetilde{\rho}_{k,J_s}| &\le C\tau^3\nu^3(\mu_{k,J_s}^2+k^2)^3|k|.\end{aligned} \tag{4.12}$$

Proof. We shall only carry out the proof on the interval $I_s=\left(\frac{(2s-1)\pi}{2},\frac{(2s+1)\pi}{2}\right)$, since the estimate on J_s defined in (2.18) can be derived by the same argument. Subtracting (4.5) from (2.15), we obtain:

$$\mu_{k,I_s}\tan\mu_{k,I_s}-\widetilde{\mu}_{k,I_s}\tan\widetilde{\mu}_{k,I_s}=-\frac{(\widetilde{\rho}_{k,I_s}-1)^2(\widetilde{\mu}_{k,I_s}^2+k^2)}{\widetilde{\rho}_{k,I_s}^2\widetilde{\mu}_{k,I_s}^2+(\widetilde{\rho}_{k,I_s}-1)^2k^2}k\tanh k. \tag{4.13}$$

Note that (3.17) is also valid here. Since $|\tanh k|\le 1$, the right hand side of (4.13) can be bounded by

$$\begin{aligned}|\mu_{k,I_s}\tan\mu_{k,I_s}-\widetilde{\mu}_{k,I_s}\tan\widetilde{\mu}_{k,I_s}| &\le \frac{(\widetilde{\rho}_{k,I_s}-1)^2}{\widetilde{\rho}_{k,I_s}^2}\frac{(\widetilde{\mu}_{k,I_s}^2+k^2)|k|}{\widetilde{\mu}_{k,I_s}^2}\\ &\le 4\tau^2\nu^2\frac{(\widetilde{\mu}_{k,I_s}^2+k^2)^3|k|}{\widetilde{\mu}_{k,I_s}^2}.\end{aligned} \tag{4.14}$$

Let $f(x)=x\tan(x)$. It can be easily verified that $|f'(x)|=\left|\frac{x+\sin(x)\cos(x)}{\cos^2(x)}\right|\ge|x|$ on I_s. Since μ_{k,I_s} and $\widetilde{\mu}_{k,I_s}\in I_s$, the mean value theorem and (4.14) yield

$$\begin{aligned}|\mu_{k,I_s}-\widetilde{\mu}_{k,I_s}| &\le \frac{2}{(2|s|-1)\pi}|\mu_{k,I_s}\tan\mu_{k,I_s}-\widetilde{\mu}_{k,I_s}\tan\widetilde{\mu}_{k,I_s}|\\ &\le \frac{8\tau^2\nu^2}{(2|s|-1)\pi}\frac{(\widetilde{\mu}_{k,I_s}^2+k^2)^3|k|}{\widetilde{\mu}_{k,I_s}^2}.\end{aligned} \tag{4.15}$$

Note $s\neq 0$. Since $|\mu_{k,I_s}+\widetilde{\mu}_{k,I_s}|\le(2|s|+1)\pi$, (4.15) leads us

$$\begin{aligned}\left|\mu_{k,I_s}^2-\widetilde{\mu}_{k,I_s}^2\right| &= |\mu_{k,I_s}+\widetilde{\mu}_{k,I_s}||\mu_{k,I_s}-\widetilde{\mu}_{k,I_s}|\\ &\le \frac{8\tau^2\nu^2(2|s|+1)}{2|s|-1}\frac{(\widetilde{\mu}_{k,I_s}^2+k^2)^3|k|}{\widetilde{\mu}_{k,I_s}^2},\end{aligned} \tag{4.16}$$

In light of $|\widetilde{\mu}_{k,I_s}|\le\frac{2|s|+1}{2|s|-1}|\mu_{k,I_s}|$, the above two estimates imply (4.10) and (4.11).

Similarly as in the proof of Lemma 3.3, $|\rho_{k,I_s}-\widetilde{\rho}_{k,I_s}|$ can be split as in (3.21) and the estimate (3.22) is still valid for A_1 and A_2, namely:

$$A_1\le C\tau^3\nu^3(\widetilde{\mu}_{k,I_s}^2+k^2)^3 \quad\text{and}\quad A_2\le C\tau^3\nu^3(\mu_{k,I_s}^2+k^2)^3.$$

Subtracting (4.3) from (2.13) leads to $A_3=\nu\tau\left|\mu_{k,I_s}^2-\widetilde{\mu}_{k,I_s}^2\right|$, and (4.16) leads to

$$A_3\le\frac{8\tau^3\nu^3(2|s|+1)}{2|s|-1}\frac{(\widetilde{\mu}_{k,I_s}^2+k^2)^3|k|}{\widetilde{\mu}_{k,I_s}^2}.$$

Inserting above A_1, A_2, and A_3 into (3.21) yields (4.12) which concludes the proof. □

Theorem 4.3. *Let* $(\mathbf{U}^n, P^n)$ *and* $(\mathbf{u}^n, \widetilde{\mathbf{u}}^n, p^n)$ *be respectively the solutions of (2.8) and of Algorithm 2, and suppose Assumption 1 holds with* $m = 8$*. Then, we have*

$$\begin{aligned} \|\mathbf{U}^n - \mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)} &\le CM_8\nu^3\tau^2, \\ \|P^n - p^n\|_{L^\infty(\Omega)} &\le CM_8\nu^3\tau^2, \quad \forall 1 \le n \le N, \\ \|\nabla \cdot \widetilde{\mathbf{u}}^n\|_{L^\infty(\Omega)} &\le CM_6\nu^2\tau^2. \end{aligned} \tag{4.17}$$

Proof. The proof is very similar to that of Theorem 3.4. We use the same error functions in (3.24), namely,

$$\begin{aligned} \mathbf{E}_{k,I_s} &:= \overline{\mathbf{u}}_{k,I_s} - \widetilde{\mathbf{u}}_{k,I_s}, \quad & \mathbf{E}_{k,J_s} &:= \overline{\mathbf{u}}_{k,J_s} - \widetilde{\mathbf{u}}_{k,J_s}, \\ e_{k,I_s} &:= \overline{p}_{k,I_s} - \widetilde{p}_{k,I_s}, \quad & e_{k,J_s} &:= \overline{p}_{k,J_s} - \widetilde{p}_{k,J_s}. \end{aligned}$$

We first compare (2.14) and (4.4) on a fixed interval I_s. Using (4.10)-(4.11), the velocity error can be bounded by

$$\begin{aligned} \|\mathbf{E}_{k,I_s}\|_{\mathbf{L}^\infty(\Omega)} \le& C\left(1 + \left|\frac{\mu_{k,I_s}}{k}\right|\right)|\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}| + C\left(1 + \left|\frac{k}{\widetilde{\mu}_{k,I_s}}\right|\right)\frac{(\widetilde{\rho}_{k,I_s} - 1)^2}{\widetilde{\rho}_{k,I_s}^2} \\ \le& C\tau^2\nu^2\frac{(\mu_{k,I_s}^2 + k^2)^4}{\mu_{k,I_s}^2}. \end{aligned}$$

Similarly, the pressure error can be bounded by

$$\begin{aligned} \|e_{k,I_s}\|_{L^\infty(\Omega)} \le& C\nu\left(\left|\frac{\mu_{k,I_s}^2 + k^2}{k}\right||\mu_{k,I_s} - \widetilde{\mu}_{k,I_s}| + \left|\frac{\mu_{k,I_s}^2 - \widetilde{\mu}_{k,I_s}^2}{k}\right|\right) \\ \le& C\tau^2\nu^3\frac{(\mu_{k,I_s}^2 + k^2)^4}{\mu_{k,I_s}^2}. \end{aligned}$$

Since ρ_{k,J_s} and $\widetilde{\rho}_{k,I_s} \in \left(\frac{1}{3}, 1\right)$ which comes from (2.19) and (4.6), the mean value theorem leads to $\left|\rho_{k,I_s}^n - \widetilde{\rho}_{k,I_s}^n\right| \le n|\rho_{k,I_s} - \widetilde{\rho}_{k,I_s}|$. So (4.12) give us

$$\left|\rho_{k,I_s}^n - \widetilde{\rho}_{k,I_s}^n\right| \le n|\rho_{k,I_s} - \widetilde{\rho}_{k,I_s}| \le C\tau^2\nu^3(\mu_{k,I_s}^2 + k^2)^3|k|.$$

By the same argument, the above estimations hold for $|\mathbf{E}_{k,J_s}|$, $\left|\rho_{k,J_s}^n - \widetilde{\rho}_{k,J_s}^n\right|$, and $|e_{k,J_s}|$.

We now compute the error of Algorithm 2 by comparing (4.9) with (2.20) and using above three inequalities. Since we have $\widetilde{\rho}_{k,I_s} \in \left(\frac{1}{3}, 1\right)$ and $\|\overline{\mathbf{u}}_{k,I_s}\|_{\mathbf{L}^\infty(\Omega)} \le C\left(1 + \left|\frac{\mu_{k,I_s}}{k}\right|\right)$,

$$\begin{aligned} \|\mathbf{U}^n - \mathbf{u}^n\|_{\mathbf{L}^\infty(\Omega)} &\le \sum_{k,s}\left\|\alpha_{k,s}\exp(iky)\Big((\rho_{k,I_s}^n - \widetilde{\rho}_{k,I_s}^n)\overline{\mathbf{u}}_{k,I_s} + \widetilde{\rho}_{k,I_s}^n\mathbf{E}_{k,I_s}\Big)\right\|_{\mathbf{L}^\infty(\Omega)} \\ &\quad + \sum_{k,s}\left\|\beta_{k,s}\exp(iky)\Big((\rho_{k,J_s}^n - \widetilde{\rho}_{k,J_s}^n)\overline{\mathbf{u}}_{k,J_s} + \widetilde{\rho}_{k,J_s}^n\mathbf{E}_{k,J_s}\Big)\right\|_{\mathbf{L}^\infty(\Omega)} \\ &\le C\tau^2\nu^3\sum_{k,s}\left(|\alpha_{k,s}|(\mu_{k,I_s}^2 + k^2)^4 + |\beta_{k,s}|(\mu_{k,J_s}^2 + k^2)^4\right) \\ &\le CM_8\nu^3\tau^2. \end{aligned}$$

Using $\left|\overline{p}_{k,I_s}\right| \le C\frac{\mu_{k,I_s}^2+k^2}{|k|}\nu$, we can obtain

$$\begin{aligned}\|P^n - p^n\|_{L^\infty(\Omega)} \le& \sum_{k,s}\left\|\alpha_{k,s}\exp(iky)\Big((\rho_{k,I_s}^n - \widetilde{\rho}_{k,I_s}^n)\overline{p}_{k,I_s} + \widetilde{\rho}_{k,I_s}^n e_{k,I_s}\Big)\right\|_{L^\infty(\Omega)}\\ &+\sum_{k,s}\left\|\beta_{k,s}\exp(iky)\Big((\rho_{k,J_s}^n - \widetilde{\rho}_{k,J_s}^n)\overline{p}_{k,J_s} + \widetilde{\rho}_{k,J_s}^n e_{k,J_s}\Big)\right\|_{L^\infty(\Omega)}\\ \le& C\tau^2\nu^3\sum_{k,s}\big(|\alpha_{k,s}|(\mu_{k,I_s}^2+k^2)^4 + |\beta_{k,s}|(\mu_{k,J_s}^2+k^2)^4\big)\\ \le& CM_8\nu^3\tau^2.\end{aligned}$$

It remains to prove the last inequality in (4.17). We compute $\nabla_k\cdot\widehat{\widetilde{\mathbf{u}}}_{k,I_s}$ from (4.4) and then use (4.3) and (3.17) to get

$$\begin{aligned}\left\|\nabla_k\cdot\widehat{\widetilde{\mathbf{u}}}_{k,I_s}\right\|_{L^\infty(\Omega)} \le& \left|\frac{(\widetilde{\rho}_{k,I_s}-1)^2}{\widetilde{\rho}_{k,I_s}^2}\frac{\widetilde{\mu}_{k,I_s}^2+k^2}{\widetilde{\mu}_{k,I_s}}\right|\\ \le& 4\tau^2\nu^2\left(\frac{2|s|+1}{2|s|-1}\right)^6(\mu_{k,I_s}^2+k^2)^3.\end{aligned}$$

Similarly, we can derive

$$\left\|\nabla_k\cdot\widehat{\widetilde{\mathbf{u}}}_{k,J_s}\right\|_{L^\infty(\Omega)} \le 4\tau^2\nu^2\left(\frac{|s|+1}{|s|}\right)^6(\mu_{k,J_s}^2+k^2)^3.$$

Summing up the above two estimates for k and s yields (4.17) which completes the proof. □

5. **Miscellaneous issues.**

5.1. **Standard pressure-correction scheme.** We observed that the normal mode analysis reveals explicit error structures of Algorithms 1 and 2. It is also interesting to take a look at a standard pressure-correction scheme, the only difference of which with the rotational pressure-correction scheme (Algorithm 1) is that in the third step, (2.3) is replaced by

$$\begin{aligned}\mathbf{u}^{n+1} &= \widetilde{\mathbf{u}}^{n+1} - \frac{2\tau}{3}\nabla\psi^{n+1},\\ p^{n+1} &= \psi^{n+1} + p^n.\end{aligned}\tag{5.1}$$

Similarly as for Algorithm 1, taking the Fourier transform in the y variable in the Algorithm (2.1)-(2.2)-(5.1) yields a family of one-dimensional problems indexed by $k \in \mathbf{Z}$;

$$\begin{cases}\dfrac{3\widetilde{\mathbf{u}}^{n+1}-4\widetilde{\mathbf{u}}^n+\widetilde{\mathbf{u}}^{n-1}}{2\tau} - \nu\Delta_k\widetilde{\mathbf{u}}^{n+1} = -\dfrac{1}{3}\nabla_k(7p^n - 5p^{n-1} + p^{n-2}),\\ -\dfrac{2\tau}{3}\nu\Delta_k\Delta_k(p^{n+1}-p^n) + \Delta_k p^{n+1} = 0,\\ \nabla_k\cdot\mathbf{u}^{n+1} = 0,\\ \mathbf{u}^{n+1} = \widetilde{\mathbf{u}}^{n+1} - \dfrac{2\tau}{3}\nabla_k(p^{n+1}-p^n),\\ \widetilde{u}^{n+1}(\pm1,t) = \widetilde{v}^{n+1}(\pm1,t) = u^{n+1}(\pm1,t) = \partial_x p^{n+1}(\pm1,t) = 0.\end{cases}$$

Assuming the normal mode solution takes the form

$$(\mathbf{u}^n, \widetilde{\mathbf{u}}^n, p^n)(x) = \widetilde{\rho}^n(\widehat{\mathbf{u}}, \widehat{\widetilde{\mathbf{u}}}, \widehat{p})(x)$$

we find that

$$\begin{cases} \partial_x^2 \widehat{\widetilde{\mathbf{u}}} - \left(k^2 + \dfrac{3\widetilde{\rho}^2 - 4\widetilde{\rho} + 1}{2\tau\widetilde{\rho}^2\nu}\right)\widehat{\widetilde{\mathbf{u}}} = \dfrac{7\widetilde{\rho}^2 - 5\widetilde{\rho} + 1}{3\widetilde{\rho}^3\nu}\nabla_k\widehat{p}, \\ -\dfrac{2\tau}{3}\dfrac{\widetilde{\rho}-1}{\widetilde{\rho}}\nu\Delta_k\Delta_k\widehat{p} + \Delta_k\widehat{p} = 0, \\ \nabla_k \cdot \widehat{\mathbf{u}} = 0, \\ \widehat{\mathbf{u}} = \widehat{\widetilde{\mathbf{u}}} - \dfrac{2\tau}{3}\dfrac{\widetilde{\rho}-1}{\widetilde{\rho}}\nabla_k\widehat{p}, \\ \widehat{\widetilde{u}}(\pm1,t) = \widehat{\widetilde{v}}(\pm1,t) = \widehat{u}(\pm1,t) = \partial_x\widehat{p}(\pm1,t) = 0. \end{cases} \tag{5.2}$$

Let us denote

$$-\widetilde{\mu}^2 = k^2 + \frac{3\widetilde{\rho}^2 - 4\widetilde{\rho} + 1}{2\tau\widetilde{\rho}^2\nu},$$

and

$$-\lambda^2 = k^2 + \frac{3}{2\tau\nu}\frac{\widetilde{\rho}}{\widetilde{\rho}-1}.$$

The symmetric solutions of the ODE system (5.2) are

$$\begin{cases} \widehat{u}(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k}, \\ \widehat{v}(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}, \\ \widehat{\widetilde{u}}(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k} + \dfrac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}\cos\widetilde{\mu}\left(\dfrac{\cosh kx}{\cosh k} - \dfrac{\cos\lambda x}{\cos\lambda}\right), \\ \widehat{\widetilde{v}}(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k} \\ \qquad\qquad - \dfrac{1}{i}\dfrac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}\cos\widetilde{\mu}\left(\dfrac{\sinh kx}{\cosh k} - \dfrac{k}{\lambda}\dfrac{\sin\lambda x}{\cos\lambda}\right), \\ \widehat{p}(x) = -\dfrac{\widetilde{\mu}^2 + k^2}{k}\nu\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k} + \dfrac{\widetilde{\mu}^2 + k^2}{\lambda}\nu\cos\widetilde{\mu}\dfrac{\sin\lambda x}{\cos\lambda}. \end{cases}$$

Since $\widehat{\widetilde{v}}(x)$ has 0 on boundary $x = \pm1$, we have

$$\widetilde{\mu}\tan\widetilde{\mu} + k\tanh k = k\frac{(3\widetilde{\rho}-1)(\widetilde{\rho}-1)^2}{3\widetilde{\rho}^3}\left(\tanh k - \frac{k}{\lambda}\tan\lambda\right).$$

We note that the normal mode solution for a scheme which is essentially the same as Algorithm (2.1)-(2.2)-(5.1) but with BDF2 replaced by Crank-Nicholson was obtained in [3].

Remark 5.1. *It is clear that $\widetilde{\rho} - 1 = O(\nu\tau)$ (as for other schemes) so we have $\lambda = O((\nu\tau)^{-1})$. Hence, as pointed out in* [3], *$(\widehat{\widetilde{u}}, \widehat{\widetilde{v}}, \widetilde{p})$ contain a spurious high oscillatory part with magnitude of order $O((\nu\tau)^2)$ for $(\widehat{\widetilde{u}}, \widehat{\widetilde{v}})$ and of order $O(\nu\tau)$ for $\widetilde{p}$.*

5.2. **The Convergence of The Consistent Splitting Method in $\mathbf{H}^1(\Omega)$ by energy method.** We note that the main difference between the error behaviors of the Algorithms 1 and 2 is that there is no spurious terms in the solutions of Algorithm 2 while there is a spurious boundary layer term of width $O((\nu\tau)^{\frac{1}{2}})$ in the solutions of Algorithm 1 which resulted in a loss of $\frac{1}{2}$-order in the divergence of $\widetilde{\mathbf{u}}^n$, and consequently, $\widetilde{\mathbf{u}}^n$ in Algorithm 1 can be at best $\frac{3}{2}$-order accurate.

As shown below, the error estimates in Theorem 4.3 are sufficient to establish second-order velocity error estimate in $\mathbf{H}^1(\Omega)$ via energy estimate without any additional regularity assumptions.

We denote by $\|\cdot\|$ the norm in $\mathbf{L}^2(\Omega)$, and by $\langle \cdot\,,\,\cdot\rangle$ the inner product in $\mathbf{L}^2(\Omega)$. Let $(\mathbf{u}(t^n), p(t^n))$ be the exact solution of (1.1) at time step t^n. If $(\mathbf{u}^n, p^n)$ is the solution of the Algorithm 2, then we denote the corresponding error by

$$\mathbf{E}^n := \mathbf{u}(t^n) - \mathbf{u}^n \quad \text{and} \quad e^n := p(t^n) - p^n.$$

¿From Theorem 4.3, we have

$$\|\mathbf{E}^n\|_{\mathbf{L}^\infty(\Omega)} + \|e^n\|_{L^\infty(\Omega)} \le C\tau^2. \tag{5.3}$$

By virtue of Taylor expansion of (1.1), we have

$$\frac{3\mathbf{u}(t^{n+1}) - 4\mathbf{u}(t^n) + \mathbf{u}(t^{n-1})}{2\tau} + \nabla(2p(t^n) - p(t^{n-1})) - \Delta\mathbf{u}(t^{n+1}) = \mathbf{R}^{n+1} + \mathbf{g}^{n+1}, \tag{5.4}$$

where $\mathbf{R}^{n+1} := \frac{1}{\tau}\int_{t^{n-1}}^{t^{n+1}} (t^{n+1}-t)^2\mathbf{u}_{ttt}(t)dt + \int_{t^{n-1}}^{t^{n+1}} (t^{n+1}-t)\nabla p_{tt}(t)dt$ is the *truncation* error. Subtracting (2.4) from (5.4) yields

$$\frac{3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}}{2\tau} - \Delta\mathbf{E}^{n+1} + \nabla(2e^n - e^{n-1}) = \mathbf{R}^{n+1}. \tag{5.5}$$

We take the inner product of (5.5) with $4\tau\mathbf{E}^{n+1}$ to get

$$\begin{aligned} &2\left\langle 3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}\,,\,\mathbf{E}^{n+1}\right\rangle + 4\tau\left\|\nabla\mathbf{E}^{n+1}\right\|^2 \\ =&4\tau\left\langle \mathbf{R}^{n+1} - \nabla(2e^n - e^{n-1})\,,\,\mathbf{E}^{n+1}\right\rangle \\ \le&\tau\left\|2e^n - e^{n-1}\right\|^2 + C\tau^4\int_{t^{n-1}}^{t^{n+1}}\left(\|\mathbf{u}_{ttt}(t)\|^2 + \|p_{tt}(t)\|^2\right)dt + 2\tau\left\|\nabla\mathbf{E}^{n+1}\right\|^2. \end{aligned}$$

Using the following algebraic identity

$$\begin{aligned} 2\left\langle 3a^{n+1} - 4a^n + a^{n-1}\,,\,a^{n+1}\right\rangle =&\left\|a^{n+1}\right\|^2 - \|a^n\|^2 + \left\|a^{n+1} - 2a^n + a^{n-1}\right\|^2 \\ &+ \left\|2a^{n+1} - a^n\right\|^2 - \left\|2a^n - a^{n-1}\right\|^2, \end{aligned}$$

and summing up n from 1 to m implies

$$\begin{aligned} &\left\|\mathbf{E}^{m+1}\right\|^2 + \left\|2\mathbf{E}^{m+1} - \mathbf{E}^m\right\|^2 + \sum_{n=1}^{m}\left\|\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\right\|^2 + 2\tau\sum_{n=1}^{m}\left\|\nabla\mathbf{E}^{n+1}\right\|^2 \\ \le&\left\|\mathbf{E}^1\right\|^2 + \left\|2\mathbf{E}^1 - \mathbf{E}^0\right\|^2 + \tau\sum_{n=1}^{m}\left\|2e^n - e^{n-1}\right\|^2 \\ &+ C\tau^4\int_0^{t^{m+1}}\left(\|\mathbf{u}_{ttt}(t)\|^2 + \|p_{tt}(t)\|^2\right)dt. \end{aligned}$$

In conjunction with (5.3), the discrete Gronwall lemma yields

$$\left\|\mathbf{E}^{m+1}\right\|^2+\left\|2\mathbf{E}^{m+1}-\mathbf{E}^m\right\|^2+\sum_{n=1}^m\left\|\mathbf{E}^{n+1}-2\mathbf{E}^n+\mathbf{E}^{n-1}\right\|^2+2\tau\sum_{n=1}^m\left\|\nabla\mathbf{E}^{n+1}\right\|^2$$
$$\leq C\tau^4,$$

which implies that under the same assumptions of Theorem 4.3, we have additionally

$$\tau\sum_{n=0}^m\left\|\mathbf{u}(t^{n+1})-\mathbf{u}^{n+1}\right\|^2_{\mathbf{H}^1(\Omega)}\leq C\tau^4,\quad \forall 1\leq m\leq N.$$

5.3. **The Gauge Method and The Gauge-Uzawa Method.** The gauge formulation introduced by E and Liu in [4] consists of rewriting (1.1) in terms of auxiliary vector field $\mathbf{a}$ and the gauge variable ϕ, which satisfy $\mathbf{u}=\mathbf{a}+\nabla\phi$. Upon replacing this relation into the momentum equation in (1.1), we get

$$\mathbf{a}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\left(\phi_t-\nu\Delta\phi\right)+\nabla p-\nu\Delta\mathbf{a}=\mathbf{f}.$$

Setting

$$p=-\phi_t+\nu\Delta\phi,$$

we obtain the gauge formulation of (1.1):

$$\begin{cases}\mathbf{a}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{a}=\mathbf{f}, & \text{in } \Omega,\\ -\Delta\phi=\nabla\cdot\mathbf{a}, & \text{in } \Omega,\\ \mathbf{u}=\mathbf{a}+\nabla\phi, & \text{in } \Omega,\\ p=-\phi_t+\nu\Delta\phi, & \text{in } \Omega.\end{cases}$$

To enforce the boundary condition $\mathbf{u}|_\Gamma=\mathbf{0}$, a set of suitable boundary conditions are

$$\partial_{\boldsymbol{\nu}}\phi|_\Gamma=0,\qquad \mathbf{a}\cdot\boldsymbol{\nu}|_\Gamma=0,\qquad \mathbf{a}\cdot\boldsymbol{\tau}|_\Gamma=-\partial_{\boldsymbol{\tau}}\phi,$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are the unit vectors in the normal and tangential directions, respectively. We now introduce the BDF2 time discretized gauge method [4, 10, 13, 19].

Algorithm 3 (The Gauge Method)**.** *Set initial values using a first-order gauge method with $\phi^0=0$ and repeat for $2\leq n\leq N=[\frac{T}{\tau}-1]$.*

Step 1: *Find $\mathbf{a}^{n+1}$ as the solution of*

$$\begin{cases}\dfrac{3\mathbf{a}^{n+1}-4\mathbf{a}^n+\mathbf{a}^{n-1}}{2\tau}+-\nu\Delta\mathbf{a}^{n+1}=\mathbf{g}^{n+1},\\ \mathbf{a}^{n+1}\cdot\boldsymbol{\nu}|_\Gamma=0,\qquad \mathbf{a}^{n+1}\cdot\boldsymbol{\tau}|_\Gamma=-2\partial_{\boldsymbol{\tau}}\phi^n+\partial_{\boldsymbol{\tau}}\phi^{n-1}.\end{cases}\tag{5.6}$$

Step 2: *Find ϕ^{n+1} as the solution of*

$$\begin{cases}-\Delta\phi^{n+1}=\nabla\cdot\mathbf{a}^{n+1},\\ \partial_{\boldsymbol{\nu}}\phi^{n+1}|_\Gamma=0.\end{cases}$$

Step 3: *Update $\mathbf{u}^{n+1}$ by*

$$\mathbf{u}^{n+1}=\mathbf{a}^{n+1}+\nabla\phi^{n+1}.\tag{5.7}$$

One may compute the pressure p^{n+1} whenever necessary as

$$p^{n+1} = -\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\tau} + \nu\Delta\phi^{n+1}. \tag{5.8}$$

Even though this formulation is consistent with (1.1), the boundary condition $\mathbf{a}^{n+1} \cdot \tau = -2\partial_{\boldsymbol{\tau}}\phi^n + \partial_{\boldsymbol{\tau}}\phi^{n-1}$ is non-variational and thus difficult to implement within a finite element context, especially in three dimensional case. To avoid the difficult associated with the boundary differentiation, the Gauge-Uzawa method was designed by Nochetto and Pyo in [9, 13] by introducing an artificial velocity function $\widetilde{\mathbf{u}}^{n+1}$ which vanishes on boundary:

$$\widetilde{\mathbf{u}}^{n+1} = \mathbf{a}^{n+1} + 2\nabla\phi^n - \nabla\phi^{n-1}. \tag{5.9}$$

If we define $\rho^{n+1} = \phi^{n+1} - \phi^n$, combining (5.7) and (5.9) yields

$$\widetilde{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \nabla(\rho^{n+1} - \rho^n). \tag{5.10}$$

We now insert (5.9) into (5.6) to obtain

$$\frac{3\widetilde{\mathbf{u}}^{n+1} - 4\widetilde{\mathbf{u}}^n + \widetilde{\mathbf{u}}^{n-1}}{2\tau} + \nabla(2p^n - p^{n-1}) - \nu\Delta\widetilde{\mathbf{u}}^{n+1} = \mathbf{g}^{n+1}.$$

Since $\nabla\cdot\, \mathbf{u}^{n+1} = 0$, by taking the divergence of (5.10), we get

$$-\Delta\rho^{n+1} = -\Delta\rho^n + \nabla\cdot\widetilde{\mathbf{u}}^{n+1}.$$

In order to remove the variable ϕ^{n+1}, we take the difference between two consecutive steps of (5.8) to get

$$p^{n+1} - p^n = -\frac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{2\tau} + \nu\Delta\rho^{n+1}.$$

Hence, a scheme equivalent to Algorithm 2 is the following:

Algorithm 4 (The Gauge-Uzawa Method). *Set initial values using a first-order Gauge-Uzawa method with $\rho^0 = 0$ and repeat for $2 \le n \le N = [\frac{T}{\tau} - 1]$.*

Step 1: *Find $\widetilde{\mathbf{u}}^{n+1}$ as the solution of*

$$\begin{cases} \dfrac{3\widetilde{\mathbf{u}}^{n+1} - 4\widetilde{\mathbf{u}}^n + \widetilde{\mathbf{u}}^{n-1}}{2\tau} + \nabla(2p^n - p^{n-1}) - \nu\Delta\widetilde{\mathbf{u}}^{n+1} = \mathbf{g}^{n+1}, \\ \widetilde{\mathbf{u}}^{n+1}|_\Gamma = \mathbf{0}. \end{cases}$$

Step 2: *Find ρ^{n+1} as the solution of*

$$\begin{cases} -\Delta\rho^{n+1} = -\Delta\rho^n + \nabla\cdot\widetilde{\mathbf{u}}^{n+1}, \\ \partial_{\boldsymbol{\nu}}\rho^{n+1}|_\Gamma = 0. \end{cases}$$

Step 3: *Update $\mathbf{u}^{n+1}$ and p^{n+1} by*

$$\begin{aligned} \mathbf{u}^{n+1} &= \widetilde{\mathbf{u}}^{n+1} + \nabla(\rho^{n+1} - \rho^n) \\ p^{n+1} &= p^n - \frac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{2\tau} + \nu\Delta\rho^{n+1}. \end{aligned} \tag{5.11}$$

Note that it is not necessary to compute $\mathbf{u}^{n+1}$ in each time iteration for this linearized problem. Since $\nabla\cdot\,\widetilde{\mathbf{u}}^{n+1} \neq 0$ while $\nabla\cdot\,\mathbf{u}^{n+1} = 0$, it is advisable to compute $\mathbf{u}^{n+1}$ for the full Navier-Stokes equations to treat the convection term.

Finally, we show that Algorithms 2 and 4 are equivalent.

Remark 5.2 (Equivalence of Algorithms 2 and 4). Subtracting two consecutive pressure equations in (5.11) and applying the Laplace operator, we obtain

$$\Delta(p^{n+1} - p^n) = \Delta(p^n - p^{n-1}) + \nabla \cdot \frac{3\widetilde{\mathbf{u}}^{n+1} - 4\widetilde{\mathbf{u}}^n + \widetilde{\mathbf{u}}^{n-1}}{2\tau} - \nu\Delta\nabla\cdot\widetilde{\mathbf{u}}^{n+1},$$

which is exactly (2.5) and (2.6) in Algorithm 2. So we conclude that Algorithms 2 and 4 are equivalent, and consequently, Algorithms 3 and 4 are also fully second-order accurate.

6. Concluding remarks.

We presented in this paper a detailed and rigorous normal mode analysis for some second-order projection-type schemes. In particular, it allowed us to establish, for the first time but only in the special domain considered here, the stability and error analysis for the second-order consistent splitting scheme.

The main advantage of the normal mode analysis is that it reveals the particular error structures, such as spurious boundary layers or spurious highly oscillatory terms and their explicit dependence on the dynamic viscosity ν. It is important to note that the projection errors decrease as ν decreases. Hence, the projection-type schemes are particularly suitable for high Reynolds number flows.

The normal mode analysis for a single normal mode of a linear problem is relatively easy compared with the energy method. However, a rigorous analysis which takes into accounts all normal modes (which are not mutually orthogonal) becomes tediously complex even for linear problems. It also requires much higher regularity on the solution than the energy method does. Although we have only considered linear problems in this paper, but by combining the techniques used here and in [3] (where the orthogonality of normal modes are implicitly assumed but in general does not hold) for nonlinear problems, it can be shown that the error estimates established in this paper would hold for the fully nonlinear Navier-Stokes equations.

Finally, it is worthy repeating, as already demonstrated in [7], that the error estimate from a normal mode analysis on the special domain may not be valid in more general domains.

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URL: `http://www.math.purdue.edu/~pjh`
E-mail address: `pjh@math.purdue.edu`
URL: `http://www.math.purdue.edu/~shen/`
E-mail address: `shen@math.purdue.edu`